Time-Dependent Reliability Calculations Approximated by a Lower and Upper Bound Method to Determine the Time-To-Failure Curve for the Optimization of Interventions

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ABSTRACT: The optimization of the timing of repair interventions on civil engineering structures is ideally based on the minimization of the total life-cycle cost. An important part of the total life-cycle cost is the expected failure cost, which can be calculated based on the lifetime function describing the probability density of the time-to-failure. However, for deteriorating structures, the determination of the lifetime function is not straightforward as it requires solving a time-dependent reliability problem. This can be computationally expensive for more complex structures and hence a challenge for the optimization algorithm that is searching for the optimal maintenance strategy. To overcome this problem, the probability of failure for time-dependent reliability problems can be approximated by time-independent lower and upper bounds that are much easier to quantify. In this work, it is investigated whether the proposed maintenance strategy changes significantly if the bounds are used instead of the more exact and numerically more expensive solution. The results show that the lower bound approach is generally a better option than the upper bound approach to achieve a close-to-optimal solution.

1. INTRODUCTION

Optimal planning of structural interventions and maintenance activities for civil engineering structures are essential to make sure government budgets are optimally used while ensuring the safety of the structure. In order to evaluate the safety level and total expected lifetime cost of a structure, the quantification of the time to failure probability, represented by the lifetime function, is necessary.

However, in the case of deteriorating structures, the time-dependent failure probability is hard to quantify accurately and is numerically expensive. As the optimal maintenance strategy is often determined by using a genetic algorithm, requiring a large number of evaluations, there is a need for an efficient way to determine the time-dependent failure probability for a structure subjected to a certain maintenance plan.

One way to solve the time-dependent reliability problem is to transform the problem into a series of time-independent reliability problems, which can be solved using system reliability calculations. However, since the lifetime of structures is typically 50 years and beyond, the series system consists of a large number of correlated elements and the problem is once again numerically expensive to solve by the classical calculation methods.

The method described in this work uses reliability bounds to approximate the solution of the series system efficiently. However, this approximation will result in a deviation from the actual failure probability and consequently all the expected costs and safety constraints related to it. The impact of the reliability bounds method on the total expected cost is studied for the case where an optimal strategy is determined for a simple deteriorating truss structure.

2. TIME-DEPENDENT RELIABILITY CALCULATIONS

When a structure with a constant resistance is subjected to a variable load, the time-dependent failure problem is transformed to a time-independent problem by considering the maximum load effect that is expected to happen during a reference period that is equal to the structure's lifetime.

However, in the case of deteriorating structures, the resistance decreases over time and the problem characterizes as highly time-dependent. A practical way to solve this time-dependent problem is by transforming it into a series system of correlated time-independent processes as proposed by Straub et al. (2020). The time-independent failure probability is evaluated every year using the First Order Reliability Method (FORM), considering an annual reference period for the maxima of the variable loads:

$$P[F_i^*] = \Phi(-\beta_i) \quad (1)$$

where $P[F_i^*]$ is the annual failure probability and β_i represents the reliability index at year i.

The probability of the structure failing in its lifetime is then the failure probability of the series system of time-independent problems, which can be approximated as follows (Straub et al. 2020):

$$P[F] = P[F_1^* \cup F_2^* \cup \ldots \cup F_n^*] \approx 1 - \Phi_n(\boldsymbol{b};\boldsymbol{\rho}) \quad (2)$$

where $P[F]$ is the cumulative probability of the structure failing before the year n and Φ_n the multivariate standard normal cumulative density function (CDF), function of vector $\boldsymbol{b} = [\beta_1, \beta_2, \ldots, \beta_n]$ and covariance matrix $\boldsymbol{\rho}$ which is composed of covariance coefficients ρ_{ij} (Straub et al. 2020):

$$\rho_{ij} = \boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j^T \text{ with } i,j \in [1, \ldots, n] \; and \; i \neq j \quad (3)$$

where $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ are the vectors of FORM sensitivities, excluding the sensitivities of the time-variant variables.

The evaluation of the multivariate standard normal CDF requires the evaluation of a multi-dimensional integral (Mori and Ellingwood, 1993), which needs to be numerically approximated. This task can become numerically expensive, especially for civil engineering structures with a lifetime that can be 50 years or more. To solve this issue, one could resort to the lower and upper bound method as described in the next section.

3. BOUNDS FOR SYSTEM RELIABILITY CALCULATIONS

Determining the failure probability of a system with multiple failure modes in series requires the evaluation of multi-dimensional integrals which can be hard to quantify. Alternatively, one can approximate the actual failure probability of the system by using boundary theory, which determines a lower and upper bound of the failure probability that are often much more straightforward to quantify. The first type of bounds that were developed for a series system are the so-called wide bounds or Cornell bounds (Cornell, 1967):

$$\max\{p_{fi}\} \leq p_F \leq 1 - \prod_{i=1}^{k}(1 - p_{fi}) \quad (4)$$

where p_F is the failure probability of the system with k failure modes and p_{fi} the failure probability of failure mode i, which can be determined by classical reliability calculation methods such as FORM. In this case, the lower and upper bounds represent a system with perfectly correlated and uncorrelated failure modes, respectively.

An improvement to these wide bounds can be made by taking into account the joint failure probability of two failure modes. This allows to determine the narrow or Ditlevsen bounds (Ditlevsen, 1979):

$$p_{f1} + \sum_{i=2}^{k} max\{p_{fi} - \sum_{j=1}^{i-1} p_{fij}, 0\} \leq p_F \leq \sum_{i=1}^{k} p_{fi} - \sum_{i=2}^{k} max\{p_{fij}\} \quad (5)$$

where p_{fij} is the joint failure probability of failure modes i and j. The failure modes are ranked according to decreasing failure probability $p_{f1} > p_{f2} > p_{f3} > \ldots$ in order to gain as narrow bounds as possible.

The values of p_{fij} can be approximated by the Feng point estimation method (Feng, 1989), which has been proven to be accurate enough to use for the narrow bounds (Song, 1992):

$$p_{fij} = [p_A + p_B]\left(1 - \frac{\arccos(\rho_{ij})}{\pi}\right) \quad (6)$$

$$p_A = \phi(-\beta_i)\phi\left(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right) \quad (7)$$

$$p_B = \phi(-\beta_j)\phi\left(-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right) \quad (8)$$

where ρ_{ij} is the correlation coefficient of failure mode i and j, and β_i and β_j the reliability indices of corresponding failure probabilities p_{fi} and p_{fj}, respectively.

One must note that there are strategies to respectively maximize and minimize the lower and upper bounds in order to decrease the bandwidth (Song and Feng 1988, Abdollahzadeh et al. 2019). However, for the application presented in this work, these turned out to have no significant influence on the obtained reliability bounds and therefore such approaches are not considered here.

4. OPTIMIZATION PROBLEM USING LIFETIME FUNCTIONS

Maintenance activities and structural interventions are ideally planned based on the solution of an optimization problem, which typically consists of three aspects:

- Design variables: the variables that are altered throughout the optimization problem, typically the timing of certain interventions.
- Objective(s): the goals of the optimization algorithm, such as minimization of costs and downtime.
- Constraint(s): certain parameters that should not be exceeded, such as a minimum required safety level or maximum amount of interventions.

Quantifying the failure probability of the structure for a given maintenance strategy is very important, both for the total expected cost of failure and for remaining within the safety threshold.

The probability of failure of a structure can be taken into account with the so-called lifetime function, which represents the distribution of the lifetime of the structure. In literature, the lifetime function of a (member of a) structure is typically represented by a Weibull or exponential distribution (Barone and Frangopol 2014, Okasha and Frangopol 2010, Sabatino et al. 2016). The parameters of the distributions are of critical importance to describe the deterioration process of the structure. However, in literature, the background of the assumed parameters is often missing.

In this work, it is proposed to determine the lifetime functions directly from the time-variant reliability calculation, by taking the derivative of the cumulative failure probability. However, because the optimization problem is typically solved using an evolutionary algorithm, which requires a high number of evaluations of the objective function, the numerically expensive equation defined in Eq. (2) may render the calculations intractable.

As introduced in section 3, this will be tackled by applying the boundary theory to

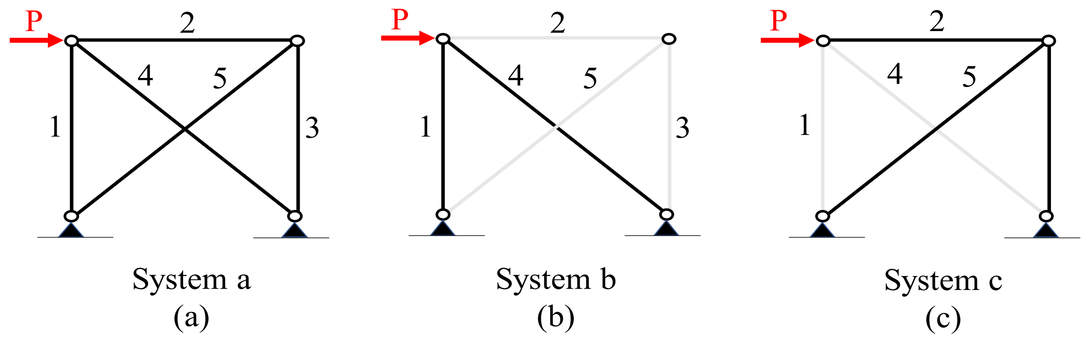

Figure 1: Truss structure; from (a) to (c): fully intact, 1st failure mode, 2nd failure mode.

approximate the lifetime function. The disadvantage of this method is that the approximation may lead to a suboptimal solution, because the evaluation of the expected failure cost is not exact.

5. CASE STUDY

In this section, a truss structure is considered with deteriorating elements subjected to a variable load with reference period of one year. The uncertainty, expressed in terms of the coefficient of variation (COV) for the resistance and load variables is changed for different cases considered. For each case, the optimal maintenance strategy is determined using the exact method and the upper and lower bound method.

5.1. Truss structure

The case considered is based on the problem described by Okasha and Frangopol (2009) and consists of a steel truss with 5 elements which is loaded by a horizontal load as illustrated in Figure 1. In case a bar fails in system a, the structure is transformed into either system b or system c. In case another bar fails from the new system, the structure collapses. Failure of the structure can be described by a series-parallel system as shown in Figure 2, where F_X^Y represents the failure of bar X in system Y as defined in Figure 1. For the sake of simplicity, it is assumed that the resistance of all elements is mutually uncorrelated.

5.2. Time-variant reliability calculation

The load P evolves in time as a rectangular block process with a reference period of one year. Each annual load value is independent and lognormally distributed with a mean value of 20 kN and constant coefficient of variation (COV), which depends on the case considered. The initial resistance R_0 of every bar is lognormally distributed with a mean of 37.5 kN in compression and 75 kN in tension and COV according to the case considered. The resistance of the bars is linearly decreasing in time with a slope d, which is lognormally distributed with mean 0.1 kN/year and a coefficient of variation of 0.2. The limit state equation of a bar is defined as:

$$g(X,t) = R_0 - dt - \xi P \quad (9)$$

where ξ is the transfer function which translates the applied horizontal load to the normal force in the bar itself.

Four different cases are considered, where the coefficient of variation of both the load (V_P) and resistance (V_{R0}) are varied as given in Table 1. The values in each case are chosen in such a way that the initial failure probability of the system is more or less equal for each case.

For every element of the series-parallel system, a FORM analysis is performed yearly over a period of 50 years to determine the values $P[F_{X\,i}^{Y*}]$ and their respective sensitivities $\boldsymbol{\alpha}_{X\,i}^{Y}$ at year i. Using the computationally expensive Eq.

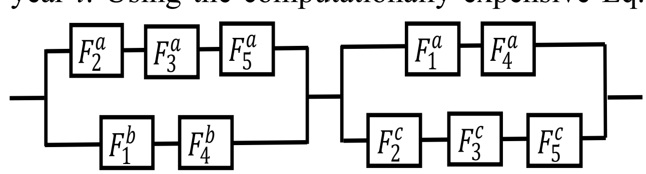

Figure 2: Series-parallel system of the considered truss structure.

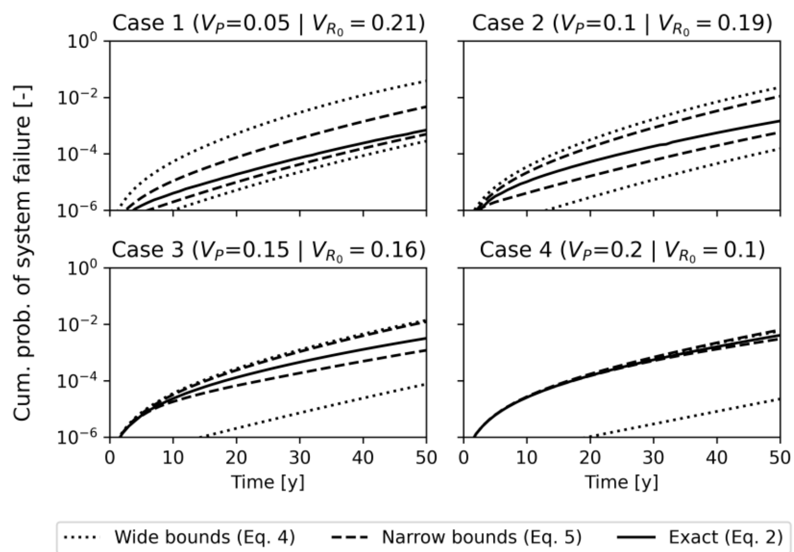

Figure 3: Cumulative failure probability of truss structure for different cases with varying coefficients of variation.

(2), the cumulative failure probability is determined for every element using the "exact" method. Combining the cumulative failure probabilities of all elements in the series-parallel system, the cumulative failure probability of the truss structure is determined.

The same calculation is repeated, but now using the reliability bounds given by Eq. (4) and Eq. (5) which reduces the calculation speed by roughly a factor of 10^4-10^5. The results are compared in Figure 3, for all coefficients of variation considered. First of all, it is observed that the narrow bounds are significantly better at approximating the exact result. As a result, only the narrow bounds will be used to approximate the failure probability. Secondly, the coefficients of variation of the load and resistance variable have a significant influence on the bandwidth of the bounds. When the uncertainty of the time-variant variable (V_P) dominates, the bounds are close to the actual solution, which is not the case when the uncertainty on the time-invariant variable dominates (V_{R0}). This is because the actual solution approaches the upper bound and the lower bound approaches the sum of the individual failure probabilities $\sum p_{fi}$, which is an approximate solution for a series of independent failure modes.

Table 1: Coefficients of variation load and resistance variables.

	V_P	V_{R_0}
Case 1	*0.05*	*0.21*
Case 2	*0.10*	*0.19*
Case 3	*0.15*	*0.16*
Case 4	*0.20*	*0.10*

5.3. Optimization problem

For the case study considered, an optimization problem is solved to determine the ideal preventive repair strategy for the structure. The optimization problem is solved three times: once with the exact lifetime function and once for the narrow lower and upper bound approaches. In the remaining part of the text, the term lower and upper bound corresponds to the corresponding narrow or Ditlevsen bound given by Eq. (5).

5.3.1. Design variables

Every 5 years between 0 and 50 years, a set of bars can be replaced by new bars. Hereby, 3 replacement sets are considered as given in Table 2. Every 5 years, one or more sets can be replaced irrespective of a set that was already replaced before or not. Each time a set of bars is replaced, an additional cost, based on the values from Okasha and Frangopol (2009), is added to the total lifetime cost according to Table 2. The new bars behave in the same way as the previous one and hence are also prone to degradation. The design variables in this problem are binary values for every possible moment of preventive repair (each 5 years), indicating replacement or no replacement for a given set of bars. The design variables are optimized such that the total lifetime cost remains as low as possible.

5.3.2. Objective function

The objective is to minimize the total lifetime cost of the structure over a period of 50 years. The total lifetime cost C_T is determined as follows:

$$C_T = C_R + C_F = \sum_{i=1}^{3} \sum_{j=1}^{k_i} C_{Ri} \exp(-\gamma t_{ij}) + H \int_0^{t_{LT}} f_{T_F} \exp(-\gamma t)\, dt \qquad (10)$$

and consists of two terms: the total repair cost C_R and expected failure cost C_F. The first term represents the total repair cost C_R, where t_{ij} is the moment of repair for strategy i, k_i the number of repair interventions for every set of bars and C_{Ri} the repair cost as indicated in Table 2. The second term is the expected failure cost C_F, where f_{T_F} is the lifetime function determined as the derivative of the cumulative system failure probability (Rackwitz, 2000), t_{LT} the considered lifetime of the structure being 50 years and the cost of failure H is considered equal to 1000, with the same monetary units as the repair cost. The time-value of money is taken into account by the discount rate γ equal to 2%.

The optimization problem is solved using a genetic algorithm with 10,000 generations and 20 solutions per generation. Note that every element's time-independent reliability calculations and cumulative failure probabilities need to be determined only once before the optimization algorithm starts. However, in more complex cases with more repair strategies and more complex failure modes, it might be necessary to evaluate the time-dependent reliability calculations at each cost evaluation. This makes the numerically extensive "exact" method less convenient, hence the use of the bounds.

Table 2: Cost of replacement set of bars.

Set i	*Bars*	*Cost C_{Ri}*
1	*1 & 3*	*0.34*
2	*2*	*0.17*
3	*4 & 5*	*0.49*

The results of the optimization algorithm for all cases are summarized in Table 3. The optimal preventive maintenance schedule is determined using either the upper or lower bound method, or the exact method. When using the bound methods, the calculated expected failure cost will deviate from the exact cost. Therefore, the final expected failure cost C_F of the proposed preventive maintenance strategy is determined by recalculating the failure probability, but now using the exact method.

The total cost is expressed in absolute values, as well as in relative terms with the solution using the exact method for the optimization algorithm as a reference. In Figure 4, the optimal preventive repair strategy resulting from the cost optimization analysis is given for each of the cases of C_R, corresponding to the values mentioned in Table 3.

Using the lower bound method always leads to a better approximation, compared to the solution obtained using the upper bound method. The deviation of the solution based on the upper bound increases with decreasing value of V_P, which can be explained by Figure 3 as the upper bound deviates further from the exact solution for lower values of V_P.

6. DISCUSSION

Several cases are considered where the variables' uncertainties are changed to investigate their influence. In case the uncertainty on the load (P) is larger than the uncertainty on the time-invariant variables (e.g. $V_P = 0.20$ vs. $V_{R0} = 0.10$), the upper bound is approached and therefore using the upper bound instead of the exact method can be a good option. However, in this case, also the bounds are rather narrow and therefore, the lower bound also becomes a viable option. Moreover, it was found that even when the uncertainty on the load dominates and the exact solution approaches

Table 3: Results of optimization problem for different cases with varying coefficients of variation.

COV load V_P	*Method (exact or bound)*	*Repair cost C_R*	*Failure cost C_F*	*Total cost C_T (abs.)*	*Total cost C_T (rel.)*
0.05	*Upper*	*0.390*	*0.093*	*0.483*	*155%*
	Lower	*0*	*0.311*	***0.311***	***100%***
	Exact	*0*	*0.311*	***0.311***	***100%***
0.10	*Upper*	*0.587*	*0.205*	*0.792*	*144%*
	Lower	*0*	*0.674*	*0.674*	*123%*
	Exact	*0.269*	*0.281*	***0.550***	***100%***
0.15	*Upper*	*0.607*	*0.496*	*1.103*	*121%*
	Lower	*0.269*	*0.663*	*0.932*	*103%*
	Exact	*0.297*	*0.612*	***0.909***	***100%***
0.20	*Upper*	*0.587*	*0.590*	*1.177*	*108%*
	Lower	*0.297*	*0.791*	***1.088***	***100%***
	Exact	*0.297*	*0.791*	***1.088***	***100%***

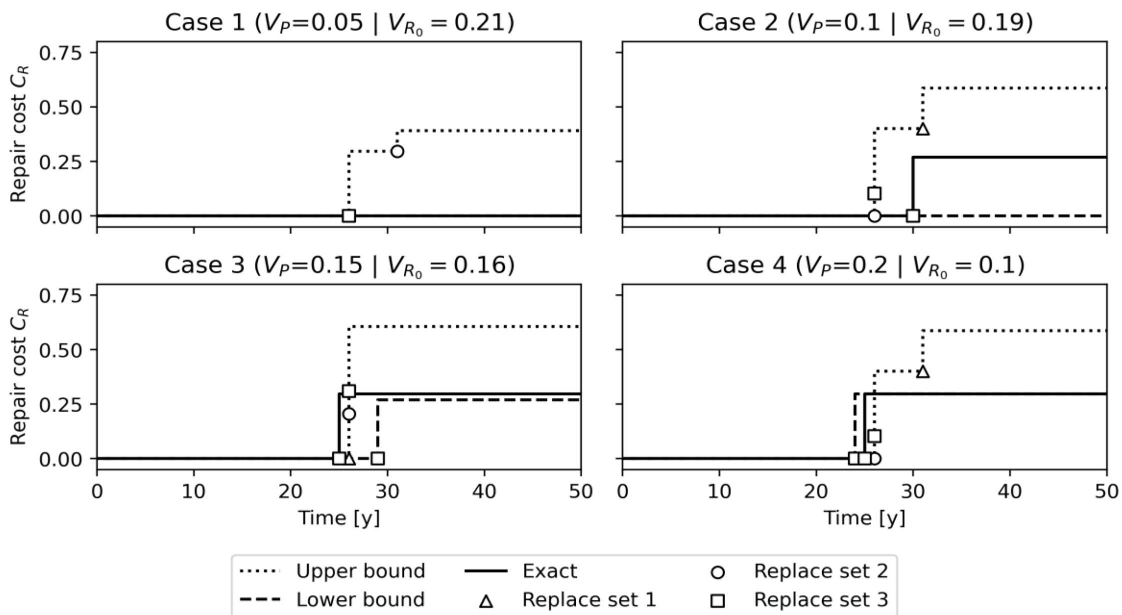

Figure 4: Proposed optimal repair strategies using the bound or exact method for different cases with varying coefficients of variation.

the upper bound, the lower bound leads to a maintenance strategy that is in line with the strategy resulting from the exact solution. However, this is only in case the narrow bound method is used.

In case the uncertainty is dominated by time-invariant variables, such as the initial resistance, the exact method approaches the lower bound and the bounds lie further apart. This means that using the upper bound to solve the optimization problem is not appropriate and can lead to a large deviation from the ideal situation. However, using the narrow lower bound method gives results that are close to the most optimal solution.

In case neither of the two dominate and the uncertainty is spread more evenly over all parameters involved, both the upper and lower bounds suggest a preventive repair strategy that deviates from the best result. Nevertheless, in this case the lower bound is still the best option.

7. CONCLUSION

To find an optimal maintenance (or repair) planning for civil engineering structures, a good optimization algorithm is needed that is computationally efficient to solve complex problems. A bottleneck here is the solution of time-dependent reliability problems, such as the failure probability in case of deteriorating structures, which require numerically expensive calculation methods.

In this work, reliability bounds are used to determine the lifetime function of the structure in an approximative way. For the considered case study in this work, it was found (in comparison to the more exact analysis) that using the narrow lower bound approximation will result in the best maintenance strategy, compared to the upper bound approximation. The solution using the lower bound deviates only up to 3% in terms of total lifetime cost in 3 of the 4 cases considered in this work. However, in one case the deviation was 23%, which is substantial. This is because the bound methods approximate the exact solution less well in case the uncertainties are more evenly distributed and not dominated by either the time-dependent or time-independent variables.

It is important to note that the considered structure could have a large influence on the conclusions made in this work. Moreover, the failure mode and considered limit state equation are expected to have an influence as well. Therefore, further work includes investigating the influence of the cost of failure/repair ratio and upscaling the case study to a more realistic problem.

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