Probabilistic failure path approach on structural system-reliability-based design optimization of fatigue-induced failure

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ABSTRACT: Reliability analysis with unknown system event definition such as sequence of member failures requires a failure-path approach to determine the component events that will induce overall system failure. In particular, redundant structures prone to fatigue-induced sequential failure needs a system-level analysis employing a failure-path approach to account for stress redistribution. Thus, there can be a high computational cost of incorporating such probabilistic constraints into a System-Reliability-based Design Optimization (SRBDO) framework against fatigue limit states. A structural system reliability analysis procedure, namely, the Branch-and-Bound method employing system reliability Bounds (termed the B^3 method) is integrated into an optimization algorithm. A gradient-based optimizer is used to find the optimum, and a modified Sequential Compounding Method (SCM) together with Chun-Song-Paulino (CSP) sensitivity analysis method is used to calculate the gradient with respect to the design variables. Additionally, a new bounding rule of the B^3 method is introduced to increase efficiency. To demonstrate the applicability, it is applied to a hypothetical structure of multilayer Daniel's system. As a result, the system failure probability of the optimal design obtained from the proposed method is found to be lower than the target probability and is verified through Monte Carlo simulation. The calculated gradient of the system failure probability accurately leads to the optimal design. It is confirmed that the proposed method can allocate limited materials throughout the structure. Moreover, the system reliability analysis of fatigue-induced sequential failure is explicitly incorporated into the design optimization, thereby resulting in cost-efficient and safer structures.

1. INTRODUCTION

The advent of advanced computational capabilities in modelling and designing of structures spur the growth of more complicated structural systems. The increase in complexity means the reliability of structures is defined by multiple components (i.e., structural members, failure mechanisms). Structural system reliability (SSR) is the probability that the structure still performs its intended function even though several components have already failed (Song et al. 2021). The failure-path approach is used to determine the component events that will induce overall system failure which is usually unknown a priori. The computational cost of performing SSR analysis and difficulty of identifying failure modes prohibits its direct integration into a design optimization framework.

Fatigue failures are common to structures that are exposed to cyclic loadings (i.e., wind loads, vehicle loads, wave loads etc.). In a structure with redundancies when one component fails, stress is redistributed to other members that can accelerate the fatigue failure or result to failure of the entire structure. Redundant structures prone to fatigue-induced sequential failure need a system-level analysis employing a failure-path approach to account for stress redistribution.

Reliability-based design optimization (RBDO) explicitly considers uncertainty in the design optimization process. Several studies have already been conducted to consider a single component fatigue limit state in an RBDO framework (Ibrahim et al. 2014, Hu et al. 2016, Yaich et al. 2018). However, few studies have been carried out to examine system-RBDO (SRBDO) considering fatigue limit states. The newly developed Branch-and-Bound method employing system reliability Bounds (termed the B^3 method) can calculate the system failure probability and search for the critical fatigue-induced failure sequences. In this study, a gradient-based SRBDO is proposed to find the optimal design of a structure considering system reliability with unknown overall system failure event definition. A modified Sequential Compounding Method (SCM) together with Chun-Song-Paulino (CSP) sensitivity analysis method is used to calculate the gradient with respect to the design variables. To further improve computational efficiency, a new bounding rule is introduced to the B^3 method. The applicability of the proposed method is tested into a hypothetical structure of multilayer Daniel's system.

2. PROPOSED METHOD

2.1. Failure path approach: B^3 method

Depending on the geometry and connectivity of a redundant structure, the failure of one structural member will not greatly affect the performance of the structure. If the structure has not collapsed (i.e., maintains stability and equilibrium) and no excessively large displacements, the structure still can perform its intended function. A probabilistic global search algorithm seeks the failure sequences of members and calculates its probability. A system failure sequence is the sequence of failed members that will induce overall system failure. Thus, a probabilistic failure path approach to SSR analysis both calculates the system reliability and searches for the most likely system failure sequences.

One approach is the so-called the branch-and-bound method (Karamchandani et al. 1992, Lee and Song 2011). The study of Lee and Song (2011) proposed the B^3 method which calculates the system failure probability of fatigue-induced sequential failures. The key idea is defining the failure sequences of fatigue-induced failures as disjoint events. The fatigue failure of *i*th member is defined when the time, T_i, for the critical crack length to occur is greater than the inspection time, T_s. Statistical independence among failure sequences is attained by imposing that the *i*th member fails before the failures of any remaining *l*th members under the given loading condition (i.e., $T_i > T_l$). Each failure sequence is the intersection of component events defined as

$$g(\mathbf{X}) = \begin{cases} T_i - T_s \\ T_i - T_l \text{ for } l \neq i \end{cases} \quad (1)$$

where $g(\cdot)$ is the limit state function and $\mathbf{X}$ is the vector of random variables. Due to this formulation, each failure sequence is mutually exclusive. The system failure probability is then defined as a series of parallel subsystems. Thus, the system failure probability $P(E_{sys})$ is the sum of each probability $P(C_k)$ of the identified failure sequence C_k

$$P(E_{sys}) = P\left(\cup_{k=1}^{N_{sfs}} C_k\right) = \sum_{i=1}^{N_{sfs}} P(C_k) \quad (2)$$

where N_{sfs} is the number of identified system failure sequences. Finding all failure sequences is computationally expensive; thus, only dominant failure sequences are used to identify a narrow gap between the lower (α_L) and upper bounds (α_U) where $P(E_{sys})$ exist (i.e., $\alpha_L \leq P(E_{sys}) \leq \alpha_U$).

The B^3 method begins with the evaluation of component failure probability when no prior member has failed yet. The branching process starts at the event/component with the highest probability followed by the failure of the other remaining components. A failure sequence is examined if an overall system failure is observed. The system failure probability of each failure sequence is calculated and used in the bounding process. When a system failure is observed, the lower bound is increased. However, if there is non-failure of the system, the upper bound is decreased. A new branch is started at the highest probability non-failure node. Additional structural analysis is executed to consider stress redistribution due to failed members. The entire process is terminated once the difference of the lower and upper bound becomes small.

Each component event of the fatigue failure is defined in terms of time to achieve the critical crack length. The time T_i of the ith member to fail is calculated as

$$T_i = \frac{1}{Cv_0\left(S_i^0\right)^m} \int_{a_i^0}^{a_{c_i}} \frac{1}{\left[\mathrm{Y}(a)\sqrt{\pi a}\right]^m} \mathrm{d}a \quad (3)$$

where S_i^0 is the far-field stress range, $\mathrm{Y}(a)$ is the geometry function with the crack length a, c and m are the material properties in the Paris equation (Lee and Song 2011), v_0 is the frequency of the external loadings, a_{ci} is the critical cracked length, and a_{ci}^0 is the initial cracked length. When prior members have failed, the stress redistribution should be considered when calculating for $T_i^{1,\ldots,i-1}$. Mathematical induction can be used to derive the expression:

$$T_i^{1,\ldots,i-1} = \frac{1}{Cv_0\left(S_i^{1,\ldots,i-1}\right)^m} \int_{a_i^0}^{a_{c_i}} \frac{1}{\left[\mathrm{Y}(a)\sqrt{\pi a}\right]^m} \mathrm{d}a - \sum_{k=1}^{i-1} \left(\frac{S_i^{1,\ldots,k-1}}{S_i^{1,\ldots,i-1}}\right)^m T_k^{1,\ldots,k-1} \quad (4)$$

2.2. SRBDO framework

The B^3 method is integrated into an SRBDO framework to determine the optimal design that satisfies system reliability threshold. The optimization problem is formulated with the objective function as the cost (i.e., weight of the structure) and constraint as the failure probability of the structure. Its formulation is as follows:

$$\mathrm{Min} \quad \mathrm{f}(\mathbf{d})$$

subject to

$$\mathrm{P}\left(E_{sys}\right) = \mathrm{P}\left(\cup_{k=1}^{N_{sfs}} C_k\right) \leq \mathrm{P}_{sys}^t \quad (5)$$

$$\mathbf{d}^\mathrm{l} \leq \mathbf{d} \leq \mathbf{d}^\mathrm{u}$$

where $\mathrm{f}(\cdot)$ is the objective function, $\mathrm{P}\left(E_{sys}\right)$ is the system failure probability, P_{sys}^t is the threshold system failure probability, $\mathbf{d}$ is the vector of the design variables, and $\mathbf{d}^\mathrm{l}$ and $\mathbf{d}^\mathrm{u}$ are the lower and upper limits of the design variables, respectively. Since B^3 method produces both the upper and lower bound of the system failure probability, an assumption $\mathrm{P}\left(E_{sys}\right) \approx \alpha_U$ is made. This assumption can be realistic because the gap between α_L and α_U is small (i.e., $\alpha_L \approx \alpha_U$). In this proposed framework, the property of the lower bound to be monotonically increasing is exploited as an additional stopping criterion of B^3:

$$\text{Stopping Criterion: } \left|\frac{\alpha_L}{\alpha_U}\right| > \epsilon \quad \text{AND} \quad \alpha_L > \mathrm{P}_{sys}^t \quad (6)$$

where ϵ is the termination threshold (usually 95%). Both the lower and upper bound are effectively used in the proposed SRBDO framework. At the exploration stage of the optimization, a wider gap between the failure bounds is acceptable since the design space is searched globally for a promising location. When a favorable region is searched locally, a narrow gap between the failure bounds is desirable.

Solutions to optimization problems (OP) often use information about the derivatives of the objective and constraint functions. The Matlab Optimization Toolbox™ offers numerous solvers to perform design optimization tasks. The next section provides the derivation of the constraint function using CSP sensitivity method.

2.3. Sensitivity calculation using CSP

The derivative of the constraint function in Eq. (5) should be calculated with respect to the design variables, $\mathbf{d}$. The study of Chun et al. (2015) developed the Chun-Song-Paulino (CSP) sensitivity calculation method for the system failure probability determined using Sequential Compounding Method (SCM). The ability of SCM to perform SSR analysis to all forms of system event definition is desirable to be integrated into the B^3 analysis.

The partial derivatives of the system failure probability, $\frac{\partial \mathrm{P}(E_{sys})}{\partial \mathbf{d}}$, requires the chain rule since the system events are defined by logical (or Boolean) function of multiple component events. It is mathematically computed as

$$\frac{\partial \mathrm{P}(E_{sys})}{\partial \mathbf{d}} = \sum_{i=1}^{n} \frac{\partial \mathrm{P}(E_{sys})}{\partial \beta_i} \cdot \frac{\partial \beta_i}{\partial \mathbf{d}} \tag{7}$$

where β_i is the *i*th component reliability index. The sensitivity measures, $\frac{\partial \beta_i}{\partial \mathbf{d}}$, are produced by First Order Reliability Method (FORM) when performing component reliability analysis. The CSP method is used to calculate the derivative of the system failure probability with respect to the component reliability indices, $\frac{\partial P(E_{sys})}{\partial \beta_i}$. SCM is first carried out before calculating the sensitivity. The concept of SCM is to determine one super-component that is equivalent to the original system of multiple components. For example, the probability of a system event with three components can be compounded sequentially:

$$\mathrm{P}(E_1 \cap E_2 \cap E_3) = \mathrm{P}(E_{1\cap 2} \cap E_3) = \mathrm{P}\left(E_{(1\cap 2)\cap 3}\right) \tag{8}$$

$$\mathrm{P}(E_1 \cup E_2 \cup \mathrm{E}_3) = \mathrm{P}(E_{1\cup 2} \cup E_3) = \mathrm{P}\left(E_{(1\cup 2)\cup 3}\right) \tag{9}$$

A compounding process is executed successively between two component events. The compounding process depends on whether the two events are defined as series or parallel. For B^3 analysis, the failure sequences are defined by intersection or parallel subsystems. The compounding process of two parallel events is calculated (Kang and Song 2010) as

$$\beta_{E_i \cap E_j} = -\Phi^{-1}\left[\Phi_2\left(-\beta_i, -\beta_j; \rho_{ij}\right)\right] \tag{10}$$

where $\beta_{E_i \cap E_j}$ is the reliability index of the compounded event, β_i, β_j are component reliability indices, ρ_{ij} is the correlation coefficient of events E_i and E_j, $\Phi(\cdot)$ is the standard normal cumulative distribution function (CDF) and $\Phi_2(\cdot)$ is the joint bivariate standard normal CDF. Before compounding any component events, each correlation coefficient is calculated by $\rho_{ij} = \alpha_i^{\mathrm{T}} \alpha_j$ where α_i, α_j are the normalized gradient vectors at the design point of the FORM analysis. After compounding two events, the correlation coefficient of the compounded event ($E_{i\cap j}$) and the remaining events (E_k) is calculated by solving an OP defined by

$$\underset{\rho_{E_i \cap E_j,k}}{\mathrm{Min}} \left| \Phi_3\left(-\beta_i, -\beta_j, -\beta_k; \rho_{ij}, \rho_{i,k}, \rho_{j,k}\right) - \Phi_2\left(-\beta_{E_i \cap E_j}, -\beta_k; \rho_{E_i \cap E_j,k}\right) \right| \tag{11}$$

subject to $-1 \leq \rho_{E_i \cap E_j,k} \leq 1$

where $\rho_{E_i \cap E_j,k}$ is the correlation coefficient of the compounded event and the remaining events.

Similar strategies of creating a super-component event have been developed (Kang and Song 2010, Gong and Zhou 2017). These approaches vary in the calculation of correlation coefficient, $\rho_{E_i \cap E_j,k}$. Ideally, the sequence of the compounding process should produce the same system failure probability. However, Gong and Zhou (2017) illustrated that the sequence of compounding affects the accuracy of the results of these equivalent component approaches in SSR analysis. Moreover, a numerical problem has been found when compounding two events that are highly correlated (Xing et al. 2021). Taking inspiration from these studies, modification to SCM is implemented into this proposed method, namely:

1. *Screening rule*. In compounding, if two events are highly correlated ($\rho_{ij} > 0.9$) the event with the higher failure probability is removed.
2. *Adaptive compounding process*. The sequence of the component events is changed adaptively with the two highly correlated events compounded first.

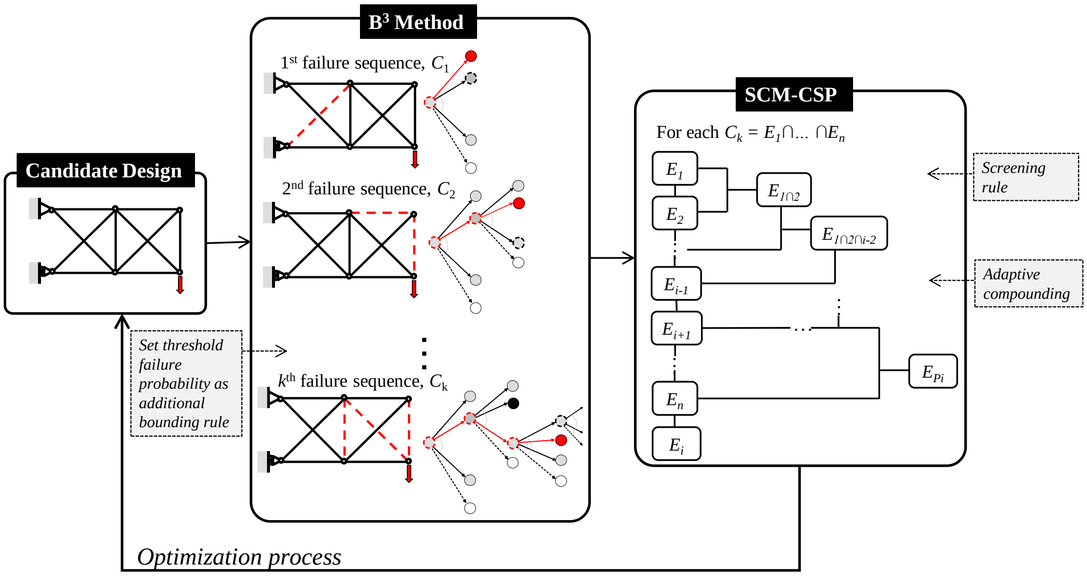

Figure 1 Proposed probabilistic failure path SRBDO framework

Once the failure probability of each failure sequence is determined, the sensitivity $\frac{\partial P(E_{sys})}{\partial \beta_i}$ is calculated. As shown in Figure 1, all components in the parallel system are compounded to E_{pi} except E_i, such as

$$E_{pi} = \cap E_p \tag{12}$$

where P is the set of all component events except E_i, i.e., $P = \{1,2, \ldots(i-1), (i+1), \ldots, n\}$ for an n-component system. The sensitivity, $\frac{\partial P(E_{sys})}{\partial \beta_i}$, is then calculated using CSP formulation (Chun et al. 2015):

$$\frac{\partial \mathrm{P}(E_{\text{parallel}})}{\partial \beta_i} = -\,\phi(-\beta_i)\cdot\Phi\left[\frac{-\beta_{pi}+\beta_i\rho_{i,P}}{\sqrt{1-\rho_{i,P}^2}}\right] \tag{13}$$

where $\phi(\cdot)$ is the standard normal probability density function (PDF).

The proposed framework starts with a candidate design as shown in Figure 1. The system failure probability by finding failure sequences is evaluated using B^3 analysis. Originally, the probability of each failure sequence in the B^3 analysis is calculated using Genz (1992) method. In this study, the modified SCM is used and CSP is applied to calculate the sensitivity of the system failure probability. This procedure is repeated in the entire optimization process until optimal solution is found.

3. NUMERICAL EXAMPLE

3.1. Problem Definition

The proposed SRBDO is tested into a system composed of brittle elements adapted from Lee and Song (2011). The structure has six members (unit weight ρ = 7,850 kg/m^3) distributed into three layers as shown in Figure 2. All members in each layer has a unit length, L, and has the same area. The design variables $\mathbf{d} = [A_1, A_2, A_3]$ are the area for each layer with lower and upper limits of 0.01 and 0.05 m^2, respectively.

The goal is to find the optimal design that will have an overall system failure probability, $\mathrm{P}(E_{sys})$, less than the threshold failure probability of 5×10^{-3}. The SRBDO problem is defined as

$$\begin{aligned} &\text{Min} \quad f(\mathbf{d}) = (A_1+2A_2+3A_3)L\rho \\ &\text{subject to} \\ &\qquad \mathrm{P}(E_{sys}) \le 5\times10^{-3} \\ &\qquad 0.01 \text{ m}^2 \le \mathbf{d} \le 0.05 \text{ m}^2 \end{aligned} \tag{14}$$

The random variables (RV) considered are: C (parameter in the Paris equation), the initial crack length, a_{io}, and the load, P, which are assumed to follow lognormal distributions with means 1.36 × 10^{-13} mm/cycle/(MPa mm)^{m}, 0.11 mm, and 1,200 kN, respectively. All RVs are assumed statistically independent, with coefficient of variation (COV) of 0.1. The deterministic parameters are: loading frequency = 500,000 cycles per year, inspection cycle T_s = 4 years, m = 3, Y(a) = 3, and critical length a_{ci} = A_i /1 m for i = 1, 2, 3.

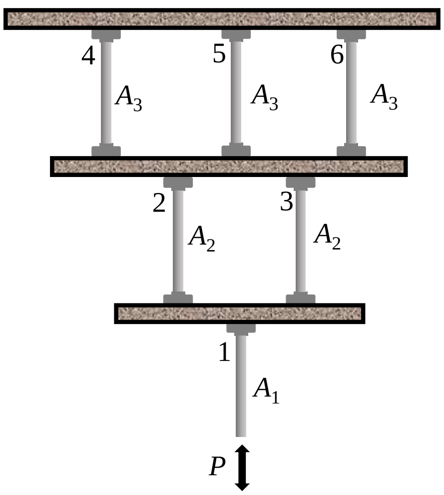

Figure 2 Multilayer Daniel's system

The overall system failure is defined when all the members in each level has failed. Additionally, the force of 1,200 kN is equally distributed to each member per level. When one member fails, the force is equally redistributed to the remaining members in a specific level.

Table 1 Optimal design

SSR Method	*Solver*	*Optimal design (m^2)*			*Weight (kg)*
Genz	*GWO*	*0.030*	*0.015*	*0.010*	*710*
Genz - FDM	*IPM*	*0.031*	*0.015*	*0.011*	*740*
	SQP	*−0.01*	*−0.01*	*−0.01*	*−284*
	ASM	*0.031*	*0.015*	*0.010*	*719*
SCM - CSP	*IPM*	*0.030*	*0.015*	*0.010*	*711*
	SQP	*0.028*	*0.014*	*0.010*	*687*
	ASM	***0.030***	***0.015***	***0.010***	***712***

The Matlab function *fmincon* with three solvers, namely, interior point method (IPM), sequential quadratic programming (SQP) and active set method (ASM) were used. Both finite difference method (FDM) and SCM-CSP method in calculating the sensitivity of the constraints were tested. A non-gradient-based algorithm named grey wolf optimizer (GWO) by Mirjalili et al. (2014) was also used for verification.

3.2. Analysis Results

The optimal designs are listed in Table 1. All the gradient-based optimizers except the SQP have converged to nearly the same optimal design as shown in Figure 3. Although all the solvers that used FDM terminated earlier compared to the one using SCM-CSP, it produces optimum with larger cost or erroneous optimum. The proposed method using ASM terminated at 16 iterations compared to 53 iterations in IPM. Thus, the use of SCM-CSP together with active set method (ASM) lead to the optimal value (which closely resembles the optimum identified by GWO). The optimal weight for this case converges to 712 kg with the areas of the members $\mathbf{d}_{\text{optimal}}$ = [0.030, 0.015, 0.010] m^2.

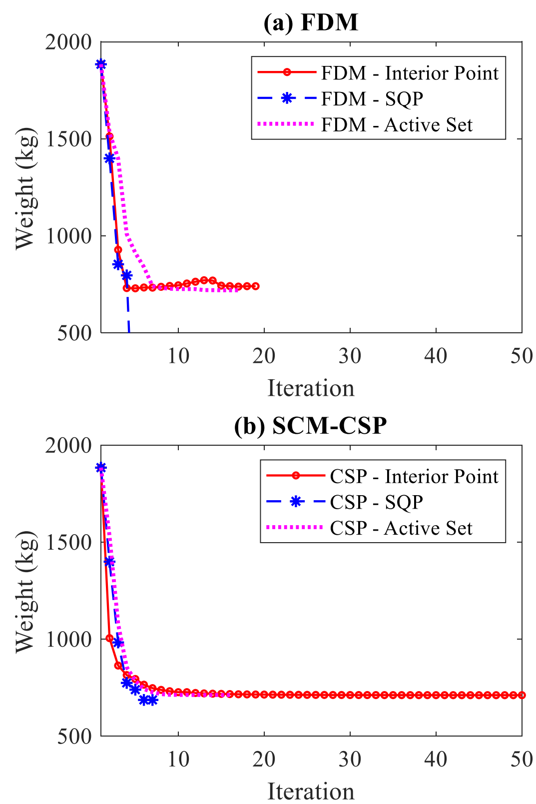

Figure 3 Convergence plots

The calculated system failure probability in each iteration is shown in Figure 4. The optimal value of the best solution (SCM-CSP – Active Set) is within the feasible region. The curve (i.e., system failure probability) when using SCM - CSP (except the SQP case) is smoother compared to the one using FDM. This demonstrates the accurate calculation of gradient when using CSP method. The effectiveness of the proposed method in searching for the optimal design guided by the accurate calculation of gradient is evident in this regard. The lower and upper bound of the B^3 analysis results at the best optimal solution are 4.62×10^{-3} and 4.86×10^{-3}, respectively. This system failure probability is comparable to the results of Monte Carlo simulation (MCS) of 4.64×10^{-3} with 10^7 samples.

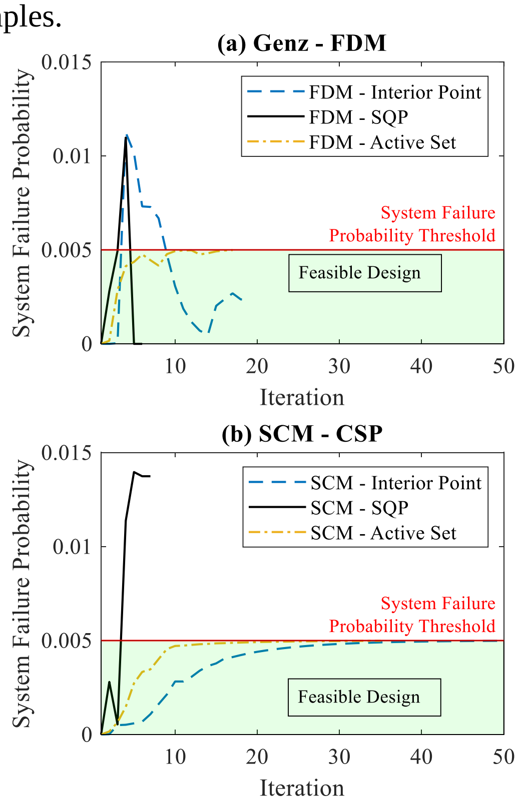

Figure 4 Probabilistic constraint evaluation

The most likely sequence of member failures found by B^3 analysis using the modified SCM are listed in Table 3. These calculated failure probabilities agree well with MCS. As shown in Figure 5, in comparison to the modified SCM the original SCM starts to deviate from the results of Genz and MCS at the lesser dominant failure sequences. This confirms the accuracy of the modified SCM with the new screening rule and adaptive compounding process proposed in this study.

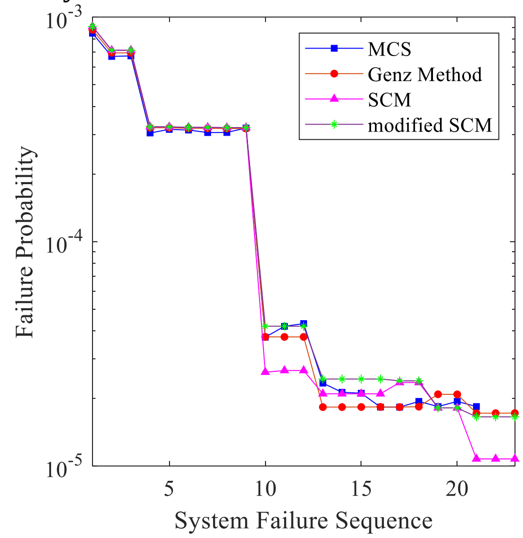

Figure 5 Calculation of failure probability of member failure sequences using different methods

Table 3 Identified dominant system failure sequences

Failure sequence	*Modified SCM ($\times 10^{-4}$)*	*MCS ($\times 10^{-4}$)*
1	*9.08*	*8.47*
3→2	*7.11*	*6.71*
2→3	*7.11*	*6.68*
6→4→5	*3.24*	*3.16*
6→5→4	*3.24*	*3.05*

Reasonably, the member with the greatest appropriated area in the optimal design is member 1 (i.e., $A_1 = 0.030$ m^2). This is expected since member 1 has the most influence in the overall system failure probability (being the most likely system failure sequence as listed in Table 3). The second most likely failure sequence (3→2) starts with member 3 failing first followed by member 2. Since there are only two members in the second level, all the forces are transferred to member 2 which eventually fails. Thus, the proposed method captures the effect of stress redistribution and allocates the materials appropriately to members with the most influence on the overall system failure.

4. CONCLUSION

A failure-path approach SSR analysis is integrated into an SRBDO framework. The proposed method uses B^3 analysis to search for fatigue induced failures together with modified SCM and CSP for probability calculation of failure sequences. Both the lower and upper bound failure probabilities from B^3 analysis are used in the optimization process. Convergence to the optimum using gradient-based optimizers was observed in a numerical example which conforms with results from non-gradient-based optimizer. The active set method as the optimization solver provided the best optimal solution. The semi-analytical formulation of the sensitivity calculation using CSP effectively guides the optimization process to the optimal design compared with FDM. The system failure calculation using the modified SCM accurately calculates the failure probability as verified with MCS. The proposed SRBDO properly allocates the limited materials to members with the most influence on the overall system failure. Stress redistribution is captured using the proposed method which is evident on the sequence of member failures. Moreover, the proposed SRBDO effectively finds an optimal design that is cost-efficient and satisfies structural system safety thresholds.

5. ACKNOWLEDGEMENT

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. RS-2022-00144434). This work is also supported by the Korea Agency for Infrastructure Technology Advancement (KAIA) grant funded by the Ministry of Land, Infrastructure and Transport (Grant 23RMPP-C163162-03).

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