Dynamic Responses and Flow Field Characteristic of Stochastic Burgers Equation

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ABSTRACT: Turbulent flow is a highly irregular motion which is usually described as stochastic process in engineering analysis and design, and Burgers equation as a turbulence model has been widely used in various fields. In order to obtain the probability density function of physical quantities in the stochastic Burgers equation and obtain more flow field characteristic information, we propose the direct probability integral method (DPIM) to solve the viscous Burgers equation with random initial conditions of Gaussian white noise. Firstly, the basic theory of Burgers equation and the numerical method for solving Burgers equation, namely predictor-corrector method is introduced. Secondly, the hybrid spectral representation and random function approach is utilized to accurately simulate the random process of Gaussian white noise by using only two random variables. Finally, the stochastic Burgers equation is calculated by DPIM, and the probability density functions of different position responses at different times are obtained, which are compared with the probability density functions obtained by Monte Carlo simulation (MCS). It is indicated that the mean and variance of velocity obtained by the two methods are very close, which validates the high accuracy and efficiency of DPIM. Combined with the probability density function curve, mean and variance, it can also be found that with the increase of time, the velocity response of Burgers equation gradually decreases due to the influence of viscosity coefficient in its diffusion term.

1. INTRODUCTION

Turbulent flow is a highly irregular motion, which is usually described as stochastic process in aircrafts, ships and engineering structural design. Burgers equation as a kind of typical model of turbulence was proposed by Dutch scientist J.M. burgers in the late 1930s. At present, the stochastic Burgers equation has been widely used and investigated in various fields, such as traffic flow and cosmology.

The traditional methods for solving the stochastic Burgers equation are using Cole-Hopf transformation, fast Legendre transformation and other methods which can transform the Burgers equation into other types. However, the solutions obtained by these methods mostly focus on the probability density function of the whole flow field, lacking the probability density information of a certain point evolving with the instant. Recently, a novel direct probability integral method (DPIM) was established by the authors' group (Chen and Yang 2019) for uncertainty propagation analysis of stochastic static and dynamic systems uniformly and efficiently. Therefore, this paper proposes the direct probability integration method to the stochastic dynamic analysis of Burgers equation with random initial conditions of Gaussian white noise. Calculation of stochastic Burgers equation by DPIM can not only obtain probability density function of physical quantities of velocity response at certain position and flow field characteristics, but also broaden the research path of turbulence behavior, has important academic and theoretical value.

2. VISCOUS BURGERS EQUATION AND ITS NUMERICAL SOLUTION

The Burgers equation was first proposed as a one-dimensional model in the dynamics of non-pressure gases by Dutch scientist J. M. Burgers in 1930s (Marco and E 1995). In the 1980s, viscous Burgers equation, as an asymptotic form of various nonlinear dissipative systems, became active again and was widely used in various fields, such as fluid mechanics, nonlinear acoustics, aerodynamics, traffic flow (Tian 2021).

The general form of viscous Burgers equation in one-dimensional space is:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \nu\dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where ν is the diffusion coefficient. In this paper, we use the predictor-corrector method to solve this equation. The basic idea of the predictor-corrector method (Lu and Guan 2016) can is dividing time t_n to time t_{n+1} into two parts. First, using the first-order accuracy scheme with good stability to calculate the approximate solution $u_j^{n+1/2}$ at $t_n + \tau/2$, and then using the second-order accuracy scheme to calculate the approximate solution $t_n + \tau$ at $[t_n, t_n + \tau]$. In the second step, the imprecise solution obtained from the first step is used. This method can be specifically expressed as:

$$\begin{cases} \dfrac{u_j^{n+1/2} - u_j^n}{\tau/2} \\ \quad = -u_j^n \dfrac{\mu\delta_x u_j^n}{h} + \nu\dfrac{\delta_x^2 u_j^{n+1/2}}{h^2}, \\ \dfrac{u_j^{n+1} - u_j^n}{\tau} \\ \quad = -u_j^{n+1/2} \dfrac{\mu\delta_x\left(u_j^n + u_j^{n+1}\right)}{2h} + \nu\dfrac{\delta_x^2\left(u_j^n + u_j^{n+1}\right)}{2h^2}. \end{cases} \tag{2}$$

where $\delta_x u_j = u_{j+1/2} - u_{j-1/2}$, $\delta_x^2 u_j = u_{j+1} - 2u_j + u_{j-1}$, $\mu u_j = \frac{1}{2}\left(u_{j+1/2} + u_{j-1/2}\right)$, h is the space step, τ is the time step.

3. HYBRID SPECTRAL REPRESENTATION AND RANDOM FUNCTION TO DESCRIBE STOCHASTIC PROCESS

In the study of stochastic Burgers equation, reasonable modeling and simulation of initial conditions, usually Gaussian white noise stochastic process, is an important basis for analyzing the dynamic response and flow field characteristics of stochastic Burgers equation. In recent years, many methods have emerged to describe stochastic process, of which the theory of spectral representation method is more perfect and the most widely used (Shinozuka and Deodatis 1991). However, spectral representation also has its shortcomings which cannot be ignored, that is, the computational complexity is huge. In order to obtain high accuracy, the series expression of stochastic process often needs to take hundreds of terms, which means that the method needs to calculate hundreds of random variables, which brings great difficulty to the actual analysis.

In order to solve this problem, Liu et al. (2015) proposed a new spectral representation method based on random function for non-stationary process simulation, which reduces hundreds of random variables to two, but still maintains high accuracy, has higher advantages than other methods. This paper chooses this method to simulate the initial condition of Gaussian white noise.

3.1. The first kind of spectral representation of stochastic processes

The first kind of spectrum of simulating real valued non-stationary random processes is expressed as:

$$f(t) = \sum_{k=1}^{N} \sqrt{2S_f \Delta\omega}\left[\cos(\omega_k t)X_k + \sin(\omega_k t)Y_k\right] \tag{3}$$

where S_f is the bilateral power spectral density function, $\Delta\omega = \omega_u / N$, ω_u is the maximum value of ω, $\omega_k = k\Delta\omega$, $\{X_k, Y_k\}$ is a set of standard orthogonal random variables and satisfies:

$$\begin{aligned} &E[X_k] = E[Y_k] = 0 \\ &E[X_j Y_k] = 0 \\ &E[X_j X_k] = E[Y_j Y_k] = \delta_{jk} \end{aligned} \quad (4)$$

3.2. Random function expression form of standard orthogonal random variables

In order to simulate the standard orthogonal random variable $\{X_k, Y_k\}$, we first need to calculate $\{\bar{X}_k, \bar{Y}_k\}$. Take

$$\begin{aligned} &\bar{X}_k = \sqrt{2}\cos(k\theta_1 + \alpha) \\ &\bar{Y}_k = \sqrt{2}\sin(k\theta_2 + \alpha), \quad k = 1, 2, \cdots N \end{aligned} \quad (5)$$

where θ_1 and θ_2 are random variables uniformly distributed on $[-\pi, \pi]$ and independent of each other, constant $\alpha \in [0, 2\pi)$ and usually $\alpha = \pi/4$. There is a mapping relationship between $\{\bar{X}_k, \bar{Y}_k\}$ and $\{X_k, Y_k\}$, that is, $\{X_k, Y_k\}$ is obtained by randomly scrambling the k=1~N terms of $\{\bar{X}_k, \bar{Y}_k\}$, and the mapping operation can be realized by using the functions *rand* ('*state*', 0) and *randperm* (N) in MATLAB software.

3.3. Simulation of Gaussian white noise

The random initial condition $u(x,0) = u_0(x)$ in this paper is Gaussian white noise, which has the following characteristics

$$\begin{aligned} &\langle u_0(x) \rangle = 0 \\ &\langle u_0(x) u_0(y) \rangle = \delta(x - y) \end{aligned} \quad (6)$$

Using the spectral representation method based on random function, we can get the autocorrelation function $R_{u_0}(x, y)$ of u_0 firstly， that is

$$\begin{aligned} R_{u_0}(x,y) &= E[u_0(x) u_0(y)] \\ &= \langle u_0(x) u_0(y) \rangle = \delta(x - y) \end{aligned} \quad (7)$$

According to the formula of power spectral density function, the unilateral power spectral density function of Gaussian white noise are obtained

$$\begin{aligned} G_{u_0} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{u_0}(\tau) e^{-i\omega\tau} \mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\tau) e^{-i\omega\tau} \mathrm{d}\tau = \frac{1}{2\pi} \end{aligned} \quad (8)$$

In Eqs. (3), S_f is the bilateral power spectral density function, and its relationship with the unilateral power spectral density function is

$$G_{u_0} = 2S_{u_0} \quad (9)$$

when N=630, $\omega_u = 80$ rad/s, the Gaussian white noise as shown in Figure 1 can be obtained. To simplify the calculation, we choose the Gaussian white noise within the range of (0,1), as shown in Figure 2.

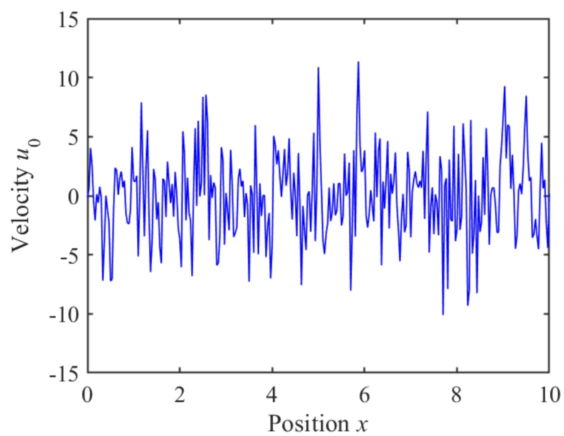


Figure 1: A sampling time series of Gaussian white noise stochastic process.

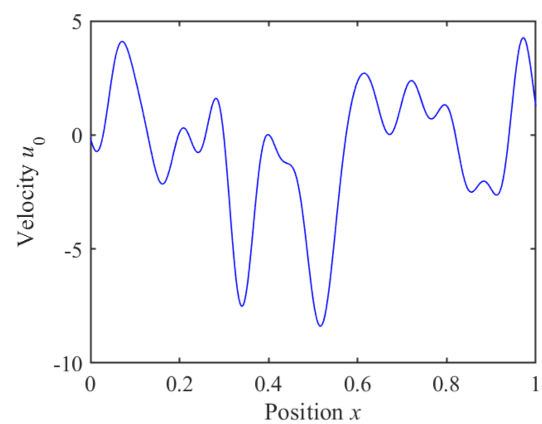


Figure 2: Gauss white noise within x value (0,1).

4. BASIC PRINCIPLE OF DIRECT PROBABILITY INTEGRATION METHOD

4.1. Principle of conservation of probability

The principle of probability conservation can be generally expressed as: probability conservation in the state evolution process of conservative stochastic systems (Li and Chen 2009). Herein, a conservative stochastic system means that neither existing stochastic factors disappear nor new stochastic factors are added in the evolution process of the stochastic system. For a time-independent random system with input random vector Θ and output random vector Y, the principle of probability conservation can be expressed as

$$\int_{\Omega_{\mathbf{Y}}} p_{\mathbf{Y}}(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\Omega_{\mathbf{\Theta}}} p_{\mathbf{\Theta}}(\mathbf{\theta})\mathrm{d}\mathbf{\theta} \tag{10}$$

where $p_{\mathbf{\Theta}}(\mathbf{\theta})$ and $p_{\mathbf{Y}}(\mathbf{y})$ represent the probability density functions of Θ and Y respectively, and $\Omega_{\mathbf{\Theta}}$ and $\Omega_{\mathbf{Y}}$ represent the sample spaces of input random variables and output random variables respectively.

4.2. Probability density integral equation

Based on the principle of probability conservation, we can obtain the relationship between the input random variable and the output random variable:

$$p_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\Theta}(\mathbf{\theta})\delta\left[\mathbf{y} - f(\mathbf{\theta})\right]\mathrm{d}\mathbf{\theta} \tag{11}$$

For time-varying systems, a similar probability density integral equation is expressed as

$$p_{\mathbf{Y}}(\mathbf{y},t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\Theta}(\mathbf{\theta})\delta\left[\mathbf{y} - f(\mathbf{\theta},t)\right]\mathrm{d}\mathbf{\theta} \tag{12}$$

4.3. Numerical solution steps of direct probability integration method

There are four steps to solve the stochastic response of Burgers equation with initial random condition applying DPIM:

1. Partition the input probability space, select the representative point sets based on Generalized-F discrepancy, and calculate the assigned probability of the representative points.
2. Utilize the representative points to calculate representative response.
3. Smooth the Dirac delta function with using Gaussian function.
4. Compute the probability density functions of the response of interest.

Through the partition of probability space and the smoothing of Dirac delta function, Eqs. (11) and (12) converted to Eqs. (13) and (14):

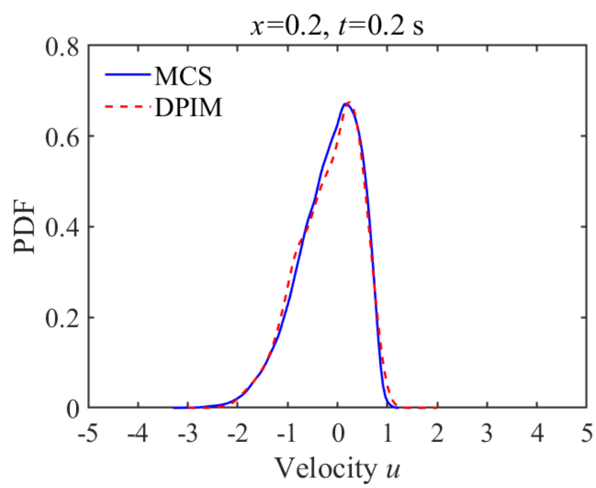
$$p_{\mathbf{Y}}(\mathbf{y}) = \sum_{q=1}^{N} \left\{ \frac{1}{\sqrt{2\pi}\sigma} e^{-[y-f(\mathbf{\theta}_q)]^2/2\sigma^2} P_q \right\} \tag{13}$$

$$p_{\mathbf{Y}}(\mathbf{y},t) = \sum_{q=1}^{N} \left\{ \frac{1}{\sqrt{2\pi}\sigma} e^{-[y-f(\mathbf{\theta}_q,t)]^2/2\sigma^2} P_q \right\} \tag{14}$$

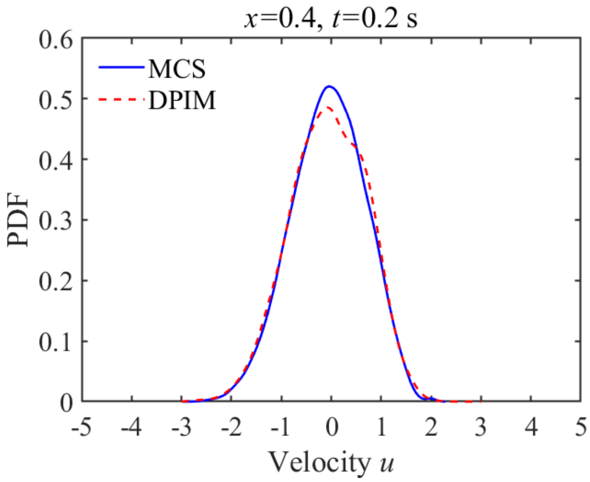
where $\mathbf{\theta}_q$ indicates the qth representative point in probability space, P_q means assigned probability of $\mathbf{\theta}_q$. We can employ Eqs. (13) and (14) to calculate the probability density function of the output random variable.

5. EXAMPLE ANALYSIS OF SOLVING STOCHASTIC BURGERS EQUATION

Assume that the initial condition in Eqs. (1) is Gaussian white noise as shown in Figure 2, and we solve the Burgers equation with initial random condition by DPIM.



(a)



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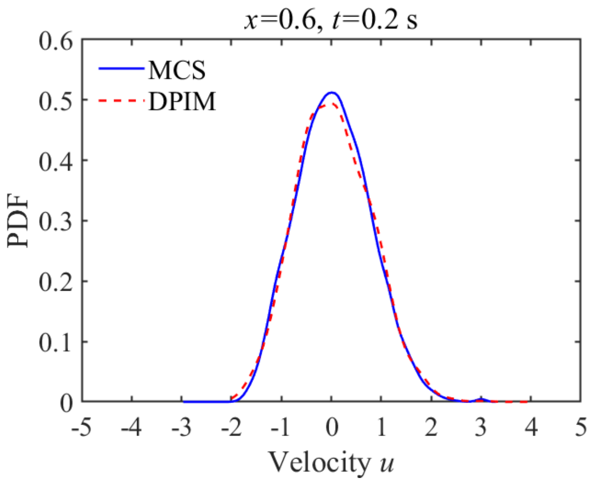

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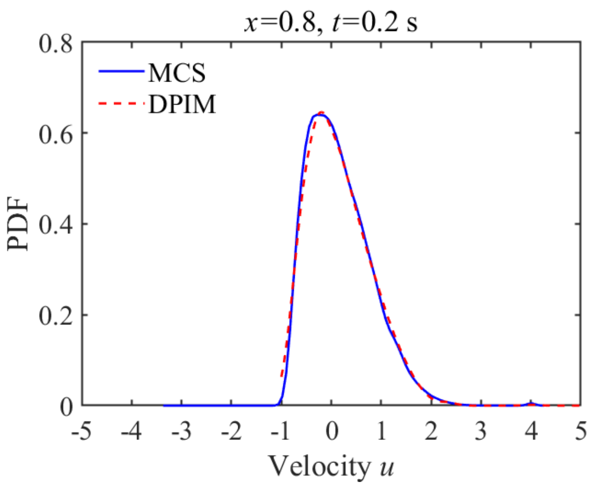


(d)

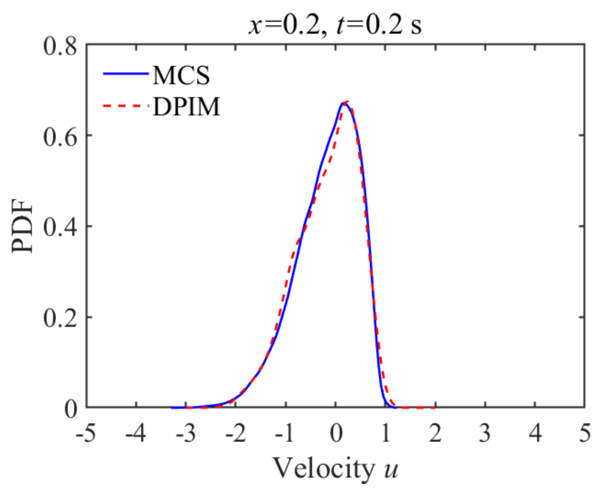
Figure 3: Probability density function curves of stochastic Burgers equation velocity with $t = 0.2$ s *at different positions* (*a*) *x*=0.2, (b) *x*=0.4, (c) *x*=0.6, (*d*) *x*=0.8.

The probability density functions of different positions at each time are computed, with taking $t = 0.2$ s as an example, as shown in Figure 3. Combining the ordinate and diagrams, it is found that on the interval (0,1), the closer to the center $x = 0.5$ the more dispersed the response distribution is. And when the position is symmetric about $x = 0.5$, the probability density function graph is also symmetric.

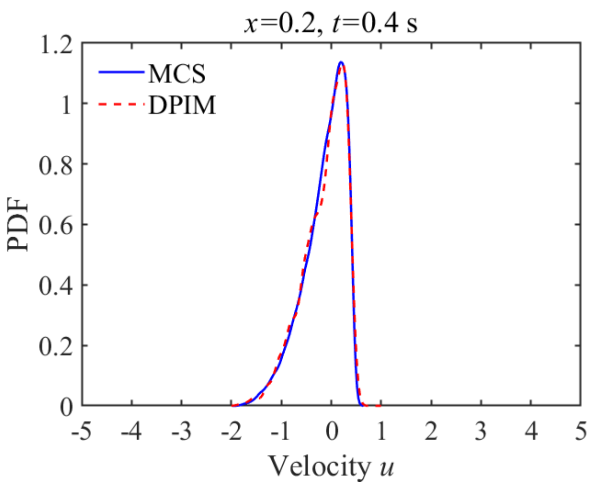
With the increase of the value of *t*, this phenomenon becomes more and more obvious, which will be demonstrated later by comparing the mean value of symmetric positions.

Then the probability density functions of velocity *u* at different times at the same position are solved, with taking $x = 0.2$ as an example, as shown in Figure 4. It is found that with the increase of *t* value, the probability density function curve of velocity tends to be large amplitude and narrow deviation, and the distribution of velocity *u* becomes centralized.

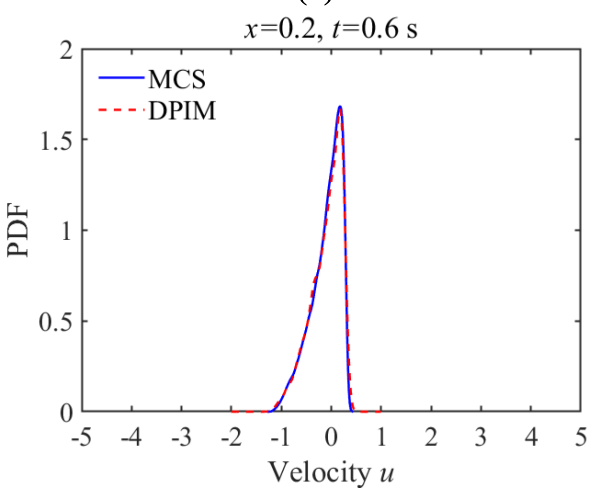
Moreover, we analyze the mean and variance of flow velocity *u* obtained by DPIM and MCS quantitatively, and calculate the relative error. The results are shown in Table 1. It is seen that the mean and variance obtained by using DPIM and MCS are very close. The maximum error is only 3.15%, not more than 5%, and most of the errors are not more than 1%, reflecting the advantages of DPIM in accuracy.



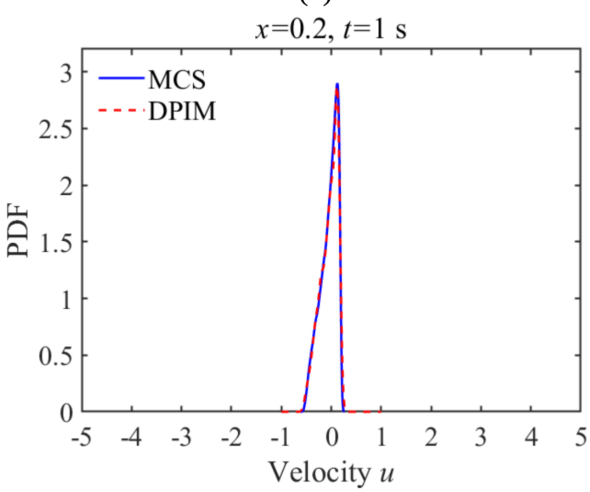
(a)



(b)



(c)



(d)

Figure 4: Probability density function of stochastic Burgers equation velocity at $x = 0.2$ *and different times* (*a*) *t*=0.2 s, (b) *t*=0.4 s, (c) *t*=0.6 s, (*d*) *t*=1.0 s.

Table 1: Mean, variance and relative error of velocity u calculated by DPIM and MCS.

		x = 0.2		x = 0.4	
		mean	variance	mean	variance
t = 0.2 s	DPIM	−0.165	0.368	−0.058	0.550
	MCS	−0.165	0.376	−0.058	0.538
	relative error	0.30%	2.02%	1.39%	2.27%
t = 0.4 s	DPIM	−0.144	0.181	−0.065	0.257
	MCS	−0.141	0.176	−0.066	0.251
	relative error	2.06%	2.96%	0.30%	2.23%
t = 0.6 s	DPIM	−0.103	0.095	−0.051	0.145
	MCS	−0.101	0.092	−0.051	0.143
	relative error	1.58%	3.04%	0.98%	0.84%
t = 0.8 s	DPIM	−0.047	0.030	−0.026	0.057
	MCS	−0.047	0.029	−0.027	0.056
	relative error	0.86%	2.03%	0.75%	1.60%
		x = 0.6		x = 0.8	
		mean	variance	mean	variance
t = 0.2 s	DPIM	0.049	0.5541	0.157	0.389
	MCS	0.050	0.5523	0.162	0.401
	relative error	1.99%	0.33%	3.14%	2.97%
t = 0.4 s	DPIM	0.057	0.2528	0.136	0.180
	MCS	0.057	0.2530	0.137	0.181
	relative error	0.88%	0.08%	0.73%	0.22%
t = 0.6 s	DPIM	0.0449	0.1427	0.097	0.093
	MCS	0.044	0.1434	0.097	0.093
	relative error	2.28%	0.49%	0.10%	0.54%
t = 0.8 s	DPIM	0.023	0.0569	0.044	0.029
	MCS	0.0222	0.0563	0.044	0.029
	relative error	3.15%	1.07%	1.37%	0.00%

Meanwhile, it is observed that with the increase of t value, the mean and variance at $x = 0.2$ and $x = 0.8$ positions of velocity in flow field are decreasing, but the mean velocity value of $x = 0.4$ and $x = 0.6$ positions shows a small increase at $t = 0.4$ s , but then resumes the decreasing trend.

Because the initial condition of Gaussian white noise is random, when the time t is small, the velocity u of turbulence field is greatly affected by the initial condition, it may increase. However, with the increase of time, the influence of the initial conditions is gradually less than that of the diffusion term of the stochastic Burgers equation itself, and the viscosity coefficient in the diffusion term will inhibit the increase in the value of velocity *u*, resulting in the phenomenon that the mean and variance decrease after $t = 0.4\text{s}$. And when the value of t is small, the phenomenon of "small increase" is more likely to occur at the position close to the center $x = 0.5$, such as $x = 0.4$ and $x = 0.6$. Note that at the position close to the boundary, like $x = 0.2$ and $x = 0.8$, the velocity *u* is not only affected by the initial conditions, but also more easily affected by the boundary conditions. Therefore, unlike the position close to the center, the possibility of "small increase" at the position close to the boundary is lower.

In addition, as mentioned earlier, the probability density function graph has the possibility of symmetry. Let's reanalyze it from the perspective of mean value. It can be seen from Table 2 that the absolute error of the absolute value of the mean value at two symmetrical positions with $x = 0.5$ is indeed getting smaller and smaller, indicating that the probability density function of the symmetrical position tends to be symmetric with the increase of time.

Table 2: Absolute error of the mean at two symmetrical positions with respect to x=0.5.

	x = 0.2/0.8	x = 0.4/0.6
t = 0.2 s	0.0083	0.0092
t = 0.4 s	0.0076	0.0081
t = 0.6 s	0.0057	0.0058
t = 1 s	0.0028	0.0035

6. CONCLUSIONS

The viscous Burgers equation with random initial condition of Gaussian white noise is solved by DPIM, and the probability density functions of velocity response of flow field at different time and different position are obtained. The graphs of the probability density function are analyzed. Moreover, the mean and variance of the velocity *u* corresponding to the stochastic viscous Burgers equation obtained by DPIM and MCS are calculated. The main conclusions are drawn as follows:

(1) At the same time, near the center, the velocity distribution of flow field represented by stochastic Burgers equation is more dispersed.
(2) At the same position, with the increase of *t* value, the velocity distribution is more centralized.
(3) Compared with the results of MCS, it is indicated that DPIM has high accuracy and efficiency for computing the PDF of stochastic dynamic response.
(4) The probability density function of the symmetrical position in stochastic Burgers equation tends to be symmetric with the increase of time.

The innovation of this paper is to introduce the newly developed efficient method of stochastic dynamic system analysis, the direct probability integration method, into the investigation of Burgers equation of stochastic fluid mechanics. The probability density functions of the velocity response of the flow field are directly calculated through the probability density integral equation, which significantly improves the computational efficiency compared with the MCS results.

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