The Method for Structural Reliability Analysis Considering the First Three Moments as Interval Variables

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**ABSTRACT:** In the traditional structural reliability method, the first three statistical moments of random variable are used to describe its uncertainty. But in practical engineering, due to the lack of statistical data, it is difficult to obtain the certain statistical moments of some random variables, and the failure probability is no longer the determined value. To solve this problem, this paper treats the uncertain first three moments as interval variables, and proposes the method to compute the interval of failure probability. This method uses the simplified third moment normal transformation model to map the limit state surface from the original space to the standard normal space. By distinguishing the statistical moment interval values corresponding to the upper and lower boundaries of the limit state strip in the normal space, limit state functions corresponding to the upper and lower boundaries of the limit state strip are established with the system reliability, and method of moments is used to calculate the interval of failure probability. In this paper, an assessment of structural reliability analysis considering the first three moments as interval variables is introduced. The example is provided to demonstrate the accuracy and efficiency of this approach in structural reliability analysis.

**INTRODUCTION**
Generally, the probabilistic reliability model is the most prevalent method to solve structural reliability problems. In the past research, many analysis approaches have been used in structural reliability analysis, which include the first order reliability method (FORM) (Hasofer and Lind 1974, Rackwitz and Fiessler 1978), second order reliability method (SORM) (Kiureghian et al. 1987, Kiureghian and Stefano 1991), Monte-Carlo simulation (Fu 1994, Melchers 1990), method of moments (Grigoriu 1983, Zhao and Ono 2001), etc. When using these methods, the cumulative distribution functions (CDF) or probability density functions (PDF) of some random variables are usually assumed to be known, so the statistical moments in the PDF are also certain. The imprecise statistical moments can only be estimated due to the lack of distribution information and the limitation of the experimental samples. The statistical moments of random variables will change due to the additional sample information, resulting in statistical moment uncertainty. The uncertainty of statistical moments can substantially affect the failure
probability and reliability index (Ben-Haim and Elishakoff 1990, Kiureghian 1989).

To consider the uncertainty of statistical moments in reliability analysis, two approaches have been developed in past research. In the first approach, the statistical moments are treated as random variables, thereby the failure probability becomes a random variable which calls conditional failure probability (Kiureghian 1989), and the overall failure probability is defined as the expectation of the conditional failure probability. And a point-estimate method based on dimension-reduction integration and Bayes theory has been proposed to estimate the probability distribution and quantile of the conditional reliability index and conditional failure probability (Zhao et al. 2018).

The second approach involves treating statistical moments as interval variables. A reliability analysis method combining the probabilistic reliability method with a convex model was proposed when information is insufficient (Elishakoff and Colombi 1993, Zhu and Elishakoff 1996). For the failure probability of basic linear function \( G(X) = R - S \), when \( R \) and \( S \) are normal random variables and independent of each other, the analytical expression of the reliability index interval is obtained (Qiu et al. 2008, Qiu et al. 2008). However, in practical engineering, \( R \) and \( S \) are not always normal distributions and function is mostly nonlinear. In more common situations, an optimization method to acquire the upper and lower bounds of the reliability index was proposed based on the FORM (Du 2008). A more efficient computing method was proposed, and two reliability analysis models were given based on the proof of monotonicity for the CDF of the random variable with respect to its statistical moments (Jiang et al. 2011, Jiang et al. 2012).

The goal of the present paper is to develop a methodology for solving the reliability analysis considering the uncertainty of the first three statistical moments. It is based on method of moments and the monotonicity analysis of third-moment normal transformation. Firstly, the monotonicity proof and the interval analysis of skewness parameters are proposed in this paper. Next, an accurate and efficient procedure for calculating the interval of failure probability and reliability index is developed. Furthermore, the calculation of structural reliability with unknown probability distributions of random variables is also one of the characteristics of this paper. The paper includes examples to demonstrate the feasibility of the proposed method. Results indicate that the method is considerably more efficient than the Monte-Carlo simulation (MCS) method while maintaining sufficient accuracy.

1. PROPOSED METHOD

1.1. Monotonicity analysis

When considering the uncertainty of the statistical moment of a random variable \( X \), it is usually treated as interval parameter \( Y \), the limit state function is transformed from the original space to the standard normal space, and it is expressed as follows (Jiang et al. 2012)

\[
G(X) = g(T(U,Y)) = G(U,Y)
\] (1)

where \( Y \) is an \( n \)-dimensional vector containing all interval statistical moments, which can be expressed as follow

\[
Y = [Y_{1}^L, Y_{1}^R], Y_i = [Y_i^L, Y_i^R], i = 1, 2, \ldots, n
\] (2)

where the superscripts \( L \) and \( R \) represent the lower boundary and upper boundary of the interval, respectively.

In traditional reliability analysis, the limit state defined by \( G(X)=0 \) is a single hyper-surface in \( u \) space. However, due to the existence of \( Y \), the limit state of the hybrid reliability defined by \( G(U,Y)=0 \) changes to a strip enclosed by two bounding hyper surfaces \( S_L \) and \( S_R \) as shown in Fig. 1.

The above analysis shows the existence of the bounding hypersurfaces, which can further explain that the reliability index will become an interval when the statistical moments of random variables are interval variables. Therefore, it is critical to find the relationship between the bounding hypersurfaces and the values of interval
variables, which will be obtained by monotonicity analysis for the third-moment normal transformation below.

\[ \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} = x_{u_i} = S_u(u_i) = a_{u_i} + a_{u_i} u_i + a_{u_i} u_i^2 \quad (3) \]

where \( x_{u_i} \) represents the standardized variable of \( x_i \), \( S_u(\cdot) \) represents that the independent variable is the function of a standard normal random variable \( u_i \), which can be written as

\[
\begin{cases}
  -a_{a_i} + \sqrt{a_{a_i}^2 - 4a_{a_i}(a_{a_i} - x_{u_i})} / 2a_{a_i}, & a_{a_i} \neq 0 \\
  x_{u_i}, & a_{a_i} = 0
\end{cases}
\quad (4)
\]

where \( a_{a_i}, a_{a_i} \) and \( a_{a_i} \) are calculated as follows

\[
a_{a_i} = -a_{a_i} = \text{sign}(\alpha_{a_i}) \sqrt[3]{2 \cos \left( \frac{\pi}{3} + \frac{|\theta|}{3} \right)} \quad (5)
\]

\[
a_{a_i} = \sqrt{1 - 2a_{a_i}^2} \quad (6)
\]

\[
\theta_i = \arctan \left( \frac{\sqrt{8 - a_{a_i}^2}}{a_{a_i}} \right) \quad (7)
\]

To hold the equality, the value of \( \alpha_{a_i} \) is limited to \([-2\sqrt{2}, 2\sqrt{2}]\). Then the monotonicity of \( u_i \) with respect to \( u_{a_i} \), \( \sigma_{a_i} \) and \( \alpha_{3_i} \) will be proved, respectively. Here the monotonicity of statistical moments when \( a_{3_i} \neq 0 \) is considered.

Firstly, the first-order derivative of \( u_i \) with respect to \( u_{a_i} \) can be obtained as

\[
\frac{\partial u_i}{\partial u_{a_i}} = \left\{ \begin{array}{ll}
  \frac{1}{\sigma_{a_i}} & \alpha_{3_i} \neq 0 \\
  \frac{1}{\sqrt{a_{a_i}^2 + 4a_{a_i}^2 + 4a_{a_i}u_{a_i}}} & \alpha_{3_i} = 0
\end{array} \right. \quad (8)
\]

In Eq. (8), it can be obtained easily that the \( u_i \) is always monotonically decreasing with respect to \( u_{a_i} \).

Secondly, the first-order derivative of \( u_i \) with respect to \( \sigma_{a_i} \) can be obtained as

\[
\frac{\partial u_i}{\partial \sigma_{a_i}} = \left\{ \begin{array}{ll}
  \frac{\mu_{a_i} - x_{u_i}}{\sigma_{a_i} \sqrt{a_{a_i}^2 + 4a_{a_i}^2 + 4a_{a_i}u_{a_i}}} & \alpha_{3_i} \neq 0 \\
  \frac{\mu_{a_i} - x_{u_i}}{\sigma_{a_i}^2} & \alpha_{3_i} = 0
\end{array} \right. \quad (9)
\]

In Eq. (9), it can be found that the sign of \( \frac{\partial u_i}{\partial \sigma_{a_i}} \) is related to the value of \( x_{u_i} \).

When \( x_{u_i} > \mu_{a_i} \), \( u_i \) is monotonically decreasing with respect to \( \sigma_{a_i} \), and when \( x_{u_i} \leq \mu_{a_i} \) the monotonicity is opposite.

Thirdly, the first-order derivative of \( u_i \) with respect to \( \alpha_{3_i} \) can be obtained as

\[
\frac{\partial u_i}{\partial \alpha_{3_i}} = \frac{(1 - u_{a_i}^2) \cdot \sqrt{1 - 2a_{a_i}^2 + 2a_{a_i}u_{a_i}}}{\sqrt{1 - 2a_{a_i}^2 \cdot (2a_{a_i}u_{a_i} + a_{a_i})}} \frac{\partial a_{a_i}}{\partial \alpha_{3_i}} \quad (10)
\]

In Eq. (10), it can be divided into three parts for facilitating the analysis.
Figure 2: Judgment of monotonicity of $a_1$ with respect to skewness

According to the Fig. 2 whose horizontal axis represents the skewness of random variables $\alpha_{3i}$ and the vertical axis represents the first-order derivative of $a_1$ with respect to $\alpha_{3i}$, it can be evaluated that $G(U, Y)$ is positive. Similarly, $(2a_1u_i + a_{1i})$ is also positive according to Eq. (4). However, the sign of $(1-u_i^2) + \frac{2a_1u_i}{\sqrt{1-2a_1^2}}$ cannot be evaluated directly, and because the value of $(1-u_i^2) + \frac{2a_1u_i}{\sqrt{1-2a_1^2}}$ is related to $x_{si}$ and $\alpha_{3i}$, this paper uses Fig. 3 to solve it.

In Fig. 3, the horizontal axis represents standardized random variables $x_{si}$ and the vertical axis represents the skewness of random variables $\alpha_{3i}$. This figure shows that when $\frac{\partial u_i}{\partial \alpha_{3i}}$, each given value of $\alpha_{3i}$ has two different values of $x_{si}$, two curves $a$ and $b$ are obtained. It is concluded that the value of $\frac{\partial u_i}{\partial \alpha_{3i}}$ is positive in the internal region surrounded by curves $a$ and $b$, while the value of $\frac{\partial u_i}{\partial \alpha_{3i}}$ is negative in the region corresponding to the left part of the curve $a$ and the right part of the curve $b$. But for any given value of $x_{si}$, $u_i$ is not monotonous with respect to $\alpha_{3i}$.

So, in this paper, curve $a$ is approximated as polygonal line $a_1$, and line $b$ is approximated as polygonal line $b_1$. When $\alpha_{3i} = 0$, two values of $x_{si}$ are -1 and 1, respectively. After analysis, the results can be obtained as

$$
\begin{align*}
\frac{\partial u_i}{\partial \alpha_{3i}} \geq 0, & -1 \leq x_{si} \leq 1 \\
\frac{\partial u_i}{\partial \alpha_{3i}} < 0, & x_{si} < -1, x_{si} > 1
\end{align*}
$$

In Eq. (11), the monotonicity of $u_i$ with respect to $\alpha_{3i}$ depends on the standardized random variables $x_{si}$, but for any given value of $x_{si}$, $u_i$ is monotonous with respect to $\alpha_{3i}$. When $-1 \leq x_{si} \leq 1$, $u_i$ is monotonically decreasing with respect to $\alpha_{3i}$, and when $x_{si} < -1$ and $x_{si} > 1$ the monotonicity is opposite.

Therefore, through the analysis above, the monotonicity analysis of the statistical moments can be divided into two categories. The first category is independent of the value of random variables, that is, $u_i$ is always monotonically decreasing function with respect to $\mu_{si}$. The second category is related to the value of random variables, for the monotonicity of $u_i$ with respect to $\sigma_{si}$, there is an inflection point so that the
monotonicity of $\sigma_u$ is opposite on both sides of the inflection point. Similarly, there are two inflection points in the monotonicity of $u_i$ with respect to $\alpha_{si}$, and on both sides of each inflection point, the monotonicity of $\alpha_{si}$ is opposite.

1.2. Structure Reliability calculation models
To obtain the Interval parameters values corresponding to the bounds of the limit state strips, it is necessary to determine the position relationship between the limit state strip and the origin point.

Since $G(X\text{\_mean}) > 0$ holds in general and the origin point in the $u$ space should be not inside the limit-state strip, the relative position can be obtained by calculating the gradient $\frac{\partial G}{\partial U_{i,\text{origin}}}$. The value of $Y$ corresponding to the limit strip can be derived by combining monotonicity and gradient, and the results are summarized in Table. 1.

![Figure 4: The relative position of the origin point and the limit state strips in u space](image)

Due to the difference in the monotonicity of statistical moments, the shape of the limit state strip and their corresponding relationship with the boundary value of statistical moment interval parameters are different. Here three different limit state strips and two structural reliability calculation models will be built according to the different monotonicity.

When only mean uncertainty is considered, after determining the values of the mean interval parameters corresponding to the lower and upper boundaries of the limit state strip, the method of moments is used to calculate the reliability index of the upper and lower boundaries, and the intervals of reliability index and failure probability can be obtained.

<table>
<thead>
<tr>
<th>$\frac{\partial G}{\partial U_{i,\text{origin}}}$</th>
<th>Monotonicity</th>
<th>$Y$ on the lower bound</th>
<th>$Y$ on the upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$&lt;0$</td>
<td>Monotonically decreasing</td>
<td>$Y^R$</td>
</tr>
<tr>
<td></td>
<td>$&lt;0$</td>
<td>Monotonically increasing</td>
<td>$Y^L$</td>
</tr>
<tr>
<td></td>
<td>$&gt;0$</td>
<td>Monotonically decreasing</td>
<td>$Y^L$</td>
</tr>
<tr>
<td></td>
<td>$&gt;0$</td>
<td>Monotonically increasing</td>
<td>$Y^R$</td>
</tr>
</tbody>
</table>

Note: Monotonicity is the monotonicity of $u$ with respect to statistical moments.

When the uncertainty of standard deviation is considered, according to the monotonicity of $u$ with respect to $\sigma$ in 2.2, there is an inflection point in $u$ space, and the monotonicity of $u$ with respect to $\sigma$ at both ends of the inflection point is opposite. Therefore, the limit state strip will rotate, and the upper and lower limit state hyper-surfaces will become enveloping surfaces. Like the second strip, when considering the skewness uncertainty, the upper and lower limit state hyper-surfaces also rotate.

In the latter two cases, the boundary of the limit state strip changes to the envelope surface, and the values of standard deviation or skewness interval parameters corresponding to each small part of the envelope surface are different, and the corresponding failure modes are also different. Therefore, the system reliability problem is formed, and the system reliability based on method of moments is used to solve it in this paper (Zhao and Ang 2003), which avoids the solution
of the correlation coefficient between failure modes.

1.3. Calculating procedure
From the above analysis, it can be seen that for the structural reliability problem considering the uncertainty of statistical moments, two different calculation models will eventually be formed. Next, the calculation procedure considering the uncertainty of statistical moments will be summarized as follows:

1. Divide the interval parameters of statistical moments into two classes, mean interval parameters $Y_{1i}$, $i=1,2,...,n$, and standard deviation and skewness interval parameters $Y_{2j}$, $j=1,2,...,m$.

2. If there are mean interval parameters, calculate the gradient $\frac{\partial G}{\partial U_i}$, and then determine the corresponding value of the upper and lower bounds of the mean interval parameter $Y_i$ in the limit state strip according to Table. 1.

3. Repeat 2 for $i=1,...,n$ to obtain the $Y_{1L}$ and $Y_{1R}$.

4. According to whether $Y_{2j}$ exists or not, to choose to calculate the interval of $P_f$ and $\beta$ or consider the $2^j$ combinations of $Y_{2j}$.

5. Obtain the limit state function of the lower boundary series system and upper boundary parallel system.

6. Calculate the interval of reliability index and the failure probability are obtained using the system reliability based on method of moments.

2. ILLUSTRATIVE EXAMPLES
A four-bar truss as shown in Fig. 5 is investigated, which is modified from the numerical examples in Ref. (Du 2008). Considering that the vertical displacement $\delta$ at the tip joint of the truss is required to be less than an allowable value the LSF can be constructed as

$$G(X) = \delta_a - \delta = \delta_a - \frac{6P_l}{E} \left( \frac{3}{A_1} + \frac{\sqrt{3}}{A_2} \right)$$

where $\delta_a = 1.7$mm is the allowable value and vertical force $P$ is 160 kN; $A_1$ is the cross-sectional area of members 1-3 and $A_2$ is the area of member 4; the Young’s Modulus $E$ of the truss is 200 GPa and the length $l$ is 500 mm. In this example, the mean of $A_1$ and the standard deviation of $A_2$ are treated as interval variable. The statistical information is given in Table. 2.

![Figure 5: Four-bar truss](image)

Table 2: Statistical information for Example 1

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Mean ($A_{1}$ mm$^2$)</th>
<th>Standard Deviation ($A_{2}$ mm$^2$)</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ ($mm^2$)</td>
<td>[635.85, 777.15]</td>
<td>70.65</td>
<td>Normal</td>
</tr>
<tr>
<td>$A_2$ ($mm^2$)</td>
<td>[169.56, 207.24]</td>
<td></td>
<td>Normal</td>
</tr>
</tbody>
</table>

Table 3: Reliability analysis results for Example 1

<table>
<thead>
<tr>
<th>Method</th>
<th>Lower bound Value</th>
<th>Upper bound Value</th>
<th>$\beta$ R.E. (%)</th>
<th>$\beta$ R.E. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCS</td>
<td>1.6164</td>
<td>3.0618</td>
<td>0.22</td>
<td>2.23</td>
</tr>
<tr>
<td>SLMA</td>
<td>1.62</td>
<td>3.13</td>
<td>0.63</td>
<td></td>
</tr>
<tr>
<td>Proposed method</td>
<td>1.6198</td>
<td>3.0811</td>
<td>0.21</td>
<td>2.63</td>
</tr>
</tbody>
</table>

In comparison, the single-loop method based on the monotonicity analysis, also known as SLMA, is used (Jiang et al. 2012). Furthermore, Monte Carlo simulation (MCS) is used as an accuracy benchmark. Table 3 shows the bounds of the reliability index obtained using these approaches, as well as the relative error (denoted by R.E.) when compared to MCS results. Table 3
shows that the relative errors of both proposed method and the SLMA methods are less than 5%, indicating that both methods are accurate for this example.

3. CONCLUSIONS
This paper proposes a reliability analysis method that considers the uncertainty of the first three statistical moments. The following findings were derived from previous research:

1. The monotonicity of the standard normal random variable $u$ with respect to the first three moments can be proved respectively based on the third-order moment normal transformation model.
2. Based on the monotonicity analysis, the study proposes a calculation procedure that requires no optimization and only two reliability analyses.
3. The results of the example are very close to MCS indicating the accuracy of the proposed method.

4. REFERENCES