Form Factors, Integrability and the AdS/CFT Correspondence

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Declaration

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This thesis is based in part on the articles [1].

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Summary

Form factors are matrix elements of local operators between scattering states, and are interesting off-shell objects in any QFT. The main objective of this thesis is to compute form factors perturbatively in the world-sheet theory describing strings in $AdS_5 \times S^5$, in the Landau–Lifshitz model and in a number of its generalizations.

The S-matrix bootstrap allows to determine completely the scattering processes of an integrable two-dimensional QFT. The world-sheet theory for type IIB superstrings in $AdS_5 \times S^5$ is believed to be integrable and the S-matrix has been computed, though we do not have a complete bootstrap program for the form factors yet. This would amount to solving the relevant set of form factor axioms. These are consistency conditions which, for a massive integrable relativistic theory, can be derived from the validity of the LSZ formalism and the hypothesis of "maximal analyticity", and allow in principle to write any form factor explicitly from the knowledge of the S-matrix. Analogous axioms for the world-sheet have been proposed, however finding a general solution for said axioms is still an open problem.

Perturbative form factor calculations have been already carried out for the $\mathfrak{su}(2)$ sector of the world-sheet string. One of the goals of this thesis is to extend this computation of the tree-level three-particle form factor to the full theory. We also discuss a particular configuration, the so-called diagonal form factors, which are related to the structure constants of "Heavy-Heavy-Light" three-point functions in the string field theory.

We also study the Landau–Lifshitz model, a non-relativistic theory that can be obtained as a thermodynamic limit of the Heisenberg $XXX_{1/2}$ spin chain, which is related to the dilatation operator in $\mathcal{N} = 4$ SYM. The LL model also emerges as a double limit of the $AdS_5 \times S^5$ string, and it has proved a useful tool in the context of the AdS/CFT correspondence to better understand the matching between the energies of on-shell string states and the anomalous dimensions of SYM operators. To study the higher-order contributions, we will work with a generalized LL model which includes all the terms allowed by symmetry at that order with generic constants. These can be fixed to match the "gauge"- or "string"-LL model, since it has been proved that the two models match only up to $\mathcal{O}(\lambda^2)$.

The Landau–Lifshitz action has also been used to explain how the world-sheet

form factors could be matched to spin-chain matrix elements and consequently to structure constants of tree-level gauge theory three-point functions. Moreover, the structure of the LL model allows the perturbative computation of the S-matrix and the form factors to all loops. The final goal of this thesis is to compute all-loop form factors in the generalized LL model derived from the world-sheet string, and compare the leading- and first-order results to the known form factors in the corresponding spin chain.

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Chapter 1

Introduction

1.1 The AdS/CFT correspondence

The idea of a correspondence between a gauge theory of particle interactions and a theory describing vibrating strings has its roots in the attempt to give an adequate description of strongly correlated quantum field theories (QFT), such as quantum chromodynamics (QCD), which describes quarks and gluons. In particular, the goal was to gain novel insights into the strong regime of QCD through the study of a dual theory at weak coupling.

After much progress on the string theory side, the turning point was Maldacena's seminal paper [2] in 1997, where he proposed a relationship (duality) between superstrings on anti-de Sitter (AdS) backgrounds and (super)conformal field theories (CFT), which is now known as the AdS/CFT correspondence or more generally as gauge/gravity duality. In particular, he noticed that the type IIB string on the ten-dimensional curved background $AdS_5 \times S^5$ was related to $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM) with gauge group $SU(N)$. The assumption is that the two theories are equivalent in the sense that we can build a precise dictionary between the objects on the string side and those on the gauge side, and computations of corresponding physical quantities yield identical results. The starting point for the comparison is the following conjectured relation between the parameters of the two theories

$$\frac{R^4}{\alpha'^2} = g_{YM}^2 N \,, \qquad 4\pi g_s = g_{YM}^2 \,, \tag{1.1.1}$$

where g_{YM} is the SYM coupling constant, N the degree of the gauge group, g_s the string coupling constant, $(2\pi\alpha')^{-1}$ the string tension, and R the radius of the AdS_5

and S^5 spaces.

A key characteristic of the gauge/string duality is that it relates strongly coupled sectors on one side to weakly coupled sectors in the corresponding theory. Thus it provides a method to study perturbatively the strongly interacting regime of gauge theories, and it also allows to explore quantum gravity through gauge theory computations. On the other hand, the strong-weak relations make testing the AdS/CFT correspondence a challenging task. Crucial steps in this direction were made in 2002 by Berenstein, Maldacena and Nastase [3] and by Minahan and Zarembo [4].

At first the analysis was focused on the low-energy limit of specific string solutions and protected SYM operators, i.e. those not receiving quantum corrections, the goal being to compare the energy of the strings with the conformal dimensions of the operators (the eigenvalues of the dilatation operator). The idea of BMN was to consider instead a particular limit[1] of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string, which corresponds to unprotected operators with large R-charge. Minahan and Zarembo noticed that the terms in the λ-expansion of the dilatation operator D are equal to the Hamiltonian of some integrable spin chain [4], which is a one-dimensional lattice model with a spin representation in every site. This means that we can find the loop corrections to the classical dimension by solving the corresponding spin chain. The computations are greatly simplified since we can utilize the useful tools of integrability built over the years, starting from Bethe's 1931 ansatz method [5] to solve the spectral problem, i.e. to find eigenvalues and eigenvectors of the spin-chain Hamiltonian.

1.2 Integrability in AdS/CFT

The integrable structure in $\mathcal{N}=4$ super Yang–Mills, and in the corresponding string theory, is of the type usually associated with two-dimensional systems, such as the Heisenberg $\mathrm{XXX}_{1/2}$ ferromagnet. This is the first example of a non-trivial four-dimensional integrable theory, and it has led to a great improvement in our understanding of both string theory and SYM, see [6] for a detailed review. In principle, integrability allows to obtain any quantity of the theory exactly (non perturbatively), though in

[1] BMN studied strings on the plane-wave background, which can be obtained as a Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$. However, in order to obtain the complete algebraic structure (two centrally extended $\mathfrak{su}(2|2)$) we actually need to start from the string on $\mathrm{AdS}_5\times\mathrm{S}^5$.

practice we expect to be able to write down the equations but not always to be able to solve them.

To see how integrability emerges in the AdS/CFT correspondence, we need to recall that in conformal theories, the structure of the two-point functions is completely determined by their conformal dimensions, i.e. the eigenvalues of the dilatation operator D. For small values of the 't Hooft coupling $\lambda = g_{YM}^2 N$, the $\mathcal{N} = 4$ super-Yang-Mills (SYM) dilatation generator can be computed perturbatively as

$$D = \sum_{r=0}^{\infty} \left(\frac{\lambda}{16\pi^2} \right)^r D_{2r} \,. \tag{1.2.1}$$

If we restrict ourselves for simplicity to single trace operators composed of just two types of complex scalars, that is an $\mathfrak{su}(2)$ sub-sector of the full theory, the one-loop part can be mapped to the Heisenberg $\text{XXX}_{1/2}$ spin-chain Hamiltonian [4]

$$H = 2 \sum_{\ell=1}^{L} (\mathbb{1} - P_{\ell,\ell+1}) \,, \tag{1.2.2}$$

where $P_{\ell,\ell+1}$ is the permutation operator acting on sites ℓ and $\ell+1$. The eigenvalues and eigenvectors of the spin-chain Hamiltonian (1.2.2) can be found by using integrability techniques such as the coordinate Bethe ansatz [5] or the algebraic Bethe ansatz (reviewed in [7]), where we can write explicitly the conserved charges to show the integrability of the model. Higher orders in λ in the expansion of the dilatation operator (1.2.1) correspond to Hamiltonian of spin-chains with longer range interactions.

1.3 Form Factors

Integrability in a two-dimensional quantum field theory greatly constrains the possible outcomes of a scattering process. We have that there is no particle production, the set of the momenta of the final state is identical to the one for the initial state, and the scattering between n particles factorizes into a product of $2 \to 2$ scattering processes. The S-matrix is the quantity describing the scattering, and thus in an integrable QFT in two dimensions the two-particles S-matrix completely determine the on-shell properties of the theory. The S-matrix can be derived from the symmetries of the theory, with the assumptions of unitarity and analyticity, thanks to the bootstrap program [8], see also [9].

The S-matrix can also be used to calculate off-shell quantities, such as the form factors, which are matrix elements of local operators between scattering states. Explicitly we can define the form factor of the operator at the origin $\mathcal{O}(\vec{x}=0)$ in terms of the particles' momenta p_i as

$$F^{\mathcal{O};i'_m,\ldots,i'_1}_{i_1,\ldots,i_n}(p'_m,\ldots,p'_1|p_1,\ldots,p_n) = {}^{i'_m,\ldots,i'_1}\langle p'_m,\ldots,p'_1|\,\mathcal{O}(0)\,|p_1,\ldots,p_n\rangle_{i_1,\ldots,i_n} \ . \quad (1.3.1)$$

According to the bootstrap approach, in principle we should be able to derive the form factors for an integrable two-dimensional QFT from the knowledge of the S-matrix and the symmetries of the theory, solving the so-called form factor axioms [10]. The axioms are a set of consistency conditions, which have been written for a relativistic theory in [11] and for the world-sheet string in [12, 13], though their general solutions are not known.

The diagonal form factors are a special class of form factors in which the two asymptotic states are taken to be identical, i.e. $n = m$ and $|p'_1,\ldots,p'_m\rangle = |p_1,\ldots,p_n\rangle$ in (1.3.1). These are particularly interesting in the context of the AdS/CFT correspondence as they are related to the structure constants of "Heavy-Heavy-Light" three-point functions [14, 15]. It was proposed in [14] that the dependence of structure constants on the length, L, of the heavy operators is given by finite volume diagonal form factors in integrable theories. This was confirmed at one-loop in [16] and, based on the Hexagon approach [17], at higher loops in [18, 19]. More generally, diagonal form factors are related to the study of non-integrable deformations of integrable theories [20] and can be used to determine the corrections to the vacuum energy, mass matrix and S-matrix.

1.4 The Landau–Lifshitz model

In [12, 13], when studying world-sheet form factors in the context of the AdS/CFT correspondence, the Landau–Lifshitz action was used to explain how they could be matched to spin-chain matrix elements and consequently to structure constants of tree-level gauge theory three-point functions. The Landau–Lifshitz (LL) model [21] is a non-relativistic σ-model on the unit sphere, which was originally introduced to describe the distribution of magnetic moments in a ferromagnet. The equations of motion includes the Heisenberg ferromagnet equation as a special case

$$\frac{\partial \vec{n}}{\partial t} = \vec{n} \times \frac{\partial^2 \vec{n}}{\partial x^2} \tag{1.4.1}$$

where $\vec{n}(x,t)$ is a three-dimensional vector living on the unit sphere, $\vec{n} \cdot \vec{n} = 1$. In large part because it was found to be integrable [22], this model has subsequently been the focus of a great deal of interest in a number of different contexts. It has played a significant role in the study of the AdS/CFT correspondence where it acted as a partial bridge between the spin-chain and string descriptions of gauge invariant operators.

In the thermodynamic limit the low-energy excitations about the ferromagnetic vacuum are described by an effective two-dimensional LL action [23, 24] and the same LL action can be found as the so-called "fast-string" limit of the bosonic string action on $\mathbb{R}\times S^3$ [24, 25, 26]. This proved a useful tool in developing the understanding of the match between the energies of on-shell string states and anomalous dimensions at this order. Generalizations of the LL action describing larger sectors of the gauge theory were studied in [27, 28, 29, 30] and a $\mathfrak{psu}(2,2|4)$ LL model arising from the thermodynamic limit of the complete one-loop $\mathcal{N} = 4$ SYM dilatation generator was constructed in [31].

Extending the analysis beyond the leading order in λ results in a generalized LL action with higher-derivative terms. The effective LL action to $\mathcal{O}(\lambda^2)$ was found in [25], however beyond $\mathcal{O}(\lambda^2)$ the two LL actions following from the spin-chain and string theory disagree. The "gauge"-LL action to order λ^3 was found in [32] by including all six-derivative terms allowed by symmetries and fixing the coefficients by matching with the energies of known solutions and was shown to disagree with the "string"-LL action following from the fast-string limit (see also [33, 34, 35]).

The LL model and its generalizations can of course be considered as two-dimensional integrable quantum field theories in their own right and their quantization studied. The quantization of the anisotropic LL model was studied by means of the quantum inverse scattering method and involves a number of subtleties [36]. An alternative approach is to formally introduce a small parameter and perform a perturbative calculation [34, 32, 35]. This can be efficiently carried out by using the Feynman diagrammatic expansion, with the small parameter acting as a loop counting parameter, and then attempting to resum all the resulting diagrams. The quantum S-matrix for the LL model was computed in this fashion in [37] and generalized in [38] to include higher-order λ corrections. In an integrable theory it is expected that the three-particle S-matrix factorizes into the product of two-particle S-matrices, however due to the subtleties of the LL model this is non-trivial and has only been explicitly demonstrated at one-loop [39], see also [40].

1.5 Plan of the thesis

We begin in *Chapter 2* by reviewing the world-sheet string theory on $AdS_5 \times S^5$ in the uniform light-cone gauge. We will discuss its large tension limit and its quantization in the decompactification limit. The world-sheet symmetries in the Hopf algebra interpretation will also be examined.

We then turn to $\mathcal{N} = 4$ SYM in *Chapter 3*, presenting the dilatation operator with the goal of highlighting how integrability emerges in the AdS/CFT correspondence. Motivated by the connection to the spin-chains, we will review the Bethe ansatz for the $XXX_{1/2}$ Heisenberg spin chain and explain how it can be interpreted algebraically.

In *Chapter 4* we introduce the form factors for a generic two-dimensional theory and analyze in detail their properties in the world-sheet string case. We will then present some perturbative results, including the complete tree-level three-particle form factor for the world-sheet string.

Finally in *Chapter 5* we introduce the Landau-Lifshitz action and study its properties. We will show how it is feasible to compute perturbatively form factors to all loops and compare our results with the spin-chain form factors calculated in [16].

Chapter 2

Superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$

String theory is the extension to one-dimensional objects (strings) of the relativistic description of the point-like particles, see [41] for an introduction to the subject. More specifically, we are interested in strings on the curved $\mathrm{AdS}_5\times\mathrm{S}^5$ background in the uniform light-cone gauge, which we will introduce in this chapter following [42]. We will also discuss the algebra $\widehat{\mathfrak{psu}}(2|2)$, i.e. the algebraic structure of the resulting string theory, and the action of the symmetries on the fields.

2.1 The $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring

2.1.1 The bosonic string

We will start from the usual action for a particle in a generic N-dimensional space with metric G_{AB}

$$\mathcal{A} = -m\int ds = -m\int \mathrm{d}\tau \left(G_{AB}\,\partial_\tau X^A(\tau)\partial_\tau X^B(\tau)\right)^{1/2}, \tag{2.1.1}$$

where $A, B = 1, ..., N$ and ∂_τ is the (partial) derivative with respect to τ. The world-line of the particle is replaced by the world-sheet, i.e. the two-dimensional surface in space-time swept by the string, parametrized by (τ,σ) or equivalently (σ^0,σ^1). If we define the world-sheet space σ in the interval $-L/2 \leq \sigma \leq L/2$, where L is the length of the string, then the time-evolution of the end-points of the string will be described by $X^A(\tau,-L/2)$ and $X^A(\tau,L/2)$. The strings satisfying $X^A(\tau,-L/2) = X^A(\tau,L/2)$ are called closed, while the ones with distinct end-points are called open. In the following, we will consider closed strings, for which the coordinate σ is periodic with period L and the world-sheet is a cylinder.

Introducing the world-sheet metric $\gamma_{ij}(\tau,\sigma)$ with Lorentzian signature $(-,+)$, the action for the classical string can be written as a σ-model [43]

$$\mathcal{A} = -\frac{1}{4\pi\alpha'} \int_{-L/2}^{L/2} \mathrm{d}\sigma \mathrm{d}\tau \sqrt{-\gamma}\gamma^{ij} G_{AB} \partial_i X^A \partial_j X^B \,, \tag{2.1.2}$$

where γ and γ^{ij} are respectively the determinant and the inverse of $\gamma_{ij}(\tau,\sigma)$, and ∂_0 (∂_1) is the partial derivative w.r.t. τ (σ). Moreover, α' is the string (Regge) slope, related to the string tension T through $T^{-1} = 2\pi\alpha'$, and the dependence of the field X^A on the world-sheet coordinates is understood. The action (2.1.2) is called the Polyakov action and describes only commuting particles, i.e. bosons. Its generalization to include fermions will be discussed in Section 2.1.3.

Let us consider now the symmetries of the Polyakov action (2.1.2). We have the usual Poincaré invariance and two additional local symmetries, namely the reparameterization invariance for the two worldsheet coordinates τ, σ and the so-called (two-dimensional) Weyl invariance, which is a local rescaling of the world-sheet metric

$$X^A(\tau,\sigma) \to X^A(\tau,\sigma) \,, \quad \gamma_{ij}(\tau,\sigma) \to e^{2\chi(\tau,\sigma)}\gamma_{ij}(\tau,\sigma) \,, \tag{2.1.3}$$

where χ is an arbitrary (scalar) function of the world-sheet coordinates.

These gauge symmetries can be used to simplify the action, removing the unphysical degrees of freedom of the theory. For example, we could fix the auxiliary metric γ_{ij} to the flat space metric[1] η_{ij}. However, with this choice we lose information, namely the equations of motion for the energy-momentum tensor[2]

$$T_{ij} \equiv \frac{-2}{\sqrt{-\gamma}} \frac{\delta \mathcal{A}}{\delta \gamma^{ij}} = 0 \,. \tag{2.1.4}$$

As a consequence we have to impose the equations (2.1.4) as additional conditions, which are called Virasoro constraints. Let us note that since the energy-momentum tensor is symmetric $T_{01} = T_{10}$ and traceless $-T_{00} + T_{11} = 0$, we have only two independent equations. For example, in flat space they are

$$T_{00} = \frac{1}{2}(\dot{X}^2 + \acute{X}^2) = 0 \,, \qquad T_{01} = \dot{X} \cdot \acute{X} = 0 \,, \tag{2.1.5}$$

[1]There are three degrees of freedom in the metric γ_{ij} and we can fix two from the reparameterization symmetry and one from the Weyl symmetry.

[2]In string theory there is usually an additional factor of -2π in the definition for convenience.

where the dot and prime stand for the derivative w.r.t. τ and σ respectively. This analysis highlights the basic features of the gauge fixing of (2.1.2), as we will see more precisely in the following for a particularly convenient choice: the light-cone gauge.

2.1.2 The uniform light-cone gauge

We will now introduce the uniform light-cone gauge for the bosonic string action (2.1.2) with the $AdS_5 \times S^5$ metric, following [42]. Let us single out two coordinates, e.g. a time-like coordinate t and an angle ϕ, and assume that the theory is invariant under shifts in these two coordinates, which is true when the action depends on t and ϕ only through their derivatives. In $AdS_5 \times S^5$ we can choose $t = X^0$, the global time coordinate on AdS_5, and $\phi = X^5$, the angle[3] parametrizing the equator of S^5. The other 8 coordinates x^μ, with $\mu = 1, \ldots, 8$, are called "transverse", where we renamed them $x^\mu = (y^\alpha, z^\alpha)$ with $y^\alpha = X^\alpha$ and $z^\alpha = X^{\alpha+5}$, $\alpha = 1, \ldots, 4$. We can write the $AdS_5 \times S^5$ metric as

$$\begin{aligned} \frac{ds^2}{R^2} &= G_{AB}\, dX^A dX^B = -G_{tt}dt^2 + G_{\phi\phi}d\phi^2 + G_{\mu\nu}\, dx^\mu dx^\nu \\ &= -G_{tt}dt^2 + G_{zz}dz^2 + G_{\phi\phi}d\phi^2 + G_{yy}dy^2 \end{aligned} \tag{2.1.6}$$

where R is the radius of the anti-de Sitter space, which is equal to the radius of the sphere, and the metric depends only on the transverse coordinates. Explicitly, we have

$$\begin{aligned} G_{tt} &= \left(\frac{1+z^2/4}{1-z^2/4}\right)^2, \qquad G_{zz} = \left(\frac{1}{1-z^2/4}\right)^2, \\ G_{\phi\phi} &= \left(\frac{1-y^2/4}{1+y^2/4}\right)^2, \qquad G_{yy} = \left(\frac{1}{1+y^2/4}\right)^2. \end{aligned} \tag{2.1.7}$$

This choice of variables is referred to as a global coordinatization of $AdS_5 \times S^5$.

We will consider the action (2.1.2)

$$\mathcal{A} = -\frac{R^2}{4\pi\alpha'} \int_{-L/2}^{L/2} \mathrm{d}\sigma \mathrm{d}\tau \sqrt{-\gamma}\gamma^{ij} G_{AB} \partial_i X^A \partial_j X^B\,, \tag{2.1.8}$$

[3] The choice of the angle is not unique, and in some cases it is more convenient to choose ϕ on AdS_5.

where now G_{AB} is the $\mathrm{AdS}_5\times\mathrm{S}^5$ metric (2.1.6). We introduce canonical momenta, conjugate to the coordinates X^A, as

$$p_A = \frac{\delta\mathcal{A}}{\delta\dot{X}^A} = -g\gamma^{0k}\partial_k X^B G_{AB}\,, \tag{2.1.9}$$

with g being the effective dimensionless string tension defined as

$$g = \frac{R^2}{2\pi\alpha'}\,. \tag{2.1.10}$$

Then we rewrite the action (2.1.8) as

$$\mathcal{A} = \int_{-L/2}^{L/2} \mathrm{d}\sigma\mathrm{d}\tau \left(p_A\dot{X}^A + \frac{\gamma^{01}}{\gamma^{00}}C_1 + \frac{1}{2g\gamma^{00}}C_2 \right)\,, \tag{2.1.11}$$

with

$$C_1 = p_A\acute{X}^A\,,\quad C_2 = G^{AB}p_Ap_B + g^2\acute{X}^A\acute{X}^B G_{AB}\,, \tag{2.1.12}$$

and we need to solve the Virasoro constraints $C_1 = 0$ and $C_2 = 0$ in the chosen gauge.

It is also convenient to introduce the light-cone coordinates x^+, x^- as the following linear combinations of t and ϕ

$$x^- = \phi - t\,,\qquad x^+ = (1-a)\,t + a\,\phi\,, \tag{2.1.13}$$

and the corresponding light-cone momenta

$$p_- = p_\phi + p_t\,,\qquad p_+ = (1-a)\,p_\phi - a\,p_t\,, \tag{2.1.14}$$

where the number a parametrizes the possible gauge choices in which p_- is equal to the sum $p_\phi + p_t$. In these coordinates the Virasoro constraints (2.1.12) become

$$\begin{aligned} C_1 &= p_+\acute{x}^- + p_-\acute{x}^+ + p_\mu\acute{x}^\mu\,,\\ C_2 &= \widetilde{G}^{--}p_-^2 + 2\widetilde{G}^{+-}p_+p_- + \widetilde{G}^{++}p_+^2 \\ &\quad + g^2G_{--}(\acute{x}^-)^2 + 2g^2G_{+-}\acute{x}^+\acute{x}^- + g^2G_{++}(\acute{x}^+)^2 + \mathcal{H}_x\,, \end{aligned} \tag{2.1.15}$$

with

$$\begin{aligned}
\widetilde{G}^{--} =& a^2 G_{\phi\phi}^{-1} + (a-1)^2 G_{tt}^{-1}, \quad & \widetilde{G}^{+-} =& a G_{\phi\phi}^{-1} + (a-1) G_{tt}^{-1}, \quad & \widetilde{G}^{++} =& G_{\phi\phi}^{-1} + G_{tt}^{-1}, \\
G_{--} =& (a-1)^2 G_{\phi\phi} - a^2 G_{tt}, \quad & G_{+-} =& -(a-1) G_{\phi\phi} - a G_{tt}, \quad & G_{++} =& G_{\phi\phi} - G_{tt}\,,
\end{aligned}$$

and the dependence on the transverse coordinates x^μ collected in

$$\mathcal{H}_x = G^{\mu\nu} p_\mu p_\nu + g^2 \acute{x}^\mu \acute{x}^\nu G_{\mu\nu}\,. \tag{2.1.16}$$

We now fix the uniform light-cone gauge by imposing the conditions[4]

$$x^+ = \tau + \frac{2\pi}{L} a\, m\, \sigma, \qquad p_+ = 1\,. \tag{2.1.17}$$

where m is an integer which counts the number of times the string winds around the circle parametrized by ϕ, since we have $\phi(L/2) - \phi(-L/2) = 2\pi\, m$. Finally, we can solve the Virasoro constraints for $\acute{x}^-$ and p_- to obtain the gauge fixed action[5]

$$\mathcal{A} = \int_{-L/2}^{L/2} \mathrm{d}\sigma \mathrm{d}\tau\, \left(p_\mu \dot{x}^\mu - \mathcal{H} \right)\,, \qquad \mathcal{H} = -p_-(x^\mu, p_\mu)\,, \tag{2.1.18}$$

where $\mathcal{H}$ is the Hamiltonian density of the gauge-fixed model.

From the invariance of the action under translations in t and ϕ, it follows the existence of two conserved quantities

$$E = -\int_{-L/2}^{L/2} \mathrm{d}\sigma\, p_t\,, \qquad J = \int_{-L/2}^{L/2} \mathrm{d}\sigma\, p_\phi\,, \tag{2.1.19}$$

which are respectively the target space-time energy and the total angular momentum of the string in the direction ϕ. From the relations (2.1.14) we also have

$$P_- = -\int_{-L/2}^{L/2} \mathrm{d}\sigma\, p_- = J - E\,, \qquad P_+ = \int_{-L/2}^{L/2} \mathrm{d}\sigma\, p_+ = (1-a) J + a\, E\,. \tag{2.1.20}$$

The name of the uniform light-cone gauge comes from the fact that p_+ is independent of σ and thus the light-cone momentum P_+ is uniformly distributed along the string.

[4]This is a generalization of the usual light-cone gauge, which corresponds to $a = 1/2$.
[5]Where we also dropped the τ-derivative of the zero mode of x^-, which is a total derivative.

Moreover, the condition (2.1.17) implies that the length of the string is $L = P_+$ and depends on the choice of the gauge a. In other words the light-cone string is defined on a cylinder of circumference P_+ and the Hamiltonian depends on P_+ only through the integration bounds. To summarize, the relations between these conserved quantities are

$$H = \int_{-L/2}^{L/2} \mathrm{d}\sigma\, \mathcal{H} = -P_- = E - J\,, \qquad L = P_+ = (1-a)J + a\,E\,. \tag{2.1.21}$$

In the AdS/CFT correspondence we are interested in calculating the space-time energy E from the Hamiltonian H, thus the usefulness of the uniform light-cone gauge is apparent from the first relation in (2.1.21).

Let us note that while we can find the derivative of x^- from the Virasoro constraint C_1 in (2.1.15)

$$\acute{x}^- = -p_\mu \acute{x}^\mu - \frac{2\pi}{L} a\, m\, p_-\,, \tag{2.1.22}$$

x^- can not be written as a local function of the transverse fields. If we integrate (2.1.22) over σ, we have instead the "level-matching" condition

$$\Delta x^- = \int_{-L/2}^{L/2} \mathrm{d}\sigma\, \acute{x}^- = \frac{2\pi}{L} a\, m\, H - \int_{-L/2}^{L/2} \mathrm{d}\sigma\, p_\mu \acute{x}^\mu = 2\pi m\,, \tag{2.1.23}$$

which implies that the total world-sheet momentum p_{ws} is conserved, where p_{ws} is the charge associated to σ-translations

$$p_{ws} = -\int_{-L/2}^{L/2} \mathrm{d}\sigma\, p_\mu \acute{x}^\mu\,. \tag{2.1.24}$$

For the strings with zero winding number the total world-sheet momentum vanishes in all physical configurations

$$p_{ws} = 0 \quad \text{for} \quad m = 0\,. \tag{2.1.25}$$

2.1.3 The action for the $\mathbf{AdS_5 \times S^5}$ string

The generalization of the Polyakov action (2.1.2) to include fermions is called supersymmetric string theory, or superstring for short, and it proved challenging, especially in the case of curved backgrounds. A formulation which also preserves the target space-time supersymmetry has been proposed by Green and Schwarz [44] and extended to curved backgrounds in [45] (for type II strings), though the construction relies on the supergravity fields of the specific background, which in general are difficult to compute. It is important to mention that we have an additional symmetry in the fermionic sector called κ-symmetry, which plays a crucial role in gauge fixing the fermionic fields (spinors), since only a quarter of the fermionic degrees of freedom are physical[6].

The coset action

A simpler way of building the action for some special backgrounds, including the $\mathrm{AdS}_5 \times \mathrm{S}^5$, has been presented in [46], using the fact that the symmetries of those super-spaces allow a super-coset description of the string model. In particular, we can write AdS_5 and S^5 respectively as the cosets

$$AdS_5 = \frac{SO(4,2)}{SO(4,1)}, \qquad S^5 = \frac{SO(6)}{SO(5)}. \tag{2.1.26}$$

The full symmetry supergroup of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string is $PSU(2,2|4)$ and its subgroup corresponding to the bosonic symmetries is $SU(2,2) \times SU(4)$, locally isomorphic to $SO(4,2) \times SO(6)$. Moreover $SO(4,1) \times SO(5)$ is precisely the subgroup of the Lorentz transformations. Thus the type IIB Green–Schwarz superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ can be built as a non-linear sigma-model on the supercoset

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}. \tag{2.1.27}$$

The resulting action, which is invariant under $PSU(2,2|4)$ by construction, includes a kinetic term and a Wess–Zumino term [47] which guarantees the invariance under κ-symmetry transformations.

The result can be written in terms of a one-form A on $\mathfrak{su}(2,2|4)$

$$A = -a^{-1}\mathrm{d}a, \qquad a \in SU(2,2|4) \tag{2.1.28}$$

[6] More precisely, we have that the equations of motion halve the fermionic degrees of freedom, and then the kappa-symmetry again reduces them by half.

or more precisely its components A^μ along the graded $\mathbb{Z}_4$ decomposition of $\mathfrak{su}(2,2|4)$, as [42]

$$\mathcal{L} = -\frac{g}{2}\left[\gamma^{ij}\mathrm{str}\left(A_i^{(2)}A_j^{(2)}\right) + \kappa\,\epsilon^{ij}\left(A_i^{(1)}A_j^{(3)}\right)\right] . \qquad (2.1.29)$$

Light-cone gauge

We will briefly discuss here the uniform light-cone gauge (2.1.17) in the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring, while we refer to [42] for a detailed analysis.Though the formalism is more involved, the procedure is still the one described in Section 2.1.2. Now we have to take into account fermions and, consequently, the κ-symmetry.

To find the light-cone-gauge-fixed superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ we apply to (2.1.29) the constraints (2.1.17)

$$x^+ = \tau + \frac{2\pi}{L} a\, m\, \sigma, \qquad p_+ = 1\,.$$

The result is a two-dimensional model defined on a cylinder with complicated non-linear interactions, though the structure of the action is still similar to the bosonic action (2.1.18), with a Lagrangian depending on the dimensionless string tension g and given by a sum of a kinetic part and a Hamiltonian density as explained in [42]. We will see this explicitly in the so-called decompactification limit.

Decompactification limit

As in the bosonic case, we have that the dependence on the light-cone momentum P_+ is only through the integration bounds, $L = P_+$, while $\mathcal{L}$ is independent of P_+. This allows us to consider the infinite length limit, i.e. $P_+ \to \infty$, with fixed string tension, which is called decompactification limit, since it results in a theory defined on a plane. The decompactified world-sheet string can be quantized canonically, while the original model on the cylinder can be reconstructed later by adding finite-volume corrections in P_+. Let us note that we need large E and J with a finite difference between the two, as we can see from (2.1.21), since we need both $P_+ \to \infty$ and finite H, to have string states with finite world-sheet energy.

The large tension expansion

An interesting limit of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string was proposed by Berenstein, Maldacena and Nastase (BMN) in [3]. In the context of the AdS/CFT correspondence, this corresponds to unprotected operators with large R-charge. At first glance this seems equivalent to a string theory on a plane-wave background [48], which can be obtained as a Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$ [49]. However, in order to obtain the complete algebraic structure (two centrally extended $\mathfrak{su}(2|2)$) we actually need to start from the action for the $\mathrm{AdS}_5\times\mathrm{S}^5$ string and take the large-tension expansion [50]. This simplifies considerably the theory, allowing for a straightforward (canonical) quantization in the light-cone gauge, as we will see in the following.

We start by rescaling the spatial world-sheet coordinate $\sigma \to g\,\sigma$ and the result is that the dimensionless string coupling g appears only as an overall factor in the action[7]. Then we rescale the bosonic and fermionic[8] fields

$$x^\mu \to \frac{1}{\sqrt{g}}x^\mu\,, \qquad p_\mu \to \frac{1}{\sqrt{g}}p_\mu\,, \qquad \chi \to \frac{1}{\sqrt{g}}\chi\,. \tag{2.1.30}$$

and expand the action (2.1.18) in the decompactification limit in powers of $1/g$ to obtain

$$\mathcal{A}_{gf} = \int_{-\infty}^{\infty} \mathrm{d}\tau\mathrm{d}\sigma \left(\mathcal{L}_2 + \frac{1}{g}\mathcal{L}_4 + \frac{1}{g^2}\mathcal{L}_6 + \cdots\right)\,, \tag{2.1.31}$$

where $\mathcal{L}_n$ contains n powers of the physical fields. The quadratic term is

$$\mathcal{L}_2 \equiv \mathcal{L}_{kin} - \mathcal{H}_2 = p_\mu\dot{X}^\mu - \frac{i}{2}\mathrm{str}\,(\Sigma_+\chi\dot{\chi}) - \mathcal{H}_2. \tag{2.1.32}$$

with $\Sigma_+ = \mathrm{diag}(\gamma_5, \gamma_5)$[9] and the Hamiltonian density

$$\mathcal{H}_2 = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{2}\acute{x}^2 + \text{fermions}\,. \tag{2.1.33}$$

[7] This is true for the fact that, before rescaling, g appeared only together with a σ-derivative, see [42].

[8] χ is a Majorana-Weyl $SO(8)$ spinor of positive chirality, describing the eight fermionic degrees of freedom in the gauge fixed theory.

[9] The matrix γ_5 is the product of the four gamma matrices satisfying the Clifford algebra

$$\{\gamma^\alpha, \gamma^\beta\} = \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = \eta^{\alpha\beta}\mathbb{I}\,.$$

.

Thus at leading order we have a Lorentz-invariant free theory of 8 bosons and 8 fermions with the same unit mass.

The quartic corrections include terms with time derivatives, though we would like to avoid the complications in the quantization, which would ensue from the corrections to the quadratic kinetic Lagrangian $\mathcal{L}_{kin}$. This problem can be solved by a field redefinition, e.g.

$$\chi \to \chi + \frac{1}{\sqrt{g}}\Phi(p,x,\chi)\,, \tag{2.1.34}$$

where $\Phi(p,x,\chi)$ is a function containing terms of cubic order and higher. The transformation (2.1.34) for an appropriate value of Φ can remove the unwanted quartic terms in the kinetic Lagrangian, leaving only terms of order six and higher. We can thus write the quartic Lagrangian simply as $\mathcal{L}_4 = -\mathcal{H}_4$, leaving the canonical Poisson structure of the quadratic Lagrangian unchanged at quartic order.

Note that another consequence of the field redefinition (2.1.34) is that we can write $\acute{x}^-$ up to sextic order as

$$\acute{x}^- = -\frac{1}{g}\left(p^\mu \acute{x}_\mu - \frac{i}{2}\text{str}\left(\Sigma_+\chi\chi'\right) + \partial_\sigma f(p,x,\chi)\right)\,, \tag{2.1.35}$$

where f is at least quartic in the fields x_μ, p_μ and χ. This implies that the last term in (2.1.35) drops out when integrating over σ and we have the same "level-matching" condition we would have in flat space (2.1.23)

$$\Delta x^- = \int_{-L/2}^{L/2} \mathrm{d}\sigma\, \acute{x}^- = p_{ws} \equiv \frac{\widetilde{p}}{g} = -\frac{1}{g}\int_{-L/2}^{L/2}\mathrm{d}\sigma\left(p^\mu \dot{x}_\mu - \frac{i}{2}\text{str}\left(\Sigma_+\chi\chi'\right)\right)\,. \tag{2.1.36}$$

Let us also mention that we can see we are considering states of small world-sheet momentum $p_{ws} \sim 1/g$ from the fact that $\widetilde{p}$ defined above is the momentum with the non-rescaled fields, which is kept constant in this limit.

A transformation similar to (2.1.34) on the fields x_μ and p_μ allows to remove also the terms of order six containing time derivatives to write $\mathcal{L}_6 = -\mathcal{H}_6$ and keep the quantization unchanged. This procedure can be repeated at every order to get

$$\mathcal{L}_{gf} = \mathcal{L}_2 - \frac{1}{g}\mathcal{H}_4 - \frac{1}{g^2}\mathcal{H}_6 - \dots\,, \tag{2.1.37}$$

where $\mathcal{H}_n$ does not contain neither terms with time derivatives nor terms without any derivative.

The $\widehat{\mathfrak{psu}}(2|2)$ algebra

The symmetry algebra of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string $\mathfrak{psu}(2,2|4)$ breaks down in this limit to (two copies of) a centrally extended $\mathfrak{psu}(2|2)$, i.e. $\widehat{\mathfrak{psu}}(2|2)$. The $\widehat{\mathfrak{psu}}(2|2)$ superalgebra is composed of the two copies of $\mathfrak{su}(2)$ $\mathbb{L}_a{}^b$ and $\mathbb{R}_\alpha{}^\beta$, the fermionic generators $\mathbb{Q}_\alpha{}^a$, their conjugates $\mathbb{Q}^{\dagger\alpha}_a$ and the central elements $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$. The generators satisfy

$$\left(\mathbb{L}_a{}^b\right)^\dagger = \mathbb{L}_b{}^a\,, \quad \sum_a \mathbb{L}_a{}^a = 0\,, \quad \left(\mathbb{R}_\alpha{}^\beta\right)^\dagger = \mathbb{R}_\beta{}^\alpha\,, \quad \sum_\alpha \mathbb{R}_\alpha{}^\alpha = 0\,, \quad \left(\mathbb{Q}_\alpha{}^b\right)^\dagger = \mathbb{Q}^{\dagger\alpha}_b\,.$$

Denoting any generator with a lower (upper) index from the first $\mathfrak{su}(2)$ as $\mathbb{J}_a$ ($\mathbb{J}^a$), and similarly for the second $\mathfrak{su}(2)$, the algebra is given by[10]

$$\begin{aligned} \left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] &= \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c\,, & \left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c\,, \\ \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma\,, & \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma\,, \end{aligned} \tag{2.1.38}$$

and

$$\begin{aligned} \{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}\,, \qquad \{\mathbb{Q}^{\dagger\alpha}_a, \mathbb{Q}^{\dagger\beta}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger\,, \\ \{\mathbb{Q}_\alpha{}^a, \mathbb{Q}^{\dagger\beta}_b\} &= \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a\delta_\beta^\alpha \mathbb{H}\,. \end{aligned} \tag{2.1.39}$$

Alternatively we can write the algebra in Chevalley-Serre form

$$[H_i, E_j] = a_{ij}E_j\,, \quad [H_i, F_j] = -a_{ij}F_j\,, \quad [E_i, F_j] = \delta_{ij}H_i\,, \tag{2.1.40}$$

with Cartan matrix

$$a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.1.41}$$

[10] It will be convenient to take $a = 1,2$ and $\alpha = 3,4$ to avoid confusion, and also denote them collectively as $A = 1,\ldots,4$.

by making the identifications for the bosonic generators

$$E_1 = \mathbb{L}_2{}^1\,, \quad F_1 = \mathbb{L}_1{}^2\,, \quad H_1 = \mathbb{L}_1{}^1 - \mathbb{L}_2{}^2$$
$$E_3 = \mathbb{R}_3{}^4\,, \quad F_3 = \mathbb{R}_4{}^3\,, \quad H_3 = \mathbb{R}_3{}^3 - \mathbb{R}_4{}^4 \tag{2.1.42}$$

and for the fermionic generators

$$E_2 = \mathbb{Q}_4{}^2\,, \quad F_2 = \mathbb{Q}_2^{\dagger 4}\,, \quad H_2 = \mathbb{L}_2{}^2 + \mathbb{R}_4{}^4 + \frac{1}{2}\mathbb{H}\,. \tag{2.1.43}$$

The world-sheet action

The gauge-fixed action (2.1.37) depends only on the bosonic and fermionic physical fields

$$Y_{a\dot{a}}\,, \quad Z_{\alpha\dot{\alpha}}\,, \quad \Upsilon_{\alpha\dot{a}}\,, \quad \Psi_{a\dot{\alpha}}\,, \tag{2.1.44}$$

which transform as bi-spinors under the bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetries and satisfy the reality conditions

$$Y^*_{a\dot{a}} = Y_{a\dot{a}}\,, \qquad Z^*_{\alpha\dot{\alpha}} = Z_{\alpha\dot{\alpha}}\,. \tag{2.1.45}$$

It will be also convenient to define $\mathcal{O}_{A\dot{A}}$ to indicate a generic field, where the index $A = 1, \ldots, 4$ includes both a and α, e.g. $\mathcal{O}_{1\dot{1}} = Y_{1\dot{1}}$, $\mathcal{O}_{1\dot{3}} = \Psi_{1\dot{3}}$, and so on.

Alternatively, these fields can be packaged into the $SU(2,2|4)$ supermatrices X and χ, which allows to write the action in a more compact way. To quadratic order in the fields, the Lagrangian is given by

$$\mathcal{L}_2 = \mathrm{str}\Big[\frac{1}{4}\dot{X}\dot{X} - \frac{1}{4}\acute{X}\acute{X} - \frac{1}{4}XX - \frac{i}{2}\Sigma_+\chi\dot{\chi} - \frac{1}{2}\Sigma_+\chi\acute{\chi}^\natural - \frac{1}{2}\chi\chi\Big]\,, \tag{2.1.46}$$

where the conjugation is $\chi^\natural = \widetilde{K}_8 \chi K_8$, and the matrices Σ, K_8 and K are defined e.g. in [51]. We also mention the quartic Lagrangian

$$\begin{aligned}\mathcal{L}_4 = &-\frac{1}{8}\mathrm{str}\Sigma_8 XX\mathrm{str}\acute{X}\acute{X} \\ &+\frac{1}{8}\mathrm{str}\chi\acute{\chi}\chi\acute{\chi}+\frac{1}{8}\mathrm{str}\chi\chi\acute{\chi}\acute{\chi}+\frac{1}{16}\mathrm{str}[\chi,\acute{\chi}][\chi^\natural,\acute{\chi}^\natural]+\frac{1}{4}\mathrm{str}\chi\acute{\chi}^\natural\chi\acute{\chi}^\natural \\ &-\frac{1}{8}\mathrm{str}\Sigma_8 XX\mathrm{str}\acute{\chi}\acute{\chi}+\frac{1}{4}\mathrm{str}[X,\acute{X}][\chi,\acute{\chi}]+\mathrm{str}X\acute{\chi}X\acute{\chi} \\ &+\frac{i}{8}\mathrm{str}[X,\dot{X}][\chi^\natural,\acute{\chi}]-\frac{i}{8}\mathrm{str}[X,\dot{X}][\chi,\acute{\chi}^\natural]\,. \end{aligned} \tag{2.1.47}$$

The relations between the two sets of fields are

$$X = \left(\begin{array}{cccc|cccc} 0 & 0 & +Z^{3\dot{4}} & +iZ^{3\dot{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & +iZ^{4\dot{4}} & -Z^{4\dot{3}} & 0 & 0 & 0 & 0 \\ -Z^{4\dot{3}} & -iZ^{3\dot{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -iZ^{4\dot{4}} & +Z^{3\dot{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & +iY^{1\dot{2}} & -Y^{1\dot{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -Y^{2\dot{2}} & -iY^{2\dot{1}} \\ 0 & 0 & 0 & 0 & -iY^{2\dot{1}} & +Y^{1\dot{1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & +Y^{2\dot{2}} & +iY^{1\dot{2}} & 0 & 0 \end{array}\right),$$

and

$$\chi = e^{\frac{i\pi}{4}} \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & +\Upsilon^{3\dot{2}} & +i\Upsilon^{3\dot{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & +i\Upsilon^{4\dot{2}} & -\Upsilon^{4\dot{1}} \\ 0 & 0 & 0 & 0 & +i\Psi^{*2\dot{3}} & -\Psi^{*1\dot{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Psi^{*2\dot{4}} & -i\Psi^{*1\dot{4}} & 0 & 0 \\ \hline 0 & 0 & +\Psi^{1\dot{4}} & +i\Psi^{1\dot{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & +i\Psi^{2\dot{4}} & -\Psi^{2\dot{3}} & 0 & 0 & 0 & 0 \\ -i\Upsilon^{*4\dot{1}} & +\Upsilon^{*3\dot{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ +\Upsilon^{*4\dot{2}} & +i\Upsilon^{*3\dot{2}} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

2.1.4 Quantization in light-cone gauge

We will now quantize the light-cone AdS$_5 \times$S^5 string in this limit, i.e. the Lagrangian $\mathcal{L}_2$ defined in (2.1.32) or (2.1.46) with the added quartic interaction terms $-\mathcal{L}_4$ in

(2.1.47). We can rewrite $\mathcal{L}_2$ in terms of the fields (2.1.44) as

$$\mathcal{L}_2 = P_{a\dot{a}}\dot{Y}^{a\dot{a}} + P_{\alpha\dot{\alpha}}\dot{Z}^{\alpha\dot{\alpha}} + i\,\Upsilon^{\dagger}_{\alpha\dot{a}}\dot{\Upsilon}^{\alpha\dot{a}} + i\,\Psi^{\dagger}_{a\dot{\alpha}}\dot{\Psi}^{a\dot{\alpha}} - \mathcal{H}_2\,, \tag{2.1.48}$$

with

$$\begin{aligned}\mathcal{H}_2 = &\frac{1}{4}P_{a\dot{a}}P^{a\dot{a}} + Y_{a\dot{a}}Y^{a\dot{a}} + Y'_{a\dot{a}}Y'^{a\dot{a}} + \frac{1}{4}P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} + Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}} \\ &+ \Upsilon^{\dagger}_{\alpha\dot{a}}\Upsilon^{\alpha\dot{a}} + \frac{\kappa}{2}\Upsilon^{\alpha\dot{a}}\Upsilon'_{\alpha\dot{a}} - \frac{\kappa}{2}\Upsilon^{\dagger\alpha\dot{a}}\Upsilon'^{\dagger}_{\alpha\dot{a}} + \Psi^{\dagger}_{a\dot{\alpha}}\Psi^{a\dot{\alpha}} + \frac{\kappa}{2}\Psi^{a\dot{\alpha}}\Psi'_{a\dot{\alpha}} - \frac{\kappa}{2}\Psi^{\dagger a\dot{\alpha}}\Psi'^{\dagger}_{a\dot{\alpha}}\,.\end{aligned} \tag{2.1.49}$$

Indices are raised and lowered using the ϵ-tensor, e.g.

$$Y_{a\dot{a}} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}Y^{b\dot{b}}\,, \qquad \Upsilon_{\alpha\dot{a}} = \epsilon_{\alpha\beta}\epsilon_{\dot{a}\dot{b}}\Upsilon^{\beta\dot{b}}\,. \tag{2.1.50}$$

From (2.1.48) we have the canonical equal-time commutation relations

$$\begin{aligned}&[\,Y^{a\dot{a}}(\sigma,\tau)\,,\,P_{b\dot{b}}(\sigma',\tau)\,] = i\,\delta^a_b\delta^{\dot{a}}_{\dot{b}}\,\delta(\sigma-\sigma')\,, \quad \{\,\Psi^{a\dot{\alpha}}(\sigma,\tau)\,,\,\Psi^{\dagger}_{b\dot{\beta}}(\sigma',\tau)\,\} = \delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\,\delta(\sigma-\sigma')\,,\\ &[\,Z^{\alpha\dot{\alpha}}(\sigma,\tau)\,,\,P_{\beta\dot{\beta}}(\sigma',\tau)\,] = i\,\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}\,\delta(\sigma-\sigma')\,, \quad \{\,\Upsilon^{\alpha\dot{a}}(\sigma,\tau)\,,\,\Upsilon^{\dagger}_{\beta\dot{b}}(\sigma',\tau)\,\} = \delta^\alpha_\beta\delta^{\dot{a}}_{\dot{b}}\,\delta(\sigma-\sigma')\,.\end{aligned}$$

The equations of motion of the free part of (2.1.48) are solved by the following mode expansion

$$\begin{aligned}Y_{a\dot{a}}(\vec{x}) &= \int\frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(a_{a\dot{a}}(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^{\dagger}_{a\dot{a}}(p)\,e^{+i\vec{p}\cdot\vec{x}}\right)\,,\\ Z_{\alpha\dot{\alpha}}(\vec{x}) &= \int\frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(a_{\alpha\dot{\alpha}}(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^{\dagger}_{\alpha\dot{\alpha}}(p)\,e^{+i\vec{p}\cdot\vec{x}}\right)\,,\\ \Psi_{a\dot{\alpha}}(\vec{x}) &= \int\frac{dp}{2\pi}\frac{1}{\sqrt{\varepsilon}}\left(b_{a\dot{\alpha}}(p)\,u(p)\,e^{-i\vec{p}\cdot\vec{x}} + b^{\dagger}_{a\dot{\alpha}}(p)\,v(p)\,e^{+i\vec{p}\cdot\vec{x}}\right)\,,\\ \Upsilon_{\alpha\dot{a}}(\vec{x}) &= \int\frac{dp}{2\pi}\frac{1}{\sqrt{\varepsilon}}\left(b_{\alpha\dot{a}}(p)\,u(p)\,e^{-i\vec{p}\cdot\vec{x}} + b^{\dagger}_{\alpha\dot{a}}(p)\,v(p)\,e^{+i\vec{p}\cdot\vec{x}}\right)\,,\end{aligned} \tag{2.1.51}$$

where the energy is $\varepsilon_p = \sqrt{1+p^2}$, and $\vec{p}\cdot\vec{x} = \varepsilon_p\tau + p\sigma$. The fermion wave functions $u_p \equiv u(p)$ and $v_p \equiv v(p)$ satisfy

$$u_p = \sqrt{\frac{\varepsilon_p+1}{2}}\,, \quad v_p = \frac{p}{2u_p}\,, \qquad u_p^2 - v_p^2 = 1\,, \quad u_p^2 + v_p^2 = \varepsilon_p\,, \tag{2.1.52}$$

so that we can define the rapidity θ as $p = \sinh\theta$ and write

$$u(p) = \cosh\frac{\theta}{2}\,, \qquad v(p) = \sinh\frac{\theta}{2}\,. \tag{2.1.53}$$

The canonical commutation relations for the creation/annihilation operators are given by

$$\begin{aligned}
\left[a^{a\dot{a}}(p), a^{\dagger}_{b\dot{b}}(p')\right] &= 2\pi\, \delta^{a}_{b}\delta^{\dot{a}}_{\dot{b}}\, \delta(p-p')\,, \qquad \{b^{a\dot{\alpha}}(p), b^{\dagger}_{b\dot{\beta}}(p')\} = 2\pi\, \delta^{a}_{b}\delta^{\dot{\alpha}}_{\dot{\beta}}\, \delta(p-p')\,. \\
\left[a^{\alpha\dot{\alpha}}(p), a^{\dagger}_{\beta\dot{\beta}}(p')\right] &= 2\pi\, \delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}}\, \delta(p-p')\,, \qquad \{b^{\alpha\dot{a}}(p), b^{\dagger}_{\beta\dot{b}}(p')\} = 2\pi\, \delta^{\alpha}_{\beta}\delta^{\dot{a}}_{\dot{b}}\, \delta(p-p')\,.
\end{aligned}$$

The quadratic Hamiltonian is then written in the standard harmonic oscillator form

$$H_2 = \int \mathrm{d}p \sum_{A,\dot{A}} \omega_p\, a^{\dagger}_{A\dot{A}}(p) a^{A\dot{A}}(p)\,, \tag{2.1.54}$$

and to build a generic N-particle state, we need to act with creation operators on the vacuum, e.g. in the bosonic case

$$|\Psi\rangle = a^{\dagger}_{b_1\dot{b}_1}(p_1)\, a^{\dagger}_{b_2\dot{b}_2}(p_2)\, \cdots\, a^{\dagger}_{b_N\dot{b}_N}(p_N)\, |0\rangle\,, \tag{2.1.55}$$

with $p_1 > p_2 > \cdots > p_{N-1} > p_N$. The energy of this state is

$$H_2|\Psi\rangle = E|\Psi\rangle\,, \qquad E = \sum_i \omega_{p_i}\,.$$

This state is also an eigenvector of the world-sheet momentum operator which takes the following form

$$\begin{aligned}
P \equiv p_{ws} &= -\frac{1}{g}\int \mathrm{d}\sigma\, \left(P_{a\dot{a}}Y'^{a\dot{a}} + P_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}} + i\Psi^{\dagger}_{\alpha\dot{a}}\Psi'^{\alpha\dot{a}} + i\Upsilon^{\dagger}_{a\dot{\alpha}}\Upsilon'^{a\dot{\alpha}}\right) \\
&= \frac{1}{g}\int \mathrm{d}p \sum_{A,\dot{A}} p\, a^{\dagger}_{A\dot{A}}(p) a^{A\dot{A}}(p)\,.
\end{aligned} \tag{2.1.56}$$

As explained above, physical states have to satisfy the level-matching condition which implies that the total world-sheet momentum vanishes

$$P\,|\Psi\rangle = 0 \;\Rightarrow \sum_i p_i = 0\,.$$

Time-evolution of the creation and annihilation operators is determined in the usual way from the Hamiltonian $H = H_2 + H_4 + \ldots$ as

$$\frac{\partial}{\partial\tau} a^{b\dot{b}}(p,\tau) = i\left[H, a^{b\dot{b}}(p,\tau)\right], \qquad \frac{\partial}{\partial\tau} b^{b\dot{\beta}}(p,\tau) = i\left[H, b^{b\dot{\beta}}(p,\tau)\right], \tag{2.1.57}$$

and analogous expressions for the other indices.

Because of the complexity of the interactions, it is more convenient to formulate the problem in terms of scattering, i.e. to consider asymptotically free states instead of describing the interaction at any time τ, as we will see in the following sections.

2.1.5 The world-sheet S-matrix

The symmetry of the quantized string we are considering is $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$ and so each particle, also called a magnon, is characterized by a $\mathfrak{psu}(2|2)^2$ index, $(A, \dot{A})$ where $A, \dot{A} = 1, \ldots, 4$, see e.g. [42, 6]. It useful to replace the momenta, p, of the massive excitations with two variables, $x^{\pm}$, such that

$$\frac{x^+}{x^-} = e^{ip}, \quad \text{and} \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g} \tag{2.1.58}$$

where g is the dimensionless string coupling defined earlier[11], $g^2 = \lambda/4\pi^2$. The dispersion relation is given by

$$E^2 = 1 + 4g^2 \sin^2\frac{p}{2}, \quad \text{or} \quad E = \frac{ig}{2}\left[x^- - \frac{1}{x^-} - x^+ + \frac{1}{x^+}\right]. \tag{2.1.59}$$

It is also useful to define a parameter u,

$$u = \frac{1}{2}\left[x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-}\right]. \tag{2.1.60}$$

The scattering of two Y-excitations with parameters $x_1^{\pm}$ and $x_2^{\pm}$ is described by the S-matrix (see App. A for an introduction)

$$\mathcal{S} = \sigma(x_1^{\pm}, x_2^{\pm})^2 \,\frac{u(x_1^{\pm}) - u(x_2^{\pm}) + i/g}{u(x_1^{\pm}) - u(x_2^{\pm}) - i/g}, \tag{2.1.61}$$

[11] The fundamental representation, and tensor products thereof, also depend on the central charge parameter ζ, see e.g. [52].

where $\sigma(x_1^\pm, x_2^\pm)$ is the so-called dressing phase, first determined by [53], and the remaining term is the BDS S-matrix [54].

We introduce the uniformizing parameters (rapidities), starting from the dispersion relation (2.1.59) and using the Jacobi elliptic functions [55], as

$$p = 2\,\mathrm{am}\,(z,k)\,, \quad \sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \quad E = \mathrm{dn}(z,k)\,, \tag{2.1.62}$$

where $k = -4g^2 < 0$. These expressions are naturally defined on the torus with real period $2\omega_1 = 4K(k)$ and imaginary period $2\omega_2 = 4iK(1-k) - 4K(k)$ with $K(k)$ the elliptic integral of the first kind. The dispersion relation is invariant under shifts of z, the analogous of the relativistic rapidity parameter, by $2\omega_1$ and $2\omega_2$. The real z-axis can be taken to be the physical region as for these values the energy is positive and the momentum real. The $x^\pm$ parameters are given by

$$x^\pm = \frac{1}{2g}\left(\frac{\mathrm{cn}(z,k)}{\mathrm{sn}(z,k)} \pm i\right)(1 + \mathrm{dn}(z,k))\,, \tag{2.1.63}$$

such that for real values of z we have $|x^\pm| > 1$ and $\mathrm{Im}(x^+) > 0$ while $\mathrm{Im}(x^-) < 0$.

2.2 World-sheet symmetries

2.2.1 Zamolodchikov–Faddeev algebra

The world-sheet string described in the previous sections is believed to be integrable, as we will discuss in Chapter 3. In describing integrable scattering it is useful to formally introduce generalized creation and annihilation operators, such that multi-particle external states are formed by their action on the vacuum [8, 56]. These can be thought of as the fully interacting generalizations of free plane-wave oscillators. For the world-sheet theory these operators were studied in [57]. Each oscillator can be thought of as an element of the vector space corresponding to the multiplet of physical excitations. For the world-sheet theory such excitations transform under the symmetries preserved by the vacuum, namely the two copies of $\mathfrak{psu}(2|2)$ and hence they carry two indices. The oscillators for a particle with world-sheet momentum p can be written as

$$\mathbb{Z}^\dagger_{A\dot{A}}(p)\,, \quad \mathbb{Z}^{A\dot{A}}(p)\,. \tag{2.2.1}$$

More accurately we should think of these oscillators as being labeled by the generalized rapidity z living on the rapidity torus

$$p(z) = \mathrm{am}(z)\,, \qquad \epsilon(z) = \mathrm{dn}(z)\,. \tag{2.2.2}$$

though we will occasionally leave this dependence implicit.

The invariant vacuum is defined by

$$\mathbb{Z}^{A\dot{A}}(z)\,|\Omega\rangle = 0\,,\ \ \forall\ A, \dot{A}, z\,, \tag{2.2.3}$$

and we define multi-particle "in-basis" and "out-basis" by

$$\begin{aligned} |z_1, z_2, \ldots, z_n\rangle^{(\mathrm{in})}_{A_1\dot{A}_1,\ldots,A_n\dot{A}_n} &= \mathbb{Z}^\dagger_{A_1\dot{A}_1}(z_1)\ldots\mathbb{Z}^\dagger_{A_n\dot{A}_n}(z_n)\,|\Omega\rangle\ , \\ |z_1, z_2, \ldots, z_n\rangle^{(\mathrm{out})}_{A_1\dot{A}_1,\ldots,A_n\dot{A}_n} &= \mathbb{Z}^\dagger_{A_n\dot{A}_n}(z_n)\ldots\mathbb{Z}^\dagger_{A_1\dot{A}_n}(z_1)\,|\Omega\rangle\ , \end{aligned} \tag{2.2.4}$$

for $p(z_1) > p(z_2) > \cdots > p(z_n)$, assuming that the particles have the same mass. These states lie in the same Hilbert space and, as they form complete bases, they can be expressed in terms of one another through the S-matrix. For two-particle states, this is given by

$$|z_1, z_2\rangle^{(\mathrm{in})}_{A\dot{A},B\dot{B}} = \mathcal{S}^{C\dot{C},D\dot{D}}_{A\dot{A},B\dot{B}}(z_1, z_2)\,|z_1, z_2\rangle^{(\mathrm{out})}_{C\dot{C},D\dot{D}}\ , \tag{2.2.5}$$

which in terms of the ZF operators corresponds to

$$\mathbb{Z}^\dagger_{A\dot{A}}(z_1)\mathbb{Z}^\dagger_{B\dot{B}}(z_2) = \mathcal{S}^{C\dot{C},D\dot{D}}_{A\dot{A},B\dot{B}}(z_1, z_2)\mathbb{Z}^\dagger_{C\dot{C}}(z_2)\mathbb{Z}^\dagger_{D\dot{D}}(z_1)\,. \tag{2.2.6}$$

2.2.2 Charges and currents

Let us consider the charges $\mathbb{Q}_A{}^B$ and the currents $\mathbb{J}_A{}^B$, $A = (a, \alpha)$, of $\mathfrak{psu}(2,2|4)$, related as usual by [58, 51]

$$\mathbb{Q}_A{}^B = \int d\sigma\, \mathbb{J}_A{}^B\,, \quad \text{with} \quad \mathbb{J}_A{}^B = e^{i\epsilon_{AB}x^-}\Omega_A{}^B \tag{2.2.7}$$

with $\epsilon_{AB} = ([A] - [B])/2$, where the grading is defined as $[a] = [\dot{a}] = 0$ and $[\alpha] = [\dot{\alpha}] = 1$. Since in the light-cone gauge we singled out two coordinates, $x^+ = \tau$ and

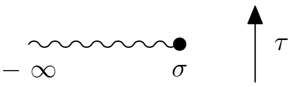


FIGURE 2.1: Contour for non-local charges.

x^-, the generators $\mathbb{Q}_A{}^B$ are naturally characterized according to their dependence on the light-cone coordinates. When the symmetry generator is independent of x^-, it is called "kinematical" and it has the property of not receiving quantum corrections, while those dependent on x^- are called "dynamical". This is because only its derivative $\acute{x}^-$ (2.1.35) is defined in terms of local fields

$$\acute{x}_- = -\frac{1}{g}\left(p^\mu \dot{x}_\mu - \frac{i}{2}\text{str}\left(\Sigma_+ \chi \chi'\right) + \dots\right) , \tag{2.2.8}$$

which means that x^- introduces a form of non-locality in the "dynamical" charges. More specifically, to determine x^- from (2.1.35) (up to a zero mode) we need to specify a contour C

$$x^- = \int_C d\sigma\, \acute{x}^- , \tag{2.2.9}$$

which, in the decompactification limit, we will take to start at negative spatial infinity (see Fig. 2.1). The functions $\Omega_A{}^B \equiv \Omega_A{}^B(X, P, \chi; g)$ are, however, local in the physical fields and their derivatives and can be expanded in the string tension g with the leading term being quadratic in the fields.

Moreover, the conservation laws in the Hamiltonian formalism read

$$\frac{\mathrm{d}\mathbb{Q}_A{}^B}{\mathrm{d}\tau} = \frac{\partial \mathbb{Q}_A{}^B}{\partial \tau} + \left[\mathbb{H}, \mathbb{Q}_A{}^B\right\} , \tag{2.2.10}$$

which means that the charges independent of $x^+ = \tau$ commute with the classical light-cone Hamiltonian. The algebra of such symmetry generators is (two copies of) the centrally-extended Lie superalgebra $\widehat{\mathfrak{psu}}(2|2)$, see (2.1.38) and (2.1.39), and thus we have

$$\mathbb{Q}_\alpha{}^\beta = \mathbb{R}_\alpha{}^\beta , \quad \mathbb{Q}_a{}^b = \mathbb{L}_a{}^b , \quad \mathbb{Q}_\alpha{}^b = \mathbb{Q}_\alpha{}^b , \quad \mathbb{Q}_a{}^\beta = \mathbb{Q}^{\dagger\beta}_a . \tag{2.2.11}$$

From the form of the dynamical supercharges in (2.2.7) and the level-matching condition (2.1.36) we can write the central charges, related to x^-, as[12]

$$\mathbb{C}=\frac{i}{2}g\,(e^{i\mathbb{P}}-1)e^{ix^-(-\infty)}\,,\quad \mathbb{C}^\dagger=-\frac{i}{2}g\,(e^{-i\mathbb{P}}-1)e^{-ix^-(-\infty)}\,, \tag{2.2.12}$$

where the most convenient choice in this case is $x^-(-\infty)=0$. We can also define $\mathbb{U}=e^{i\mathbb{P}/2}$ and rewrite the central charges as

$$\mathbb{C}=\frac{i}{2}g\,(\mathbb{U}^2-1)\,,\quad \mathbb{C}^\dagger=-\frac{i}{2}g\,(\mathbb{U}^{-2}-1)\,. \tag{2.2.13}$$

2.2.3 Hopf Algebra interpretation

We now want to review the action of the symmetries on fields and the algebraic structure in the two-dimensional world-sheet theory. Essentially we will recap the Hopf algebra description of the $\widehat{\mathfrak{psu}}(2|2)$ theory [55, 59, 60, 57], however we will more closely follow the framework discussed in [61, 62, 63] which was previously discussed in the context of the light-cone gauge fixed theory in [51]. While the world-sheet theory is different than usual relativistic integrable quantum field theories in that non-localities already appear in the definition of the global charges, nonetheless much of the same algebraic structure appears.

Coproduct for $\widehat{\mathfrak{psu}}(2|2)$

The global $\widehat{\mathfrak{psu}}(2|2)$ charges, which we will write collectively as

$$\mathbb{Q}_{\hat{I}}=\{\mathbb{Q}_A{}^B,\mathbb{H},\mathbb{C},\mathbb{C}^\dagger\}\,,$$

acting on the Hilbert space of the theory are simply given as equal time integrals of the currents. Because of the non-locality of the currents there is a non-trivial braiding with fields

$$\mathbb{J}_{\hat{I}}(\sigma,\epsilon)\Phi_A(\sigma_0,0)=\hat{\Upsilon}_{\hat{I}}^{\hat{J}}[\Phi_A](\sigma_0,\epsilon)\mathbb{J}_{\hat{J}}(\sigma,0)\,,\quad \text{for}\quad \sigma>\sigma_0\,, \tag{2.2.14}$$

[12] $\mathbb{P}\equiv p_{\rm ws}$ is the world-sheet momentum.

with $\hat{Y}_{\hat{I}}^{\hat{J}}[\Phi_A] = Y_{\hat{I}A}^{\hat{J}B}\Phi_B$ and an implicit time ordering with fields at later times to the left, i.e.

$$\lim_{\epsilon\to 0}\left[\mathbb{J}_{\hat{I}}(\sigma,\epsilon)\Phi_A(\sigma_0,0) = \hat{Y}_{\hat{I}}^{\hat{J}}[\Phi_A](\sigma_0,\epsilon)\mathbb{J}_{\hat{J}}(\sigma,0)\right]\ ,\quad \text{for}\quad \sigma > \sigma_0\,.$$

For the world-sheet theory the non-local part of $J_{\hat{I}}$ is the integration path used to define x^-, and the explicit form of the braiding can be found from the expressions for the currents, e.g. (2.2.7),

$$e^{i\epsilon_{\hat{I}}\int_{-\infty}^{\sigma}d\sigma'\acute{x}^-}\Omega_{\hat{I}}(\sigma)\Phi(\sigma_0) = \left[e^{i\epsilon_{\hat{I}}\int_{-\infty}^{\infty}d\sigma'\acute{x}^-}\Phi(\sigma_0)e^{-i\epsilon_{\hat{I}}\int_{-\infty}^{\infty}d\sigma'\acute{x}^-}\right]e^{i\epsilon_{\hat{I}}\int_{-\infty}^{\sigma}d\sigma'\acute{x}^-}\Omega_{\hat{I}}(\sigma)\,,$$

where $\sigma > \sigma_0$. One choice of the contours defining x^- is given by the wavy line in Figure 2.2, which represents graphically the equation (2.2.14) and it is meant to reach $\pm\infty$ to the right and left respectively, though the contours can be freely deformed in the absence of poles.

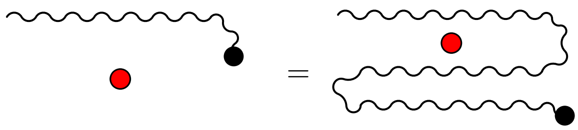


FIGURE 2.2: Braiding of currents with local fields in the world-sheet (i.e. σ-τ plane, see Fig. 2.1), where we assume the lines on the left to continue to $\sigma = -\infty$ and likewise those on the right to ∞. The red vertex corresponds to the world-sheet position of the local field and the black vertex to the position of the local part of the current.

The action on the fields, $\hat{Q}_{\hat{I}}[\Phi_A(\sigma_0)]$, is given by a separate integration contour $\gamma(\sigma_0)$ starting and ending at $\sigma = -\infty$ which surrounds the point σ_0, represented by the solid line in Figure 2.3. This is to be distinguished from the equal time contour used to define global charges acting on asymptotic states and so we use the ˆ notation.

Correspondingly the action of the charges on products of fields, represented diagrammatically in Fig. 2.4, can be calculated from the braiding of the current with the fields. It is explicitly given by

$$\hat{Q}_{\hat{I}}[\Phi_{A_1}(\sigma_1)\Phi_{A_2}(\sigma_2)] = \hat{Q}_{\hat{I}}[\Phi_{A_1}(\sigma_1)]\Phi_{A_2}(\sigma_2) + \hat{Y}_{\hat{I}}^{\hat{J}}[\Phi_{A_1}(\sigma_1)]\hat{Q}_{\hat{J}}[\Phi_{A_2}(\sigma_2)]\,.$$

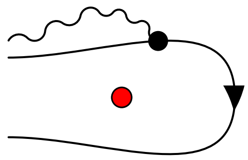


FIGURE 2.3: Integration contour (solid line) defining the action of a charge on a field, assuming that the lines on the left continue to $\sigma = -\infty$, while the wavy line is once again the integration contour defining x^-.

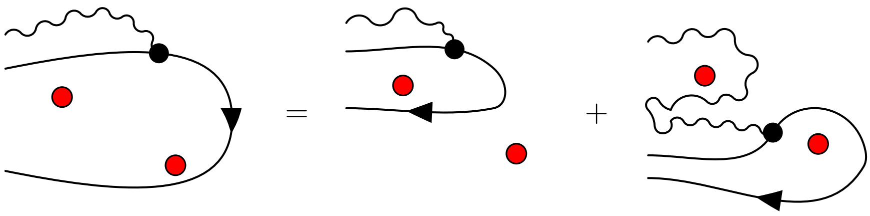


FIGURE 2.4: Action of a charge on products of fields, derived from the braiding of the current with the fields.

If we think of the fields at a specific world-sheet point $\Phi_{A_1}(\sigma_1)$ as defining a vector space V_1 we can think of products of fields at different points, e.g. $\Phi_{A_1}(\sigma_1)\Phi_{A_2}(\sigma_2)$ as defining tensor products, $V_1 \otimes V_2$. As the world-sheet theory involves both bosonic and fermionic fields we will use a graded tensor product [13]

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd) .$$

where $|a| = 0$ if the element of the algebra is even and $|a| = 1$ if it is odd. The action of charges on products of fields yields an operator $\Delta(\hat{\mathrm{Q}}_{\hat{I}})$ acting on $V_1 \otimes V_2$ which in turn defines the coproduct

$$\Delta(\widehat{\mathrm{Q}}_{\hat{I}}) = \widehat{\mathrm{Q}}_{\hat{I}} \otimes \mathbb{1} + \hat{\Upsilon}_{\hat{I}}^{\hat{J}} \otimes \hat{\mathrm{Q}}_{\hat{J}} .$$

From the form of the currents in the world-sheet theory we have the following expressions for the coproduct and the braiding operator $\hat{\Upsilon}_{\hat{I}}^{\hat{J}}$ which has the non-zero elements

[13] At different points we use the letters $a, b, c, \ldots$ to denote both elements of an algebra $\mathcal{A}$ and $\mathfrak{psu}(2|2)$ indices. From the context the meaning should be clear.

$\{\hat{\Upsilon}^{DB}_{AC}, \hat{\Upsilon}^{\mathbb{H}}_{\mathbb{H}}, \hat{\Upsilon}^{\mathbb{C}}_{\mathbb{C}}, \hat{\Upsilon}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger}\}$ given by:

$$\begin{aligned} \Delta(\hat{\mathbb{Q}}_A{}^B) &= \hat{\mathbb{Q}}_A{}^B \otimes \mathbb{1} + \hat{\Upsilon}^{DB}_{AC} \otimes \hat{\mathbb{Q}}_D{}^C\,, & \hat{\Upsilon}^{DB}_{AC} &= \delta^D_A \delta^B_C\, e^{i\epsilon_{AB}\mathbb{P}}\,, \\ \Delta(\hat{\mathbb{C}}) &= \hat{\mathbb{C}} \otimes \mathbb{1} + \hat{\Upsilon}^{\mathbb{C}}_{\mathbb{C}} \otimes \hat{\mathbb{C}}\,, & \hat{\Upsilon}^{\mathbb{C}}_{\mathbb{C}} &= e^{i\mathbb{P}}\,, \\ \Delta(\hat{\mathbb{C}}^\dagger) &= \hat{\mathbb{C}}^\dagger \otimes \mathbb{1} + \hat{\Upsilon}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger} \otimes \hat{\mathbb{C}}^\dagger\,, & \hat{\Upsilon}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger} &= e^{-i\mathbb{P}}\,, \\ \Delta(\hat{\mathbb{H}}) &= \hat{\mathbb{H}} \otimes \mathbb{1} + \hat{\Upsilon}^{\mathbb{H}}_{\mathbb{H}} \otimes \hat{\mathbb{H}}\,, & \hat{\Upsilon}^{\mathbb{H}}_{\mathbb{H}} &= \mathbb{1}\,, \end{aligned} \tag{2.2.15}$$

where $\epsilon_{AB} = ([A] - [B])/2$. We also have the coproduct for the braiding operator

$$\Delta(\hat{\Upsilon}^{\hat{J}}_{\hat{I}}) = \hat{\Upsilon}^{\hat{K}}_{\hat{I}} \otimes \hat{\Upsilon}^{\hat{J}}_{\hat{K}}\,,$$

or explicitly

$$\begin{aligned} \Delta(\hat{\Upsilon}^{DB}_{AC}) &= \hat{\Upsilon}^{FB}_{AE} \otimes \hat{\Upsilon}^{DE}_{FC} = \delta^F_A \delta^B_E\, e^{i\epsilon_{AB}\mathbb{P}} \otimes \delta^D_F \delta^E_C\, e^{i\epsilon_{DF}\mathbb{P}}\,, \\ \Delta(\hat{\Upsilon}^{\mathbb{C}}_{\mathbb{C}}) &= \hat{\Upsilon}^{\mathbb{C}}_{\mathbb{C}} \otimes \hat{\Upsilon}^{\mathbb{C}}_{\mathbb{C}} = e^{i\mathbb{P}} \otimes e^{i\mathbb{P}}\,, \\ \Delta(\hat{\Upsilon}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger}) &= \hat{\Upsilon}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger} \otimes \hat{\Upsilon}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger} = e^{-i\mathbb{P}} \otimes e^{-i\mathbb{P}}\,, \\ \Delta(\hat{\Upsilon}^{\mathbb{H}}_{\mathbb{H}}) &= \hat{\Upsilon}^{\mathbb{H}}_{\mathbb{H}} \otimes \hat{\Upsilon}^{\mathbb{H}}_{\mathbb{H}} = \mathbb{1} \otimes \mathbb{1}\,. \end{aligned}$$

These follow from the simple contour argument in Fig. 2.5 and are equivalent to

$$\Delta(e^{im\mathbb{P}}) = e^{im\mathbb{P}} \otimes e^{im\mathbb{P}}\,, \quad m \in \mathbb{Z}\,.$$

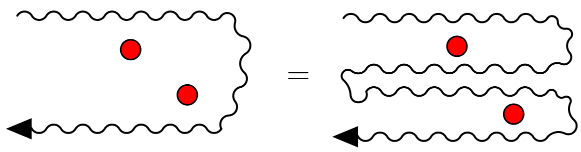


FIGURE 2.5: Braiding Coproduct: analyticity allows to freely deform the contour around the fields.

We will be interested in studying the algebra $\mathcal{A}$ generated by $\hat{\mathbb{Q}}_{\hat{I}}$ and $\hat{\Upsilon}^{\hat{J}}_{\hat{I}}$ or equivalently the algebra generated by the global charges $\mathbb{Q}_{\hat{I}}$ and the global braiding $\Upsilon^{\hat{J}}_{\hat{I}}$, defined by the contour from $\sigma = -\infty$ to ∞, which has the same coproduct. The algebra $\mathcal{A}$ implicitly comes with the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the identity operator id defined such that the coproduct satisfies coassociativity

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta\,. \tag{2.2.16}$$

We define the action of products of algebra elements so that the coproduct is an algebra homomorphism i.e. for all $a, b \in \mathcal{A}$

$$\Delta(ab) = \Delta(a)\Delta(b)\,.$$

In addition to the coproduct we can equip this algebra with a counit and antipode.

The counit as vacuum expectation

The action of the counit can be thought of as the vacuum expectation of the various global operators and so we have

$$\epsilon(\mathbb{1}) = 1\,, \qquad \epsilon(\mathrm{Q}_{\hat{I}}{}^{\hat{J}}) = 0\,, \qquad \epsilon(\mathrm{Y}_{\hat{I}}^{\hat{J}}) = \delta_{\hat{J}}^{\hat{I}}\,. \tag{2.2.17}$$

Equivalently we can define the counit as the action of the operators on the identity i.e. for $a \in \mathcal{A}$

$$\epsilon(a) = \hat{a}(\mathbb{1})\,, \tag{2.2.18}$$

which gives the same result. We can define the action of the generators so that the counit is also an algebra homomorphism: $\epsilon(ab) = \epsilon(a)\epsilon(b)$ for every $a, b \in \mathcal{A}$. This definition of the counit is consistent with the property,

$$(\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta\,. \tag{2.2.19}$$

The antipode as Euclidean Rotation

In [63] the antipode for the current Hopf algebra was interpreted as the action of a rotation in the $\sigma - \tau$ plane by an angle π or $-\pi$ on elements of the algebra, i.e. for an element a of $\mathcal{A}$ they defined an antipode s as

$$s(a) = R_\pi(a) \; . \tag{2.2.20}$$

Even for the case of the non-Lorentz invariant world-sheet theory we can repeat this identification by defining the rotation to act on the symmetry charges by a combination of time-reversal transformation followed by a parity transformation plus a reversal of the integration path defining the charge.

We first consider the action of the antipode on the symmetry charges by writing the action of the product of a time-reversal transformation followed by a parity transformation acting on the local part of the currents $\Omega^A{}_B$

$$\mathcal{PT}(\Omega_A{}^B(\sigma)) = \Omega_A{}^B(-\sigma) \, , \tag{2.2.21}$$

which can be seen explicitly at the level of the quadratic currents, see App. B. For the bosonic charges $\mathbb{B} = \{\mathbb{R}_\alpha{}^\beta, \mathbb{L}_a{}^b, \mathbb{H}\}$ this simply implies $s(\mathbb{B}) = -\mathbb{B}$. For the fermionic currents, and for the central charges $\mathbb{C}$, $\mathbb{C}^\dagger$, the issue is complicated slightly due to the presence of the zero-mode of the light-cone coordinate x^-. The contour defining x^- is deformed, resulting in an additional factor:

$$s(\mathbb{Q}_A{}^B) = -e^{-i\epsilon_{AB}\mathbb{P}}\mathbb{Q}_A{}^B \, , \quad s(\mathbb{C}) = -e^{-i\mathbb{P}}\mathbb{C} \, , \quad s(\mathbb{C}^\dagger) = -e^{i\mathbb{P}}\mathbb{C}^\dagger \; . \tag{2.2.22}$$

For the global (or for that matter the usual) braiding operators $\mathrm{Y}_{\hat{I}}^{\hat{J}}$ either rotation simply acts as $\mathcal{PT}$ or equivalently reverses the orientation of the contour defining x^-. Hence the action of antipode gives the inverse acting on the braiding operator

$$\mathrm{Y}_{\hat{K}}^{\hat{I}} s(\mathrm{Y}_{\hat{I}}^{\hat{J}}) = \delta_{\hat{K}}^{\hat{J}} \, . \tag{2.2.23}$$

Noting that the factors such as $e^{-i\mathbb{P}}$ that appear in the antipodal action on the global charges are related to the global braiding operator, we can write the action on the global charges as

$$s(\mathbb{Q}_{\hat{I}}) = -s(\mathrm{Y}_{\hat{I}}^{\hat{J}})\mathbb{Q}_{\hat{J}} \; . \tag{2.2.24}$$

This is exactly the same formula as in [63]. Due to the time-reversal in the definition the action on products of elements of the algebra is

$$s(ab) = (-1)^{[a][b]} s(b)s(a)\,, \tag{2.2.25}$$

where we have included the grading factor to account for the odd elements of the algebra and thus we note that the antipode is an antihomomorphism. We also have the trivial action of the antipode on the identity element, i.e. $s(\mathbb{1}) = \mathbb{1}$. Indeed for any operator which commutes with the world-sheet momentum, which includes all of the braiding operators and all the global charges, we do have $s^2(a) = a$. It is straightforward to check that the world-sheet antipode satisfies

$$m(s \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes s)\Delta = \mathbb{1}\epsilon\,. \tag{2.2.26}$$

We also note that due to the time reversal in the definition of the antipode we have that

$$\Delta(s(\mathrm{Y}^{\hat{J}}_{\hat{I}})) = s(\mathrm{Y}^{\hat{J}}_{\hat{K}}) \otimes s(\mathrm{Y}^{\hat{K}}_{\hat{I}})\,. \tag{2.2.27}$$

In fact, this relation is trivial, since the matrix structure of the braiding factors is essentially trivial. However, writing it in this way makes it straightforward to show the identity

$$\Delta s = (s \otimes s)\Delta^{\mathrm{op}} \tag{2.2.28}$$

where $\Delta^{\mathrm{op}} = P\Delta$ with $P(a \otimes b) = (-1)^{[a][b]}(b \otimes a)$ is the skew comultiplication. We can use Δ^{op} to define the skew antipode s' as

$$m(s' \otimes \mathrm{id})\Delta^{\mathrm{op}} = m(\mathrm{id} \otimes s')\Delta^{\mathrm{op}} = \mathbb{1}\epsilon\,. \tag{2.2.29}$$

Following [63], we can think of this from the world-sheet perspective as a rotation in the opposite direction to s, i.e. $s' = R_{-\pi}$. The identity (2.2.29) is consistent for the world-sheet theory as

$$s'(\mathbb{Q}_{\hat{I}}) = -\mathbb{Q}_{\hat{J}} s'(\mathrm{Y}^{\hat{J}}_{\hat{I}}) \quad \text{and} \quad s'(\mathrm{Y}^{\hat{K}}_{\hat{I}})\mathrm{Y}^{\hat{J}}_{\hat{K}} = \mathrm{Y}^{\hat{K}}_{\hat{I}} s'(\mathrm{Y}^{\hat{J}}_{\hat{K}}) = \delta^{\hat{J}}_{\hat{I}}\,. \tag{2.2.30}$$

This definition gives the natural relation $ss' = s's = 1$.

2.2.4 Adjoint actions

Essentially everything above is well known for the world-sheet theory. As we are interested in studying form factors and their generalizations, we wish to understand the action of the symmetries on the fields of the theory as well as on asymptotic states. This requires the generalization of the commutator of global charges to the adjoint action of a quantum group [62, 63]. Given an element of the algebra $a \in \mathcal{A}$, we can define further elements of algebra $a_i, a^i \in \mathcal{A}$ from the coproduct

$$\Delta(a) = \sum_i a_i \otimes a^i \,, \tag{2.2.31}$$

where the index i runs over the number of terms that appear in the coproduct of a. We can now define a graded adjoint and skew-adjoint action[14] on fields in the world-sheet theory

$$\begin{aligned} \mathrm{ad}_a(\Phi) &= \sum_i (-1)^{[a^i][\Phi]}\, a_i\, \Phi\, s(a^i)\,, &(2.2.32)\\ \mathrm{ad}^{\mathrm{op}}_a(\Phi) &= \sum_i (-1)^{[a_i]([\Phi]+[a^i])}\, a^i\, \Phi\, s'(a_i)\,. &(2.2.33)\end{aligned}$$

In the current basis the adjoint action of any of the global charges is

$$\begin{aligned} \mathrm{ad}_{\mathbb{Q}_{\hat{I}}}(\Phi) &= \mathbb{Q}_{\hat{I}}\Phi - (-1)^{[\hat{J}][\Phi]}\, \mathrm{ad}_{\mathrm{Y}^{\hat{J}}_{\hat{I}}}(\Phi)\mathbb{Q}_{\hat{J}} &(2.2.34)\\ \mathrm{ad}^{\mathrm{op}}_{\mathbb{Q}_{\hat{I}}}(\Phi) &= (-1)^{[\hat{I}][\Phi]}\Phi s'(\mathbb{Q}_{\hat{I}}) + \mathbb{Q}_{\hat{J}}\Phi s'(\mathrm{Y}^{\hat{J}}_{\hat{I}})\,. &(2.2.35)\end{aligned}$$

Thus we can see that the adjoint action for a global charge corresponds to $\hat{\mathbb{Q}}_{\hat{I}}[\Phi]$, i.e. the r.h.s of (2.2.34) is precisely the action of the charge on a field defined by the contour starting and ending at negative spatial infinity which can be read off from the diagram in Fig. 2.3,

$$\hat{\mathbb{Q}}_{\hat{I}}[\Phi] = \mathrm{ad}_{\mathbb{Q}_{\hat{I}}}(\Phi)\,. \tag{2.2.36}$$

[14] These are in fact left-adjoint actions, we could additionally define right adjoint actions but these will not be needed.

Alternatively, $\mathrm{ad}^{\mathrm{op}}$ gives the action defined using a contour starting and ending at positive spatial infinity.

Example

Let us explicitly give an example of the adjoint action on world-sheet fields. If we take Φ to be one of the fundamental world-sheet fields in an $\mathfrak{su}(2|2)$ sector, i.e. $\Phi_A = (Y_{a\dot{1}}, \Upsilon_{\alpha\dot{1}})$, then for the $\mathfrak{psu}(2|2)$ charges we have

$$\mathrm{ad}_{\mathbb{Q}_A{}^B}(\Phi_C) = \mathbb{Q}_A{}^B\Phi_C - (-1)^{[C]([A]+[B])} e^{i\epsilon_{AB}\mathbb{P}}\Phi_C e^{-i\epsilon_{AB}\mathbb{P}}\mathbb{Q}_A{}^B \,, \tag{2.2.37}$$

while for the central charges

$$\mathrm{ad}_{\mathbb{C}}(\Phi_C) = \mathbb{C}\Phi_C - e^{i\mathbb{P}}\Phi_C e^{-i\mathbb{P}}\mathbb{C} \,, \qquad \mathrm{ad}_{\mathbb{C}^\dagger}(\Phi_C) = \mathbb{C}^\dagger\Phi_C - e^{-i\mathbb{P}}\Phi_C e^{i\mathbb{P}}\mathbb{C}^\dagger \,, \tag{2.2.38}$$

and $\mathrm{ad}_{\mathbb{H}}(\Phi_C) = [\mathbb{H}, \Phi_C]$. For the braiding operators we have

$$\begin{aligned} &\mathrm{ad}_{\mathrm{Y}^{CD}_{AB}}(\Phi) = e^{i\epsilon_{AB}\mathbb{P}}\Phi e^{-i\epsilon_{AB}\mathbb{P}}\delta^C_A\delta^D_B \,, && \mathrm{ad}_{\mathrm{Y}^{\mathbb{H}}_{\mathbb{H}}}(\Phi) = \Phi \,, \\ &\mathrm{ad}_{\mathrm{Y}^{\mathbb{C}}_{\mathbb{C}}}(\Phi) = e^{i\mathbb{P}}\Phi e^{-i\mathbb{P}} \,, && \mathrm{ad}_{\mathrm{Y}^{\mathbb{C}^\dagger}_{\mathbb{C}^\dagger}}(\Phi) = e^{-i\mathbb{P}}\Phi e^{i\mathbb{P}} \,. \end{aligned} \tag{2.2.39}$$

Properties

The graded adjoint action satisfies the properties

$$\begin{aligned} \mathrm{ad}_{a_1}(\mathrm{ad}_{a_2}\Phi) &= \mathrm{ad}_{a_1 a_2}(\Phi) \,, \\ \mathrm{ad}^{\mathrm{op}}_{a_1}(\mathrm{ad}^{\mathrm{op}}_{a_2}\Phi) &= \mathrm{ad}^{\mathrm{op}}_{a_1 a_2}(\Phi) \,, \end{aligned} \tag{2.2.40}$$

as well as

$$\begin{aligned} \mathrm{ad}_a(\Phi_1\Phi_2) &= \sum_i (-1)^{[\Phi_1][a^i]} \mathrm{ad}_{a_i}(\Phi_1)\mathrm{ad}_{a^i}(\Phi_2) \,, \\ \mathrm{ad}^{\mathrm{op}}_a(\Phi_1\Phi_2) &= \sum_i (-1)^{[a^i][a_i]+[\Phi_1][a_i]} \mathrm{ad}^{\mathrm{op}}_{a^i}(\Phi_1)\mathrm{ad}^{\mathrm{op}}_{a_i}(\Phi_2) \,. \end{aligned} \tag{2.2.41}$$

2.2.5 A linear basis and a dual algebra

In [63] the generators $\mathbb{Q}_{\hat{I}}$ and $\mathbb{Y}_{\hat{I}}^{\hat{J}}$ were called the current basis. We will consider instead the so-called linear basis, labeling the generators of the $\mathfrak{su}(2|2)$ algebra combined with group element operator $\mathbb{U} = e^{i\mathbb{P}/2}$ by $\{\tilde{\mathbb{Q}}_I\} = \{\mathbb{R}_\alpha{}^\beta, \mathbb{L}_a{}^b, \mathbb{Q}_\alpha{}^b, \mathbb{Q}_a^{\dagger\beta}, \mathbb{H}, \mathbb{U}\}$. We do not include separately the central extensions $\mathbb{C}$ and $\mathbb{C}^\dagger$ as they are related to the the operator $\mathbb{U}$ as in (2.2.13), a relation which can be viewed as following from the demand of cocommutativity of the Hopf algebra [60]. We take the graded symmetric products of generators with weight one as a basis for the universal enveloping algebra adjoined with central elements, $\mathcal{A} = U(\widehat{\mathfrak{psu}}(2|2))$,

$$e_{\rm a} = \left\{ (\tilde{\mathbb{Q}}_1)^{t_1} (\tilde{\mathbb{Q}}_2)^{t_2} \dots (\tilde{\mathbb{Q}}_{15})^{t_{15}} (\tilde{\mathbb{Q}}_{16})^{t_{16}} \right\}_{\rm sym} \tag{2.2.42}$$

where ${\rm a} = \{t_1, t_2, \dots\}$ is a multi-index and $t_i \geq 0$ for $i = 1, \dots, 15$ while t_{16} can be either positive or negative. We can now write the Hopf algebra of charges in terms of structure constants,

$$\begin{aligned} e_{\rm a} e_{\rm b} &= m_{\rm ab}^{\rm c} e_{\rm c} , \qquad \Delta(e_{\rm a}) = \mu_{\rm a}^{\rm bc} e_{\rm b} \otimes e_{\rm c} , \\ s(e_{\rm a}) &= s_{\rm a}^{\rm b} e_{\rm b} , \qquad \epsilon(e_{\rm a}) = \epsilon_{\rm a} . \end{aligned} \tag{2.2.43}$$

It is also useful to define constants $\epsilon^{\rm a}$ such that $\epsilon^{\rm a} e_{\rm a} = \mathbb{1}$. The relation to the current basis global charges where ${\rm a}_I = \{t_{J\neq I} = 0, t_I = 1\}$ is obviously $\mathbb{Q}_{\rm I} = e_{{\rm a}_{\rm I}}$ and similarly for $\mathbb{H}$. On the other hand, the braiding operators are given by $\mathbb{Y}_J^I = \mu_{\rm c_J}^{\rm ab_I} e_{\rm a}$, where for example,

$$\mathbb{Y}_{AC}^{DB} = \delta_A^D \delta_C^B \, e_{{\rm a}_{\mathbb{U}}^{[{\rm A}]-[{\rm B}]}} \tag{2.2.44}$$

where ${\rm a}_{\mathbb{U}}^{[{\rm A}]-[{\rm B}]} = \{0, \dots, [{\rm A}] - [{\rm B}]\}$. The central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ can of course still be expressed as linear combinations of elements of the new basis

$$\begin{aligned} \mathbb{C} &= \frac{ig}{2} \left(e_{\{t_i=0, t_{16}=2\}} - \mathbb{1} \epsilon_{\{t_i=0, t_{16}=2\}} \right) , \\ \mathbb{C}^\dagger &= \frac{ig}{2} \left(e_{\{t_i=0, t_{16}=-2\}} - \mathbb{1} \epsilon_{\{t_i=0, t_{16}=-2\}} \right) . \end{aligned} \tag{2.2.45}$$

The dual algebra

Having defined the structure constants for the universal enveloping it is interesting to use them to define the dual algebra for $U(\widehat{\mathfrak{psu}}(2|2))$. Generically, given a linear basis $e_{\rm a}$ satisfying (2.2.43) for the algebra $\mathcal{A}$, we can give the following definition of the dual algebra $\mathcal{A}^*$ in terms of the dual basis $e^{\rm a}$ [15]

$$m^*(e^{\rm a} \otimes e^{\rm b}) = e^{\rm a} e^{\rm b} = \mu^{\rm ab}_{\rm c} e^{\rm c} \,, \qquad \Delta^*(e^{\rm a}) = (-1)^{[{\rm b}][{\rm c}]} m^{\rm a}_{\rm bc} e^{\rm c} \otimes e^{\rm b} \,,$$
$$s^*(e^{\rm a}) = (s^{-1})^{\rm a}_{\rm b} e^{\rm b} \,, \qquad \epsilon^*(e^{\rm a}) = \epsilon^{\rm a} \,. \tag{2.2.46}$$

We furthermore use the constants $\epsilon_{\rm a}$ to define the unit operator in the dual algebra $\mathbb{1}_{\rm d} = \epsilon_{\rm a} e^{\rm a}$. In the case of Lie algebras which are cocommutative the dual algebras are simply commutative. In the more interesting case of Yangian symmetries the dual algebra can be used to construct the quantum double and the universal R-matrix[16]. It would of course be interesting to study the dual of the algebra of Yangian symmetries found in the world-sheet theory, however we will restrict ourselves to the global world-sheet symmetries in this thesis. Here the dual algebra is nonetheless somewhat non-trivial due to the coproduct.

We can be slightly more explicit regarding the dual algebra. We will use the basis defined by $e^{\rm c} = (e_{\rm c})^*$. To find the commutators of the dual generators we must first determine their product which is given by coproduct coefficients, $\mu^{\rm bc}_{\rm a}$. For example, the coproduct coefficients involving $\mathbb{L}_a{}^b$ and $\mathbb{L}_c{}^d$ appear in

$$\Delta(\{\mathbb{L}_a{}^b \mathbb{L}_c{}^d\}) = \mathbb{L}_a{}^b \otimes \mathbb{L}_c{}^d + \mathbb{1} \otimes \{\mathbb{L}_a{}^b \mathbb{L}_c{}^d\} + \{\mathbb{L}_a{}^b \mathbb{L}_c{}^d\} \otimes \mathbb{1} + \mathbb{L}_c{}^d \otimes \mathbb{L}_a{}^b \tag{2.2.47}$$

hence

$$(\mathbb{L}_a{}^b)^* (\mathbb{L}_c{}^d)^* = (\{\mathbb{L}_a{}^b \mathbb{L}_c{}^d\})^* = (\mathbb{L}_c{}^d)^* (\mathbb{L}_a{}^b)^* \tag{2.2.48}$$

and so $[(\mathbb{L}_a{}^b)^*, (\mathbb{L}_c{}^d)^*] = 0$. Similarly $[(\mathbb{R}_\alpha{}^\beta)^*, (\mathbb{R}_\gamma{}^\delta)^*] = 0$ and $\mathbb{H}^*$ is a central element. Note that this is quite different from the case of the Yangian dual where the elements $(\mathbb{L}_a{}^b)^*$, $(\mathbb{R}_\alpha{}^\beta)^*$, ... would have non-trivial commutators and would act as

[15] We use the definition of the dual algebra as in [64, 65, 62], with the graded version as in [66] which differs by a permutation from that in [63].

[16] We will introduce the idea of integrability and the (algebraic) Bethe ansatz in the next chapter, more specifically for the R-matrix see Section 3.2.3.

generators for the full algebra. In the present case, however, as

$$\Delta(\mathbb{Q}_A{}^B) = \mathbb{Q}_A{}^B \otimes \mathbb{1} + \mathbb{U}^{[A]-[B]} \otimes \mathbb{Q}_A{}^B \,, \tag{2.2.49}$$

we do have the non-trivial commutator

$$[(\mathbb{U}^{[A]-[B]})^*, (\mathbb{Q}_A{}^B)^*] \;=\; \begin{cases} (\mathbb{Q}_A{}^B)^* & , \quad [A] \neq [B] \\ 0 & , \quad [A] = [B] \,. \end{cases} \tag{2.2.50}$$

while for the group-like element we have

$$(\mathbb{U}^n)^*(\mathbb{U}^m)^* = \begin{cases} (\mathbb{U}^n)^* & , \quad n = m \\ 0 & , \quad n \neq m \,. \end{cases} \tag{2.2.51}$$

From the definition of the unit in the dual algebra we see explicitly that

$$\mathbb{1}_{\mathrm{d}} = (\mathbb{1})^* + \sum_{n\neq 0} (\mathbb{U}^n)^* \,. \tag{2.2.52}$$

The definition in (2.2.46) gives the coproduct for the dual algebra in terms of the multiplication coefficients $m^{\mathrm{c}}_{\mathrm{ab}}$. Some explicit examples for $\widehat{\mathfrak{psu}}(2|2)$ are

$$\begin{aligned}
\Delta^*((\mathbb{L}_a{}^b)^*) &= \mathbb{1}^* \otimes (\mathbb{L}_a{}^b)^* + (\mathbb{L}_a{}^b)^* \otimes \mathbb{1}^* - (\mathbb{L}_a{}^c)^* \otimes (\mathbb{L}_c{}^b)^* + (\mathbb{L}_c{}^b)^* \otimes (\mathbb{L}_a{}^c)^* \\
&\quad -(\mathbb{Q}_\alpha{}^a)^* \otimes (\mathbb{Q}^{\dagger\alpha}_b)^* - (\mathbb{Q}^{\dagger\alpha}_b)^* \otimes (\mathbb{Q}_\alpha{}^a)^* + \dots \\
\Delta^*((\mathbb{R}_\alpha{}^\beta)^*) &= \mathbb{1}^* \otimes (\mathbb{R}_\alpha{}^\beta)^* + (\mathbb{R}_\alpha{}^\beta)^* \otimes \mathbb{1}^* - (\mathbb{R}_\alpha{}^\gamma)^* \otimes (\mathbb{R}_\gamma{}^\beta)^* + (\mathbb{R}_\gamma{}^\beta)^* \otimes (\mathbb{R}_\alpha{}^\gamma)^* \\
&\quad -(\mathbb{Q}_\alpha{}^a)^* \otimes (\mathbb{Q}^{\dagger\beta}_a)^* - (\mathbb{Q}^{\dagger\beta}_a)^* \otimes (\mathbb{Q}_\alpha{}^a)^* + \dots \\
\Delta^*((\mathbb{Q}_\alpha{}^a)^*) &= \mathbb{1}^* \otimes (\mathbb{Q}_\alpha{}^a)^* + (\mathbb{Q}_\alpha{}^a)^* \otimes \mathbb{1}^* + \tfrac{3}{4}(\mathbb{L}_c{}^a)^* \otimes (\mathbb{Q}_\alpha{}^c)^* - \tfrac{3}{4}(\mathbb{Q}_\alpha{}^c)^* \otimes (\mathbb{L}_c{}^a)^* \\
&\quad -\tfrac{3}{4}(\mathbb{R}_\alpha{}^\gamma)^* \otimes (\mathbb{Q}_\gamma{}^a)^* + \tfrac{3}{4}(\mathbb{Q}_\gamma{}^a)^* \otimes (\mathbb{R}_\alpha{}^\gamma)^* + \dots \\
\Delta^*((\mathbb{Q}^{\dagger\alpha}_a)^*) &= \mathbb{1}^* \otimes (\mathbb{Q}^{\dagger\alpha}_a)^* + (\mathbb{Q}^{\dagger\alpha}_a)^* \otimes \mathbb{1}^* - \tfrac{3}{4}(\mathbb{L}_a{}^c)^* \otimes (\mathbb{Q}^{\dagger\alpha}_c)^* + \tfrac{3}{4}(\mathbb{Q}^{\dagger\alpha}_c)^* \otimes (\mathbb{L}_a{}^c)^* \\
&\quad +\tfrac{3}{4}(\mathbb{R}_\gamma{}^\alpha)^* \otimes (\mathbb{Q}^{\dagger\gamma}_a)^* - \tfrac{3}{4}(\mathbb{Q}^{\dagger\gamma}_a)^* \otimes (\mathbb{R}_\gamma{}^\alpha)^* + \dots \,.
\end{aligned}$$

For the Hamiltonian we have

$$\Delta^*(\mathbb{H}^*) = \mathbb{1}^* \otimes \mathbb{H}^* + \mathbb{H}^* \otimes \mathbb{1}^* + (\mathbb{Q}_\alpha{}^a)^* \otimes (\mathbb{Q}^{\dagger\alpha}_a)^* + (\mathbb{Q}^{\dagger\alpha}_a)^* \otimes (\mathbb{Q}_\alpha{}^a)^* + \dots \tag{2.2.53}$$

while the duals of the braiding elements satisfy

$$\Delta^*(\mathbb{U}^*) = \mathbb{1}^* \otimes \mathbb{U}^* + \mathbb{U}^* \otimes \mathbb{1}^* + (\mathbb{U}^{-1})^* \otimes (\mathbb{U}^2)^* + (\mathbb{U}^2)^* \otimes (\mathbb{U}^{-1})^* + \ldots . \quad (2.2.54)$$

It is also interesting to consider higher powers of the braiding element as these are related to the duals of the central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ in the original algebra. In particular the coupling g appears in the coproduct, e.g.

$$\Delta^*((\mathbb{U}^2)^*) =$$
$$\mathbb{1}^* \otimes (\mathbb{U}^2)^* + (\mathbb{U}^2)^* \otimes \mathbb{1}^* - ig\epsilon^{\alpha\beta}\epsilon_{ab}(\mathbb{Q}_\alpha{}^a)^* \otimes (\mathbb{Q}_\beta{}^b)^* - ig\epsilon^{\alpha\beta}\epsilon_{ab}(\mathbb{Q}_\beta{}^b)^* \otimes (\mathbb{Q}_\alpha{}^a)^* .$$

The antipode and counit can be explicitly found in a similar way from the standard definitions above.

The quantum double

Following Drinfeld [64] one can identify the quantum double $\mathcal{D}(\mathcal{A})$ as a Hopf algebra containing as subalgebras both $\mathcal{A}$ and $\mathcal{A}^*$. This was generalized to $\mathbb{Z}_2$-graded Hopf algebras in [66]. The Hopf algebra structure on $\mathcal{D}(\mathcal{A})$ can be found starting with the definition of a formal product of basis elements of $\mathcal{A}$ and $\mathcal{A}^*$:

$$\psi(e_{\mathrm{b}} \otimes e^{\mathrm{a}}) = e_{\mathrm{b}} e^{\mathrm{a}} \quad (2.2.55)$$

which defines an isomorphism between $\mathcal{D}(\mathcal{A})$ and $\mathcal{A} \otimes \mathcal{A}^*$ considered as graded vector spaces. One can subsequently define a graded Hopf algebra structure on $\mathcal{D}(\mathcal{A})$ by making use of this inherited structure [66]. For example given the natural coproduct on the tensor spaces $\hat{\Delta} : \mathcal{A}^* \otimes \mathcal{A} \to \mathcal{A}^* \otimes \mathcal{A} \otimes \mathcal{A}^* \otimes \mathcal{A}$ defined by

$$\hat{\Delta} = (I \otimes P \otimes I) \circ (\Delta^* \otimes \Delta) \quad (2.2.56)$$

and $\hat{\Delta}^{\mathrm{op}} : \mathcal{A} \otimes \mathcal{A}^* \to \mathcal{A} \otimes \mathcal{A}^* \otimes \mathcal{A} \otimes \mathcal{A}^*$ defined by

$$\hat{\Delta}^{\mathrm{op}} : (I \otimes P \otimes I) \circ (\Delta \otimes \Delta^*) . \quad (2.2.57)$$

we can define $\bar{\Delta} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{A})$

$$\bar{\Delta} = (\psi \otimes \psi) \circ \hat{\Delta}^{\rm op} \circ \psi^{-1} \,. \tag{2.2.58}$$

Furthermore we can define multiplication for $\mathcal{D}(\mathcal{A})$ (see again [64] or [66] for the superalgebra version). A key step is defining the vector space isomorphism $M : \mathcal{A}^* \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}^*$

$$M = (I \otimes I \otimes {\rm Str}) \circ \hat{\Delta}^{\rm op} \circ P \circ ({\rm Str} \otimes I \otimes I) \circ (s^* \otimes I \otimes I \otimes I) \circ \hat{\Delta} \tag{2.2.59}$$

where ${\rm Str}(e^{\rm a} \otimes e_{\rm b}) = (-1)^{[{\rm a}][{\rm b}]}{\rm Str}(e_{\rm b} \otimes e^{\rm a}) = \delta^a_b$. In terms of original Hopf algebra structure constants this is given by

$$M(e^{\rm a} \otimes e_{\rm b}) = M^{\rm aa'}{}_{\rm bb'} e_{\rm a'} \otimes e^{\rm b'} \,, \tag{2.2.60}$$

with

$$M^{\rm aa'}{}_{\rm bb'} = (-1)^{[{\rm c}][{\rm f}]+[{\rm r}]} \mu^{\rm ef}_{\rm b} \mu^{\rm a'r}_{\rm f} m^{\rm c}_{\rm rb'} m^{\rm a}_{\rm cd} (s^{-1})^{\rm d}_{\rm e} \,. \tag{2.2.61}$$

Multiplication in the double, $\bar{m} : \mathcal{D}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$, is given by

$$\bar{m} = \psi \circ (m \circ m^*) \circ (I \otimes M \otimes I) \circ (\psi^{-1} \otimes \psi^{-1}) \,, \tag{2.2.62}$$

and an antipode for $\mathcal{D}(\mathcal{A})$ can be defined as

$$\bar{s} = \psi \circ M \circ P(s \otimes s^*) \circ \psi^{-1} \,. \tag{2.2.63}$$

Finally we need to understand how to reorder elements of the double so that the elements $e_{\rm a} e^{\rm b}$ do indeed form a basis. To this end we define

$$e^{\rm a} e_{\rm b} = \psi \circ M(e^{\rm a} \otimes e_{\rm b}) \tag{2.2.64}$$

such that

$$e^{\rm a} e_{\rm b} = M^{\rm aa'}{}_{\rm bb'} e_{\rm a'} e^{\rm b'} \,, \tag{2.2.65}$$

from which one can compute commutation relations for the generators of the original algebra from those of its dual .

A standard application, indeed one of the main motivations, of the quantum double is the construction of the universal R-matrix. Given a Hopf algebra $\mathcal{A}$ and its dual $\mathcal{A}^*$ we can formally write the R-matrix as

$$\mathcal{R} = \sum_{\rm a} e_{\rm a} \otimes e^{\rm a} \tag{2.2.66}$$

where the sum goes over all basis elements. This is defined so that, due to the relation (2.2.65), it automatically satisfies the condition of quasi-triangularity for a $\mathbb{Z}_2$-graded Hopf algebra

$$\mathcal{R}\Delta(e_{\rm a}) = \Delta^{\rm op}(e_{\rm a})\mathcal{R} \tag{2.2.67}$$

and hence $\mathcal{R}$ satisfies the graded Yang-Baxter equation. In terms of the linear basis coefficients this condition corresponds to

$$(-1)^{[\rm b][f]}\mu_{\rm a}^{\rm bc} m_{\rm fb}^{\rm d} e^{\rm f} e_{\rm c} = (-1)^{[\rm b][c]+[f][c]}\mu_{\rm a}^{\rm cb} m_{\rm bf}^{\rm d} e_{\rm c} e^{\rm f} \, . \tag{2.2.68}$$

Focusing on the $\widehat{\mathfrak{psu}}(2|2)$ case, if $[{\rm a}] = 0$ and $\mu_{\rm a}^{\rm bc} = \delta_{\mathbb{1}}^{\rm b}\delta_{\rm a}^{\rm c} + \delta_{\mathbb{1}}^{\rm c}\delta_{\rm a}^{\rm b}$ then

$$[e_{\rm a}, e^{\rm b}] = (m_{\rm ca}^{\rm b} - m_{\rm ac}^{\rm b})e^{\rm c} \, . \tag{2.2.69}$$

Hence

$$\begin{aligned}
\left[\mathbb{L}_a{}^b, (\mathbb{L}_c{}^d)^*\right] &= \delta_d^b(\mathbb{L}_c{}^a)^* - \delta_a^c(\mathbb{L}_b{}^d)^* \\
\left[\mathbb{L}_a{}^b, (\mathbb{Q}_\alpha{}^c)^*\right] &= \delta_c^b(\mathbb{Q}_\alpha{}^a)^* - \frac{1}{2}\delta_b^a(\mathbb{Q}_\alpha{}^c)^* \\
\left[\mathbb{L}_a{}^b, (\mathbb{Q}_c^{\dagger\alpha})^*\right] &= -\delta_a^c(\mathbb{Q}_b^{\dagger\alpha})^* + \frac{1}{2}\delta_a^b(\mathbb{Q}_c^{\dagger\alpha})^*
\end{aligned} \tag{2.2.70}$$

and similarly for the commutators of the cocommutative elements $\mathbb{R}_\alpha{}^\beta$ with the dual generators while the generator $\mathbb{H}$ remains central in the double, i.e. $[\mathbb{H}, e^{\rm d}] = 0$. This is the standard structure for the quantum double of a Lie algebra. Alternatively if

$$\mu_{\rm a}^{\rm bc} = \delta_{\rm a}^{\rm b}\delta_{\mathbb{1}}^{\rm c} + \delta_{\rm a}^{\rm c}\delta_{\mathbb{U}^\alpha}^{\rm b} \tag{2.2.71}$$

then

$$e^{\mathrm{d}'}e_{\mathrm{a}} - (-1)^{[\mathrm{a}][\mathrm{d}]}e_{\mathrm{a}}e^{\mathrm{d}} = (m^{\mathrm{d}}_{\mathrm{af}} - (-1)^{[\mathrm{a}][\mathrm{f}]}m^{\mathrm{d}}_{\mathrm{fa}}\mathbb{U}^{\alpha})e^{f} , \tag{2.2.72}$$

where $e^{\mathrm{d}'} = (\mathbb{U}^{-\alpha}e_{\mathrm{d}})^*$. For example we consider the (anti)commutators involving the generator $\mathbb{Q}_\alpha{}^a$ so that $\mathbb{U}^\alpha = \mathbb{U}$ and we find

$$\begin{aligned}
(\mathbb{U}^{-1}\mathbb{L}_a{}^b)^*\mathbb{Q}_\alpha{}^c - \mathbb{Q}_\alpha{}^c(\mathbb{L}_a{}^b)^* &= \frac{1}{2}(\mathbb{1}+\mathbb{U})(\mathbb{Q}^{\dagger\alpha}_b)^*\delta^a_c \\
(\mathbb{U}^{-1}\mathbb{R}_\alpha{}^\beta)^*\mathbb{Q}_\gamma{}^a - \mathbb{Q}_\gamma{}^a(\mathbb{R}_\alpha{}^\beta)^* &= \frac{1}{2}(\mathbb{1}+\mathbb{U})(\mathbb{Q}^{\dagger\beta}_a)^*\delta^\alpha_\gamma \\
(\mathbb{U}^{-1}\mathbb{H})^*\mathbb{Q}_\alpha{}^a - \mathbb{Q}_\alpha{}^a(\mathbb{H})^* &= \frac{1}{4}(\mathbb{1}+\mathbb{U})(\mathbb{Q}^{\dagger\alpha}_a)^* .
\end{aligned} \tag{2.2.73}$$

In the limit where $\mathbb{U} \to \mathbb{1}$ we can see that these become the usual relations between the generators of a cocommutative Lie algebra and its dual. Several (anti)commutators which would be vanishing in the cocommutative case are non-trivial in the world-sheet theory, for example

$$\begin{aligned}
(\mathbb{U}^{-1}\mathbb{Q}_\alpha{}^a)^*\mathbb{Q}_\beta{}^b + \mathbb{Q}_\beta{}^b(\mathbb{Q}_\alpha{}^a)^* &= (\mathbb{1}-\mathbb{U})\delta^b_a\delta^\alpha_\beta \\
(\mathbb{U})^*\mathbb{Q}_\alpha{}^a + \mathbb{Q}_\alpha{}^a(\mathbb{U}^2)^* &= \frac{ig}{4}\epsilon_{\alpha\beta}\epsilon^{ab}(\mathbb{1}-\mathbb{U})(\mathbb{Q}_\beta{}^b)^* .
\end{aligned} \tag{2.2.74}$$

However, it is straightforward to see that the generator $\mathbb{U}$ remains central with respect to the dual generators.

Given a specific theory there is of course the question of how to embed the known symmetries into the quantum double. Here we have essentially taken the double of the entire algebra. Alternatively one may, for example, take the positive Borel sub-algebra and attempt to identify the negative Borel sub-algebra with the dual algebra. While this is important in attempting to find the universal R-matrix for the world sheet theory we will not discuss such issues here.

It is very important to note that we have not considered the double of the world-sheet Yangian algebra. In particular if we take the generators above as level-zero generators, $\mathbb{Q}^I_0$, and include level-one generators, $\mathbb{Q}^I_1$, with a non-trivial coproduct coefficients $\mu^{\mathbb{J}_0\mathbb{J}_0}_{\mathbb{J}_1}$, this implies an entirely different dual algebra. In particular now the product $(\mathbb{Q}^I_0)^*(\mathbb{Q}^J_0)^*$ contains the dual of the level-one generator $(\mathbb{Q}^I_1)^*$.

Action on fields

We now want to define the action of the algebra and the dual algebra on the fields in the theory, as in [63]. We can rewrite the definition of the adjoint action given in (2.2.32) in terms of the linear basis

$$\begin{aligned} \mathrm{ad}_{e_{\mathrm{a}}}(\Phi) &= (-1)^{[\mathrm{c}][\Phi]}\, \mu_{\mathrm{a}}^{\mathrm{bc}} s_{\mathrm{c}}^{\mathrm{d}}\, e_{\mathrm{b}} \Phi e_{\mathrm{d}} \,, \\ \mathrm{ad}^{\mathrm{op}}_{e_{\mathrm{a}}}(\Phi) &= (-1)^{[\mathrm{b}][\Phi]}\, \mu_{\mathrm{a}}^{\mathrm{bc}} (s')_{\mathrm{b}}^{\mathrm{d}}\, e_{\mathrm{c}} \Phi e_{\mathrm{d}} \,. \end{aligned} \tag{2.2.75}$$

From these definitions and the usual consistency conditions for a Hopf algebra, given in App. D, we have

$$\begin{aligned} e_a \Phi &= (-1)^{[\mathrm{c}][[\Phi]}\, \mu_{\mathrm{a}}^{\mathrm{bc}} \mathrm{ad}_{e_{\mathrm{b}}}(\Phi) e_{\mathrm{c}} \\ \Phi s(e_a) &= (-1)^{[\mathrm{a}][[\Phi]}\, \mu_{\mathrm{a}}^{\mathrm{bc}} s(e_{\mathrm{b}}) \mathrm{ad}_{e_{\mathrm{c}}}(\Phi) \,. \end{aligned} \tag{2.2.76}$$

This gives rise to an alternative definition of the quantum double in terms of the braiding of fields. Given fields transforming in some representation of the algebra Λ_1 and Λ_2 their braiding is

$$\Phi_{\Lambda_1}(\sigma_1)\Phi_{\Lambda_2}(\sigma_2) = \mathcal{R}_{\rho_{\Lambda_1}\rho_{\Lambda_2}} \Phi_{\Lambda_2}(\sigma_2)\Phi_{\Lambda_1}(\sigma_1) \,, \quad \text{for} \quad \sigma_1 > \sigma_2 \,. \tag{2.2.77}$$

We assume the existence of some highest weight states which cannot be found by acting on other fields by adjoint action of the algebra. To be precise we need a definition of raising and lowering operators in the algebra, but for the moment we will take a highest weight field, Φ, to be annihilated by the dual generators in the sense that they provide a one-dimensional representation defined by

$$e^{\mathrm{a}}(\Phi(\sigma)) = \epsilon^{\mathrm{a}} \Phi(\sigma) \,. \tag{2.2.78}$$

Descendant fields are then found by repeated adjoint or skew-adjoint action,

$$\begin{aligned} \Phi_{\mathrm{a}_1,\mathrm{a}_2,\ldots,\mathrm{a}_n} &= \mathrm{ad}_{e_{\mathrm{a}_1}} \circ \mathrm{ad}_{e_{\mathrm{a}_2}} \circ \ldots \mathrm{ad}_{e_{\mathrm{a}_n}}(\Phi) \\ \Phi^{\mathrm{op}}_{\mathrm{a}_1,\mathrm{a}_2,\ldots,\mathrm{a}_n} &= \mathrm{ad}^{\mathrm{op}}_{e_{\mathrm{a}_1}} \circ \mathrm{ad}^{\mathrm{op}}_{e_{\mathrm{a}_2}} \circ \ldots \mathrm{ad}^{\mathrm{op}}_{e_{\mathrm{a}_n}}(\Phi) \,. \end{aligned} \tag{2.2.79}$$

Due to the property of the adjoint and skew-adjoint actions, (2.2.40), the tower of descendants form a representation of $\mathcal{A}$ which we collectively denote Φ_Λ or $\Phi^{\rm op}_\Lambda$. That is to say, given specific fields in these vector spaces Φ_λ and $\Phi^{\rm op}_\lambda$ we have

$$\begin{aligned} {\rm ad}_{e_{\rm a}}(\Phi_\lambda) &= \rho_\Lambda(e_{\rm a})^\nu_\lambda\, \Phi_\nu \\ {\rm ad}^{\rm op}_{e_{\rm a}}(\Phi_\lambda) &= \rho^{\rm op}_\Lambda(e_{\rm a})^\nu_\lambda\, \Phi^{\rm op}_\nu \ , \end{aligned} \tag{2.2.80}$$

knowing their behaviour under rotations, $R_\pi[\Phi(0)] = e^{\rm a}[\Phi(0)]e_{\rm a}$. These definitions can also be used to derive the commutators of generators of the original algebra with those of its dual,

$$\mu^{\rm bc}_{\rm a} m^{\rm d}_{\rm fb} e^{\rm f} e_{\rm c} = \mu^{\rm cb}_{\rm a} m^{\rm d}_{\rm bf} e_{\rm c} e^{\rm f} \ , \tag{2.2.81}$$

and if $\mu^{\rm bc}_{\rm a} = \delta^{\rm b}_{\mathbb{1}}\delta^{\rm c}_{\rm a} + \delta^{\rm c}_{\mathbb{1}}\delta^{\rm b}_{\rm a}$ then

$$[e_{\rm a}, e^{\rm b}] = (m^{\rm b}_{\rm ca} - m^{\rm b}_{\rm ac})e^{\rm c} \ . \tag{2.2.82}$$

In particular we can define the highest weight operators as those which are invariant under rotations

$$e^{\rm a}\Phi_\lambda(0) = \epsilon^{\rm a}\Phi_\lambda(0) \ . \tag{2.2.83}$$

For the string world sheet we will be interested in operators formed from local products of fundamental world-sheet fields and these operators will transform trivially under such rotations. However, descendants created by acting with supercharges will be non-local and so will no longer be invariant.

2.2.6 Asymptotic symmetries

The asymptotic states provide a representation of the world-sheet symmetries. Given symmetry generators $e_{\rm a}$ forming an algebra $\mathcal{A}$, here taken to be the $\widehat{\mathfrak{psu}}(2|2)$ corresponding to the undotted indices, we define their action on the invariant vacuum[17]

$$e_{\rm a}|\Omega\rangle = \epsilon_{\rm a}|\Omega\rangle\,, \qquad \langle\Omega|e_{\rm a} = \langle\Omega|\epsilon_{\rm a}\,. \tag{2.2.84}$$

The ZF operators are further characterized by their intertwining relations with the elements of $\mathcal{A}$ which act via the adjoint action

$$\mathrm{ad}_{e_{\rm a}}(\mathbb{Z}^\dagger_{A\dot{A}}(z)) = \rho^z(e_{\rm a})^B_A\mathbb{Z}^\dagger_{B\dot{A}}(z)\,, \tag{2.2.85}$$

and the action of the dual algebra which is defined as

$$e^{\rm a}\cdot(\mathbb{Z}^\dagger_{A\dot{A}}(z)) = \rho^z(e^{\rm a})^B_A\mathbb{Z}^\dagger_{B\dot{A}}(z)\,. \tag{2.2.86}$$

Alternatively we can specify the action of the algebra on the one-particle states, for instance

$$e_{\rm a}|z\rangle_{A\dot{A}} \equiv \mathrm{ad}_{e_{\rm a}}(\mathbb{Z}^\dagger_{A\dot{A}}(z))|\Omega\rangle = \rho^z(e_{\rm a})^B_A|z\rangle_{B\dot{A}}\,. \tag{2.2.87}$$

For $\widehat{\mathfrak{psu}}(2|2)$ this one particle representation was studied in [50, 52] and can be given as (here we simply suppress the dotted index)

$$\begin{aligned}
&\mathbb{L}_a{}^b|z\rangle_c = \delta^b_c|z\rangle_a - \tfrac{1}{2}\delta^b_a|z\rangle_c\,, && \mathbb{R}_\alpha{}^\beta|z\rangle_c = 0\,,\\
&\mathbb{L}_a{}^b|z\rangle_\gamma = 0\,, && \mathbb{R}_\alpha{}^\beta|z\rangle_\gamma = \delta^\beta_\gamma|z\rangle_\alpha - \tfrac{1}{2}\delta^\beta_\alpha|z\rangle_\gamma\,,\\
&\mathbb{Q}_\alpha{}^a|z\rangle_b = a\delta^a_b|z\rangle_\alpha\,, && \mathbb{Q}_\alpha{}^a|z\rangle_\beta = b\epsilon^{ab}\epsilon_{\alpha\beta}|z\rangle_b\,,\\
&\mathbb{Q}^{\dagger\alpha}_a|z\rangle_b = c\epsilon_{ab}\epsilon^{\alpha\beta}|z\rangle_\beta\,, && \mathbb{Q}^{\dagger\alpha}_a|z\rangle_\beta = d\delta^\alpha_\beta|z\rangle_a\,,
\end{aligned} \tag{2.2.88}$$

[17] It is also conventional to take the vacuum to be annihilated by elements of the dual algebra

$$e^{\rm a}|\Omega\rangle = \epsilon^{\rm a}|\Omega\rangle\,, \qquad \langle\Omega|e^{\rm a} = \langle\Omega|\epsilon^{\rm a}\,.$$

where the parameters a, b, c, d satisfy the constraint $ad - bc = 1$ and the central charges are given by

$$\mathbb{H}|z\rangle_A = H|z\rangle_A \,, \quad \mathbb{C}|z\rangle_A = C|z\rangle_A \,, \quad \mathbb{C}^\dagger|z\rangle_A = \overline{C}|z\rangle_A \,, \tag{2.2.89}$$

with $H = ad + bc$ and $C = ab$, $\overline{C} = cd$. We will consider unitary representations for which we have $d^* = a$ and $c^* = b$. Importantly the constraint, or shortening condition, implies that the multiplet is on-shell. For a unitary representation, and with the usual string choice for the phase, we have in terms of the particle momentum p

$$C = ig\frac{e^{ip} - 1}{2} \,, \qquad H^2 = 1 + 4g^2 \sin\frac{p}{2} \,. \tag{2.2.90}$$

The action on multi-particle states can now be straightforwardly described by repeated use of the coproduct (2.2.15), or in terms of the coefficients (2.2.43),

$$e_{\mathrm{a}}|z_1, \ldots, z_n\rangle_{A_1,\ldots,A_n} = \rho^{z_1,\ldots,z_n}(e_{\mathrm{a}})_{A_1\ldots A_n}^{B_1\ldots B_n}|z_1, \ldots, z_n\rangle_{B_1,\ldots,B_n} \tag{2.2.91}$$

with

$$\rho^{z_1,\ldots,z_n}(e_{\mathrm{a}})_{A_1\ldots A_n}^{B_1\ldots B_n} = \mu_{\mathrm{a}}^{\mathrm{b}_1\mathrm{c}_1}\mu_{\mathrm{c}_1}^{\mathrm{b}_2\mathrm{c}_2}\ldots\mu_{\mathrm{c}_{n-2}}^{\mathrm{b}_{n-1}\mathrm{b}_n}\rho^{z_1}(e_{\mathrm{b}_1})_{A_1}^{B_1}\rho^{z_2}(e_{\mathrm{b}_2})_{A_2}^{B_2}\ldots\rho^{z_n}(e_{\mathrm{b}_n})_{A_n}^{B_n} \,. \tag{2.2.92}$$

In particular, for the $\mathfrak{psu}(2|2)$ charges

$$\mathbb{Q}_A{}^B|z_1, \ldots, z_n\rangle_{A_1,\ldots,A_n} = \sum_{i=1}^{n} e^{i\sum_{j=1}^{i-1}\epsilon_{AB}p_j}\rho^{z_i}(\mathbb{Q}_A{}^B)_{A_i}^{B_i}|z_1, \ldots, z_n\rangle_{A_1,\ldots,B_i,\ldots,A_n} \tag{2.2.93}$$

and similarly for the central charges

$$\begin{aligned} \mathbb{H}|z_1, \ldots, z_n\rangle_{A_1,\ldots,A_n} &= \sum_{i=1}^{n} H_i|z_1, \ldots, z_n\rangle_{A_1,\ldots,A_n} \\ \mathbb{C}|z_1, \ldots, z_n\rangle_{A_1,\ldots,A_n} &= \sum_{i=1}^{n} e^{i\sum_{j=1}^{i-1}p_j}C_i|z_1, \ldots, z_n\rangle_{A_1,\ldots,A_n} \end{aligned} \tag{2.2.94}$$

using $\rho^{z_i}(\mathbb{H}) = H_i$ and $\rho^{z_i}(\mathbb{C}) = C_i$. In Section 4.2.3 we will discuss the symmetries for the form factors, off-shell objects which we will introduce in Chapter 4.

Chapter 3

Integrability in the AdS/CFT correspondence

In the context of the AdS/CFT correspondence, it emerged a type of integrability which is usually associated with two-dimensional systems. More specifically, when studying the conformal dimensions of some operators in $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM), it has been found that the loop corrections can be found by solving a specific spin chain, i.e. a one-dimensional lattice model in which every site carries a (finite) representation of the corresponding (sub)group, as we will see in more detail in the following. In this chapter we will briefly review $\mathcal{N} = 4$ SYM, to explain the connection to spin chains and thus integrability.

3.1 $\mathcal{N} = 4$ supersymmetric Yang–Mills theory

3.1.1 The $\mathcal{N} = 4$ SYM action

$\mathcal{N} = 4$ supersymmetric Yang–Mills is a conformal gauge theory in four dimensions (one in time and three in space) with the maximal amount of supersymmetry allowed (four). Its action can be derived by dimensional reduction of the $\mathcal{N} = 1$ SYM action in ten dimensions

$$\mathcal{A} = -\frac{2}{g_{YM}^2} \int d^4x \ \mathrm{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^I D^\mu \phi^I - \frac{1}{4} [\phi^I, \phi^J][\phi^I, \phi^J] \right. \\ \left. + \frac{1}{2} \bar{\chi} \Gamma^\mu D_\mu \chi - \frac{i}{2} \bar{\chi} \Gamma_I [\phi^I, \chi] \right] , \tag{3.1.1}$$

where the field strength $F_{\mu\nu}$ and the covariant derivative D_μ are defined from the gauge boson $A_\mu(x)$ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \qquad D_\mu Z = \partial_\mu Z - i[A_\mu, Z], \tag{3.1.2}$$

with $Z(x)$ any of the covariant fields. The other fields in (3.1.1) are six massless real scalars ϕ^I ($I = 1, ..., 6$) and a 16-component ten-dimensional Majorana-Weyl spinor χ, while (Γ_μ, Γ_I) are the Dirac matrices in 10 dimensions. In particular, χ is equivalent to eight four-dimensional Weyl spinors (fermions), four chiral $\psi_\alpha{}^a$ and four anti-chiral $\overline{\psi}_{\dot{\alpha}a}$, with $\alpha, \dot{\alpha} = 1, 2$ and $a = 1, ..., 4$ [1]. In order to have $SU(N)$ as the gauge group, we choose all the fields in its adjoint representation. In (3.1.1) the gauge indices are understood and taking the trace of covariant combinations guarantees the invariance of the action under $U(N)$.

The conformal symmetry is an extension of the Poincaré symmetry: not only is the theory invariant under Lorentz transformations and space-time translations, but also under special conformal transformations and dilatations. Being conformal implies that the theory has no inherent mass scale and this remains true at the quantum level, thanks to the supersymmetry[2]. From the Lagrangian, it can be seen that there is an additional symmetry which accounts for the invariance under $SO(6)$ rotation of the scalars ϕ^I: it is called R-symmetry.

Conformal symmetry (and thus Poincaré symmetry), supersymmetry and R-symmetry are all subgroups of the $\mathcal{N} = 4$ superconformal group $PSU(2,2|4)$, which is the full symmetry group of the theory. In particular, the bosonic part is $SU(2,2) \times SU(4) \simeq SO(2,4) \times SO(6)$, where the first factor is the conformal group and the second the R-symmetry.

3.1.2 $\mathcal{N} = 4$ SYM symmetries

Let us analyze in more detail the algebraic structure of $\mathcal{N} = 4$ super Yang–Mills. There are the 15 generators of the conformal algebra $SU(2,2)$: the four-vector P_μ generate the space-time translations, the anti-symmetric tensor $M_{\mu\nu}$ the $SO(1,3) \simeq SU(2) \times SU(2)$ Lorentz transformations, the four-vector K_μ the special conformal

[1] See below for the conventions for the indices.

[2] For $\mathcal{N} = 4$ super Yang–Mills the β function is zero at every order in perturbation theory, see [67].

transformations and the scalar D the dilatations. Their commutation relations are as follows

$$[D, P_\mu] = -iP_\mu \qquad [D, M_{\mu\nu}] = 0 \qquad [D, K_\mu] = +iK_\mu$$
$$[M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu) \qquad [M_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu)$$
$$[P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu}D)\,. \tag{3.1.3}$$

There are also 16 generators of supersymmetry transformations $Q_{\alpha\, a}$ and $\dot{Q}^a_{\dot\alpha}$, called supercharges. They are fermionic, i.e. $\alpha, \dot\alpha = 1, 2$ are the spinor indices of the Lorentz algebra, one for each $SU(2)$, while a superscript (subscript) a labels the fundamental (anti-fundamental) representation of the R-symmetry algebra $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$. The anti-commutation relations are

$$\{Q_{\alpha\, a}, \dot{Q}^b_{\dot\alpha}\} = \sigma^\mu_{\alpha\dot\alpha}\delta_a{}^b P_\mu\,, \quad \{Q_{\alpha\, a}, Q_{\alpha\, b}\} = 0 \quad \{\dot{Q}^a_{\dot\alpha}, \dot{Q}^b_{\dot\alpha}\} = 0\,. \tag{3.1.4}$$

The supersymmetry transformations commute with the translations, $[P_\mu, Q_{\alpha\, a}] = 0$, $[P_\mu, \dot{Q}^b_{\dot\alpha}] = 0$, and we have

$$[M^{\mu\nu}, Q_{\alpha\, a}] = i\sigma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma\, a}\,, \quad [M^{\mu\nu}, \dot{Q}^a_{\dot\alpha}] = i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\dot{Q}^a_{\dot\gamma}\,, \tag{3.1.5}$$

where $\sigma^{\mu\nu}_{\alpha\beta} = \sigma^{[\mu}_{\alpha\dot\alpha}\sigma^{\nu]}_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$.

In order to complete the superconformal algebra we need

$$[D, Q_{\alpha\, a}] = -\frac{i}{2}Q_{\alpha\, a}\,, \qquad [D, \dot{Q}^a_{\dot\alpha}] = -\frac{i}{2}\dot{Q}^a_{\dot\alpha}\,, \tag{3.1.6}$$

which means that $Q_{\alpha\, a}$ and $\dot{Q}^a_{\dot\alpha}$ have dimension one half; and

$$[K^\mu, Q_{\alpha\, a}] = \sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\dot{S}_{\dot\beta\, a} \qquad [K^\mu, \dot{Q}^a_{\dot\alpha}] = \sigma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}S^a_\beta\,, \tag{3.1.7}$$

where S^a_β and $\dot{S}_{\dot\beta\, a}$ are new supercharges, bringing the total supersymmetry generators to 32. Thus we see that combining supersymmetry and (special) conformal transformations introduces the superconformal charges S^a_α and $\dot{S}_{\dot\alpha\, a}$ of dimension $-1/2$ with anticommutators similar to $Q_{\alpha\, a}$ and $\dot{Q}^a_{\dot\alpha}$:

$$
\begin{gathered}
[S^a_\alpha, \dot{S}_{\dot\alpha\, b}] = \sigma^\mu_{\alpha\dot\alpha}\delta^a{}_b K_\mu \qquad [S^a_\alpha, S^b_\alpha] = 0 \quad [\dot{S}_{\dot\alpha\, a}, \dot{S}_{\dot\alpha\, b}] = 0 \\
[K_\mu, S^a_\alpha] = 0 \qquad [K_\mu, \dot{S}_{\dot\alpha\, a}] = 0 \\
[D, S^a_\alpha] = \frac{i}{2} S^a_\alpha \qquad [D, \dot{S}_{\dot\alpha\, a}] = \frac{i}{2}\dot{S}_{\dot\alpha\, a}\,.
\end{gathered} \tag{3.1.8}
$$

Finally we have

$$
\begin{aligned}
\{Q_{\alpha\, a}, S^b_\beta\} &= -i\varepsilon_{\alpha\beta}\sigma^{IJ}{}_a{}^b R_{IJ} + \sigma^{\mu\nu}_{\alpha\beta}\delta_a{}^b M_{\mu\nu} - \frac{1}{2}\varepsilon_{\alpha\beta}\delta_a{}^b D \\
\{\dot{Q}^a_{\dot\alpha}, \dot{S}_{\dot\beta\, b}\} &= +i\varepsilon_{\dot\alpha\dot\beta}\sigma^{IJ\, a}{}_b R_{IJ} + \sigma^{\mu\nu}_{\dot\alpha\dot\beta}\delta^a{}_b M_{\mu\nu} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\delta^a{}_b D\,,
\end{aligned} \tag{3.1.9}
$$

where R_{IJ} are the R-symmetry generators, which commute with all the others, while $\{Q_{\alpha\, a}, \dot{S}_{\dot\beta\, b}\} = 0 = \{\dot{Q}^a_{\dot\alpha}, S^b_\beta\}$. We will be interested in the so-called 't Hooft limit, i.e. $g_{YM} \to 0$, $N \to \infty$, while $\lambda = g^2_{YM} N$ is kept fixed. This limit greatly simplifies the perturbative computation since only planar Feynman diagrams survive the limit, the others being suppressed by powers of $1/N$.

Equivalently, we can write the representations of the (complexified) Lorentz group in terms of those of the tensor product $SU(2)_\mathbb{C} \times SU(2)_\mathbb{C}$, as we have seen in Section 2.1.3. Thus e.g. the generators of the translations and conformal transformations are $P_{\alpha\dot\alpha} = P_\mu \sigma^\mu_{\alpha\dot\alpha}$ and $K^{\alpha\dot\alpha} = K_\mu(\sigma^\mu)^{\alpha\dot\alpha}$ respectively, and their commutator is

$$
[K^{\alpha\dot\alpha}, P_{\beta\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta} L^\alpha{}_\beta + \delta^\alpha_\beta \dot{L}^{\dot\alpha}{}_{\dot\beta} + \delta^\alpha_\beta \delta^{\dot\alpha}_{\dot\beta} D\,. \tag{3.1.10}
$$

3.1.3 Primary operators and scaling dimensions

In conformal theories there is a special class of operators called chiral primaries that have protected dimensions. To understand the meaning of this statement, let us first consider the action of the dilatation generator D on a local operator $\mathcal{O}(x)$, $\mathcal{O}(x) \to \lambda^{-iD}\mathcal{O}(x)\lambda^{iD}$,

$$
[D, \mathcal{O}(x)] = i\left(-\Delta + x\,\frac{\partial}{\partial x}\right)\mathcal{O}(x)\,, \tag{3.1.11}
$$

where Δ is called the dimension of $\mathcal{O}(x)$. In other terms, $\mathcal{O}(x)$ transforms as $\mathcal{O}(x) \to \lambda^{-\Delta}\,\mathcal{O}(\lambda x)$ under the rescaling $x \to \lambda x$. The action of D on the operator $[K_\mu, \mathcal{O}]$, with

$\mathcal{O}(x)$ a generic (composite) local operator, is

$$[D, [K_\mu, \mathcal{O}]] = [[D, K_\mu], \mathcal{O}] + [K_\mu, [D, \mathcal{O}] \tag{3.1.12}$$

$$= i[K_\mu, \mathcal{O}] - i\Delta[K_\mu, \mathcal{O}]\,, \tag{3.1.13}$$

where in the first line we used the Jacobi identity. This means that $[K_\mu, \mathcal{O}]$ has dimension $\Delta - 1$. Similarly, for the supercharges S^a_α and $\dot{S}_{\dot{\alpha}\, a}$ we have that $[S^a_\alpha, \mathcal{O}]$ has dimension $\Delta - 1/2$. Thus K_μ, S^a_α, and $\dot{S}_{\dot{\alpha}\, a}$ have the property of lowering the dimension when they act on $\mathcal{O}$. The same is true for half of the $L^\alpha{}_\beta$, when $\alpha < \beta$, half of the $\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}$, when $\dot{\alpha} < \dot{\beta}$, and half of the R_{IJ}, when $I < J$, and they are all called lowering operators. Vice versa, the other generators raise the dimension of an operator $\mathcal{O}$ and are thus called raising operators: P_μ, Q^α_a, $\dot{Q}^{\dot{\alpha}\, a}$ and the other half of $L^\alpha{}_\beta$, $\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}$, and R_{IJ} (i.e. when $\alpha > \beta$, $\dot{\alpha} > \dot{\beta}$, and $I > J$ respectively).

Since local operators in (unitary) QFT must have positive dimension[3], there exists a local operator $\widetilde{\mathcal{O}}(x)$ such that $[W, \widetilde{\mathcal{O}}] = 0$ for any lowering operator W. Then $\widetilde{\mathcal{O}}(x)$ is called primary and its commutators with the raising operators of the algebra are called its descendants. An example of a primary operator in SYM is the Konishi operator $\mathcal{O}(x) = Tr[\phi^I(x)\phi^I(x)]$.

A primary operator $\widetilde{\mathcal{O}}(x)$ and its descendants constitute an irreducible representation of $PSU(2,2|4)$, of which $\widetilde{\mathcal{O}}(x)$ is the highest weight. Moreover, the primary operators, and thus the corresponding representations, can be characterized by six charges: $[\Delta, j_L, j_R, r_1, r_2, r_3]$, which are respectively the conformal dimension, left and right spin, and three charges labelling its $SU(4)$-representation.

3.1.4 Correlation functions

There is a special subset of primary operators, called BPS (after Bogomol'nyi, Prasad and Sommerfield) operators, which also commute with some of the raising operators Q^α_a and $\dot{Q}^{\dot{\alpha}\, a}$. From the commutators (3.1.9) we see that this entails a relation between spin, R-charge and scaling dimension which implies that the latter does not receive quantum corrections, since spin and R-charge are protected quantities.

[3]More precisely: they must have non-negative dimension and the identity is the only operator of zero dimension.

The fact that BPS operators have protected scaling dimensions Δ is of particular importance because knowing the dimension of conformal operators allows to completely determine their two-point functions. Moreover, conformal symmetry constrains the form of the correlation functions so that we can determine any of them from two sets of quantities: the operators' scaling dimensions Δ_I and the so-called structure constants C_{IJK} between any three operators $\{\mathcal{O}_I, \mathcal{O}_J, \mathcal{O}_K\}$.

To be more specific, let us restrict ourselves for simplicity to the composite operators built from the six scalars ϕ_I as

$$\mathcal{O}_{\underline{I}}(x) = \mathrm{Tr}\,[\phi_{I_1}\phi_{I_2}\cdots\phi_{I_L}]\,, \qquad \underline{I} = \{I_1,\ldots,I_L\}\,, \tag{3.1.14}$$

called single-trace operators of length L. It is clear that the (3.1.14) are primary operators and have dimension $\Delta^{(0)} = L$. We note that if we allow composite operators with any number of traces, the n-point functions between them will be suppressed by powers of $1/N$ w.r.t. the ones between single-trace operators, so it make sense to consider only the latter ones in the 't Hooft limit.

Conformal symmetry forces one-point functions to be constant and two-point function to be of the form

$$\langle \mathcal{O}_{\underline{I}}(x)\,\overline{\mathcal{O}}_{\underline{J}}(y)\rangle = \frac{M_{\underline{IJ}}}{|x-y|^{\Delta_{\underline{I}}+\Delta_{\underline{J}}}}\,, \tag{3.1.15}$$

where $M_{\underline{IJ}} \neq 0$ only when $\Delta_{\underline{I}} = \Delta_{\underline{J}}$, and $\overline{\mathcal{O}}_{\underline{J}}$ is hermitian conjugate of $\mathcal{O}_{\underline{J}}$, which for real operators means simply that the order of the fields in the trace is reversed. The dimensions of the operators involved determine also the three-point function, up to a factor, $C_{\underline{IJK}}$, called structure constant:

$$\langle\, \mathcal{O}_{\underline{I}}(x)\,\mathcal{O}_{\underline{J}}(y)\,\mathcal{O}_{\underline{K}}(z)\,\rangle = \frac{C_{\underline{IJK}}}{|x-y|^{\Delta_{\underline{I}}+\Delta_{\underline{J}}-\Delta_{\underline{K}}}\,|y-z|^{\Delta_{\underline{J}}+\Delta_{\underline{K}}-\Delta_{\underline{I}}}\,|z-x|^{\Delta_{\underline{K}}+\Delta_{\underline{I}}-\Delta_{\underline{J}}}}\,. \tag{3.1.16}$$

We also have that all the higher-point functions can be determined from the two- and three-point ones thanks to the operator product expansion (OPE)

$$\mathcal{O}_{\underline{I}}(x)\,\mathcal{O}_{\underline{J}}(y) = \frac{M_{\underline{IJ}}}{|x-y|^{\Delta_{\underline{I}}+\Delta_{\underline{J}}}} + \sum_k \frac{C_{\underline{IJ}}{}^{\underline{K}}}{|x-y|^{\Delta_{\underline{I}}+\Delta_{\underline{J}}-\Delta_{\underline{K}}}} F(x-y,\partial_y)\mathcal{O}_{\underline{K}}(y)\,, \tag{3.1.17}$$

where[4] the sum runs over all primary operators and the differential operator $F = 1 + o(x - y)$ is necessary to include their descendants.

In general, when quantizing a classical CFT, the classical dimension $\Delta^{(0)}$ receives radiative corrections, and, after renormalization[5], we can write the two-point function as (3.1.15) with the $\Delta_{\underline{I}}$ expanded in terms of the effective coupling g:

$$\Delta_{\underline{I}} = \sum_{n=0}^{\infty} g^{2n}\, \Delta_{\underline{I}}^{(n)}\,, \qquad g = \frac{\sqrt{\lambda}}{2\pi}\,. \tag{3.1.18}$$

The quantity $\Delta - \Delta^{(0)}$ is called anomalous dimension, that is the quantum correction to the classical dimension, and in the case of the BPS operators mentioned above it vanishes, i.e. $\Delta = \Delta^{(0)}$.

3.1.5 Spin chains and $\mathcal{N} = 4$ SYM

In Section 3.1.4 we saw why the conformal dimension of an operator is an especially interesting quantity in any conformal theory. In $\mathcal{N} = 4$ SYM, an observation by Minahan and Zarembo [4] spurred a major breakthrough in the computation of anomalous dimensions, namely the identification of the first order correction to $\Delta^{(0)}$ of a particular class of operators with the Hamiltonian of the Heisenberg spin chain.

Let us consider only BMN operators with scalar insertions[6], i.e. linear combinations of $\mathrm{Tr}[\phi_{I_1} \cdots \phi_{I_L}]$. In this case, it can be shown from a one-loop Feynman diagram computation that the first order correction to the classical dimension $\Delta^{(1)}$ can be written as

$$\Delta^{(1)} = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(K_{l,l+1} + 2 - P_{l,l+1}\right)\,, \tag{3.1.19}$$

where $K_{l,l+1}$ stands for $K_{i_l\, i_{l+1}}^{j_l\, j_{l+1}} = \delta_{i_l\, i_{l+1}} \delta^{j_l\, j_{l+1}}$, and similarly $P_{l,l+1}$ is the shorthand of $P_{i_l\, i_{l+1}}^{j_l\, j_{l+1}} = \delta_{i_l}^{j_{l+1}} \delta_{i_{l+1}}^{j_l}$.

$K_{l,l+1}$ and $P_{l,l+1}$ can be interpreted as operators acting on the sites l and $l+1$ of a spin chain (nearest-neighbour interaction) and are called trace operator and permutation operator respectively: $K\,|a\rangle\,|b\rangle = (a \cdot b)$ and $P\,|a\rangle\,|b\rangle = |b\rangle\,|a\rangle$. With this

[4] The indices of the structure constants are raised and lowered with the matrix M: $C_{\underline{IJK}} = \sum_{\underline{K}} C_{\underline{IJ}}{}^{\underline{K}} M_{\underline{KL}}$.

[5] For a more detailed and pedagogical analysis we refer to the lecture notes [68].

[6] It should be noted that at one loop there is no mixing with the other operators.

interpretation, $\Delta^{(1)}$ becomes the Hamiltonian of an $\mathfrak{so}(6)$ spin chain [4][7] and its eigenvalues can be found using the Bethe ansatz. This analysis has been generalized to the full $\mathfrak{psu}(2,2|4)$ in [69].

If we limit ourselves instead to only an $\mathfrak{su}(2)$ sub-sector, i.e. considering only

$$Y = \frac{\phi^1 + i\phi^2}{\sqrt{2}} \quad \text{and} \quad Z = \frac{\phi_5 + i\phi_6}{\sqrt{2}}, \tag{3.1.20}$$

we have that $K = 0$ and $\Delta^{(1)}$ is now the Hamiltonian for the XXX Heisenberg model (3.2.4). Let us note that at higher loops there are interaction terms between the scalars ϕ_I and other fields of the theory, which means that one can consider the $SO(6)$ sector only at one loop. However, there are some special sub-sectors of $PSU(2,2|4)$ for which there is no operator mixing at any loop [69], and these are called closed sectors. An example is the $SU(2)$ sub-sector seen above, or the non-compact $SL(2)$ sector built from only one complex scalar Z and its covariant derivatives.Another closed sector is the $SU(2|3)$ one, containing two fermions and three bosons, which exhibits an important feature at two loops, namely that the length of the single trace operators is not preserved [70]. For example, in the action there is a term which translates into the scattering of two fermions into three bosons, which means that the corresponding Hilbert space contains operators of different lengths.

Thus the first loop correction to the conformal dimension of BMN operators can be found solving a spin chain with nearest-neighbour interactions, and the spin representation in each site depends on the chosen sector. Higher-loop corrections have been studied in [71] and subsequent papers[8]. It has been found that the spin chains contributing to higher loops have long range interactions, with the range increasing by one for any additional loop order. Moreover, operators with different lengths mix together if they have the same classical dimension, e.g. three scalar fields with two fermions. Nevertheless, these spin chains are still considered integrable, and have been proven so in many explicit cases, starting from [70].

[7] See also [29] for a detailed analysis.
[8] See the review [6] for further references.

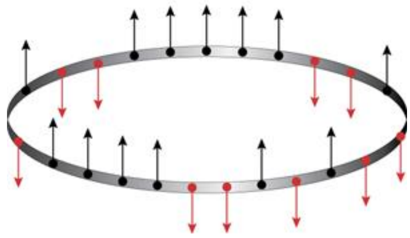


FIGURE 3.1: The $\mathrm{XXX}_{1/2}$ spin chain [72]

3.2 The Bethe ansatz for the $\mathbf{XXX}_{1/2}$ spin chain

3.2.1 The Heisenberg XXX spin chain

Motivated by the previous analysis, we now discuss the spin chain and the solutions to its spectral problem. The XXX Heisenberg model, represented in Fig. 3.1, is a one-dimensional lattice in which every site carries a (finite) representation of $SU(2)$ labelled by its highest weight s. The associated symmetry algebra $\mathfrak{su}(2)$ is generated by the S^i ($i = 1, 2, 3$) satisfying

$$\left[S^i, S^j\right] = i\,\epsilon^{ijk} S^k\,, \tag{3.2.1}$$

with ϵ^{ijk} completely antisymmetric and $\epsilon^{123} = 1$. In the first non-trivial case $s = 1/2$, where each spin can be either up ($\uparrow$) or down ($\downarrow$), we can write $S^i_n = \sigma^i_n/2$, with σ^i denoting the three Pauli matrices for the site n acting on the vector space $V_n = \mathbb{C}^2$:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{3.2.2}$$

with commutators

$$\left[\sigma^i_n, \sigma^j_m\right] = 2i\,\epsilon^{ijk}\,\sigma^k_n\,\delta_{mn}\,, \tag{3.2.3}$$

where the Kronecker delta δ_{mn} means that operators at different sites commute.

The Hamiltonian of the $\mathrm{XXX}_{1/2}$ spin chain with L sites is

$$H = \sum_{n=1}^{L} \left(1 - \sigma^i_n \sigma^i_{n+1}\right)\,, \tag{3.2.4}$$

with the identification $\sigma^i_{n+L} = \sigma^i_n$ defining periodic boundary conditions. Since interactions occur only between two adjacent sites, models of this type are called nearest-neighbour spin chains. More precisely, the interaction in the Hamiltonian (3.2.4) amounts to the exchange of two nearby spins, that is we can rewrite it in terms of the operators identity $\mathbb{I}$ and permutation $\mathbb{P}$ as

$$H = \sum_{n=1}^{L} \left(\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}\right) , \tag{3.2.5}$$

where the operators act on any two spins a, b respectively as $\mathbb{I}(a \otimes b) = a \otimes b$ and $\mathbb{P}(a \otimes b) = b \otimes a$.

If we define the total spin along the direction i as

$$S^i = \sum_{n=1}^{L} S^i_n = \frac{1}{2} \sum_{n=1}^{L} \sigma^i_n , \tag{3.2.6}$$

it is easy to see that the Hamiltonian commutes with any of them:

$$\left[H, S^i\right] = 0 \qquad i = 1, 2, 3 ; \tag{3.2.7}$$

i.e. the three S^i are the generators of a global $\mathfrak{su}(2)$ symmetry algebra. Let us note that this is a consequence of the fact that for each site the three terms in (3.2.4) have the same coefficient regardless of the direction i, hence the term "XXX", while this is not the case in a more general model[9].

The commutator (3.2.7) implies that we can choose one special spin direction, conventionally S^3, and find spin-chain configurations (states) $|\Psi\rangle$ that satisfy both

$$H |\Psi\rangle = E_\Psi |\Psi\rangle \quad \text{and} \quad S^3 |\Psi\rangle = s_\Psi |\Psi\rangle , \tag{3.2.8}$$

i.e. $|\Psi\rangle$ is an eigenstate of the Hamiltonian and the selected spin operator, as we will see in the next section. Moreover, we can rearrange the other two S^i into the so-called spin-lowering and raising operators respectively:

$$S^+ = S^1 + i\, S^2 \quad \text{and} \quad S^- = S^1 - i\, S^2 , \tag{3.2.9}$$

[9]We can define a general (anisotropic) nearest-neighbour spin chain introducing coefficients k^i as $H = k^0 - \sum_n \sum_i k^i S^i_n S^i_{n+1}$; when $k^1 = k^2 \neq k^3$ this is called a XXZ spin chain, if they are all different it is denoted XYZ, and we recover our XXX model when $k^1 = k^2 = k^3$.

which satisfy

$$\left[S^+, S^-\right] = 2S^3 \qquad \left[S^3, S^\pm\right] = \pm S^\pm \qquad \left[H, S^\pm\right] = 0\,. \tag{3.2.10}$$

The name "lowering/raising operator" comes from the fact that we can generate other eigenstates with the same energy but a different spin by acting one or more times with S^+ or S^-. Explicitly, for $|\Psi_\pm\rangle = S^\pm |\Psi\rangle$ we have from (3.2.10)

$$H|\Psi_\pm\rangle = S^\pm H|\Psi\rangle + 0 = E_\Psi |\Psi_\pm\rangle \qquad S^3|\Psi_\pm\rangle = S^\pm S^3|\Psi\rangle \pm S^\pm|\Psi\rangle = (s_\Psi \pm 1)|\Psi_\pm\rangle$$

3.2.2 The coordinate Bethe ansatz

We will now discuss how to solve the spectral problem

$$H|\Psi\rangle = E_\Psi |\Psi\rangle\,, \tag{3.2.11}$$

i.e. how to find the eigenstates $|\Psi\rangle$ of the Hamiltonian H, and their corresponding eigenvalues E_Ψ. The standard procedure to obtain them, i.e. diagonalizing the Hamiltonian, becomes cumbersome for large L and does not help in the analysis of the interesting limit $L \to \infty$. Bethe was able to find a more efficient method in 1931 [5], by making an "educated guess" (ansatz) for the eigenfunctions[10], as we will see in the following[11].

Let us first define the ground state (or vacuum) of the spin chain $|0\rangle$, that is the eigenstate of the Hamiltonian with lowest energy (e.g. $H|0\rangle = 0$), as the state in which all spins are up[12]:

$$S^+|0\rangle = 0 \quad \longleftrightarrow \quad |0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle\,. \tag{3.2.12}$$

Then the excitations, called magnons, are obtained by acting with the lowering operator S^- on the vacuum $|0\rangle$. To simplify the notation, it is convenient to label the excitations as $|n\rangle = |\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow\rangle$ where the spin at the n-th site is flipped, $|n,m\rangle =$

[10] Eigenfunctions and eigenstates are often used interchangeably, but while the latter is an abstract object, the former depends on the chosen basis, say coordinates or momenta.

[11] See also the reviews [73] and lectures [74, 7, 75].

[12] It would be clearly equivalent to consider as the ground state the opposite state with all spins down $|\downarrow\downarrow\cdots\downarrow\rangle$, switching the role of S^+ and S^- in the following expressions.

$|\uparrow \cdots \uparrow\downarrow\uparrow \cdots \downarrow \cdots \uparrow\rangle$ where the spins at the n-th and m-th sites are flipped, and so on.

We will briefly review Bethe's original idea, now referred to as the coordinate Bethe ansatz. We begin with the observation that the Hamiltonian does not change the length L of the state nor the number M of excitations, it only changes the positions of the spins. This fact constrains the eigenstates to be a linear combination of the states with M magnons

$$|\Psi_M\rangle = \sum_{1\leq n_1<\cdots<n_M\leq L} \psi(n_1,\ldots,n_M)\,|n_1,\ldots,n_M\rangle\ , \tag{3.2.13}$$

where the functions $\psi(n_1,\ldots,n_M)$ are the quantity we need to determine.

In the case of only one magnon we can write the eigenstate as

$$|\Psi_1\rangle = \sum_{n=1}^{L} e^{ipn}\,|n\rangle \tag{3.2.14}$$

and the periodic boundary condition $S^i_{n+L} = S^i_n$ on the Hamiltonian implies

$$e^{ipL} = 1\,. \tag{3.2.15}$$

We can then interpret Ψ_1 as a one-particle state with momentum p, quantized according to (3.2.15), and energy

$$E_{\Psi,1} = 2 - 2\cos(p) = \frac{1}{2}\frac{1}{u^2+1/4}\,, \tag{3.2.16}$$

where in the last expression we introduced the rapidity u, defined as

$$u = \frac{1}{2}\cot\left(\frac{p}{2}\right) \quad \longleftrightarrow \quad e^{ip} = \frac{u+i/2}{u-i/2}\,. \tag{3.2.17}$$

Similarly, we can start from a generic two-particle state

$$|\Psi_2\rangle = \sum_{1\leq n_1<n_2\leq L} \psi(n_1,n_2)\,|n_1,n_2\rangle \tag{3.2.18}$$

and make the following ansatz for the wave-function

$$\psi(n_1, n_2) = e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2) e^{ip_1 n_2 + ip_2 n_1}\,, \tag{3.2.19}$$

with the phase $S(p_1, p_2)$ of the form

$$S(p_1, p_2) = -\frac{1 - 2e^{ip_1} + e^{i(p_1+p_2)}}{1 - 2e^{ip_2} + e^{i(p_1+p_2)}} = \frac{u_1(p_1) - u_2(p_2) + i}{u_1(p_1) - u_2(p_2) - i}\,. \tag{3.2.20}$$

One can verify that (3.2.18) with the ansatz (3.2.19) is an eigenstate with its eigenvalue equal to the sum of the energies of the two particles and $S(p_1, p_2)$ can be interpreted as a scattering matrix (S-matrix) describing the interaction between the two particles with momenta p_1 and p_2. Imposing the periodic boundary conditions, we see that the momenta are now quantized according to

$$e^{ip_1 L} = S(p_1, p_2)\,, \qquad e^{ip_2 L} = S(p_2, p_1)\,. \tag{3.2.21}$$

In general we can write the $\psi(n_1, \ldots, n_M)$ as a sum over permutations $\{\tau\}$

$$\psi(n_1, \ldots, n_M) = \sum_{\{\tau\}} A(\tau) \exp\left(\sum_{k=1}^{M} ip_{\tau(k)} n_k\right)\,, \tag{3.2.22}$$

$$\text{where} \quad A(\tau) = \text{sign}(\tau) \prod_{j<k} \left(1 - 2e^{ip_k} + e^{i(p_k+p_j)}\right)\,, \tag{3.2.23}$$

with the conditions

$$e^{ip_j L} = \prod_{k=1, k\neq j}^{M} S(p_j,\, p_k)\,. \tag{3.2.24}$$

These are n independent equations for n variables $p_1, \ldots, p_n$, and are called Bethe equations. The spectral problem then reduces to solving the Bethe equations, which is easier than diagonalizing the Hamiltonian and also suitable to be applied to large systems (large L).

The Bethe equations are often written in terms of the rapidities u_i

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k=1, k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \tag{3.2.25}$$

and their solutions are called Bethe roots, with momenta and energies given by

$$P = -i \log \left(\frac{u + i/2}{u - i/2} \right) , \qquad E_{\Psi,M} = \frac{1}{2} \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4} \,. \tag{3.2.26}$$

3.2.3 The algebraic Bethe ansatz

The original work by Bethe [5] provides a key example of what is now called quantum integrability, as we will see in the following, though it is not apparent in the formalism of the coordinate Bethe ansatz. Therefore, we will review a different approach to solving integrable systems, such as the Heisenberg spin chain (3.2.4), called quantum inverse scattering or algebraic Bethe ansatz (ABA), following [7].

We begin by introducing a new object called the Lax operator, which acts on $V_n \otimes V$, where $V_n = \mathbb{C}^2$ is the space on which our representation of S_n^i (i.e. the Pauli matrices) acts and $V = \mathbb{C}^2$ is a new auxiliary (non-physical) space. The Lax operator $L_{n,a}(u)$ is a linear operator that can be written as a 2×2 matrix on V, with the matrix elements acting on V_n, in the following way

$$L_{n,a}(u) = \begin{pmatrix} u + i\,S_n^3 & i\,S_n^- \\ i\,S_n^+ & u - i\,S_n^3 \end{pmatrix} , \tag{3.2.27}$$

and u is a complex variable called the spectral parameter. We can also rewrite the Lax operator in terms of the permutation operator $\mathbb{P}$ and identity $\mathbb{I}$ used in (3.2.5), with an obvious generalization to the auxiliary space since it is again $\mathbb{C}^2$, and obtain

$$L_{n,a}(u) = \left(u - \frac{i}{2} \right) \mathbb{I}_{n,a} + i\,\mathbb{P}_{n,a} \,, \tag{3.2.28}$$

where the index a corresponds to the auxiliary space V, and n to V_n. Similarly, we introduce another auxiliary space $V' = \mathbb{C}^2$ and an operator $R_{a,b}$ on $V \otimes V'$, called the R-matrix, defined as

$$R_{a,b}(u) = u\,\mathbb{I}_{a,b} + i\,\mathbb{P}_{a,b} \,. \tag{3.2.29}$$

One can then prove the following RLL relation

$$R_{a,b}(u - v) L_{n,a}(u) L_{n,b}(v) = L_{n,b}(u) L_{n,a}(v) R_{a,b}(u - v) \,, \tag{3.2.30}$$

where the products on both sides are defined on $V_n \otimes V \otimes V'$[13].

It is also useful to define the "monodromy matrix"

$$T_{L,a}(u) = L_{L,a}(u) \cdots L_{1,a}(u) \tag{3.2.31}$$

for which the RLL relation (3.2.30) implies

$$R_{a,b}(u-v)T_{L,a}(u)T_{L,b}(v) = T_{L,b}(u)T_{L,a}(v)R_{a,b}(u-v)\,. \tag{3.2.32}$$

The trace of the monodromy matrix over the auxiliary space is called the transfer matrix $T(u) = \mathrm{Tr}_a(T_{L,a})$ and one can prove that transfer matrices commutes for any value of the spectral parameter $[T(u), T(v)] = 0$. Moreover $T(i/2)$ is proportional to the Hamiltonian (plus a constant term):

$$H = 2L - 2i\,\frac{d}{du}\log T(u)\bigg|_{u=\frac{i}{2}}\,, \tag{3.2.33}$$

which means $[T(u), H)] = 0$, that is the transfer matrix commute with the Hamiltonian for any u. Moreover, the monodromy matrix is a polynomial of order L, $T_{L,a}(u) = \sum_{n=1}^{L} V_n u^n$ with $V_L = 1$, so $T(u)$ is a polynomial of order L with a constant coefficient, which means that $[T(u), T(v)] = 0$ implies the existence of $L-1$ commuting operators V_n, which include the Hamiltonian. We can then choose one particular spin, e.g. S^3, to obtain a family of L commuting operators, and in this sense one can prove the (quantum) integrability of the Heisenberg spin chain.

To conclude we will mention how to obtain the spectrum of the theory in this quantum inverse scattering method: the idea is to diagonalize the transfer matrix $T(u)$, instead of the Hamiltonian. To be more specific, if we write the monodromy matrix as

$$T_{L,a}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}\,, \tag{3.2.34}$$

with matrix elements being operators on the Hilbert space of the $XXX_{1/2}$ spin chain $V_1 \otimes \cdots \otimes V_L = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, then the transfer matrix is $T(u) = A(u) + D(u)$.

[13] The operators can be extended to act on the full $V_n \otimes V \otimes V'$ if we simply assume that they act as the identity on the third space, e.g. $L_{n,a} \otimes \mathbb{I}_b$.

Considering again the vacuum $|0\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$, it follows from the definitions that

$$A(u)\,|0\rangle = \left(u + \frac{i}{2}\right)^L |0\rangle \;, \qquad D(u)\,|0\rangle = \left(u - \frac{i}{2}\right)^L |0\rangle \;, \qquad C(u)\,|0\rangle = 0\,, \tag{3.2.35}$$

and from equation (3.2.32) that $[B(u), B(v)] = 0$, $[C(u), C(v)] = 0$. Moreover, we will define the states Ψ_M as

$$|\Psi_M\rangle = B(u_1)B(u_2)\cdots B(u_M)\,|0\rangle \;. \tag{3.2.36}$$

Then it can be shown [76] that the Ψ_M are eigenstates of the transfer matrix

$$T(u)\,|\Psi_M\rangle = t(u)\,|\Psi_M\rangle \tag{3.2.37}$$

with eigenvalues

$$t(u) = \left(u + \frac{i}{2}\right)^L \prod_{k=1}^{M} \frac{u - u_k - i}{u - u_k} + \left(u - \frac{i}{2}\right)^L \prod_{k=1}^{M} \frac{u - u_k + i}{u - u_k} \tag{3.2.38}$$

if and only if the parameters $\{u_i\}$ (Bethe roots) satisfy the Bethe equations (3.2.25)

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}\,. \tag{3.2.39}$$

Then $B(u)$ and $C(u)$ can be considered the creation and annihilation operators respectively, with energy and momentum of the state $|\Psi\rangle$ given by (3.2.26)

$$P = -i\log\left(\frac{u + i/2}{u - i/2}\right)\,, \qquad E_{\Psi,M} = \frac{1}{2}\sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}\,. \tag{3.2.40}$$

Chapter 4

Form Factors

4.1 Definitions

The form factors are matrix elements of local operators between scattering states. We will introduce them in a generic two-dimensional theory and then specialize to the world-sheet string theory described in Section 2.1.3. The form factor axioms are a set of consistency conditions which if solved would complete the bootstrap program for the form factors in any integrable theory. The form factor axioms for this model have been written in [12], however, finding a general solution for them remains an open problem. In the following we will present instead some perturbative results, and, in particular, we will write the complete three-particle form factor up to one loop.

4.1.1 Generalized form factors

Let us consider a 1+1-dimensional theory with coordinates $\mathbf{x} = (\tau, \sigma)$ and particles' two-momenta denoted by $\mathbf{p}_j = (\epsilon_j, p_j)$, where $j = 1, \ldots, n$ if there are n particles. In order to fully label a particle, we also need a flavour index i, which e.g. in the case of the light-cone string of Sec. 2.1.3 will run over the transverse (bosonic and fermionic) world-sheet fields. Thus we will denote a generic scattering state with n particles $|p_1, \ldots, p_n\rangle_{i_1,\ldots,i_n}$.

Given a local operator $\mathcal{O}(\mathbf{x})$, which can be any combination of the fields of the theory and their derivatives located at the world-sheet point $\mathbf{x} = (\tau, \sigma)$, we define its generalized form factor[1] as

$$ {}_{i'_m,\ldots,i'_1}\langle p'_m,\ldots,p'_1|\,\mathcal{O}(\mathbf{x})\,|p_1,\ldots,p_n\rangle_{i_1,\ldots,i_n} = e^{i(\mathbf{p}'_1+\ldots+\mathbf{p}'_m-\mathbf{p}_1-\ldots-\mathbf{p}_n)\cdot\mathbf{x}} \times F^{\mathcal{O};i'_m,\ldots,i'_1}_{i_1,\ldots,i_n}(p'_m,\ldots,p'_1|p_1,\ldots,p_n)\,. \quad (4.1.1) $$

In a relativistic theory, or in any theory in which crossing can be defined such as the world-sheet string, the definition (4.1.1) is clearly equivalent to one where only matrix elements between a particle state and the vacuum $|\Omega\rangle$ are considered. Thus we can also write

$$ \langle\Omega|\,\mathcal{O}(\mathbf{x})\,|p_1,\ldots,p_n\rangle_{i_1,\ldots,i_n} = e^{-i(\mathbf{p}_1+\ldots+\mathbf{p}_n)\cdot\mathbf{x}}F^{\mathcal{O}}_{i_1,\ldots,i_n}(p_1,\ldots,p_n)\,, \quad (4.1.2) $$

or in momentum space[2]

$$ \langle\Omega|\,\widetilde{\mathcal{O}}(\mathbf{q})\,|p_1,\ldots,p_n\rangle_{i_1,\ldots,i_n} = (2\pi)^2\delta^{(2)}(\mathbf{q}-\mathbf{p}_1-\ldots-\mathbf{p}_n)F^{\mathcal{O}}_{\underline{i}}(p_{\underline{i}})\,, \quad (4.1.4) $$

where $\underline{i}=\{i_1,\ldots,i_n\}$ and $p_{\underline{i}}=\{p_1,\ldots,p_n\}$. Moreover, we specify the ordering of the momenta for the scattering states, following the conventions of [12, 13]. Assuming $p_1>p_2>\cdots>p_n$, we define the "in"-scattering states $|p_1,\ldots,p_n\rangle^{(\text{in})}_{i_1,\ldots,i_n}$ as incoming states $|p_1,\ldots,p_n\rangle_{i_1,\ldots,i_n}$, and the "out"-scattering states $|p_1,\ldots,p_n\rangle^{(\text{out})}_{i_1,\cdots,i_n}$ as outgoing states $|p_n,\ldots,p_1\rangle_{i_n,\ldots,i_1}$.

The discussion above motivates the following definition. The form factors of an operator $\mathcal{O}(\mathbf{x})$ are the auxiliary functions $f^{\mathcal{O}}_{\underline{i}}(p_{\underline{i}})$ equal to the matrix elements of the operator at the origin, $\mathcal{O}\equiv\mathcal{O}(\mathbf{0})$, between the vacuum (as the "out"-state) and a scattering state with the "in"-ordering of momenta

$$ f^{\mathcal{O}}_{\underline{i}}(p_{\underline{i}}) = \langle\Omega|\,\mathcal{O}\,|p_1,\ldots,p_n\rangle^{(\text{in})}_{i_1,\ldots,i_n}\,. \quad (4.1.5) $$

This definition can be extended to any other ordering by analytical continuation.

[1]We will follow the conventions of [12, 13]. The term generalized is used to differentiate between this generic form and the more convenient formulation that follows, where the out-state is the vacuum and the momenta are ordered.

[2]The two expressions are of course related by a Fourier transformation, i.e.

$$ \widetilde{\mathcal{O}}(\mathbf{q}) = \int d^2\mathbf{x}\,e^{i\mathbf{q}\cdot\mathbf{x}}\mathcal{O}(\mathbf{x})\,. \quad (4.1.3) $$

4.1.2 Diagonal form factors

In the study of form factors, it is particularly interesting to consider a different configuration in which the particles in the in-state and out-state are identical (in both flavour and momenta). This property characterizes the so-called diagonal form factors, which are related to the structure constants of the three-point functions of conformal operators (3.1.16), as we will see below.

We will begin with the following matrix elements of a local operator $\mathcal{O}(\mathbf{x})$, with $p_1 > p_2 > \cdots > p_n$,

$$ {}^{(\text{out})}_{i_1,\ldots,i_n}\langle p_1,\ldots,p_n|\,\mathcal{O}(\mathbf{x})\,|p_1,\ldots,p_n\rangle^{(\text{in})}_{i_1,\ldots,i_n} \ . \tag{4.1.6} $$

For simplicity of notation, we will restrict ourselves to the $n = 1$ case, i.e. the one-particle diagonal form factor. We have

$$ F_D^{\mathcal{O}}(p) = \langle p|\,\mathcal{O}(\mathbf{x})\,|p\rangle = \langle p|\,e^{i\tau\mathcal{H}}e^{i\sigma\mathbb{P}}\mathcal{O}(0)e^{-i\sigma\mathbb{P}}e^{-i\tau\mathcal{H}}\,|p\rangle = \langle p|\,\mathcal{O}(0)\,|p\rangle \ , \tag{4.1.7} $$

that is $F_D^{\mathcal{O}}$ depends only on the external state, not on the position of the insertion, and this is clearly also true for any other diagonal form factor. One can ask how to extend the definition of the (4.1.5), assuming that we can define crossing. If we cross the outgoing particle and write e.g.

$$ \langle p|\,\mathcal{O}(0)\,|p\rangle = \langle \Omega|\,\mathcal{O}(0)\,|\overline{p},\,p\rangle \ , $$

where $\overline{p}$ is the momentum of the ingoing anti-particle obtained by crossing the outgoing particle with momentum p, we are omitting a possible divergence coming from the disconnected term proportional to $\langle p|p\rangle$[3] One way of regularizing this expression is shifting the outgoing momenta to $p' = p + \epsilon$, then taking the limit $p' \to p$. This means that we can define the one-particle diagonal form factor, similarly to (4.1.5), as

$$ f_D^{\mathcal{O}}(p) = \lim_{\epsilon\to 0}\langle \Omega|\,\mathcal{O}(0)\,|\overline{p}+\epsilon,\,p\rangle \ . \tag{4.1.8} $$

In the case of more than one particle the result can depend on the way in which the limit is performed, as it was noted in [20]. To clarify this statement we will now

[3]When the fields representing the "in" and "out"-states are Wick-contracted among themselves.

discuss briefly the two-particle case. Let p_1 and p_2 be the momenta of the two particles and assume $p_1 > p_2$, so that the "in"- and "out"-scattering states are $|p_1, p_2\rangle^{(\text{in})} = |p_1, p_2\rangle$ and $|p_1, p_2\rangle^{(\text{out})} = |p_2, p_1\rangle$ respectively. We will shift the momenta of the particles in the "out" state to p_1' and p_2', cross both particles, and take the limit $p_i' \to p_i$, or equivalently $\epsilon_i = p_i' - p_i \to 0$, as before. Thus we obtain the analogue of (4.1.8)

$$f_D^{\mathcal{O}}(p_1, p_2) = \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \langle \Omega | \, \mathcal{O}(0) \, | \overline{p}_2 + \epsilon_2, \, \overline{p}_1 + \epsilon_1, \, p_1, \, p_2 \rangle \; , \tag{4.1.9}$$

though here there is more than one way of taking the double limit. In order to be consistent we need to make a choice on how to regularize the diagonal form factors. In the literature, both the symmetric and the connected regularizations are commonly used. In the symmetric regularization scheme, all the ϵ_i are taken equal, so that (4.1.9) becomes simply

$$f_{D,s}^{\mathcal{O}}(p_1, p_2) = \lim_{\epsilon \to 0} \langle \Omega | \, \mathcal{O}(0) \, | \overline{p}_2 + \epsilon, \, \overline{p}_1 + \epsilon, \, p_1, \, p_2 \rangle \; . \tag{4.1.10}$$

In the connected scheme, one takes instead the finite part of the r.h.s. of (4.1.9), i.e. the terms which do not depend on ϵ_i.

Equivalently, we can also write the diagonal form factors in terms of the rapidities, as in [77, 16] for the spin-chain, see Sec. 3.2.2. In general the rapidities are the uniformizing parameters of the dispersion relation of the theory. In a massive relativistic theory, the relation between momenta and rapidities is simply $p = m \sinh u$ and the crossing symmetry relates outgoing particles with rapidity u to incoming anti-particles with rapidity $\overline{u} = u + i\pi$ respectively. As before, we shift the rapidities in the "out"-state by a small amount $u_i \to u_i + \epsilon_i$, and we need to take the limit $\epsilon_i \to 0$ of

$$\langle 0 | \mathcal{O} | \overline{u}_1 + \epsilon_1, \ldots, \overline{u}_n + \epsilon_n, u_1, \ldots, u_n \rangle^{\text{in}} \; . \tag{4.1.11}$$

In general we can write

$$f^{\mathcal{O}}(\overline{u}_1 + \epsilon_1, \ldots, \overline{u}_n + \epsilon_n, u_1, \ldots, u_n) = \prod_{i=1}^{n} \frac{1}{\epsilon_i} \sum_{i_1}^{n} \cdots \sum_{i_n}^{n} a_{i_1 \ldots i_n}(u_1, \ldots, u_n) \epsilon_{i_1} \ldots \epsilon_{i_n} + \text{ terms vanishing as } \epsilon_i \to 0 \; , \tag{4.1.12}$$

where $a_{i_1 \ldots i_n}$ is a completely symmetric tensor. Then, in the symmetric regularization scheme, we define the diagonal form factor by taking all the ϵ_i's to be the same, $\epsilon_i = \epsilon$ for each $i = 1, \ldots, n$, and then setting $\epsilon \to 0$. Alternatively, if we choose the connected scheme, we write the diagonal form factors as the finite part of (4.1.12)

$$f^{\mathcal{O}}_{D,c}(u_1, \ldots, u_N) \equiv n!\, a_{12\ldots n}\ . \tag{4.1.13}$$

Let us note that the diagonal form factors calculated in the two regularization schemes are not independent, and the relation between them has been discussed in [77].

Finite volume

Diagonal form factors are related to the structure constants of a special class of three-point functions, called "Heavy-Heavy-Light" (HHL) [14, 15], where the "heavy" (i.e. long) operators correspond to classical string solutions and the "light" one to a vertex operator on the string world-sheet. It was proposed in [14] that the dependence of structure constants on the length, L, of the heavy operators is given by finite volume diagonal form factors in integrable theories, which was confirmed at one-loop in [16].

To introduce the idea, we will consider one-particle diagonal form factors in a generic 1+1-dimensional theory on a finite volume L, e.g. $-L/2 < \sigma < L/2$ in our notation[4]. In a finite volume there are no singularities in the definition of diagonal form factors, though the result will of course depend on the choice of the normalization. The momenta are quantized according to the Bethe equations (3.2.24)

The states can be labelled by the quantization number, $|p_1, p_2, \ldots\rangle_L = |n_1, n_2, \ldots\rangle$, and a convenient normalization is $\langle n_i | n_j \rangle = \delta_{ij}$. In order to compare these with the infinite volume states, let us consider (again in the one particle case)

$$\sum_n |n\rangle \langle n| \approx \int \frac{dp}{2\pi} L\, |p\rangle_L \langle p|_L = \int \frac{dp}{2\pi} |p\rangle \langle p| \tag{4.1.14}$$

[4] We assume L large enough so that we can ignore exponential corrections coming from the vacuum polarization.

which means $|p\rangle_L = \frac{1}{\sqrt{\rho_1}}|p\rangle$ with $\rho_1 = L$. Thus we have that the finite volume one-particle diagonal form factor is

$$f^{\mathcal{O}}_{L,p} = {}_L\langle p|\,\mathcal{O}(0)\,|p\rangle_L = \frac{1}{\rho_1}\langle p|\,\mathcal{O}(0)\,|p\rangle = \frac{1}{\rho_1} f^{\mathcal{O}}_p\,, \tag{4.1.15}$$

i.e the volume dependence comes from the normalization factor only, as observed in [77] for the general case.

4.1.3 Form factor axioms

A set of axioms for the form factors has been proposed in [10], extending to off-shell quantities the analysis which allowed to formulate the S-matrix bootstrap. These axioms are (phenomenological) consistency properties derived from the symmetries of the theory which in principle can be used to write down the exact form factors for any integrable theory, starting from the knowledge of the S-matrix. Babujian, Fring, Karowski and Zapletal [78] have been able to write the axioms for the form factor bootstrap using only the validity of the LSZ formalism (Lehmann, Symanzik, Zimmermann) [79] and the hypothesis of "maximal analyticity". In other words, the assumption is that all singularities must be of physical origin, i.e. form factors (and the S-matrix) are analytic functions everywhere except at the points corresponding to physical intermediate states, hence the term "maximal analyticity". We refer to [11] for a review of the relativistic case, while in the following we will focus only on the axioms for the world-sheet-model form factors presented in [12].

We will write the axioms for the world-sheet form factors in terms of the rapidities z defined in (2.1.62) (see Sec. 2.1.5)

$$p = 2\,\mathrm{am}\,z\,, \quad \sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \quad E = \mathrm{dn}(z,k)\,. \tag{4.1.16}$$

We assume the form factors $f^{\mathcal{O}}_{i_1,\ldots,i_n}(z_1,\ldots,z_n)$ to be meromorphic functions of the torus parameters z_α of each external particle, with $\alpha = 1,\ldots,n$. The proposed form factor axioms are

- *Permutation*: the form factors are symmetric under permutation of any two consecutive variables z_l, z_{l+1} together with the permutation of their corresponding spaces which is performed through the action of the two-particles S-matrix

$$f_{\dots,i'_{l+1},i'_{l},\dots}(\dots,z_{l+1},z_{l},\dots)=f_{\dots,i_{l},i_{l+1},\dots}(\dots,z_{l},z_{l+1},\dots)\mathcal{S}^{i_{l}i_{l+1}}_{i'_{l}i'_{l+1}}(z_{l},z_{l+1})\,; \quad (4.1.17)$$

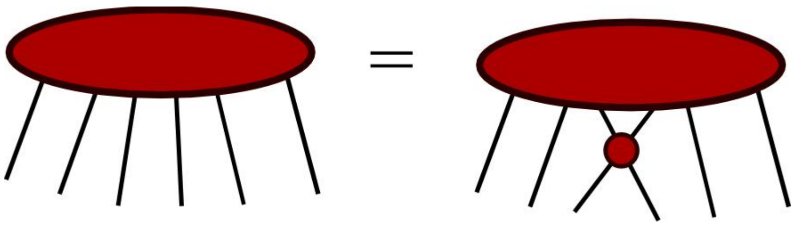


FIGURE 4.1: Graphical representation of the permutation axiom.

- *Periodicity*: the form factors transform as follows under cyclic permutations

$$f_{i_1,i_2,\dots,i_n}(z_1+\omega_2,z_2,\dots,z_n)=f_{i_2,\dots,i_n,i_1}(z_2,\dots,z_n,z_1-\omega_2)\,, \quad (4.1.18)$$

where $2\omega_2=4iK(1-k)-4K(k)$ with $K(k)$ the elliptic integral of the first kind, see Sec. 2.1.5;

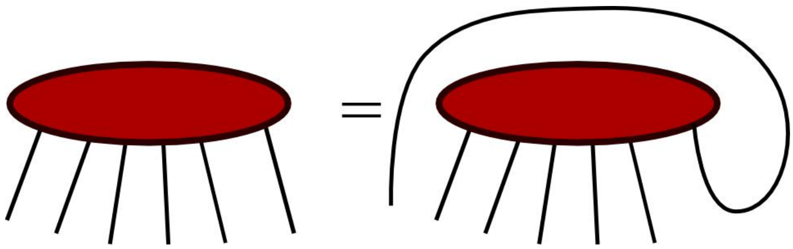


FIGURE 4.2: Graphical representation of the periodicity axiom.

- *One-particle poles*: the form factors have poles in each subchannel corresponding to one-particle intermediate states going on-shell, e.g. when $\mathbf{p}_{12}\equiv\mathbf{p}(z_1)+\overline{\mathbf{p}}(z_2)$ vanishes[5]

$$\begin{aligned}\operatorname*{Res}_{\mathbf{p}_{12}=0} f_{i_1,\dots,i_n}(z_1,z_2,z_3,\dots,z_n) &= 2iC_{i_1i'_2}f_{i'_3,\dots,i'_n}(z_3,\dots,z_n)\\ &\times\left[\delta^{i'_2}_{i_2}\cdots\delta^{i'_n}_{i_n}-\mathcal{S}^{i'_2\ \ i'_n}_{j_{n-3}i_n}(z_n,z_2)\dots\mathcal{S}^{j_1i'_3}_{i_2i_3}(z_3,z_2)\right], \quad (4.1.19)\end{aligned}$$

[5]The bar over p in $\overline{\mathbf{p}}$ is used to indicate that it is the momentum of the anti-particle.

where $C_{i_1 i_2}$ is the charge conjugation matrix;

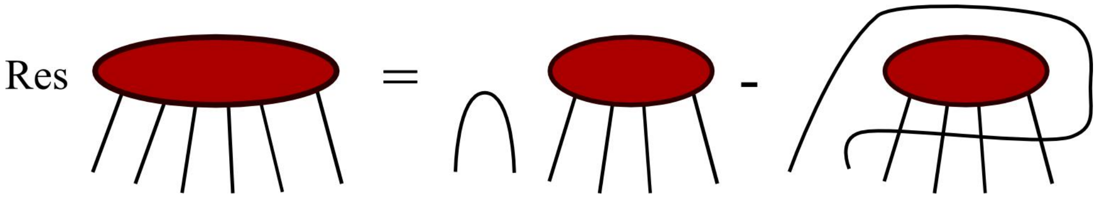


FIGURE 4.3: Graphical representation of the axiom regarding one-particle poles.

- *Bound state poles*: the form factors have poles originating from the bound states present in the world-sheet theory, and the residues of these poles are given by form factors with such bound states as external particles. To clarify this statement, let us consider for simplicity a rank one subsector of the full theory and a two-particle bound state in this subsector. We assume there is a pole in the (scalar) S-matrix at values z_1' and z_2' such that $u(z_2') - u(z_1') = 2i/g$, where u is the rapidity parameter defined in (2.1.60), and we denote the residue of the S-matrix at this pole as

$$\operatorname*{Res}_{z_1',z_2'} \mathcal{S}_{12}(z_1,z_2) = R_{(12)} \,. \tag{4.1.20}$$

Then the statement is that the corresponding form factor also has a pole at the values z_1' and z_2' and the residue is

$$\operatorname*{Res}_{z_1',z_2'} f(z_1,z_2,z_3,\ldots,z_n) = \sqrt{2iR_{(12)}} f(z_{(12)},z_3,\ldots,z_n) \,, \tag{4.1.21}$$

where $z_{(12)}$ is the rapidity parameter for the bound state[6].

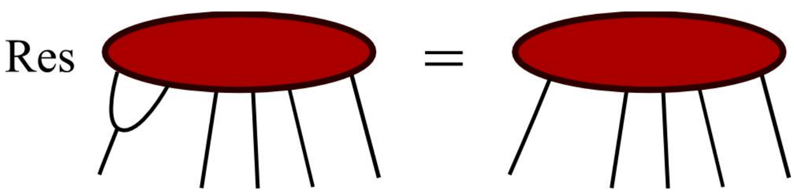


FIGURE 4.4: Graphical representation of the axiom regarding bound-state poles.

[6]We refer to [80] for a detailed discussion of bound states in this context.

4.2 Perturbative calculations

In order to compute perturbatively form factors of a local operator $\mathcal{O}(x)$ we recall the LSZ formula [79] for a generic theory $\mathcal{S}[\Phi]$

$$f_{\underline{i}}^{\mathcal{O}}(p_1,\ldots,p_n) = \left(\frac{1}{\sqrt{2}}\right)^n \lim_{p_i^2\to m_i^2} \prod_{i=1}^{n} \left(\frac{p_i^2 - m_i^2}{i}\right) G_n^{\mathcal{O}}(q,p_1,\ldots,p_n)\,, \tag{4.2.1}$$

where

$$(2\pi)^2\,\delta^2\left(q+\sum_{i=1}^{n} p_i\right) G_n^{\mathcal{O}}\,(q,p_1,\ldots,p_n) = \int \prod_{i=1}^{n} \mathrm{d}x_i \mathrm{d}y\, e^{-i\sum_i p_i x_i} e^{-iqy}\,\langle 0|\,T\left[\mathcal{O}(y)\Phi(x_1)\cdots\Phi(x_n)\right]|0\rangle\;, \tag{4.2.2}$$

with T denoting the time-ordered product and the delta function fixing q to be equal to $-\sum_i p_i$. We can then obtain the form factors perturbatively, order by order, by adapting (4.2.1) to the case at hand and drawing the corresponding Feynman diagrams to simplify the calculations. The Feynman diagrams for the form factors are similar to the S-matrix ones, except for the fact that the external momenta may not be on-shell. We also need to consider the contractions with the operator $\mathcal{O}$, which translates in the Feynman formalism to taking into account all the possible locations for the operator insertion, as we will see in the explicit computations below.

4.2.1 Perturbative world-sheet form factors

It is a slightly non-trivial question how to unambiguously define the perturbative world-sheet form factors. They will certainly depend on the gauge fixing choice, specifically on the gauge parameter a, a feature which is similarly seen in the computation of the world-sheet S-matrix. In the case of the S-matrix the gauge parameter can be seen to explicitly cancel in the Bethe equations and thus it drops out of the computation of physical string energies. For form factors, being partially off-shell quantities, there is the additional freedom to make field redefinitions which will modify the local operator. For example, if we consider a field redefinition of the form

$$\Phi^{A\dot{A}}(\sigma) = \tilde{\Phi}^{A\dot{A}}(\sigma) + \frac{1}{\sqrt{\lambda}} F(\tilde{\Phi}^{B\dot{B}}(\sigma))\tilde{\Phi}^{A\dot{A}}(\sigma) \tag{4.2.3}$$

where F is an arbitrary quadratic function of the fields, the definition of the asymptotic states and the S-matrix will remain unchanged. However, the transformation (4.2.3) will change the form of higher-order terms in the action and in general any off-shell quantities.

World-sheet propagators

We will now compute the world-sheet propagators which we will need in the following. As the simplest example of off-shell observables we consider the two-point functions of world-sheet fields in momentum space[7]. At leading order we find for the transverse AdS and sphere bosons[8]

$$\langle Z^{\alpha\dot{\alpha}}(\mathbf{p})Z^{\beta\dot{\beta}}(-\mathbf{p})\rangle = \frac{i}{\mathbf{p}^2-1}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\,,\quad \langle Y^{a\dot{a}}(\mathbf{p})Y^{b\dot{b}}(-\mathbf{p})\rangle = \frac{i}{\mathbf{p}^2-1}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} \tag{4.2.5}$$

while for the fermions we can define

$$\Theta_{\Upsilon}^{\alpha\dot{a}}(\mathbf{p}) = \begin{pmatrix} \Upsilon^{\alpha\dot{a}}(\mathbf{p}) \\ \Upsilon^{*\alpha\dot{a}}(\mathbf{p}) \end{pmatrix}\,,\quad \Theta_{\Psi}^{a\dot{\alpha}}(\mathbf{p}) = \begin{pmatrix} \Psi^{a\dot{\alpha}}(\mathbf{p}) \\ \Psi^{*a\dot{\alpha}}(\mathbf{p}) \end{pmatrix} \tag{4.2.6}$$

and then write

$$\langle \Theta_{\Upsilon}^{\mathrm{T}\,\alpha\dot{a}}(\mathbf{p})\Theta_{\Upsilon}^{\beta\dot{b}}(-\mathbf{p})\rangle = \frac{i\epsilon^{\alpha\beta}\epsilon^{\dot{a}\dot{b}}}{\mathbf{p}^2-1}\begin{pmatrix} p & -1+\epsilon \\ 1+\epsilon & p \end{pmatrix}\,, \tag{4.2.7}$$

$$\langle \Theta_{\Psi}^{\mathrm{T}\,a\dot{\alpha}}(\mathbf{p})\Theta_{\Upsilon}^{b\dot{\beta}}(-\mathbf{p})\rangle = \frac{i\epsilon^{ab}\epsilon^{\dot{\alpha}\dot{\beta}}}{\mathbf{p}^2-1}\begin{pmatrix} p & -1+\epsilon \\ 1+\epsilon & p \end{pmatrix}\,. \tag{4.2.8}$$

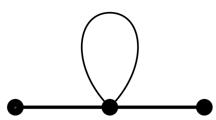


FIGURE 4.5: Feynman diagram for the one-loop propagator correction.

[7] We define the Fourier transform with the conventions

$$\Phi^{A\dot{A}}(\mathbf{q}) = \int d^2\sigma\, e^{i\mathbf{q}\cdot\sigma}\Phi^{A\dot{A}}(\sigma)\,. \tag{4.2.4}$$

[8] There is an additional factor of the coupling, $2\pi/\sqrt{\lambda}$, that we are omitting in these expressions.

These propagators also provide a concrete example of the use of actions differing by field redefinitions. We can consider the one-loop wavefunction renormalization of fundamental fields computed from the action (2.1.47), and compare it with the result found from the action (in the same gauge $a = 1/2$) used to compute the one-loop S-matrix and two-loop dispersion relation in [81]. Calculating the simple one-loop correction to the propagator, shown schematically in Fig. 4.5, we find that the correction to the bosonic propagator get contributions from both scalars and fermions running in the loop. From the scalars we find a contribution proportional to

$$\int \frac{d^2k}{(2\pi)^2} \frac{k_\sigma^2}{\boldsymbol{k}^2 - 1} \tag{4.2.9}$$

where k_σ is the spatial component of the loop momentum. In principle this divergent integral would lead to the renormalization of the wavefunction, however for the action (2.1.47) there is an extra factor from fermions running in the loop such that the total one-loop contribution is zero. This differs from the result of [81], where they found a non-vanishing UV divergent tadpole integral contributing to the bosonic propagators. Note that the correction to the fermionic propagators can be computed in a similar way, however it vanishes for both actions. Thus we see that our choice of the action (2.1.46) and (2.1.47) simplifies the computations at the price of limiting ourselves to quartic terms, since the sextic ones are not currently explicitly known[9]. For this reason the computation of the one-loop S-matrix for the action (2.1.46) would be a challenging task. We now turn instead to the computation of tree-level world-sheet form factors.

One-particle form factors

We will begin with defining the form factor coefficients $f^{A\dot{A}}_{B\dot{B}}(p_1)$ of the one-particle form factors as the matrix elements

$$\langle \Omega | \mathcal{O}^{A\dot{A}}(\mathbf{q}) | \Phi_{B\dot{B}}(p_1) \rangle = (2\pi)^2 \delta^{(2)}(\boldsymbol{q} - \boldsymbol{p}_1) f^{A\dot{A}}_{B\dot{B}}(p_1)\,, \tag{4.2.10}$$

[9] The issue here is that to find this more convenient Lagrangian we need to perform a change of variables in the Hamiltonian formalism, thus the relation between the two Lagrangians is not straightforward.

where $\mathcal{O}^{A\dot{A}}$ indicates one of the fields of the theory depending on the choice of the indices $A, \dot{A} = 1, \ldots, 4$ as mentioned at the end of Section 2.1.3. We can straightforwardly compute

$$f^{a\dot{a}}_{B\dot{B}} = \delta^a_B \delta^{\dot{a}}_{\dot{B}} \frac{1}{\sqrt{2\epsilon_1}}\,, \qquad f^{\alpha\dot{\alpha}}_{B\dot{B}} = \delta^a_B \delta^{\dot{a}}_{\dot{B}} \frac{1}{\sqrt{2\epsilon_1}}\,,$$
$$f^{\alpha\dot{a}}_{B\dot{B}} = \frac{\delta^\alpha_B \delta^{\dot{a}}_{\dot{B}}}{\sqrt{\epsilon_1}} \begin{pmatrix} u(p_1) \\ v(p_1) \end{pmatrix}\,, \qquad f^{a\dot{\alpha}}_{B\dot{B}} = \frac{\delta^a_B \delta^{\dot{a}}_{\dot{B}}}{\sqrt{\epsilon_1}} \begin{pmatrix} u(p_1) \\ v(p_1) \end{pmatrix}\,, \tag{4.2.11}$$

where $u(p_1)$ and $v(p_1)$ are the fermionic wavefunctions, appearing in the asymptotic expressions of the fermionic fields presented in Appendix C.

Three-particle ff: bosonic operators

Similarly to the one-particle case, we define for convenience the coefficients $f^{A\dot{A}}_{B\dot{B},C\dot{C},D\dot{D}}$ through

$$\langle \Omega | \mathcal{O}^{A\dot{A}}(\mathbf{q}) | \Phi_{B\dot{B}}(p_1) \Phi_{C\dot{C}}(p_2) \Phi_{D\dot{D}}(p_3) \rangle = - \frac{(2\pi)^2 \delta^{(2)}(\boldsymbol{q} - \sum_{i=1}^{3} \boldsymbol{p}_i)}{\sqrt{8\epsilon_1\epsilon_2\epsilon_3}\; p^2_{123} - 1} f^{A\dot{A}}_{B\dot{B},C\dot{C},D\dot{D}}(p_1, p_2, p_3)\,, \tag{4.2.12}$$

where $p_{ijk} = p_i + p_j + p_k$.

We will start from the case in which the operator corresponds to one of the transverse directions of the sphere. When the external on-shell particles also correspond to excitations on the sphere the form factor can be easily calculated. It has the form

$$f^{a\dot{a}}_{b\dot{b},c\dot{c},d\dot{d}}(p_1, p_2, p_3) \;=\; A^Y_1 \delta^a_b \delta^{\dot{a}}_{\dot{b}} \epsilon_{cd} \epsilon_{\dot{c}\dot{d}} + A^Y_2 \delta^a_c \delta^{\dot{a}}_{\dot{c}} \epsilon_{bd} \epsilon_{\dot{b}\dot{d}} + A^Y_3 \delta^a_d \delta^{\dot{a}}_{\dot{d}} \epsilon_{bc} \epsilon_{\dot{b}\dot{c}}\;,$$

with

$$\begin{aligned} A^Y_1 &= 2(p^2_1 + p_1 p_2 + p_1 p_3 - p_2 p_3)\,, \\ A^Y_2 &= 2(p^2_2 + p_2 p_1 + p_2 p_3 - p_1 p_3)\,, \\ A^Y_3 &= 2(p^2_3 + p_3 p_2 + p_1 p_3 - p_2 p_1)\,. \end{aligned} \tag{4.2.13}$$

If we allow AdS excitations in the external state, we have

$$f^{a\dot{a}}_{b\dot{b},\alpha\dot{\alpha},\beta\dot{\beta}}(p_1,p_2,p_3) = B^Y \delta^a_b \delta^{\dot{a}}_{\dot{b}} \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$$
$$\text{with} \quad B^Y = -2(p_1+p_2)(p_1+p_3)\,. \tag{4.2.14}$$

We can also have that two of the external particles are fermionic, while any other configuration of the external particles not mentioned is not allowed by the symmetries of the theory. For example we can have the external states corresponding to the fields $Y_{b\dot{b}}$, $\Psi_{c\dot{\gamma}}$ and $\Psi_{d\dot{\delta}}$. In this case we have three possible $\mathfrak{su}(2)$ structures $\delta^a_c\epsilon_{db}$, $\delta^a_b\epsilon_{dc}$ and $\delta^a_b\epsilon_{bc}$ appearing in $f^{a\dot{a}}_{b\dot{b},c\dot{\gamma},d\dot{\delta}}$ however only two are independent. To remove the ambiguity we choose the coefficient of $\delta^a_b\epsilon_{bc}$ to be vanishing, and we find

$$\begin{aligned}
f^{a\dot{a}}_{b\dot{b},c\dot{\gamma},d\dot{\delta}}(p_1,p_2,p_3) &= (C^Y_1\delta^a_c\epsilon_{bd} + C^Y_2\delta^a_d\epsilon_{bc})\delta^{\dot{a}}_{\dot{b}}\epsilon_{\dot{\gamma}\dot{\delta}} \\
f^{a\dot{a}}_{b\dot{b},\gamma\dot{c},\delta\dot{d}}(p_1,p_2,p_3) &= (C^Y_3\delta^{\dot{a}}_{\dot{c}}\epsilon_{\dot{d}\dot{b}} + C^Y_4\delta^{\dot{a}}_{\dot{d}}\epsilon_{\dot{c}\dot{b}})\delta^a_b\epsilon_{\gamma\delta} \\
f^{a\dot{a}}_{\beta\dot{\beta},\gamma\dot{c},d\dot{\delta}}(p_1,p_2,p_3) &= C^Y_5\delta^a_d\delta^{\dot{a}}_{\dot{c}}\epsilon_{\gamma\beta}\epsilon_{\dot{\beta}\dot{\delta}}
\end{aligned} \tag{4.2.15}$$

with the following coefficients, written in terms of the rapidities z_i for convenience,

$$\begin{aligned}
C^Y_1 &= \cosh\frac{z_1-z_2}{2}\left(\sinh\frac{z_1+2z_2-3z_3}{2} + \sinh\frac{z_1-2z_2+z_3}{2} - 2\sinh\frac{z_1+2z_2+z_3}{2}\right) \\
C^Y_2 &= \cosh\frac{z_1-z_3}{2}\left(\sinh\frac{z_1+z_2-2z_3}{2} + \sinh\frac{z_1-3z_2+2z_3}{2} - 2\sinh\frac{z_1+z_2+2z_3}{2}\right) \\
C^Y_3 &= \cosh\frac{z_1-z_2}{2}\left(\sinh\frac{z_1+2z_2-3z_3}{2} + \sinh\frac{z_1-2z_2+z_3}{2} + 2\sinh\frac{z_1+2z_2+z_3}{2}\right) \\
C^Y_4 &= \cosh\frac{z_1-z_3}{2}\left(\sinh\frac{z_1+z_2-2z_3}{2} + \sinh\frac{z_1-3z_2+2z_3}{2} + 2\sinh\frac{z_1+z_2+2z_3}{2}\right) \\
C^Y_5 &= 4\cosh z_2\cosh z_3\cosh\frac{z_2-z_3}{2}\,.
\end{aligned} \tag{4.2.16}$$

When the operator corresponds to one of the transverse directions of the AdS space, i.e. $Z^{\alpha\dot{\alpha}}$, we have essentially the same results as in the $Y^{a\dot{a}}$ case. For external on-shell particles corresponding to excitations of the AdS space, we have

$$f^{\alpha\dot{\alpha}}_{\beta\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}}(p_1,p_2,p_3) = A^Z_1\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}\epsilon_{\gamma\delta}\epsilon_{\dot{\gamma}\dot{\delta}} + A^Z_2\delta^\alpha_\gamma\delta^{\dot{\alpha}}_{\dot{\gamma}}\epsilon_{\beta\delta}\epsilon_{\dot{\beta}\dot{\delta}} + A^Z_3\delta^\alpha_\delta\delta^{\dot{\alpha}}_{\dot{\delta}}\epsilon_{\beta\gamma}\epsilon_{\dot{\beta}\dot{\gamma}} \tag{4.2.17}$$

with $A_1^Z = -A_1^Y$, $A_2^Z = -A_2^Y$, and $A_3^Z = -A_3^Y$. Allowing sphere excitations in the external state, the only non-zero result is

$$f^{\alpha\dot{\alpha}}_{\beta\dot{\beta},a\dot{a},b\dot{b}}(p_1,p_2,p_3) = B^Z \delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}} \epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \tag{4.2.18}$$

with $B^Z = -B^Y$. Finally including fermions in the external states gives

$$\begin{aligned}
f^{\alpha\dot{\alpha}}_{\beta\dot{\beta},\gamma\dot{c},\delta\dot{d}}(p_1,p_2,p_3) &= (C_1^Z \delta^\alpha_\gamma \epsilon_{\beta\delta} + C_2^Z \delta^\alpha_\delta \epsilon_{\beta\gamma})\delta^{\dot{\alpha}}_{\dot{\beta}}\epsilon_{\dot{c}\dot{d}} \\
f^{\alpha\dot{\alpha}}_{\beta\dot{\beta},c\dot{\gamma},d\dot{\delta}}(p_1,p_2,p_3) &= (C_3^Z \delta^{\dot{\alpha}}_{\dot{\gamma}} \epsilon_{\dot{\delta}\dot{\beta}} + C_4^Z \delta^{\dot{\alpha}}_{\dot{\delta}} \epsilon_{\dot{\gamma}\dot{\beta}})\delta^{\alpha}_{\beta}\epsilon_{cd} \\
f^{\alpha\dot{\alpha}}_{b\dot{b},c\dot{\gamma},\delta\dot{d}}(p_1,p_2,p_3) &= C_5^Z \delta^\alpha_\delta \delta^{\dot{\alpha}}_{\dot{\gamma}} \epsilon_{cb}\epsilon_{\dot{b}\dot{d}}
\end{aligned} \tag{4.2.19}$$

with $C_1^Z = -C_1^Y$, $C_2^Z = -C_2^Y$, and $C_3^Z = -C_3^Y$.

Three-particle ff: fermionic operators

We will now consider the three-particle form factors in the case in which the operator is a fermion. For $\Psi^{a\dot{\alpha}}/\sqrt{2}$ we have the following structure

$$\begin{aligned}
f^{a\dot{\alpha}}_{b\dot{b},c\dot{c},d\dot{\delta}}(p_1,p_2,p_3) &= \left(D_1^{\Psi Y}\delta^a_b\epsilon_{cd} + D_2^{\Psi Y}\delta^a_c\epsilon_{bd}\right)\delta^{\dot{\alpha}}_{\dot{\delta}}\epsilon_{\dot{b}\dot{c}} \\
f^{a\dot{\alpha}}_{\beta\dot{\beta},\gamma\dot{\gamma},d\dot{\delta}}(p_1,p_2,p_3) &= \left(D_1^{\Psi Z}\delta^{\dot{\alpha}}_{\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} + D_2^{\Psi Z}\delta^{\dot{\alpha}}_{\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}}\right)\delta^a_d\epsilon_{\beta\gamma} \\
f^{a\dot{\alpha}}_{b\dot{b},\gamma\dot{\gamma},\delta\dot{d}}(p_1,p_2,p_3) &= D_3^{\Psi}\delta^a_b\delta^{\dot{\alpha}}_{\dot{\gamma}}\epsilon_{\gamma\delta}\epsilon_{\dot{b}\dot{d}} \\
f^{a\dot{\alpha}}_{b\dot{\beta},c\dot{\gamma},d\dot{\delta}}(p_1,p_2,p_3) &= E_1^{\Psi}\delta^a_c\delta^{\dot{\alpha}}_{\dot{\delta}}\epsilon_{bd}\epsilon_{\dot{\beta}\dot{\gamma}} + E_2^{\Psi}\delta^a_d\delta^{\dot{\alpha}}_{\dot{\beta}}\epsilon_{bc}\epsilon_{\dot{\gamma}\dot{\delta}} + E_3^{\Psi}\delta^a_d\delta^{\dot{\alpha}}_{\dot{\delta}}\epsilon_{bc}\epsilon_{\dot{\beta}\dot{\gamma}}\,.
\end{aligned} \tag{4.2.20}$$

The coefficients for asymptotic states involving two Y bosons are

$$\begin{aligned}
D_1^{\Psi Y} = \frac{1}{8}\Big(&\mathrm{e}^{z_3/2}(p_1^2 - p_2^2 + 4p_3^2 + p_1(6p_3 + \epsilon_1 - \epsilon_2) + p_2(2p_3 + \epsilon_1 - \epsilon_2))(1 + q_\sigma - q_\tau) \\
&- \mathrm{e}^{-z_3/2}(p_1^2 - p_2^2 + 4p_3^2 + p_1(6p_3 - \epsilon_1 + \epsilon_2) + p_2(2p_3 - \epsilon_1 + \epsilon_2))(-1 + q_\sigma + q_\tau)\Big) \\
D_2^{\Psi Y} = \frac{1}{8}\Big(&\mathrm{e}^{z_3/2}(p_1^2 - p_2^2 - 4p_3^2 - p_1(2p_3 - \epsilon_1 + \epsilon_2) + p_2(6p_3 - \epsilon_1 + \epsilon_2))(1 + q_\sigma - q_\tau) \\
&- \mathrm{e}^{-z_3/2}(p_1^2 - p_2^2 - 4p_3^2 - p_1(2p_3 + \epsilon_1 - \epsilon_2) - p_2(6p_3 + \epsilon_1 - \epsilon_2))(-1 + q_\sigma + q_\tau)\Big)
\end{aligned}$$

where q_σ and q_τ are respectively the space and time components of the off-shell momentum which is related to the on-shell momenta via momentum conservation. Similarly to the case with bosonic operators, the coefficients for states with two Z bosons are related to those with two Y bosons as $D_1^{\Psi Z} = -D_1^{\Psi Y}$, $D_2^{\Psi Z} = -D_2^{\Psi Y}$. D_3^{Ψ} can be found considering the case of mixed external bosons

$$D_3^{\Psi} = q_\sigma p_3 (\mathrm{e}^{z_3/2}(1 + q_\sigma - q_\tau) + \mathrm{e}^{-z_3/2}(-1 + q_\sigma + q_\tau)) \,. \tag{4.2.21}$$

For asymptotic states involving three fermions we have

$$\begin{aligned} E_1^{\Psi} \;=\; & \frac{1}{8} e^{-(z_1+z_2+z_3)/2} \Big[q_\sigma \big[(e^{z_1+z_2} - e^{z_3})(p_1+p_2)^2 + (e^{z_1+z_3} - e^{z_2})(p_1+p_3)^2 \\ & \qquad + (e^{z_2+z_3} - e^{z_1})(p_2+p_3)^2 \big] \\ & + (-1+q_\tau) \big[(e^{z_1+z_2} + e^{z_3})(p_1+p_2)^2 + (e^{z_1+z_3} + e^{z_2})(p_1+p_3)^2 \\ & \qquad + (e^{z_2+z_3} + e^{z_1})(p_2+p_3)^2 \big] \Big] \end{aligned} \tag{4.2.22}$$

and $E_2^{\Psi} = E_3^{\Psi} = -E_1^{\Psi}$. Finally, we need to consider the cases in which the operator is $\mathrm{Y}^{\alpha\dot{a}}/\sqrt{2}$. We have

$$\begin{aligned} f^{\alpha\dot{a}}_{b\dot{b},c\dot{c},\delta\dot{d}}(p_1,p_2,p_3) &= \left(D_1^{YY} \delta^{\dot{a}}_{\dot{d}} \epsilon_{\dot{c}\dot{d}} + D_2^{YY} \delta^{\dot{a}}_{\dot{c}} \epsilon_{\dot{b}\dot{c}} \right) \delta^{\alpha}_{\delta} \epsilon_{bc} \\ f^{\alpha\dot{a}}_{\beta\dot{\beta},\gamma\dot{\gamma},\delta\dot{d}}(p_1,p_2,p_3) &= \left(D_1^{YZ} \delta^{\alpha}_{\beta} \epsilon_{\gamma\delta} + D_2^{YZ} \delta^{\alpha}_{\gamma} \epsilon_{\beta\delta} \right) \delta^{\dot{a}}_{\dot{d}} \epsilon_{\dot{\beta}\dot{\gamma}} \\ f^{\alpha\dot{a}}_{b\dot{b},\gamma\dot{\gamma},d\dot{\delta}}(p_1,p_2,p_3) &= D_3^{Y} \delta^{\dot{a}}_{\dot{b}} \delta^{\alpha}_{\gamma} \epsilon_{\dot{\gamma}\dot{\delta}} \epsilon_{bd} \\ f^{\alpha\dot{a}}_{\beta\dot{b},c\dot{\gamma},d\dot{\delta}}(p_1,p_2,p_3) &= E_1^{Y} \delta^{\alpha}_{\gamma} \delta^{\dot{a}}_{\dot{c}} \epsilon_{\beta\delta} \epsilon_{\dot{b}\dot{d}} + E_2^{Y} \delta^{\alpha}_{\gamma} \delta^{\dot{a}}_{\dot{b}} \epsilon_{\beta\delta} \epsilon_{\dot{c}\dot{d}} + E_3^{Y} \delta^{\alpha}_{\delta} \delta^{\dot{a}}_{\dot{c}} \epsilon_{\beta\gamma} \epsilon_{\dot{b}\dot{d}} \,, \end{aligned} \tag{4.2.23}$$

with $D_1^{YY} = D_1^{\Psi Y}$, $D_2^{YY} = D_2^{\Psi Y}$, $D_1^{YZ} = D_1^{\Psi Z}$, $D_2^{YZ} = D_2^{\Psi Z}$, $D_3^{Y} = -D_3^{\Psi}$ for the external states with one fermion and two bosons and $E_1^{Y} = -E_2^{Y} = -E_3^{Y} = E_1^{\Psi}$ for three external fermions.

4.2.2 The form factors in the near-flat limit

We will now turn to the so-called near-flat limit of the $AdS_5 \times S^5$ string sigma-model, or Maldacena–Swanson limit [82], where it is feasible to compute the form factors at one loop, following [12]. It corresponds to focusing on the sector of world-sheet excitations with light-cone momenta $p_\pm = (E \pm p)/2$, which scale as $p_\pm \sim g^{\mp 1/2}$. The near-flat string can be derived from the $AdS_5 \times S^5$ string through an expansion in the radius $R \gg 1$ and a boost of the world-sheet coordinates with parameter $\lambda^{1/4}$

[82]. In this limit we have

$$p_- = e^{-z}\,, \quad \text{and} \quad p_+ = e^{z}\left(1 - \frac{e^{-4z}}{48}\right)\,, \tag{4.2.24}$$

which corresponds to the correct limit of the exact dispersion relation, see [82, 83]. This model has the same number of degrees of freedom of the plane-wave string discussed until now and presents simplified interactions since the couplings for the right-movers are suppressed $\partial_+ \sim \lambda^{-1/4}$, while for left-movers $\partial_- \sim \lambda^{1/4}$. The resulting near-flat-space Lagrangian can be written as [84, 83]

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}(\partial\mathbf{Y})^2 - \frac{1}{2}\mathbf{Y}^2 + \frac{1}{2}(\partial\mathbf{Z})^2 - \frac{1}{2}\mathbf{Z}^2 + \frac{1}{2}\psi\frac{\partial^2+1}{\partial_-}\psi \\
&+ \gamma\,(\mathbf{Y}^2 - \mathbf{Z}^2)\left[(\partial_-\mathbf{Y})^2 + (\partial_-\mathbf{Z})^2\right] + i\gamma\,(\mathbf{Y}^2 - \mathbf{Z}^2)\psi\partial_-\psi \\
&+ i\gamma\,\psi\,(\partial_- Y_{i'}\Gamma_{i'} + \partial_- Z_i\Gamma_i) + \left[Y_{j'}\Gamma_{j'} - Z_j\Gamma_j\right]\psi \\
&- \frac{\gamma}{24}\left[\psi\Gamma_{i'j'}\psi\,\psi\Gamma_{i'j'}\psi - \psi\Gamma_{ij}\psi\,\psi\Gamma_{ij}\psi\right]\,,
\end{aligned} \tag{4.2.25}$$

where $\gamma = \pi/\sqrt{\lambda}$. The bosonic fields Y, Z are the same transverse excitations of Section 2.1.3, while the $SO(8)$ Majorana-Weyl spinor ψ describes the eight fermionic degrees of freedom.

Because the interaction terms contain ∂_- derivatives but are free from ∂_+ derivatives, it is convenient to quantize the model with the light-cone coordinate σ^+ considered as time. Thus the mode expansions of the fields are

$$Y_i(\mathbf{x}) = \int \frac{dp_-}{2\pi}\frac{1}{\sqrt{2p_-}}\left[a_{i'}(p_-)\,e^{-i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{i'}(p_-)\,e^{+i\mathbf{p}\cdot\mathbf{x}}\right]\,, \tag{4.2.26}$$

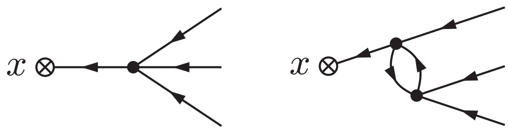
$$Z_i(\mathbf{x}) = \int \frac{dp_-}{2\pi}\frac{1}{\sqrt{2p_-}}\left[a_{i}(p_-)\,e^{-i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{i}(p_-)\,e^{+i\mathbf{p}\cdot\mathbf{x}}\right]\,, \tag{4.2.27}$$

$$\psi(\mathbf{x}) = \int \frac{dp_-}{2\pi}\frac{1}{\sqrt{2}}\left[b(p_-)\,e^{-i\mathbf{p}\cdot\mathbf{x}} + b^\dagger(p_-)\,e^{+i\mathbf{p}\cdot\mathbf{x}}\right]\,, \tag{4.2.28}$$

from which we have the tree-level wave-functions[10] $Z_Y = Z_Z = 1/(2p_-)$ and $Z_\psi = 1/2$. The free bosonic and fermionic propagators are respectively

$$\frac{i}{\mathbf{p}^2 - 1} \quad \text{and} \quad \frac{ip_-}{\mathbf{p}^2 - 1}\,, \tag{4.2.29}$$

[10] There are no corrections at one-loop, but there are at higher-loop orders, see [83].



(A) Tree level (B) One loop

FIGURE 4.6: Feynman diagrams for the near-flat-space three-particle form factors.

and the free dispersion relation is

$$2p_+ = \frac{1}{2p_-} \,. \tag{4.2.30}$$

This also implies that the form factor axioms of Sec. 4.1.3 apply with all p's replaced by p_-'s. Thus p_+ can be considered the energy of the particle and p_- its momentum, which in the following we will relabel η for convenience.

Starting from the field $Y = (Y_1 + iY_2)/\sqrt{2}$, we have at tree level (see Fig. 4.6a)

$$F^{Y}_{\bar{Y}YY}(\bar{\eta}_1, \eta_2, \eta_3) = \frac{-\sqrt{2}\gamma}{\sqrt{\eta_1\eta_2\eta_3}} \frac{\eta_{23}^2}{\mathbf{p}_{123}^2 - 1} \tag{4.2.31}$$

and at one loop (see fig. 4.6b)

$$\begin{aligned} F^{Y}_{\bar{Y}YY}(\bar{\eta}_1, \eta_2, \eta_3) = \frac{\sqrt{8}i\gamma^2}{\sqrt{\eta_1\eta_2\eta_3}(\mathbf{p}_{123}^2 - 1)} & \left[\eta_{23}^2 \left(\eta_{23}^2 B(\eta_{23}) + \eta_{1\bar{2}}^2 B(\eta_{12}) + \eta_{1\bar{3}}^2 B(\eta_{13})\right)\right. \\ & + 4\left(\eta_2^2 - \eta_3^2 - 2\eta_1\eta_3\right)\eta_1\eta_2 B(\eta_{12}) \\ & \left. + 4\left(\eta_3^2 - \eta_2^2 - 2\eta_1\eta_2\right)\eta_1\eta_3 B(\eta_{13})\right] \,, \end{aligned} \tag{4.2.32}$$

where we use again the bar to distinguish the anti-particles. If we consider the field on AdS $Z = (Z_1 + iZ_2)/\sqrt{2}$, we have

$$F^{Z}_{\bar{Z}ZZ} = F^{Y}_{\bar{Y}YY} \,, \tag{4.2.33}$$

and an analogous calculation gives (tree-level)

$$F^{Y}_{\bar{Z}ZY}(\bar{\eta}_1, \eta_2, \eta_3) = \frac{\sqrt{2}\gamma}{\sqrt{\eta_1\eta_2\eta_3}} \frac{2\eta_1\eta_2 - \eta_{13}\eta_{23}}{\mathbf{p}_{123}^2 - 1} \tag{4.2.34}$$

and (one-loop)

$$F^{Y}_{\overline{Z}ZY}(\overline{\eta}_1,\eta_2,\eta_3) = \frac{\sqrt{8}i\gamma^2}{\sqrt{\eta_1\eta_2\eta_3}(\mathbf{p}_{123}^2-1)}\,[-4\eta_1\eta_2\eta_{13}\eta_{23}B(\eta_{12})$$
$$(\eta_{13}\eta_{23}-2\eta_1\eta_2)\left(\eta_{23}^2B(\eta_{23})+\eta_{13}^2B(\eta_{13})+\eta_{12}^2B(\eta_{12})\right)\big] \;. \quad (4.2.35)$$

The bubble integral $B(\mathbf{p})$ is defined in general as

$$B(\mathbf{p}) = \int \frac{d^2k}{(2\pi)^2}\frac{1}{[\mathbf{k}^2-1+i\varepsilon][(\mathbf{p}-\mathbf{k})^2-1+i\varepsilon]}\,, \quad (4.2.36)$$

and can be explicitly calculated in this case to give

$$B(\eta_1,\eta_2) = \frac{i}{2\pi}\frac{\eta_1\eta_2}{\eta_1^2-\eta_2^2}\ln\left|\frac{\eta_2}{\eta_1}\right| - \frac{\eta_1\eta_2}{4(\eta_1+\eta_2)\,|\eta_1-\eta_2|}\left(\frac{\eta_1}{|\eta_1|}+\frac{\eta_2}{|\eta_2|}\right). \quad (4.2.37)$$

We can also consider a composite operator such as $\mathcal{O}_2(x) = Y(x)Y(x)/2$, where T indicates the time-ordered product. The simplest case is

$$F^{\mathcal{O}_2}_{YY}(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{2\sqrt{\eta_1\eta_2}} + \frac{-i\gamma}{\sqrt{\eta_1\eta_2}}\eta_{12}^2\,B(\eta_1\eta_2)\,, \quad (4.2.38)$$

where $B(\eta_1,\eta_2)$ is the bubble integral defined above. The first term is the tree level contribution and the second the one-loop correction. The computation for the four-particle form factor is more involved and we mention only the tree level result

$$F^{\mathcal{O}_2}_{\overline{Z}ZYY}(\overline{\mathbf{p}}_1,\mathbf{p}_2,\mathbf{p}_3) = \frac{\gamma}{\sqrt{\eta_1\eta_2\eta_3\eta_4}}\left[\frac{\eta_1\eta_2+\eta_3\eta_{123}}{\mathbf{p}_{123}^2-1}+\frac{\eta_1\eta_2+\eta_4\eta_{124}}{\mathbf{p}_{124}^2-1}\right]. \quad (4.2.39)$$

4.2.3 Perturbative symmetries

While we can calculate the world-sheet form factors directly for any given operator they are of course not all independent but in fact are related by the $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$ world-sheet symmetries.

For example let us focus on a single copy of $\widehat{\mathfrak{psu}}(2|2)$ by fixing the dotted index of our world-sheet fields and consider the computation of matrix elements of the adjoint action of the world-sheet supercharge $Q_A{}^B$ on fundamental world-sheet fields

$$\langle\Omega|\,\mathrm{ad}_{\mathbb{Q}_A{}^B}(\Phi^C)\,|\Phi_{D_1}\Phi_{D_2}\ldots\rangle = \\ -(-1)^{[C]([A]+[B])}\,\langle\Omega|\,\Phi^C e^{-i\epsilon_{AB}\mathbb{P}}\mathbb{Q}_A{}^B\,|\Phi_{D_1}\Phi_{D_2}\ldots\rangle\ . \quad (4.2.40)$$

We can thus compute the adjoint action of the charges on fields by using their action on asymptotic states. In order to compare with the perturbative results we must expand the exact expression in large $g = \sqrt{\lambda}/2\pi$ and we find in particular

$$a = d = u(p)\ , \qquad b = c = -v(p) \quad (4.2.41)$$

which satisfy the relation $ad - bc = 1$ such that $ad + bc = \epsilon$ and $ab = cd = -p/2$. Moreover we expand the exponentials $e^{i\mathbb{P}}$ by rescaling the world-sheet momenta $p \to p/g$. For the one-particle form factors these relations are essentially trivial and follow from the free theory. Hence we find for example

$$\begin{aligned} \langle\Omega|\,\mathrm{ad}_{\mathbb{Q}_\alpha{}^b}(Y^{c\dot{1}})\,|Y_{\delta\dot{1}}(p_1)\rangle &= \frac{v(p_1)}{\sqrt{2\epsilon_1}}\epsilon^{bc}\epsilon_{\alpha\delta}\ ,\\ \langle\Omega|\,\mathrm{ad}_{\mathbb{Q}^\dagger_a{}^\beta}(Y^{c\dot{1}})\,|Y_{\delta\dot{1}}(p_1)\rangle &= -\frac{u(p_1)}{\sqrt{2\epsilon_1}}\delta^c_a\delta^\beta_\delta\\ \langle\Omega|\,\mathrm{ad}_{\mathbb{Q}_\alpha{}^b}(Y^{\gamma\dot{1}})\,|Y_{d\dot{1}}(p_1)\rangle &= \frac{u(p_1)v(p_1)}{\sqrt{\epsilon_1}}\delta^b_d\delta^\gamma_\alpha\ ,\\ \langle\Omega|\,\mathrm{ad}_{\mathbb{Q}^\dagger_a{}^\beta}(Y^{\gamma\dot{1}})\,|Y_{d\dot{1}}(p_1)\rangle &= -\frac{u(p_1)v(p_1)}{\sqrt{\epsilon_1}}\epsilon_{ad}\epsilon^{\beta\gamma}\ , \end{aligned} \quad (4.2.42)$$

which can been seen to be consistent with the actions

$$\begin{aligned} \mathrm{ad}_{\mathbb{Q}_\alpha{}^b}(Y^{c\dot{1}}) &= \frac{1}{\sqrt{2}}\epsilon^{bc}\epsilon_{\alpha\gamma}Y^{*\gamma\dot{1}}, & \mathrm{ad}_{\mathbb{Q}^\dagger_a{}^\beta}(Y^{c\dot{1}}) &= -\frac{1}{\sqrt{2}}\delta^c_a Y^{\beta\dot{1}}\\ \mathrm{ad}_{\mathbb{Q}_\alpha{}^b}(Y^{*\gamma\dot{1}}) &= \frac{i}{\sqrt{2}}\delta^\gamma_\alpha \acute{Y}^{b\dot{1}}, & \mathrm{ad}_{\mathbb{Q}^\dagger_a{}^\beta}(Y^{\gamma\dot{1}}) &= -\frac{i}{\sqrt{2}}\epsilon_{ac}\epsilon^{\beta\gamma}\acute{Y}^{c\dot{1}} \end{aligned} \quad (4.2.43)$$

which follow from computing the (anti)commutators of the quadratic charges (B.6) using the commutation relations for the free fields (C.2), (C.3) and simply ignoring the non-local $e^{\pm iX^-/2}$ terms[11]. Of course the charges will also contain quartic local terms which will generate cubic corrections to the adjoint action on the fundamental fields. These are not apparent for the one-particle form factors but will only occur

[11] The relevant expressions are presented in App. B and App. C.

for the three-particle case. As currently these quartic terms are not known we can not compare the two calculations, though we can check that our results are compatible with the known symmetries by examining the singularity structure of our form factors. For example, if we consider the action

$$\mathrm{ad}_{\mathbb{Q}_a^{\dagger\beta}}(Y^{c\dot{1}}) = -\frac{1}{\sqrt{2}}\delta_a^c(\mathrm{Y}^{\beta\dot{1}} + \mathcal{O}^{\beta\dot{1}} + \dots) \tag{4.2.44}$$

where $\mathcal{O}^{\beta\dot{1}}$ is some operator cubic in the world-sheet fields and their derivatives. Hence a three-particle form factor such as $\langle\Omega|\,\mathrm{ad}_{\mathbb{Q}_a^{\dagger\beta}}(Y^{c\dot{1}})\left|Y_{d\dot{1}}(p_1)Y_{e\dot{1}}(p_2)\mathrm{Y}_{\gamma\dot{2}}(p_3)\right\rangle$ will have two contributions. However only the term

$$-\frac{1}{\sqrt{2}}\delta_a^c\,\langle\Omega|\,\mathrm{Y}^{\beta\dot{1}}\left|Y_{d\dot{1}}(p_1)Y_{e\dot{1}}(p_2)\mathrm{Y}_{\gamma\dot{2}}(p_3)\right\rangle \tag{4.2.45}$$

will have a propagator pole when we analytically continue p_3 to the out state by crossing and then take $p_3 \to p_1$ as the term arising from $\mathcal{O}^{\beta\dot{1}}$ is a contact term for three external particles. We can compare the residue of this pole with the corresponding terms arising from using the formula (4.2.40). To be explicit, for example choosing $a = c = e = 1$, $d = 2$ and $\beta = \gamma = 4$, one can show that

$$\begin{aligned}
&-\frac{1}{\sqrt{2}}\left\langle \mathrm{Y}^{3\dot{1}}(p_3)\right|\mathrm{Y}^{4\dot{1}}(q)\left|Y_{2\dot{1}}(p_1)Y_{1\dot{1}}(p_2)\right\rangle\Big|_{p_3\to p_1} = \\
&v(p_1)\left\langle Y^{2\dot{1}}(p_3)\right|\mathrm{Y}^{1\dot{1}}(q)\left|Y_{2\dot{1}}(p_1)Y_{1\dot{1}}(p_2)\right\rangle - v(p_3)\left\langle \mathrm{Y}^{3\dot{1}}(p_3)\right|Y^{1\dot{1}}(q)\left|\mathrm{Y}_{3\dot{1}}(p_1)Y_{1\dot{1}}(p_2)\right\rangle \\
&\qquad -\frac{ip_2 v(p_1)}{2}\left\langle \mathrm{Y}^{3\dot{1}}(p_3)\middle|\mathrm{Y}_{3\dot{1}}(p_1)\right\rangle\left\langle\Omega\right|Y^{1\dot{1}}(q)\left|Y_{1\dot{1}}(p_2)\right\rangle\Big|_{p_3\to p_1}
\end{aligned}$$

where the last term originates from the additional disconnected piece one picks up when using crossing on the p_3 particle and then expanding to leading order the factors of $\exp(i\mathbb{P}/2)$. It is important to include the Jacobian factor of $\epsilon_1 p_2 - \epsilon_2 p_1$ from the different energy momentum delta-functions in the connected and disconnected pieces and when evaluating the action of the charges on the outgoing states, e.g. $\left\langle \mathrm{Y}^{3\dot{1}}(p_3)\right|\mathbb{Q}_1^{\dagger 4}$, we must use the crossed, outgoing, asymptotic particle representation.

Given that we can single out the contribution from the action of the quadratic symmetry generators on the fields, rather than using the symmetries to determine the different form factors we can think of the three particle form factors as determining the unknown cubic terms in the world-sheet symmetry generators,

$$\mathcal{O}^{\beta\dot{a}} \equiv \mathrm{ad}_{\mathrm{Q}^{\dagger\beta}_{a}}(Y^{a\dot{a}}) + \frac{1}{\sqrt{2}}\mathrm{Y}^{\beta\dot{1}}. \tag{4.2.46}$$

It would of course be interesting to compare this result with explicit expressions calculated directly from the world-sheet action.

Chapter 5

The Landau–Lifshitz model

The Landau–Lifshitz model was first proposed to describe time evolution of magnetism in solids [21], and it was later found to be integrable [22]. Its interest in the context of AdS/CFT stems from the fact that it can be obtained as a thermodynamic limit of the Heisenberg $XXX_{1/2}$ spin chain and also emerges as a double limit of the $AdS_5 \times S^5$ string, as we will see in detail in the following, discussing its natural extension to higher orders. We will also compute perturbatively diagonal form factors in this model and make contact with the known spin-chain results [16]. Finally, we will briefly discuss the Form Factor Perturbation Theory (FFPT) in this context.

5.1 The Landau–Lifshitz model and the AdS/CFT correspondence

5.1.1 The LL action

The Landau–Lifshitz (LL) model is a non-relativistic σ-model on the sphere S^2:

$$\mathcal{A} = \int d\tau \int_0^{2\pi} d\sigma\, \mathcal{L} = \int d\tau \int_0^{2\pi} d\sigma \left[C_\tau(\vec{n}) - \frac{1}{8} (\partial_\sigma \vec{n})^2 \right] , \tag{5.1.1}$$

with $\vec{n}^2 = 1$ and the Wess–Zumino term $C_\tau(\vec{n})$ defined as

$$C_\tau(\vec{n}) = -\frac{1}{2} \int_0^1 dz\, \varepsilon_{ijk} n^i \partial_z n^j \partial_\tau n^k \,, \tag{5.1.2}$$

where n^i ($i = 1, 2, 3$) are the components of $\vec{n}$. Equivalently we can express the Wess–Zumino term locally for $n_3 \neq -1$ as

$$C_\tau(\vec{n}) = \frac{n_2 \partial_\tau n_1 - n_1 \partial_\tau n_2}{2 + 2n_3}\,, \tag{5.1.3}$$

with the condition $n_3 = \sqrt{1 - n_1^2 - n_2^2}$. The corresponding equations of motion are the Landau–Lifshitz equations

$$\partial_\tau n_i = \frac{1}{2} \varepsilon_{ijk} n_j \partial_\sigma^2 n_k\,. \tag{5.1.4}$$

5.1.2 Spin chains and the LL model

In the thermodynamic limit, the low-energy oscillations around the ferromagnetic vacuum of the $\mathrm{XXX}_{1/2}$ spin chain are described by an effective action which is precisely the Landau–Lifshitz action (5.1.1), as we will show in this section following [23] and [24].

In order to describe the low energy limit of the spin chain, it is useful to introduce coherent states, i.e. the eigenvectors of the annihilation operator of a quantum harmonic oscillator. Let us consider a spin s representation of $\mathfrak{su}(2)$, with its three generators $\vec{S} = (S^1, S^2, S^3)$ as discussed in Section 3.2.1. Let $|\Psi\rangle = |s, s\rangle$ be the highest-weight state of the representation: it satisfies

$$S^3 |\Psi\rangle = s |\Psi\rangle\,, \quad \vec{S}^2 |\Psi\rangle = s(s+1) |\Psi\rangle\,. \tag{5.1.5}$$

We can then define the coherent state $|\Psi_n\rangle$ from the unit vector $\vec{n}$ as

$$|\Psi_n\rangle = e^{i\theta(\vec{n}_0 \times \vec{n})\cdot \vec{S}} |\Psi\rangle\,, \tag{5.1.6}$$

where $\vec{n}_0$ is a unit vector along the quantization axis and θ is the co-latitude $\vec{n}_0 \cdot \vec{n} = \cos\theta$. This definition is motivated by the fact that the $|\Psi_n\rangle$ in (5.1.6) have the characteristic property

$$\left\langle \Psi_n \middle| S^i \middle| \Psi_n \right\rangle = s\, n^i\,. \tag{5.1.7}$$

Let us also note that if $\vec{n}$ and $\vec{n}'$ differ only by a rotation around $\vec{n}_0$ then two coherent states built from them, $|\Psi_n\rangle$ and $|\Psi_{n'}\rangle$, will differ by a phase. Thus, for any choice of the quantization axis, the observable states will be in a one-to-one correspondence with the elements of the coset $SU(2)/U(1) \simeq S^2$. In particular, we are interested in the fundamental representation $s = 1/2$, with the generators given by the Pauli matrices $S^i = \sigma^i/2$, and we will choose the third axis as the rotation axis ($\vec{n}_0 = \hat{z}$). The idea is that we can rewrite the phase space path integral for the $\mathrm{XXX}_{1/2}$ Hamiltonian[1] in this (overcomplete) basis of coherent states and obtain

$$\mathcal{A} = \int dt \sum_{l=1}^{L} \left[\vec{C}(\vec{n}_l) \cdot \partial_t \vec{n}_l - \frac{\lambda}{32\pi^2} (\vec{n}_{l+1} - \vec{n}_l)^2 \right] \tag{5.1.8}$$

where $dC = \vec{n} \cdot d\vec{n} \times d\vec{n}$, i.e. $\vec{C}$ is a monopole potential on S^2 and the corresponding expression is called a Wess–Zumino term. We refer to [23] for a detailed derivation.

We now want to take the low-energy limit of the spin-chain action written as in (5.1.8). Since at large L and for low energy excitations n_i varies slowly from site to site, we can introduce the field $\vec{n}(\tau, \sigma)$ such that $\vec{n}\,(\tau, 2\pi l/L) = \vec{n}_l(\tau)$, for $l = 1, ..., L$. The result is a rescaled Landau–Lifshitz action (5.1.1)

$$\mathcal{A} = \frac{L}{2\pi} \int d\tau \int d\sigma \sum_{l=1}^{L} \left[C_\tau(\vec{n}) - \frac{\lambda}{8L^2} (\partial_\sigma \vec{n})^2 + \ldots \right] \tag{5.1.9}$$

where the dots represent the higher derivatives, suppressed by $1/L$, and the Wess–Zumino term has been defined above in (5.1.2) or (5.1.3). The rescaling also introduces the coupling constant $\tilde{\lambda} = \lambda/L^2$, that is the t'Hooft coupling λ does not appear by itself in (5.1.9) as noted in [3][2]. With $\tilde{\lambda}$ held fixed, the factor of L appearing in front of the action plays the role of $\hbar$ and so the tree-level results correspond to $L \to \infty$. Quantizing the theory and including loop effects corresponds to including finite L corrections; however, as we have also dropped higher derivative terms in our expansion, it is not possible to recover the complete finite-L result of the spin-chain via this method.

This analysis was extended to larger sectors of the gauge theory [27, 28, 29, 30]

[1] The path integral for a generic Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{Z} = \int [dU] e^{i\mathcal{A}[U]}, \quad \text{with} \quad \mathcal{A}[U] = \int dt \left(\langle U | \, i \frac{d}{dt} \, | U \rangle - \langle U | \mathcal{H} | U \rangle \right).$$

[2] Usually the rescaled t'Hooft coupling is written as $\tilde{\lambda} = \lambda/J^2$, where J is the R-charge of the operator of which one is computing the anomalous dimension, though for large $J \sim L$ and a small number of insertions (impurities) the difference is negligible.

and a $\mathfrak{psu}(2,2|4)$ LL model arising from the thermodynamic limit of the complete one-loop $\mathcal{N} = 4$ SYM dilatation generator was constructed in [31].

5.1.3 The LL model as a limit of the light-cone string action

In this section we will see that there is also a limit of the light-cone AdS string (introduced in Section 2.1.3) that reproduces the Landau–Lifshitz action (5.1.1). This is the so-called "fast-string" limit of a bosonic string moving on $\mathbb{R} \times S^3$ [24, 25, 26], as we will explain in the following.

We will start from the bosonic part of the near-plane-wave string action in light-cone coordinates expanded to quartic order in the fields (2.1.46)

$$\mathcal{A} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \int_0^{2\pi} d\sigma\, \mathcal{L} \tag{5.1.10}$$

with

$$\begin{aligned} \mathcal{L} = \;& \frac{1}{2}(\partial_i X)^2 - \frac{1}{2}X^2 - \frac{1}{4}Z^2(\partial_i Z)^2 - \frac{1}{4}Y^2(\partial_i Y)^2 + \frac{1}{4}\left(Y^2 - Z^2\right)\left(\dot{X}^2 + \acute{X}^2\right) + \\ & - \frac{1-2a}{8}\left(X^2\right)^2 + \frac{1-2a}{4}\left(\partial_i X \cdot \partial_j X\right)^2 - \frac{1-2a}{8}\left((\partial_i X)^2\right)^2 , \end{aligned} \tag{5.1.11}$$

where the 8 bosons are denoted collectively $X = (Z, Y)$, Z^μ are the 4 AdS bosons and Y^μ are the 4 bosons on the sphere ($\mu = 0, \ldots, 3$). Note that the world-sheet is the usual cylinder with coordinates τ and σ ($0 \leq \sigma \leq 2\pi$), with indices $i, j = 0, 1$, that is we are not starting here from the action in the decompactification limit $P_+ \to \infty$ of Section 2.1.3, which we will use in Section 5.2.1 to quantize the LL model. In order to obtain the Landau–Lifshitz action (5.1.1), we need to consider string solutions with e.g. $Z^\mu = 0$ and $Y^2 = Y^3 = 0$, that is on $\mathbb{R} \times S^2$, and also choose the light-cone gauge $a = 1$. Thus from the action (5.1.10) we can write

$$\mathcal{A} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \left\{ \partial V \partial \overline{V} - V\overline{V} + 2V\acute{V}\overline{V}\acute{\overline{V}} - \frac{1}{2}\left((\partial V)^2 (\partial \overline{V})^2 - V^2 \overline{V}^2\right) \right\}$$

where we collected Y^0 and Y^1 in a single complex boson V and the bar denotes complex conjugation

$$V = \frac{Y^0 + iY^1}{\sqrt{2}} , \qquad \overline{V} = \frac{Y^0 - iY^1}{\sqrt{2}} . \tag{5.1.12}$$

In order to select fast moving strings, we will introduce v as $V = v\, e^{-i\tau}$ and rescale the world-sheet variables introducing a parameter κ

$$\tau \to \kappa\tau\,, \qquad \sigma \to \sqrt{\kappa}\sigma\,. \tag{5.1.13}$$

Then we expand the Lagrangian for $\kappa \to \infty$ keeping only the leading terms[3] to obtain

$$\begin{aligned}\mathcal{A} = \frac{\sqrt{\lambda\kappa}}{2\pi}\int d\tau \int_0^{2\pi/\sqrt{\kappa}} d\sigma\, \Big[\, & i(v^*\dot{v} - v\dot{v}^*) - |\acute{v}|^2 + 2|v|^2|\acute{v}| \\ & - \frac{1}{2}\left(2i|v|^2(v^*\dot{v} - v\dot{v}^*) + v^{*2}\acute{v}^2 + v^2\acute{v}^{*2}\right)\Big]\,.\end{aligned} \tag{5.1.14}$$

Finally, we can perform the following change of variables [12]

$$v = \sqrt{\frac{\pi}{\sqrt{\lambda\kappa}}}\,\varphi\left(1 + \frac{3}{4}\frac{2\pi}{\sqrt{\lambda\kappa}}|\varphi|^2\right) \tag{5.1.15}$$

and expand for large $\sqrt{\lambda\kappa}$ to find

$$\begin{aligned}\mathcal{A} = \frac{1}{2}\int d\tau \int_0^{2\pi/\sqrt{\kappa}} d\sigma \Bigg[& i(\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*) - |\acute{\varphi}|^2 + \frac{2\pi i}{\sqrt{\lambda\kappa}}|\varphi|^2(\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*) \\ & - \frac{2\pi}{\sqrt{\lambda\kappa}}(\varphi^*\acute{\varphi} + \varphi\acute{\varphi}^*)^2\Bigg]\,.\end{aligned} \tag{5.1.16}$$

In order to make contact with the LL action, we will rewrite (5.1.1), or equivalently (5.1.8), to match (5.1.16). Remembering that the three n^i in (5.1.1) are not independent, since $\vec{n}^2 = 1$, we can then introduce a complex field $\varphi = (n^1 + in^2)/2$, while $n^3 = \sqrt{1 - (n^1)^2 - (n^2)^2} = \sqrt{1 - 4|\varphi|^2}$, and rewrite the action (5.1.8) as

$$\mathcal{A} = \frac{L}{4\pi}\int d\tau \int_0^{2\pi} d\sigma \left[\frac{i(\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*)}{1 + \sqrt{1 - 4|\varphi|^2}} + \tilde{\lambda}|\acute{\varphi}|^2 + \tilde{\lambda}\frac{(\varphi^*\acute{\varphi} + \varphi\acute{\varphi}^*)^2}{1 - 4|\varphi|^2}\right]\,. \tag{5.1.17}$$

We notice that $\tilde{\lambda}$ appears paired with spatial derivatives so that, if we rescale the spatial coordinate as $\sigma \to \sigma\sqrt{\tilde{\lambda}}$, the $\tilde{\lambda}$ coefficients will drop out. Finally we redefine

[3]That is the terms proportional to κ^{-1}, as there are no terms proportional to κ^0

$\varphi \to \varphi\sqrt{L/2\pi}$ to expand the rescaled (5.1.17) in large L and obtain

$$\mathcal{A} = \frac{1}{2}\int d\tau \int_0^{\frac{2\pi}{\sqrt{\tilde\lambda}}} d\sigma \left[i(\varphi^*\dot\varphi - \varphi\dot\varphi^*) - |\acute\varphi|^2 + \frac{2\pi i}{L}|\varphi|^2(\varphi^*\dot\varphi - \varphi\dot\varphi^*) \right. \\ \left. - \frac{2\pi}{L}(\varphi^*\acute\varphi + \varphi\acute\varphi^*)^2 \right] \tag{5.1.18}$$

thus proving that (5.1.16) is a rescaled LL action. Moreover, we find that for $\kappa = \tilde\lambda^{-1}$ the action (5.1.16) derived from the light-cone string action coincides with the action (5.1.8) obtained as the thermodynamic limit of the $\mathrm{XXX}_{1/2}$ spin chain. This is true because we are considering only the leading order of the interactions, while higher-order corrections are different for the two actions, as we will see in Section 5.2.2.

5.1.4 The SU(3) LL model

We mentioned in the last section that we can take the "fast-string" limit of bosonic strings on $\mathbb{R} \times S^3$, and to reproduce (5.1.1) we limited ourselves to one complex boson $V = (Y^0 + iY^1)/\sqrt{2}$. In this section we will see that, if we do not assume $Y^2 = Y^3 = 0$, we can obtain the limit of a more general LL model on S^3.

Let us first write the $SU(3)$ LL action in terms of a vector $\mathbf{U}$ as [28]

$$\mathcal{L} = -i\,U_i^*\partial_\tau U_i - \frac{1}{2}|D_\sigma U_i|^2 \tag{5.1.19}$$

where U_i ($i = 1, 2, 3$) are the three components of $\mathbf{U}$ and D_σ is defined as

$$D_\sigma U_i = \partial_\sigma U_i - i\,C_\sigma U_i\,, \quad \text{with} \quad C_\sigma = -i\,U_i^*\partial_\sigma U_i\,. \tag{5.1.20}$$

If $U_3 = 0$, the Lagrangian (5.1.19) reduces to the standard one (5.1.1) with the change of variables

$$n_i = \mathbb{U}^\dagger\sigma_i\mathbb{U}\,, \qquad \mathbb{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}\,. \tag{5.1.21}$$

Alternatively, one can define the matrix $N_{ab} = 3\,U_a^*U_b - \delta_{ab}$, with the following properties

$$\mathrm{Tr}\,N = 0\,, \quad N^\dagger = N\,, \quad N^2 = N + 2\,, \tag{5.1.22}$$

and rewrite the Lagrangian (5.1.19) in a compact form:

$$\mathcal{L} = \frac{i}{18}\int_0^1 dz\ \mathrm{Tr}\left(N\left[\partial_z N, \partial_\tau N\right]\right) - \frac{1}{36}\,\mathrm{Tr}\left(\partial_\sigma N \partial_\sigma N\right) . \tag{5.1.23}$$

Now we can follow the same steps as in Section 5.1.3:

- We begin with the quartic near-plane-wave string action (5.1.10) in the gauge $a = 1$ with $Z^\mu = 0$ and Y^μ rearranged into the two complex variables

$$V = \frac{Y^0 + iY^1}{\sqrt{2}}\,, \qquad W = \frac{Y^2 + iY^3}{\sqrt{2}}\,;$$

- We introduce v and w as $V = v\,e^{-i\tau}$, $W = w\,e^{-i\tau}$, rescale the world-sheet variables

$$\tau \to \kappa\tau\,, \qquad \sigma \to \sqrt{\kappa}\sigma\,,$$

  and expand the action around $1/\kappa$ as in (5.1.14), keeping only the leading terms to select fast moving strings;

- We write the following relations extending to SU(3) the reasoning of [12]

$$\begin{aligned} v &= \sqrt{\frac{\pi}{\sqrt{\lambda\kappa}}}\,\varphi\left(1 + \frac{3}{4}\frac{2\pi}{\sqrt{\lambda\kappa}}\left(|\varphi|^2 + |\psi|^2\right)\right)\,, \\ w &= \sqrt{\frac{\pi}{\sqrt{\lambda\kappa}}}\,\psi\left(1 + \frac{3}{4}\frac{2\pi}{\sqrt{\lambda\kappa}}\left(|\varphi|^2 + |\psi|^2\right)\right)\,; \end{aligned}$$

- Finally we expand the action, written in terms of φ and ψ, for large $\sqrt{\lambda\kappa}$ and obtain

$$\begin{aligned} \mathcal{A} = \frac{1}{2}\int d\tau \int_0^{2\pi/\sqrt{\kappa}} d\sigma \Bigg[& i\left(\dot{\varphi}\varphi^* - \varphi\dot{\varphi}^* + \dot{\psi}\psi^* - \psi\dot{\psi}^*\right) - |\acute{\varphi}|^2 - |\acute{\psi}|^2 \\ & + \frac{2\pi i}{\sqrt{\lambda\kappa}}\left(|\varphi|^2 + |\psi|^2\right)\left(\dot{\varphi}\varphi^* - \varphi\dot{\varphi}^* + \dot{\psi}\psi^* - \psi\dot{\psi}^*\right) \\ & - \frac{2\pi}{\sqrt{\lambda\kappa}}\left(\acute{\varphi}\varphi^*\acute{\psi}\psi^* + \varphi\acute{\varphi}^*\psi\acute{\psi}^* + 2\acute{\varphi}\acute{\varphi}^*\varphi\varphi^* + 2\acute{\psi}\acute{\psi}^*\psi\psi^*\right) \\ & - \frac{2\pi}{\sqrt{\lambda\kappa}}\left(\acute{\varphi}\varphi^* + \psi\acute{\psi}^*\right)^2 - \frac{2\pi}{\sqrt{\lambda\kappa}}\left(\varphi\acute{\varphi}^* + \acute{\psi}\psi^*\right)^2 \Bigg] . \end{aligned} \tag{5.1.24}$$

To compare the SU(3) LL action (5.1.23) with (5.1.24), we parametrize N as

$$N = \sum_{s=1}^{8} n_s \mathcal{X}_s\,, \quad \mathcal{X} = \{E_1, F_1, H_1, E_2, F_2, H_2, E_{12}, F_{12}\} \tag{5.1.25}$$

where $[E_a, F_a] = H_a$ for $a = 1,2$ (Chevalley–Serre basis) and $E_{12} = [E_1, E_2]$, $F_{12} = [F_1, F_2]$. The constraints (5.1.22) for N imply that not all the n_s's are independent. In particular, the tracelessness is already satisfied by the choice (5.1.25), and the hermiticity property becomes $n_2 = n_1^*$, $n_5 = n_4^*$, $n_3 = n_3^*$, $n_6 = n_6^*$, $n_8 = -n_7^*$. Finally using $N^2 = N + 2$, and choosing as independent variables $n_1 \equiv \varphi$ and $n_5 \equiv \psi$, we obtain

$$\begin{gathered} n_3 = \frac{-\psi\psi^* + \varphi\varphi^*\left(1 \pm \sqrt{\Delta}\right)}{\varphi\varphi^* + \psi\psi^*}\,, \quad n_6 = \frac{\varphi\varphi^* - \psi\psi^*\left(1 \pm \sqrt{\Delta}\right)}{\varphi\varphi^* + \psi\psi^*}\,, \\ n_7 = \varphi\psi^* \frac{1 \pm \sqrt{\Delta}}{\varphi\varphi^* + \psi\psi^*}\,, \qquad \Delta = 1 - (\varphi\varphi^* + \psi\psi^*)\,. \end{gathered} \tag{5.1.26}$$

Let us mention also that the relation with the U_i's, except for an overall arbitrary phase, is

$$U_2 = \sqrt{1 - (\varphi\varphi^* + \psi\psi^*)}\,, \quad U_1 = \frac{\varphi^*}{U_2}\,, \quad U_3 = \frac{\psi^*}{U_2} \tag{5.1.27}$$

Now we can write the matrix N in terms of φ and ψ, rescale the spatial coordinate $\sigma \to \sigma\sqrt{\tilde{\lambda}}$, redefine the fields

$$\varphi \to \varphi\sqrt{\frac{L}{2\pi}}\,, \qquad \psi \to \psi\sqrt{\frac{L}{2\pi}}\,, \tag{5.1.28}$$

and finally expand the action for large L. Thus the SU(3) LL action (5.1.19) becomes

$$\begin{aligned} \mathcal{A} = \frac{1}{2}\int d\tau \int_0^{\frac{2\pi}{\sqrt{\tilde{\lambda}}}} d\sigma \Big[& i\left(\dot{\varphi}\varphi^* - \varphi\dot{\varphi}^* + \dot{\psi}\psi^* - \psi\dot{\psi}^*\right) - |\acute{\varphi}|^2 - |\acute{\psi}|^2 \\ & + \frac{2\pi i}{L}\left(|\varphi|^2 + |\psi|^2\right)\left(\dot{\varphi}\varphi^* - \varphi\dot{\varphi}^* + \dot{\psi}\psi^* - \psi\dot{\psi}^*\right) \\ & - \frac{2\pi}{L}\Big(\left(\acute{\varphi}\varphi^* + \psi\acute{\psi}^*\right)^2 + \left(\varphi\acute{\varphi}^* + \acute{\psi}\psi^*\right)^2 + \left(\acute{\varphi}\varphi^*\acute{\psi}\psi^* + \varphi\acute{\varphi}^*\psi\acute{\psi}^*\right) \\ & + 2\left(\acute{\varphi}\acute{\varphi}^*\varphi\varphi^* + \acute{\psi}\acute{\psi}^*\psi\psi^*\right)\Big)\Big]\,. \end{aligned} \tag{5.1.29}$$

which agrees with (5.1.24) for $L = \sqrt{\lambda}\kappa$, i.e. $\kappa = \tilde{\lambda}^{-1}$ as in the SU(2) case.

5.2 The generalized LL model

5.2.1 Perturbative Quantization

We will now turn to the description and quantization of the natural extension to higher orders of the standard LL action (5.1.1). In order to perform a perturbative expansion, following [37] and [38, 39], we will use the more convenient form of the action of Section 5.1.3. We will introduce the necessary ingredients to use the Feynman formalism to compute the two-dimensional S-matrix and form factors, which we will study in Section 5.3.

First we take the decompactification limit $L \to \infty$ while keeping λ fixed, since the quantities we are interested in are defined on the two-dimensional plane rather than on the cylinder. Then we rescale the spatial coordinate so that it has period L and also the time coordinate to simplify our expressions:

$$x = \frac{L\sigma}{2\pi}\,, \qquad t = \frac{\lambda\tau}{8\pi^2}\,. \tag{5.2.1}$$

Secondly, as was done in the Hamiltonian perturbation expansion [34], and used in computing the LL S-matrix [37, 38, 39], it is convenient to solve the constraint $\vec{n}\cdot\vec{n} = 1$ by introducing a complex field φ given by

$$\varphi = \frac{n^1 + in^2}{\sqrt{2+2n^3}}\,, \qquad |\varphi|^2 = \frac{1}{2}(1-n^3) \tag{5.2.2}$$

which is valid away from the point $n^3 = -1$. An advantage of this particular transformation is that it generates an action with a canonical kinetic term:

$$\begin{aligned} \mathcal{A} &= \int dt \int_0^L dx \left[\frac{i}{2}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*) - |\partial_x\varphi|^2 - V(\varphi)\right] \\ \text{where}\quad V(\varphi) &= \frac{2-|\varphi|^2}{4(1-|\varphi|^2)}\left[(\varphi^*\partial_x\varphi)^2 + c.c.\right] + \frac{|\varphi|^4|\partial_x\varphi|^2}{2(1-|\varphi|^2)}\,. \end{aligned} \tag{5.2.3}$$

The only dependence of the action on L is now in the range of integration and we can take the decompactification limit. As the potential is clearly quite non-linear in φ, we will consider quantizing this theory near the $\varphi = 0$, i.e. $n^3 = 1$, vacuum by expanding the action in small φ. In the gauge theory this vacuum is given by the BPS state $\mathrm{Tr}(Z^L)$ while in the string theory this corresponds to expanding about the BMN vacuum [3].

Due to the non-relativistic form of the quadratic action the field $\varphi(t,x)$ can be expanded in negative energy modes only[4]

$$\varphi(t,x) = \int \frac{dp}{2\pi}\, a_p e^{-i\omega_p t + ipx} \tag{5.2.4}$$

where the particle energy $\omega_p = p^2$ and the conjugate field is given by

$$\varphi^*(t,x) = \int \frac{dp}{2\pi}\, a_p^\dagger e^{i\omega_p t - ipx}\ . \tag{5.2.5}$$

The operators a_p and $a_p^\dagger$ are annihilation and creation operators for particles of momentum p and satisfy the usual commutation relations

$$[a_p, a_{p'}^\dagger] = 2\pi\delta(p - p')\ , \tag{5.2.6}$$

and the ground state is annihilated by the field operator $\varphi(t,x)\,|0\rangle = 0$. An essential feature of this model, emphasized in [37], is that due to the non-relativistic form of the kinetic term, the propagator has a single pole in ω in momentum space

$$\widetilde{D}(\omega,p) = \frac{i}{\omega - p^2 + i0} \tag{5.2.7}$$

and correspondingly in position space it is purely retarded

$$D(t,x) = \theta(t)\sqrt{\frac{\pi}{it}}\exp\left(\frac{ix^2}{4t}\right)\ . \tag{5.2.8}$$

This results in a number of important simplifications in the perturbative calculation, in particular the direction of the arrow on the propagator is essential as any diagram with a closed loop containing propagators whose arrows point in the same direction vanishes. This implies the non-renormalization of the vacuum energy and one-particle propagator and the very helpful fact that the two-body S-matrix is given by a sum of bubble diagrams; we will see that a similar simplification occurs also for form factors.

[4] Our normalization of the creation and annihilation operators is the same as [37, 39] and differs from [38] by $\sqrt{2\pi}$.

5.2.2 The action of the generalized LL model

We will now consider higher-loop terms in the dilatation operator expansion in λ: they correspond to spin-chain Hamiltonians with interactions at longer range and can be related to a generalized LL action with higher-derivative terms (with increasingly higher power of λ). On the string theory side, we can repeat the process of Section 5.1.3 while keeping higher-order terms and thus compare the two generalized LL actions. It has been shown that they agree up to $\mathcal{O}(\lambda^2)$ [25], but not at higher orders [33, 32, 34, 35].

The most general form of the LL action (5.1.9) up to order $\mathcal{O}(\lambda^3)$, including all the derivative terms allowed by the symmetries, is [32]

$$\begin{aligned}\mathcal{A} = \frac{L}{2\pi}\int d\tau \int_0^{2\pi} d\sigma \Big[C_\tau(\vec{n}) - \frac{\tilde{\lambda}}{8} b_0 (\partial_\sigma \vec{n})^2 - \frac{\tilde{\lambda}^2}{32}(b_1(\partial_\sigma^2 \vec{n})^2 + b_2(\partial_\sigma \vec{n})^4) \\ - \frac{\tilde{\lambda}^3}{64}\left(b_3(\partial_\sigma^3 \vec{n})^2 + b_4(\partial_\sigma \vec{n})^2(\partial_\sigma^2 \vec{n})^2 + b_5(\partial_\sigma \vec{n}\cdot \partial_\sigma^2 \vec{n})^2 + b_6(\partial_\sigma \vec{n})^6\right)\Big] . \quad (5.2.9)\end{aligned}$$

Let us note that the kinetic term $C_\tau(\vec{n})$ does not receive any corrections. The action (5.2.9) works as an ansatz for the two specific LL actions we are interested in, which can be then determined by fixing the coefficients $b_1, \ldots, b_6$. This can be done by computing the energies of specific solutions in terms of the b's and comparing them with known gauge theory and string theory results. As mentioned before, we find the same result up to $\mathcal{O}(\lambda^2)$

$$b_0 = 1\,, \quad b_1 = -1\,, \quad b_2 = \frac{3}{4}\,. \qquad (5.2.10)$$

At the next order, the value of b_3 is fixed to be 1 by demanding BMN-like scaling for the magnon energy and we have agreement with the known gauge theory anomalous dimensions to $\mathcal{O}(\lambda^3)$ only if

$$b_4 = -\frac{7}{4}\,, \quad b_5 = -\frac{23}{2}\,, \quad b_6 = \frac{3}{4}\,. \qquad (5.2.11)$$

However, the values that match with the string results are

$$b_4 = -\frac{7}{4}\,, \quad b_5 = -\frac{25}{2}\,, \quad b_6 = \frac{13}{16}\,, \qquad (5.2.12)$$

that is b_5 and b_6, which appear at $\mathcal{O}(\lambda^3)$, are different for the two LL models derived from the gauge theory and string theory respectively. In general, it is interesting to check for which values of the b_i's the theory is integrable, and one way to do this is to verify that we have factorized scattering as we will see in Section 5.2.4. At order g^2 we find that there are no constraints on b_1 and b_2, while at order g^4 the vanishing of the generic $3 \to 3$ S-matrix requires cancellation between terms involving different b_i's, which can be seen to be satisfied in both the gauge case (5.2.11) and the string case (5.2.12).

We can rewrite the action (5.2.9) in a more convenient way for future calculations as explained in the previous section. We rescale the world-sheet coordinates as in (5.2.1), introduce the complex scalar field φ, define the parameter $g = \sqrt{\lambda}/4\pi$ and expand in powers of the field to find

$$\mathcal{A} = \int d^2x \; \Big\{ \frac{i}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) - b_0 |\partial_x \varphi|^2 - g^2 b_1 |\partial_x^2 \varphi|^2 - 2g^4 b_3 |\partial_x^3 \varphi|^2 \\ - V_{\text{quartic}} - V_{\text{sextic}} + \cdots \Big\} \tag{5.2.13}$$

where the terms in the potential are to leading order

$$\begin{aligned} V_{\text{quartic}} &= \frac{b_0}{2}(\varphi^{*2}(\partial_x\varphi)^2 + \varphi^2(\partial_x\varphi^*)^2) + \mathcal{O}(\lambda) \\ V_{\text{sextic}} &= -\frac{b_0}{4}\varphi\varphi^*(\varphi^*\partial_x\varphi + \varphi\partial_x\varphi^*)^2 + \mathcal{O}(\lambda) \end{aligned} \tag{5.2.14}$$

and we give the higher order terms in App. F. The form of the potential is not identical to that of [38], yet the difference is due to total derivative terms and, as we will check below, gives rise to the same S-matrix. As at leading order, the spin-chain length now only appears in the range of integration and so we can again take the decompactification limit. However, the rescaling of the time coordinate does not remove the dependence on λ (now g), which now appears explicitly even at quadratic order. This results in a modification of the dispersion relation in addition to the new, higher derivative, interaction terms.

5.2.3 Feynman rules

While the quadratic higher-order-in-g terms result in a corrected dispersion relation

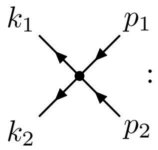
$$\omega(p) = b_0 p^2 + g^2 b_1 p^4 + 2g^4 b_3 p^6 \tag{5.2.15}$$

the corresponding propagator still only has a single pole in ω

$$\tilde{D}(\omega, p) = \frac{i}{\omega - b_0 p^2 - g^2 b_1 p^4 - g^4 b_3 p^6 + i0} \tag{5.2.16}$$

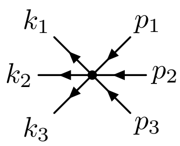
and so remains purely retarded. This ensures that we have the same non-renormalization theorems and simplifications in the diagrammatic expansion as in the leading-order LL model. For example the quantum S-matrix is still simply given by a sum over bubble diagrams, [38], but now with more complicated vertices, as we will see in the following.

The quartic vertex is



$$\begin{aligned} &: 2ib_0(k_1k_2 + p_1p_2) - 2ig^2\Big[4(3b_1 + 2b_2)p_1p_2k_1k_2 \\ &\quad +b_1\left(k_1^2k_2^2 + p_1^2p_2^2 - 2(p_1+p_2)(k_1+k_2)(k_1k_2 + p_1p_2)\right)\Big] \\ &-2ig^4\Big[(2b_4 + b_5)((k_1+k_2)(p_1+p_2) - 2b_5(k_1k_2 + p_1p_2))k_1k_2p_1p_2 \\ &\quad +b_3\left(3(k_1k_2p_1p_2(2(k_1^2 + k_2^2 + p_1^2 + p_2^2) + 6(k_1+k_2)(p_1+p_2)\right. \\ &\quad -12(k_1k_2 + p_1p_2)) + (k_1^2k_2^2 + p_1^2p_2^2)(p_1+p_2)(k_1+k_2) \\ &\quad -(k_1k_2 + p_1p_2)(p_1^2 + p_2^2)(k_1^2 + k_2^2) - (k_2^3 + k_1^3)(p_1+p_2)p_1p_2 \\ &\quad \left. -(p_2^3 + p_2^3)(k_1+k_2)k_1k_2) - 2k_1^3k_2^3 - 2p_1^3p_2^3\right)\Big] \end{aligned} \tag{5.2.17}$$

where there is understood to be an overall momentum conservation delta-function imposing $p_1 + p_2 = k_1 + k_2$. Finally we will also make use of the sextic vertex to calculate the three-particle S-matrix and form factors:



$$\begin{aligned} &: ib_0\Big[3(k_1k_2 + k_1k_3 + k_2k_3) + 3(p_1p_2 + p_1p_3 + p_2p_3) \\ &\quad -2(k_1 + k_2 + k_3)(p_1 + p_2 + p_3)\Big] + \mathcal{O}(g^2) \end{aligned} \tag{5.2.18}$$

where we have only written the leading term in $\mathcal{O}(g^2)$. The subleading terms can be extracted straightforwardly from the sextic potential (F.1).

5.2.4 The Landau-Lifshitz S-matrix

The calculation of the quantum S-matrix from the quartic vertex was carried out for the leading-order LL model in [37] and was done for the generalized LL model in [38]. Here we briefly recap this calculation as it both provides a check on the form of our action and is closely related to that of form factors. The two-body S-matrix is defined by

$$\langle k_1 k_2| \mathcal{S} |p_1 p_2\rangle = \langle k_1 k_2| \mathrm{Texp}\left(-i\int d^2x\, H_I\right)|p_1 p_2\rangle \tag{5.2.19}$$

where H_I is the interaction Hamiltonian, the asymptotic states are given simply by

$$|p_1 p_2\rangle = a^\dagger_{p_1} a^\dagger_{p_2}|0\rangle \ , \quad \langle k_1 k_2| = \langle 0| a_{k_1} a_{k_2} \tag{5.2.20}$$

and in the perturbative expansion we only keep amputated, connected terms. Due to spatial- and time-translational invariance of the action the S-matrix elements (5.2.19) naturally come with overall energy $p_i^0 = \omega(p_i)$, momentum $p_i^1 = p_i$, and delta-functions

$$(2\pi)^2 \delta^{(2)}(p_1^\mu + p_2^\mu - k_1^\mu - k_2^\mu) = \mathcal{J}\delta_+(p_1, p_2, k_1, k_2)\,, \tag{5.2.21}$$

where the Jacobian factor is $\mathcal{J} = (\partial\omega(p_1)/\partial p_1 - \partial\omega(p_2)/\partial p_2)^{-1}$ and

$$\delta_+(p_1, p_2, k_1, k_2) = (2\pi)^2\left(\delta(p_1 - k_1)\delta(p_2 - k_2) + \delta(p_1 - k_2)\delta(p_2 - k_1)\right)\ .$$

We can also define the T-matrix by $S(p_1, p_2) = 1 + T(p_1, p_2)$, where

$$\langle k_1 k_2| \mathcal{S} |p_1 p_2\rangle = S(p_1, p_2)\delta_+(p_1, p_2, k_1, k_2)\ , \tag{5.2.22}$$

to include the Jacobian factor.

The action (5.2.13) has an implicit small parameter, from the expansion in powers

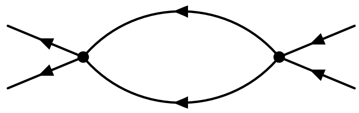


FIGURE 5.1: One-loop bubble diagram contribution to two-body S-matrix.

of the fields, with which we can organize a diagrammatic expansion. For the two-body S-matrix the leading term is the tree-level quartic contribution, which gives,

$$\begin{aligned} T^{(0)}(p_1, p_2) =& \frac{2ip_1p_2}{p_1-p_2} - \frac{2ig^2}{b_0}\frac{(5b_1+4b_2)p_1^2p_2^2}{p_1-p_2} + \frac{2ig^4p_1^2p_2^2}{b_0^2(p_1-p_2)}\times \\ & \times \left[10b_1^2(p_1^2+p_1p_2+p_2^2) + 8b_1b_2(p_1^2+p_1p_2+p_2^2) \right. \\ & \left. - b_0\left(b_5(p_1-p_2)^2 + 2b_4(p_1+p_2)^2 + 7b_3(3p_1^2-2p_1p_2+3p_2^2)\right)\right] . \end{aligned} \tag{5.2.23}$$

Using the appropriate string values for the coefficients we obtain the LL limit of the known string and spin-chain S-matrices to this order in g [38].

As was shown in [37], due to the nature of the LL propagator only bubble diagrams contribute to the S-matrix calculation, and these can be calculated by simple contour integration. These observations remain true in the generalized model [38]. Considering higher orders in g results in higher powers of momenta in both the propagators and numerators. However there are no additional powers of the energy, ω, and so the contour argument goes through. We use the full vertex (5.2.17) and propagator (5.2.16) before expanding in g to evaluate the diagram in Fig. 5.1. The resulting loop integral is naively UV divergent with power-like divergences: these can be treated by use of dimensional regularization, which for practical purposes amounts to essentially ignoring them [37]. To order g^4, and using values for b_i's that reproduce the tree-level BDS S-matrix we find, as in [38],

$$T^{(1)}_{\text{gauge}} = -\frac{2p_1^2p_2^2}{(p_1-p_2)^2}(1+4g^2p_1p_2 - 4g^4p_1p_2(p_1-p_2)^2) \tag{5.2.24}$$

which as we will see later agrees with the one-loop BDS result to $\mathcal{O}(g^4)$. The corresponding result with the string theory coefficients is quite similar but differs at $\mathcal{O}(g^4)$,

$$T^{(1)}_{\text{string}} = -\frac{2p_1^2p_2^2}{(p_1-p_2)^2}(1+4g^2p_1p_2 - 2g^4p_1p_2(p_1-p_2)^2) . \tag{5.2.25}$$

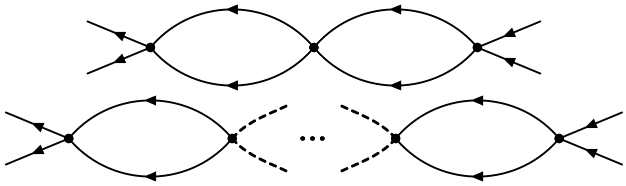


FIGURE 5.2: Higher loop bubble diagrams for the two-body S-matrix.

FIGURE 5.3: Contact and dog-diagram contribution to three-body S-matrix.

As in the leading-order calculation this can be extended to two- and higher-loop by evaluating higher loop bubble diagrams Fig. 5.2. Each bubble can be essentially evaluated independently and so the result is a geometric series which can be easily resummed.

As the theory is known to be integrable we of course expect the generalized LL model to exhibit factorized scattering. This implies that the three-body S-matrix is only non-vanishing when the out-going momenta are a permutation of the incoming momenta. For the LL model and its generalization, as there are sextic terms in the potential, such a factorization is not immediately apparent and results from a non-trivial cancellation between diagrams. Factorization of scattering at one-loop for the standard LL-model was shown in [39] and was further studied in [40]. To check tree-level factorization for the generalized LL-model to $\mathcal{O}(g^4)$ we computed the $3 \to 3$ scattering by evaluating the diagrams Fig. 5.3 and then checked numerically that for generic external momenta the scattering vanished to order g^4. The result is that there are no constraints on b_1 and b_2 (order g^2), while the vanishing of the generic $3 \to 3$ S-matrix requires cancellation between terms involving different b_i's (order g^4). For example fixing b_0 through b_4 as above, we find the condition $1 - 2b_5 - 32b_6 = 0$, which is naturally satisfied in both the gauge case (5.2.11) and the string case (5.2.11). When the set of outgoing momenta is a permutation of the incoming momenta specific internal propagators in diagrams of the form Fig. 5.3 will go on-shell and so there are additional non-vanishing contributions from delta-functions arising from using the principle value prescription

$$\frac{1}{z+i0} = -i\delta(z) + \text{P.V.}\left[\frac{1}{z}\right] . \tag{5.2.26}$$

We now turn to the analogous computations for form factors.

5.3 Diagonal form factors in the Landau–Lifshitz model

We will focus on the diagonal form factors introduced in Section 4.1.2, which we want to compute in the case of the Landau–Lifshitz model. In other words, we will consider the form factors of the operator $\mathcal{O}$ in terms of the momenta p_i

$$F_D^{\mathcal{O}}(p_1, \ldots, p_n) = {}^{\text{out}}\langle p_1 + \epsilon_1, \ldots, p_n + \epsilon_n | \widetilde{\mathcal{O}}(\omega_q, q) | p_1, \ldots, p_n \rangle^{\text{in}} , \tag{5.3.1}$$

where we used the Fourier transform of the operator

$$\mathcal{O}(t,x) = \int \frac{d^2 q}{(2\pi)^2} \, \widetilde{\mathcal{O}}(\omega_q, q) e^{iqx - i\omega_q t} . \tag{5.3.2}$$

Then, as mentioned in Sec. 4.1.2, in order to obtain the diagonal form factors, we will need to take the limit of (5.3.1) following the symmetric or connected prescription. The conservation of overall energy and momentum implies

$$\omega_q = \sum_{i=1}^{N} \omega(p_i + \epsilon_i) - \omega(p_i) \quad \text{and} \quad q = \sum_{i=1}^{N} \epsilon_i . \tag{5.3.3}$$

Thus we have that $q \to 0$ and $\omega_q \to 0$ in the diagonal limit, since the terms with ϵ_i will disappear regardless of the choice of the prescription.

Let us also note that we can not write (5.3.1) in terms of the usual form factors (4.1.5) since we do not have a way of defining crossing in the LL model. We will drop the subscript D in the following when there is no ambiguity.

5.3.1 $|\varphi|^2$-Operator

We will begin with the operator

$$\Phi_1 = |\varphi|^2 , \tag{5.3.4}$$

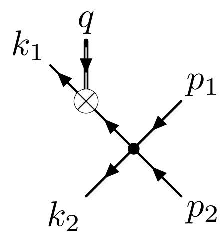


FIGURE 5.4: Tree-level contribution to the two-particle form factor.

which corresponds to an operator with one impurity in the spin-chain language (see Sec. 5.3.4). It is apparent that the vacuum expectation, or zero-particle form factor, is vanishing $F^{\Phi_1}(\emptyset) = 0$, and the one-particle form factors are essentially trivial: they receive no loop corrections, and with our normalizations are given by

$$F^{\Phi_1}(p) = 1 \tag{5.3.5}$$

which corresponds to a definition of the external states without normalization factors involving the particle energy. Both of these facts follow from the vanishing of loop diagrams with arrows forming a closed loop and are correspondingly true in both the LL model and the generalization to higher order in g.

For the two-particle form factors $F^{|\varphi|^2}(p_1, p_2)$, however, we have non-trivial results. Starting with the tree level, we must evaluate the diagram in Fig. 5.4 which at $\mathcal{O}(g^0)$ gives

$$\frac{-2b_0\left[(k_1-q)k_2+p_1p_2\right]}{\omega_{k_1}-\omega_q-b_0(k_1-q)^2+i0} \tag{5.3.6}$$

which is clearly singular in the diagonal limit. However after summing over the diagrams with the insertion on the other legs the limit becomes regular and one finds

$$F^{(0)\Phi_1}_{s,c}(p_1, p_2)\Big|_{\mathcal{O}(g^0)} \quad = \quad -2\kappa_{s,c}\frac{p_1^2+p_2^2}{(p_1-p_2)^2}\ . \tag{5.3.7}$$

This result is the same regardless of whether it is calculated using the symmetric or the connected prescription up to the overall normalizations. In the connected prescription one finds $\kappa_c = 1$ while in the symmetric prescription it is $\kappa_s = 2$. Given the corrected propagator (5.2.16) and vertex (5.2.17) for the generalized LL-model we can extend this to higher orders in g^2:

$$\begin{aligned} F_{s,c}^{(0)\Phi_1}(p_1,p_2) &= 2\kappa_{s,c}\Big[-\frac{p_1^2+p_2^2}{(p_1-p_2)^2}+g^2\frac{2(4b_2+5b_1)p_1p_2(p_1^2-p_1p_2+p_2^2)}{b_0(p_1-p_2)^2} \\ &+g^4\frac{2p_1p_2}{b_0^2(p_1-p_2)^2}\big[(-10b_1-8b_1b_2+b_0(21b_3+2b_4+b_5))(p_1^4+p_2^4) \\ &\quad+(5b_1^2+4b_1b_2+b_0(-63b_3+2b_4-5b_5))(p_1^3p_2+p_1p_2^3) \\ &\quad+(-20b_1^2-16b_1b_2+8b_0(14b_3+b_5))p_1^2p_2^2\big]\Big] \end{aligned} \tag{5.3.8}$$

where the values of the coefficients for the different prescriptions $\kappa_{s,c}$ are as above. Using the specific choices for the coefficients b_i we find that in the connected prescription

$$F_c^{(0)\Phi_1}(p_1,p_2) = -\frac{2(p_1^2+p_2^2)}{(p_1-p_2)^2}-\frac{8g^2p_1p_2(p_1^2-p_1p_2+p_2^2)}{(p_1-p_2)^2} \tag{5.3.9}$$

to order $\mathcal{O}(g^2)$ for both the string theory and BDS gauge theory cases while

$$F_c^{(0)\Phi_1}(p_1,p_2)\Big|_{\mathcal{O}(g^4)} = \frac{4p_1p_2}{(p_1-p_2)^2}\times\begin{cases}(p_1^4-2p_1^3p_2+4p_1^2p_2^2-2p_1p_2^2+p_2^4),\\ (2p_1^4-7p_1^3p_2+12p_1^2p_2^2-7p_1p_2^2+2p_2^4)\,,\end{cases}$$

where the first line is the string case and the second the gauge case, reflecting the three-loop difference at the level of the form factor.

One-loop result In order to compute the one-loop results we must consider the diagrams shown in Fig. 5.5. The procedure for evaluating these diagrams is essentially identical to that used in the case of the S-matrix, reviewed in the previous section. We regularize any power-like divergences by dimensional regularization and evaluate the integrals by using the residue theorem. In simplifying our formulas we explicitly assume that $p_1 > p_2$. The same assumption will be made in all loop-calculations that we perform for form factors. The choice of the prescription for taking the diagonal limit superficially appears to make a more significant difference at loop level as there are different contributions from individual diagrams. Using the symmetric prescription we find that diagrams with the form factor inserted on external legs and internal legs contribute equally at order $\mathcal{O}(g^0)$, while at $\mathcal{O}(g^2)$ the sum of the two

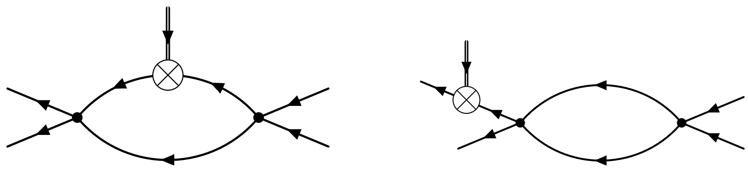


FIGURE 5.5: One-loop diagrams for the two-particle form factor.

contributions cancels part of the denominator, giving

$$F^{(1)\Phi_1}_{s,c}(p_1,p_2)\Big|_{\mathcal{O}(g^2)} = -\frac{4\kappa_{s,c} i p_1 p_2 (p_1^2+p_2^2)}{(p_1-p_2)^3} + \frac{8\kappa_{s,c} g^2 i p_1 p_2 (p_1^2+p_2^2)}{(p_1-p_2)} \,. \tag{5.3.10}$$

In the connected prescription the diagrams with the insertion on the external legs do not contribute at all but the diagrams with the insertion on the internal legs contributes the same as in the symmetric case. Hence we find that the connected scheme gives half of symmetric result just as at tree-level.

All-loop result To extend these results to all-loop we need only to consider chains of bubble diagrams, as noted above. There are again essentially two classes of diagrams: those with the insertion on the external leg and those with the insertion on an internal loop leg. For each bubble we can perform the loop integration by evaluating the residues. For the symmetric prescription we find equal contributions from the insertions on the external legs and from the n internal insertions with the final result that at n-loops we have,

$$F^{(n)\Phi_1}_{s}(p_1,p_2)\Big|_{\mathcal{O}(g^0)} = 4(n+1)\frac{i^{n+2} p_1^n p_2^n (p_1^2+p_2^2)}{(p_1-p_2)^{n+2}} \,. \tag{5.3.11}$$

Alternatively, for the connected prescription we find for the contribution with the insertion on the external legs

$$2(n-1)\frac{i^{n} p_1^n p_2^n (p_1^2+p_2^2)}{(p_1-p_2)^{n+2}} \,. \tag{5.3.12}$$

Taking the connected diagonal limit for the diagrams with internal insertions is slightly complicated but it can be numerically checked that it gives

$$4n\frac{i^{n+2} p_1^n p_2^n (p_1^2+p_2^2)}{(p_1-p_2)^{n+2}} \tag{5.3.13}$$

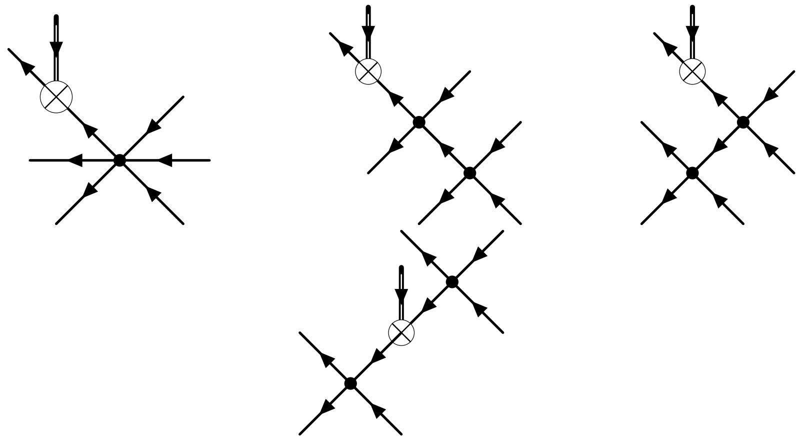


FIGURE 5.6: Diagrams for the three-particle form factor.

so that

$$F_c^{(n)\Phi_1}(p_1,p_2)\Big|_{\mathcal{O}(g^0)} = 2(n+1)\frac{i^{n+2}p_1^n p_2^n(p_1^2+p_2^2)}{(p_1-p_2)^{n+2}}\,, \tag{5.3.14}$$

which is again simply half the symmetric prescription. In both cases we can sum up the contributions from each loop order to give the all-loop quantum form factor:

$$F_{s,c}^{\Phi_1}(p_1,p_2)\Big|_{\mathcal{O}(g^0)} = -\frac{2\kappa_{s,c}(p_1^2+p_2^2)}{(p_1-p_2)^2}\frac{1}{\left[1-ip_1p_2/(p_1-p_2)\right]^2}\,. \tag{5.3.15}$$

Three-particle form factor It is straightforward, if somewhat cumbersome, to extend to higher numbers of particles in the external states. In this case we must include the contributions from the graphs in Fig. 5.6. From a perturbative perspective this is of interest as it includes contributions from the sextic vertex. Furthermore in this case the dependence on the prescription for taking the diagonal limit is more pronounced. It is convenient to define the functions of external momenta

$$p_{ij} = p_i - p_j\,, \quad \chi_{i,j,k} = p_ip_j - p_jp_k + p_kp_i \tag{5.3.16}$$

such that for the connected prescription the result can be written as

$$F_c^{\Phi_1}(p_1,p_2,p_3)\Big|_{\mathcal{O}(g^0)} = \frac{4}{p_{12}^2p_{13}^2p_{23}^2}\Big[p_1^4(p_2^2+p_3^2)+p_2^4(p_1^2+p_3^2)+p_3^4(p_1^2+p_2^2)$$
$$-2p_1p_2p_3(p_1\chi_{1,2,3}+p_2\chi_{2,3,1}+p_3\chi_{3,1,2})\Big]\,,$$

while the symmetric prescription gives

$$F_s^{\Phi_1}(p_1, p_2, p_3)\Big|_{\mathcal{O}(g^0)} = \\ \frac{24}{p_{12}^2 p_{13}^2 p_{23}^2}\Big[(p_1^2 p_2^2 + p_3^2 p_1^2 + p_2^2 p_3^2)(p_1^2 + p_2^2 + p_3^2 - p_1 p_2 - p_3 p_1 - p_2 p_3)\Big] .$$

Further Quadratic Operators It is possible to consider other quadratic-in-field operators by adding derivatives. One such operator which will be relevant to our later considerations is

$$\Phi_2 = \varphi^* \acute{\varphi} - \varphi \acute{\varphi}^* \tag{5.3.17}$$

for which we can calculate $F_{c,s}^{\Phi_2}(\emptyset) = 0$, $F_{c,s}^{\Phi_2}(p_1) = 2ip_1$ and

$$F_{c,s}^{\Phi_2}(p_1, p_2)\Big|_{\mathcal{O}(g^0)} = -\frac{4i\kappa_{c,s} p_1 p_2}{(p_1 - p_2)^2} \frac{(p_1 + p_2)}{[1 - ip_1 p_2/(p_1 - p_2)]^2} . \tag{5.3.18}$$

One feature of this calculation is that as the insertion operator involves derivatives when it is inserted inside a loop, as in Fig. 5.5, it gives rise to additional numerator factors. In this case care must be taken in the labelling of the loop momenta passing through the insertion. In particular we must sum over contributions corresponding to inserting the operator on the top line with loop momentum ℓ and the bottom line with momentum $-\ell + p_1 + p_2$ as these are not equal.

There are of course many other possible operators one could consider. If there were two derivatives such terms could act as possible higher order corrections to the $|\varphi|^2$ operator, for example

$$\mathcal{O}_{\text{corr}} = |\varphi|^2 + g^2\Big[\alpha_1(\partial_x^2 \bar{\varphi}^*)\varphi + \alpha_2 \bar{\varphi}^*(\partial_x^2 \varphi) + \alpha_3(\partial_x \bar{\varphi}^*)(\partial_x \varphi)\Big] . \tag{5.3.19}$$

Of course as the correction terms are related by total derivatives, for diagonal form factors we would expect the three correction terms to give the same contributions and so there is only one parameter at this order. As we will see, such corrections are likely to play a role in understanding the relation to gauge theory structure constants.

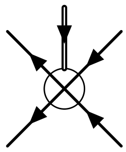


FIGURE 5.7: Two-particle tree-level form factor diagram.

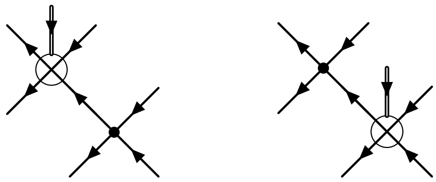


FIGURE 5.8: Three-particle tree-level form factor diagrams.

5.3.2 $|\varphi|^4$-Operator

We now turn to the $|\varphi|^4$ operator which will correspond to an operator with two impurities in the spin-chain language. The zero-particle form factor is again obviously vanishing as is the one-particle diagonal form factor. The two particle diagonal form factor at tree-level is simply

$$F^{(0)\Phi_3}(p_1, p_2)\Big|_{\mathcal{O}(g^0)} = 4 \tag{5.3.20}$$

corresponding to Fig. 5.7. The loop corrections in the LL-model are given by essentially the same diagrams as in the S-matrix calculation, Fig. 5.2, with one of the interaction vertices replaced by the operator insertion. These diagrams can again be resummed to give

$$F^{\Phi_3}(p_1, p_2)\Big|_{\mathcal{O}(g^0)} = \frac{4}{[1 - ip_1p_2/(p_1 - p_2)]^2} \tag{5.3.21}$$

where the result does not depend on the prescription used in taking the diagonal limit.

Three-particle form factor For the $|\varphi|^4$ operator it is particularly straightforward to extend to three-particles by evaluating the diagrams shown in Fig. 5.8. However now the result does depend on the presciption used to define the diagonal limit in much the same fashion as the two-particle form factors of $|\varphi|^2$

$$F_{s,c}^{0,\Phi_3}(p_1,p_2,p_3)\Big|_{\mathcal{O}(g^0)} = 8(\kappa_{s,c}-1) - \frac{8\kappa_{s,c}}{p_{12}^2 p_{23}^2 p_{31}^2}\left[p_{31}^4\chi_{2,3,1} + p_{23}^4\chi_{1,2,3} + p_{12}^4\chi_{3,1,2}\right] ,$$

where $\kappa_{s,c}$ is the prescription dependent constant we introduced above.

These results can be extended to the generalized LL-model by including the higher-loop gauge theory corrections to the interaction vertex and propagator. One finds, at tree-level in the two-dimensional theory, to $\mathcal{O}(g^2)$ that

$$F_{s,c}^{0,\Phi_3}(p_1,p_2,p_3)\Big|_{\mathcal{O}(g^2)} = \frac{8(4b_2+5b_1)\kappa_{s,c}}{b_0\ p_{12}^2 p_{23}^2 p_{31}^2}\left[p_2^2p_{31}^2\chi_{1,2,3} + p_1^2p_{23}^2\chi_{3,1,2} + p_3^2p_{12}^2\chi_{2,3,1}\right] .$$

The result at $\mathcal{O}(g^4)$ can be computed similarly ($p_123 = p_1+p_2+p_3$)

$$\begin{aligned}
F_{s,c}^{0,\Phi_3}(p_1,p_2,p_3)\Big|_{\mathcal{O}(g^4)} &= \frac{2-\kappa_{s,c}}{2b_0^2}(5b_1)^2 + 20b_1b_2 - b_0(35b_3+6b_4+b_5))p_1p_2p_3p_{123} \\
+ \frac{16\kappa_{s,c}}{b_0^2p_{12}^2p_{23}^2p_{31}^2}&\Big[(10b_1^2+8b_1b_2-b_0(21b_3+2b_4+b_5))p_1^7(p_2^3-p_2p_3(p_2+p_3)+p_3^3) \\
&- (5b_1^2+4b_1b_2+b_0(-63b_3+2b_4-5b_5))p_1^6(p_2^4+p_3^4) \\
&- \frac{1}{2}(85b_1^2+68b_1b_2+b_0(49b_3-30b_4+11b_5))p_1^6(p_2^3p_3+p_3^3p_2) \\
&+ (95b_1^2+76b_1b_2+b_0(-77b_3-26b_4+b_5))p_1^6p_2^2p_3^2 \\
&+ 4(5b_1^2+4b_1b_2-2b_0(14b_3+b_5))p_1^5p_2^5 \\
&- \frac{1}{2}(15b_1^2+12b_1b_2+b_0(-119b_3+2b_4-9b_5))p_1^5(p_2^4p_3+p_3^4p_2) \\
&+ \frac{1}{2}(-25b_1^2-20b_1b_2+b_0(105b_3+2b_4+7b_5))p_1^5(p_2^3p_3^2+p_3^3p_2^2) \\
&+ (35b_1^2+28b_1b_2-b_0(161b_3+2b_4+11b_5))p_1^4p_2^4p_3^2 \\
&+ (-15b_1^2-12b_1b_2+b_0(49b_3+2b_4+3b_5))p_1^4p_2^3p_3^3 \\
&+ \text{cylic permutations of particle indices}\Big] . \qquad (5.3.22)
\end{aligned}$$

Further Quartic Operators Just as for the quadratic operators we can consider additional operators by distributing derivatives across the fields. There is now an even greater number of possibilities, though we will consider here only the operator

$$\Phi_4 = |\varphi|^2(\varphi^*\acute{\varphi} - \varphi\acute{\varphi}^*) . \qquad (5.3.23)$$

It can be seen immediately that at tree-level the two-particle diagonal form factor simply acquires an additional factor of $i(p_1 + p_2)$. This is actually the case also at any loop order since when inserted in a chain of bubbles (as in Fig. 5.2 but with the operator replacing an interaction vertex) the loop-momenta from the vertex contribution always cancel and the additional momentum factor can be pulled out. Hence we have

$$F^{\Phi_4}(p_1, p_2)\Big|_{\mathcal{O}(g^0)} = \frac{4i(p_1 + p_2)}{[1 - ip_1p_2/(p_1 - p_2)]^2} \,. \tag{5.3.24}$$

5.3.3 The spin-chain S-matrix and its LL limit

The Landau-Lifshitz theory can be found by considering the low-energy limit of the spin-chain which can be done either at the level of the action or at the level of computed quantities. We have seen the former limit in Section 5.1.2, and we will discuss here the latter for the S-matrix and form factors. This can then be repeated for the higher-loop results where the spin-chain Hamiltonian is significantly more complicated or even unknown.

Let us briefly recall the basic quantities describing the $\mathrm{XXX}_{1/2}$ spin-chain introduced in Section 3.2.1. We saw in Sec. 3.1.5 that, in a $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM, the one-loop term in the expansion in $g = \sqrt{\lambda}/4\pi$ of the dilatation operator is the (rescaled) nearest-neighbour Hamiltonian (3.2.4)

$$H = \frac{\lambda}{16\pi^2} \sum_{n=1}^{L} \left(1 - \sigma_n^i \sigma_{n+1}^i\right) \,. \tag{5.3.25}$$

Its eigenstates $|\Psi_M\rangle$ are characterized by their magnon-number M, i.e. the number of excitation from the ground state $|0\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$

$$|\Psi_M\rangle = \sum_{1\leq n_1 < \cdots < n_M \leq L} \psi(n_1, \ldots, n_M)\, |n_1, \ldots, n_M\rangle \,, \tag{5.3.26}$$

where $|n_1, n_2, \ldots\rangle = |\uparrow\downarrow_{n1}\uparrow \cdots \downarrow_{n2} \cdots \uparrow\rangle$ (spin flipped in the $n_1, n_2, \ldots, n_M$ positions). The energies are given as a sum over single-magnon energies, e.g. in terms of the rapidities u_k

$$E_{\Psi,M} = \sum_{k=1}^{M} \varepsilon(p_k) = \frac{1}{2} \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4} \,, \qquad u(p) = \frac{1}{2} \cot\left(\frac{p}{2}\right) \,, \tag{5.3.27}$$

where we have defined $\varepsilon(p) \equiv E_{\Psi,1}(u(p))$ for convenience. The spin-chain S-matrix which appears in the multi-particle wavefunctions $\psi(n_1, \ldots, n_M)$ as two-magnons are exchanged is given by

$$S(p_1, p_2) = \frac{u(p_2) - u(p_1) + i}{u(p_2) - u(p_1) - i}\,. \tag{5.3.28}$$

To take the LL-limit we rescale the magnon energy, $\varepsilon \to \kappa^2\varepsilon$, and consider the small κ limit. For the rapidity variable we have that at leading order

$$u(p) \simeq \frac{1}{\kappa\sqrt{\varepsilon}} = \frac{1}{p} \tag{5.3.29}$$

where the momentum is given by $p = \kappa\sqrt{\varepsilon}$ and hence the S-matrix is

$$S(p_1, p_2) = \frac{(p_2)^{-1} - (p_1)^{-1} + i}{(p_2)^{-1} - (p_1)^{-1} - i}\,. \tag{5.3.30}$$

This is the quantum S-matrix for the LL-model and written in this fashion there is no small parameter. It can be perturbatively computed by resumming all loop orders in the LL model. Note that though the momenta are not taken to be small it does not reproduce the complete spin-chain S-matrix. To extract just the tree-level result we reintroduce the small parameter by rescaling the momenta $p_i \to \gamma p_i$ and then take the small γ limit so that

$$S(p_1, p_2) = 1 + \sum_{i=0}^{\infty} \gamma^{i+1} T^{(i)}(p_1, p_2) \tag{5.3.31}$$

with

$$T^{(0)}(p_1, p_2) = \frac{2ip_1p_2}{p_1 - p_2}\,, \tag{5.3.32}$$

which is the same as the leading term in the tree-level T-matrix computed perturbatively (5.2.23).

The extension to higher-loops in $g = \sqrt{\lambda}/4\pi$ can be described in terms of the generalized $u(p)$ functions

$$u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1 + 16g^2\sin^2\frac{p}{2}}\,. \tag{5.3.33}$$

The all-order magnon energy is given by

$$2g^2\varepsilon(p) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} - 1 \tag{5.3.34}$$

and the S-matrix is

$$S(p_1, p_2) = \frac{u(p_2) - u(p_1) + i}{u(p_2) - u(p_1) - i}\sigma(u_1, u_2)^2 \tag{5.3.35}$$

where $\sigma(u_1, u_2)$ gives the well-known dressing phase. As the dressing phase does not contribute until $\mathcal{O}(g^8)$ it can be ignored for our purposes.

To study the low-energy limit to three-loops we again rescale $\varepsilon \to \kappa^2\varepsilon$ but additionally we define $\tilde{g} = \kappa g$ which is essentially the effective coupling that appears in the BMN and other fast string expansions. We expand the magnon energy to $\mathcal{O}(\tilde{g}^4)$ so that

$$\kappa^2\varepsilon = 4\sin^2 \frac{p}{2} - 16\frac{\tilde{g}^2}{\kappa^2} \sin^4 \frac{p}{2} + -128\frac{\tilde{g}^4}{\kappa^4} \sin^4 \frac{p}{2} \,. \tag{5.3.36}$$

In the limit of small κ this implies

$$p = \kappa\sqrt{\varepsilon}\left(1 + \tilde{g}^2\frac{\varepsilon}{2} - \tilde{g}^4\frac{\varepsilon^2}{8}\right) \tag{5.3.37}$$

or $\varepsilon = \tilde{p}^2 - \tilde{g}^2\tilde{p}^4 + 2\tilde{g}^4\tilde{p}^6$ where $\tilde{p} = p/\kappa$. Taking the same limit for $u(p)$ we find

$$u(p) = \frac{1}{\kappa}\tilde{u}(\tilde{p}) = \frac{1}{\kappa}\left(\frac{1}{\tilde{p}} + 2\tilde{g}^2\tilde{p} - 2\tilde{g}^4\tilde{p}^3\right) \tag{5.3.38}$$

so that

$$\begin{aligned} S(p_1, p_2) &\equiv \frac{\tilde{u}(p_2) - \tilde{u}(p_1) + i}{\tilde{u}(p_2) - \tilde{u}(p_1) - i} \\ &= \frac{1 + \dfrac{ip_1p_2}{p_1 - p_2}(1 + 2\tilde{g}^2p_1p_2 - 2\tilde{g}^4p_1p_2(p_1^2 - p_1p_2 + p_2^2))}{1 - \dfrac{ip_1p_2}{p_1 - p_2}(1 + 2\tilde{g}^2p_1p_2 - 2\tilde{g}^4p_1p_2(p_1^2 - p_1p_2 + p_2^2))} \,. \end{aligned} \tag{5.3.39}$$

This is the quantum S-matrix for the generalized LL-model. As in the LL-model, in order to define the perturbative two-dimensional expansion we again rescale the momenta $p_i \to \gamma p_i$. However, in order to keep the correct scaling result we write[5] $\tilde{g} = g/\gamma$ so that in the small γ limit we have

$$T^{(n)}(p_1, p_2) = 2\Big[\frac{ip_1p_2}{(p_1-p_2)}(1 + 2g^2p_1p_2 - 2g^4p_1p_2(p_1^2 - p_1p_2 + p_2^2))\Big]^{n+1} \quad (5.3.40)$$

This result can be compared with the perturbative results above, (5.2.23) and (5.2.24), and it can be seen that they agree.

5.3.4 Form Factors from the XXX spin-chain

Infinite volume diagonal spin-chain form factors, $f^{\mathcal{O}}(p_1, \ldots, p_n)$, have been calculated in [16] by extracting them from finite volume matrix elements. We will compare the low-energy limit of these results with those calculated directly from the LL-model and then consider the generalization to higher orders in g. Let us note that we will find additional factors of the S-matrix when compared to $f^{\mathcal{O}}$ as computed in [16] because of different conventions in the definitions of the diagonal form factors. We defined the $F_D^{\mathcal{O}}$ in Sec. 4.1.2 as expectation values between "in" and "out" states, while they considered both states as "in" states. This difference in the ordering of the momenta/rapidities will then amount to different factors of the S-matrix through the relation

$$^{\text{out}}\langle k_1, \ldots, k_M| = {}^{\text{in}}\langle p_1, \ldots, p_N|S(p_1, \ldots, p_N; k_1, \ldots, k_M)\,. \quad (5.3.41)$$

Operators with one impurity

The operators with one impurity, which correspond to the gauge theory operators $\text{Tr}(Z\bar{Z})$, $\text{Tr}(Z\bar{X})$, $\text{Tr}(X\bar{Z})$ and $\text{Tr}(X\bar{X})$, are described by the spin-chain operators acting on the n-th spin-chain site:

$$E_n^{11} = \frac{1}{2}(\mathbb{1} + \sigma_n^z)\,, \quad E_n^{12} = \sigma_n^+\,, \quad E_n^{21} = \sigma_n^-\,, \quad E_n^{22} = \frac{1}{2}(\mathbb{1} - \sigma_n^z)\,. \quad (5.3.42)$$

[5] This careless use of notation gives sensible results as we naturally think of both small parameters corresponding to the same large volume expansion, $\kappa \simeq \gamma \simeq L^{-1}$.

For example, denoting $o_1(n) = E_n^{11}$ the vacuum, one-particle and two-particle diagonal form factors computed in [16] were

$$f^{o_1}(\emptyset) = 1\,, \quad f^{o_1}(p) = \epsilon(p)\,, \quad f^{o_1}(p_1, p_2) = (\varepsilon(p_1) + \varepsilon(p_2))\phi_{12} \tag{5.3.43}$$

where $\varepsilon(p)$ is the magnon energy as above and

$$\phi_{12} = \frac{2}{1 + (u(p_1) - u(p_2))^2}\,. \tag{5.3.44}$$

These can be compared to the previous perturbative results by using the map between the spin-chain and the LL-model via the coherent state representation, whereby the spin-chain operator o_1 corresponds to the LL operator

$$o_1 = \frac{1}{2}(1 + \sigma_n^z) \leftrightarrow 1 - |\varphi|^2\,. \tag{5.3.45}$$

We can see that compared to the LL operator considered in Sec. 5.3.1 there is an additional identity operator. This contribution gives rise to the non-trivial vacuum expectation value but can be ignored for higher-particle form factors. To extract the prediction for the LL-model we must also perform the low-energy rescaling described above i.e. $\varepsilon \to \kappa^2 \varepsilon$ with $\kappa \to 0$. However in this limit all the multi-particle form factors (5.3.43) will vanish due to the normalization of the one-particle states which results in the factors of the magnon energy. To get a well defined limit we rescale by a factor of $\sqrt{\varepsilon(p_i)}$ for each external leg. This results in the one-particle form factor being equal to 1 which corresponds to the normalization used in the perturbative calculation. For the two-particle case we find after this rescaling

$$\frac{f^{o_1}(p_1, p_2)}{\varepsilon_1 \varepsilon_2} \to \left(\frac{1}{\varepsilon(p_1)} + \frac{1}{\varepsilon(p_2)}\right) \frac{4}{1 + (1/p_1 - 1/p_2)^2} \tag{5.3.46}$$

where on the r.h.s. we understand the dispersion relation to be that of the LL-model i.e. $\varepsilon(p) = p^2$. As a rule of thumb we see that the LL limit of infinite volume spin-chain quantities is taken by replacing $u(p) \to 1/p$ while keeping constant terms that occur with differences of u's. For example in the quantities ϕ_{ij} we have

$$\phi_{ij} \to \phi_{ij}^{\mathrm{LL}} = \frac{2p_i^2 p_j^2}{(p_i - p_j)^2} \left[1 + \frac{p_i^2 p_j^2}{(p_i - p_j)^2} \right]^{-1} , \tag{5.3.47}$$

where the terms that arise in the small momentum expansion corresponds to world-sheet loop effects in the LL-model. For the two-particle form factor it is apparent that this result (5.3.46) still does not match the LL result (5.3.15). However this is again a consequence of the definition of the states used in defining the form factor and in fact

$$\frac{1}{\varepsilon_1 \varepsilon_2} S_{\mathrm{LL}}(p_1, p_2) f^{-o_1}(p_1, p_2) \to F_s^{|\varphi|^2}(p_1, p_2) \tag{5.3.48}$$

with the S-matrix factor due to the different ordering of momenta in the in- and out-states.

A formula for multi-particle form factors was also proposed in [16]. For the operator o_1, we have

$$f^{o_1}(p_1, \dots, p_n) = \sum_{\sigma \in S_n} \varepsilon_{\sigma(1)} \phi_{\sigma(1)\sigma(2)} \phi_{\sigma(2)\sigma(3)} \cdots \phi_{\sigma(n-1),\sigma(n)} , \tag{5.3.49}$$

where the sum is over the set of all permutations of the n-indices, S_n. As in the two-particle case in order to have a non-vanishing answer in the LL-limit we must rescale by a factor of $(\varepsilon_1 \cdots \varepsilon_n)^{-1}$ and thus taking the limit we find

$$\begin{aligned} \frac{f^{o_1}(p_1, \dots, p_n)}{\varepsilon_1 \cdots \varepsilon_n} &\to 2^{n-1} \sum_{\sigma \in S_n} \frac{p_{\sigma(1)}^2 \cdots p_{\sigma(n-1)}^2}{(p_{\sigma(1)} - p_{\sigma(2)})^2 \cdots (p_{\sigma(n-1)} - p_{\sigma(n)})^2} \times \\ &\times \frac{1}{1 + \left[p_{\sigma(1)} p_{\sigma(2)} \Big/ \Big(p_{\sigma(1)} - p_{\sigma(2)} \Big) \right]^2} \cdots \frac{1}{1 + \left[p_{\sigma(n-1)} p_{\sigma(n)} \Big/ \Big(p_{\sigma(n-1)} - p_{\sigma(n)} \Big) \right]^2} . \end{aligned}$$

To compare with the tree-level Landau-Lifshitz result for three particles computed in Sec. 5.3.1 we expand in powers of the momenta and take the leading results. Up to an overall sign agreement is found. At tree-level the S-matrix is simply 1, however we would expect to see factors of the S-matrix by keeping higher orders in the momenta corresponding to loop effects in the perturbative calculation.

Higher-loop form factors As seen above, one can straightforwardly calculate higher-loop form factors in the generalized Landau-Lifshitz model. An $\mathcal{O}(g^2)$ prediction for these form factors was given in [18] where they were related to the computation of certain structure constants. The explicit perturbative computation involves several contributions: corrections to the states due to the two-loop gauge theory corrections to the dilatation generator and modifications due to operator insertions capturing the effects of one-loop gauge theory Feynman diagrams [85, 86]. As a result, in the form factor picture the operator itself must be viewed as acquiring $\mathcal{O}(g^2)$ corrections

$$o_1(g) = o_1 + g^2 o_1' \,. \tag{5.3.50}$$

Somewhat remarkably, the “sum over products" form of the tree-level result (5.3.49) remains, with the corrections coming in the individual components. Specifically

$$f^{o_1(g)}(u_1, \ldots, u_n) = \sigma_1 \varphi_{12} \varphi_{23} \cdots \varphi_{n-1,n} + \text{permutations} \tag{5.3.51}$$

where

$$\sigma_i = \frac{1}{u_i^2 + 1/4} + \frac{8g^2 u_i^2}{(u_i^2 + 1/4)^3} \tag{5.3.52}$$

and

$$\varphi_{ij} = \frac{2}{(u_i - u_j)^2 + 1} + \frac{4g^2(u_i^2 - u_j^2)}{(u_i^2 + 1/4)(u_j^2 + 1/4)((u_i - u_j)^2 + 1)}. \tag{5.3.53}$$

For the one-particle form factor, $f^{o_1(g)}(u(p_1))$, at the leading order $\mathcal{O}(g^0)$, we rescaled by the energy of the external particle to find agreement with the perturbative LL calculation. As the σ_i's do not correspond to the g-corrected expression for the particle energy which is instead given by

$$\epsilon(u) = \frac{1}{u^2 + 1/4} + g^2 \frac{12u^2 - 1}{4(u^2 + 1/4)^3} \,, \tag{5.3.54}$$

we must add a correction to the operator. Computing the small momentum limit we have

$$\frac{1}{\epsilon} f^{o_1(g)}(u) \to 1 + 5g^2 p^2 \tag{5.3.55}$$

hence by considering the generalized LL operator $\Phi_1(g) = \varphi^* \varphi + 5g^2 \acute{\varphi} \acute{\varphi}^*$ we have that

$$\frac{1}{\epsilon} f^{o_1(g)}(u) \to F^{\Phi_1(g)}(p) \, . \tag{5.3.56}$$

For the two-particle form factor, again dividing by factors of the particle energy and expanding in powers of the momenta, we have

$$\frac{1}{\varepsilon_1 \varepsilon_2} f^{o_1(g)}(p_1, p_2) \to \frac{2(p_1^2 + p_2)^2}{(p_1 - p_2)^2} \left(1 + g^2 (3p_1^2 + 4p_1 p_2 + 3p_2^2)\right) \, . \tag{5.3.57}$$

This can be seen to not agree with (5.3.9) and also does not reproduce $F^{\Phi_1(g)}(p_1, p_2)$ when the coefficients b_1 and b_2 appearing in the generalized LL action are set to their string/BDS value. One can include further corrections to the operator, which as long at they are at least quartic in the fields will not change the one-particle form factor result,

$$\Phi_1(g) = \varphi^* \varphi + 5g^2 \acute{\varphi} \acute{\varphi}^* + \alpha_1 g^2 (\varphi^{*2} \acute{\varphi}^2 + \varphi^2 \acute{\varphi}^{*2}) + \alpha_2 g^2 \varphi \varphi^* \acute{\varphi} \acute{\varphi}^*, \tag{5.3.58}$$

however there do not appear to be values of α_1 and α_2 that correctly reproduce the limit of the two-particle form factor of $o_1(g)$ and it seems that a more general deformation or extra contribution is required. [6]

Operators with two impurities

We can additionally consider the spin-chain operators with two impurities

$$o_2^1 = E_n^{11} E_{n+1}^{11} \, , \quad o_2^2 = E_n^{12} E_{n+1}^{21} \, , \quad o_2^3 = E_n^{21} E_{n+1}^{12} \tag{5.3.59}$$

[6] In [86] an operator correction reproducing the effect of the insertions to the heavy operator for the one-particle form factor constructed, responsible for the "δ_H" correction, was given. It corresponds to $\Phi_1' = \varphi^* \varphi + 2g^2 \acute{\varphi} \acute{\varphi}^* + g^2 (\varphi^{*2} \acute{\varphi}^2 + \varphi^2 \acute{\varphi}^{*2})$. However this does not reproduce the full two-particle form factor.

where the operators now sit on two spin-chain lattice sites and the infinite volume form factors were again extracted from spin-chain matrix elements in [16]. For each operator, $\mathcal{O} \in \{o_2^1, o_2^2, o_2^3\}$, they can be written as a combination of two terms

$$f^{\mathcal{O}}(u_1, \ldots, u_n) \quad = \quad f_E^{\mathcal{O}}(u_1, \ldots, u_n) + f_S^{\mathcal{O}}(u_1, \ldots, u_n) \tag{5.3.60}$$

with each term given as a sum over permutations

$$f_E^{\mathcal{O}}(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} \left[\varepsilon_{\sigma(1)} \phi_{\sigma(1)\sigma(2)} \cdots \phi_{\sigma(n-1)\sigma(n)} \mathfrak{f}_n^{\mathcal{O}} \right] \tag{5.3.61}$$

and

$$f_S^{\mathcal{O}}(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} \left[\sum_{i=1}^{n-1} \varepsilon_{\sigma(1)} \phi_{\sigma(1)\sigma(2)} \cdots \psi^{\mathcal{O}}_{\sigma(i-1)\sigma(i)} \cdots \phi_{\sigma(n-1)\sigma(n)} \varepsilon'_{\sigma(n)} \right] \tag{5.3.62}$$

where ε is the energy and ε' its derivative with respect to the rapidity variable u, ϕ_{ij} is as in (5.3.44) and

$$\begin{aligned}
\mathfrak{f}_i^{o_2^1} &= 2\,, & \psi_{ij}^{o_2^1} &= -(u_i - u_j)(u_i u_j - 1/4)\phi_{ij}\,, \\
\mathfrak{f}_i^{o_2^2} &= \frac{u_i - i/2}{u_i + i/2}\,, & \psi_{ij}^{o_2^2} &= (u_i - u_j)(u_i - i/2)(u_j - i/2)\phi_{ij}\,, \\
\mathfrak{f}_i^{o_2^3} &= \frac{u_i + i/2}{u_i - i/2}\,, & \psi_{ij}^{o_2^3} &= (u_i - u_j)(u_i + i/2)(u_j + i/2)\phi_{ij}\,.
\end{aligned} \tag{5.3.63}$$

Of course one can consider linear combinations of these operators and one such combination in which we will be interested is

$$o_2^4 = E_n^{22} E_{n+1}^{22} = \mathbb{1} - o_1(n) - o_1(n+1) + o_2^1(n) \tag{5.3.64}$$

and for which we have

$$\mathfrak{f}_i^{o_2^4} = 0\,, \qquad \psi_{ij}^{o_2^4} = -(u_i - u_j)(u_i u_j - 1/4)\phi_{ij}\,. \tag{5.3.65}$$

Taking the continuum limit by using the replacement rule (5.3.45) it is easy to see that the operator o_2^4 corresponds to the LL operator $|\varphi|^4$ up to derivative terms which we neglect. Thus we can compare the form factors for this operator with those previously calculated perturbatively. The tree-level results can be found by simply making a small momentum expansion. Explicitly, this gives

$$\varepsilon' \to -2p^3 \,, \quad \text{and} \quad \psi_{ij}^{o_4^1} \to \frac{2}{(p_i - p_j)} \,, \tag{5.3.66}$$

in addition to $\varepsilon \to p^2$ and $\phi_{ij} \to \phi_{ij}^{\rm LL}$. It is easy to see that the two-particle form factor, once rescaled, has in the small momentum limit the trivial result

$$\frac{1}{\varepsilon_1\varepsilon_2} f^{o_2^4}(p_1, p_2) \to F_c^{(0)|\varphi|^4}(p_1, p_2)\Big|_{\mathcal{O}(g^0)} = 4 \,, \tag{5.3.67}$$

while the three-particle case gives

$$\frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} f^{o_2^4}(p_1, p_2, p_3) \to -F_c^{(0)|\varphi|^4}(p_1, p_2, p_3)\Big|_{\mathcal{O}(g^0)} \,, \tag{5.3.68}$$

which means it agrees with the tree-level LL result up to a sign.

Furthermore the loop effects are reproduced by using our rule of thumb of replacing $u \to p^{-1}$ and keeping those constants that are added to differences of u's. In particular this does not retain the factor of $1/4$ in the definition of $\psi_{ij}^{o_2^4}$. These factors can be reproduced in the LL model by adding derivative terms to the operator.

5.4 Form Factor Perturbation Theory

One interesting application of diagonal form factors is to the study of perturbations of integrable models. Form factor perturbation theory (FFPT) is such an approach to studying non-integrable massive theories, introduced in [20] with a particular focus on deformations of relativistic integrable models which themselves can be viewed as deformations of conformal field theories. However, as the authors of [20] make clear, their approach is quite general. Given an integrable model with action $\mathcal{A}_0^{\rm int}$ they study the theory with action

$$\mathcal{A} = \mathcal{A}_0^{\rm int} - \sum_j g_j \int d^2x \, \Phi_j(x) \tag{5.4.1}$$

where $\Phi_j(x)$ are the deforming operators. An assumption behind this approach is that, at least for small values of g_j, asymptotic particle states are a good basis for studying the deformed theory and that while the integrable model has a different spectrum it acts as a useful starting point. We will be interested in calculating the S-matrix of the deformed theory

$$S(p_1, \ldots, p_n; k_1, \ldots, k_m) = {}^{\text{out}}\langle k_1, \ldots, k_m | p_1, \ldots, p_n \rangle^{\text{in}} . \tag{5.4.2}$$

In order to preserve the normalization of the vacuum, in [20] the authors introduced a counter-term corresponding to the vacuum energy so that

$${}^{\text{out}}\langle 0|0\rangle^{\text{in}} = {}^{\text{out}}_{0}\langle 0|0\rangle^{\text{in}}_{0} = 1 , \tag{5.4.3}$$

where, for example, $|0\rangle^{\text{in}}_{0}$ is the "in"-vacuum state in the undeformed theory. They further introduced counter-terms to preserve the one-particle normalization. Here we define the operators $\mathcal{O}^{(i)}(0,0)$, $i = 1, 2$, in terms of their form factors in the unperturbed, integrable theory,

$$F^{\mathcal{O}^{(1)}}(p_1, \ldots, p_n) = \delta_{n,1} \; , \quad \text{and} \quad F^{\mathcal{O}^{(2)}}(p_1, \ldots, p_n) = i p_1 \delta_{n,1} \tag{5.4.4}$$

such that[7]

$${}^{\text{out}}\langle k|p\rangle^{\text{in}} = {}^{\text{out}}_{0}\langle k|p\rangle^{\text{in}}_{0} = 2\pi\delta(p-k) . \tag{5.4.5}$$

As described in [20], a perturbative expansion for the S-matrix can be found by expanding (5.4.2) in terms of matrix elements of time-ordered products of the deformations and inserting sums over asymptotic states of the undeformed theory. In principle this gives an expansion to all orders in the couplings g_j, with higher orders involving progressively more sums over intermediate states much as in covariant perturbation theory. Here we will only consider the leading-order terms

$$\begin{aligned} {}^{\text{out}}\langle k_1, \ldots, k_m | p_1, \ldots, p_n \rangle^{\text{in}} \;\; &\simeq \;\; {}^{\text{out}}_{0}\langle k_1, \ldots, k_m | p_1, \ldots, p_n \rangle^{\text{in}}_{0} \\ -i(2\pi)^2 \delta^{(2)}(\textstyle\sum k_i - \sum p_j) & {}^{\text{out}}_{0}\langle k_1, \ldots, k_m | \Big[\displaystyle\sum_j g_j \Phi_j - \sum_i \delta\mathcal{E}^{(i)} \mathcal{O}^{(i)} - \delta\mathcal{E}_{\text{vac}} \Big] | p_1, \ldots, p_n \rangle^{\text{in}}_{0} \end{aligned} \tag{5.4.6}$$

[7]As we will be considering the Landau-Lifshitz model which is not Lorentz invariant we modify several of the definitions of [20]; for example we don't use the usual Lorentz invariant one-particle normalizations.

where the coefficients $\delta\mathcal{E}^{(i)}$ and $\delta\mathcal{E}_{\text{vac}}$ are determined by demanding that (5.4.3) and (5.4.5) are satisfied.

5.4.1 Marginal Deformations

The use of integrable models in the study of the AdS/CFT correspondence has been very fruitful but the vast majority of interesting theories are almost certainly non-integrable. The corresponding world-sheet theories will likely involve multi-particle production with a corresponding increase in analytical complexity of the world-sheet S-matrix. Leigh-Strassler marginal deformations are one particularly simple class of deformations of $\mathcal{N} = 4$ SYM that preserve $\mathcal{N} = 1$ superconformal symmetry [87] and which are parameterized by two complex parameters h and $q = \exp(2\pi i\beta)$; thus, including the gauge coupling, there is a three-dimensional space of finite theories. The case with $h = 0$ is often called the β-deformed theory and, particularly for real β, it has received a very significant amount of attention as the gravitational dual is known [88] and the model is believed to be integrable – the string Lax pair was constructed in [89], the all-loop asymptotic Bethe ansatz was proposed in [90] and Y-system in [91]. The integrability of the β-deformed theory can be understood as arising from a Drinfeld-Reshetikhin twist of the undeformed theory combined with twisted boundary conditions [92].

For complex β, the one-loop dilatation operator restricted to two holomorphic scalar fields corresponds to the $\mathfrak{su}(2)_q$ XXZ deformed spin chain and so is integrable [93], however this does not extend beyond this subsector of fields [94]. For special values of $h \neq 0$ and $q \in \mathbb{C}$ the one-loop Hamiltonian is integrable [95] which can be understood in terms of Hopf twists of the real-β case [96]. More generally for generic values of q and h the theory is not believed to be integrable. For general q and h the R-matrix constructed by applying the Hopf algebraic transformation will not satisfy the Yang-Baxter equations and so the usual methods of integrable spin-chains will not be applicable, such deformations may however be studied by use of FFPT. In some sense the one-loop marginal deformations in the $\mathfrak{su}(2)$ sector which we study below are too simple to be of much interest, however they will allow us the check the general formula against known results and so demonstrate its reliability to this order.

5.4.2 Deformed Landau-Lifshitz

Here will consider the Landau-Lifshitz model following from the low-energy limit of the general Leigh-Strassler deformed one-loop spin-chain given in [95]. The LL model for complex-β but $h = 0$ was considered in [97][8] where checks of the match between the spin-chain and string descriptions were carried out. In keeping with our previous considerations we will truncate to the case of two holomorphic scalars such that the spin-chain Hamiltonian is given by

$$H^D = \frac{\lambda}{16\pi|q|}\sum_{\ell=1}^{L}\Bigg[\left(\frac{1+qq^*}{2}+hh^*\right)\mathbb{1}\otimes\mathbb{1}-\left(\frac{1+qq^*}{2}-hh^*\right)\sigma^z_\ell\otimes\sigma^z_{\ell+1} \\ -2q\,\sigma^-_\ell\otimes\sigma^+_{\ell+1}-2q^*\sigma^+_\ell\otimes\sigma^-_{\ell+1}\Bigg]\,. \quad (5.4.7)$$

Using the parameterization $q = \exp(\beta_I + i\beta_R)$, $2hh^*e^{-\beta_I} = \Lambda^2$ and taking the Landau-Lifshitz limit we find that in order to have a sensible behaviour the deformation parameters must be taken to be small with $\tilde{\beta}_R = \frac{\beta_R L}{2\pi}$, $\tilde{\beta}_I = \frac{\beta_I L}{2\pi}$ and $\tilde{\Lambda} = \frac{L\Lambda}{2\pi}$ fixed. With this scaling the resulting deformed Landau-Lifshitz action is

$$\mathcal{A} = \mathcal{A}^{\rm LL} - \frac{\lambda}{16\pi L}\int d\tau d\sigma\Big[\tilde{\beta}_R^2\left((n^1)^2+(n^2)^2\right)+2\tilde{\beta}_R\left(n^1\acute{n}^2-n^2\acute{n}^1\right) \\ +\tilde{\beta}_I^{\,2}\left(1-(n^3)^2\right)+\tilde{\Lambda}^2\left(1+(n^3)^2\right)\Big]$$

where $\mathcal{A}^{\rm LL}$ is the Landau-Lifshitz action (5.1.1). Setting $\tilde{\Lambda} = 0$ and using

$$(n^1, n^2, n^3) = (\sin 2\theta\cos 2\eta, \sin 2\theta\sin 2\eta, \cos 2\theta) \quad (5.4.8)$$

one reproduces the result from [97]. Instead we rescale the coordinates as in (5.2.1) so that the spatial coordinate has period L, use the complex field φ defined in (5.2.2) and expand the action to quartic powers in the field

$$\mathcal{A} = \mathcal{A}^{\rm LL} - \int dxdt\left[\frac{\Lambda^2}{2}+(\beta_I^2-\Lambda^2)\Phi_1 - i\beta_R\Phi_2 \\ +(\Lambda^2-\beta_I^2)\Phi_3 + i\beta_R\Phi_4\right] \quad (5.4.9)$$

[8] In [97] the authors use the complex parameter $\beta_{\mathbb{C}} = \beta_d + i\kappa_d$ where $\beta_d = \beta_R/2\pi$ and $\kappa_d = \beta_I/2\pi$.

where the deformations are given by the operators considered previously

$$\Phi_1 = |\varphi|^2\,, \quad \Phi_2 = (\varphi^* \acute{\varphi} - \varphi \acute{\varphi}^*)\,, \quad \Phi_3 = |\varphi|^4\,, \quad \Phi_4 = |\varphi|^2(\varphi^* \acute{\varphi} - \varphi \acute{\varphi}^*)\,. \tag{5.4.10}$$

We will use the form factor perturbation procedure to describe the corrections to the S-matrix due to these deformations.

Integrable deformations As the Hamiltonian (5.4.7) with $\Lambda = 0$ is in fact integrable, the full Bethe equations are known and we will be able to compare our results with those previously calculated [97],

$$e^{-i\beta_R L}\Big[\frac{\tilde{u}_k + i/2}{\tilde{u}_k - i/2}\Big]^L = \prod_{j=1, j\neq k}^{M} \frac{\tilde{u}_k - \tilde{u}_j + i\dfrac{\tanh\beta_I}{2\tanh(\beta_I/2)}\Big(1 + 4\tilde{u}_k\tilde{u}_j \tanh^2(\beta_I/2)\Big)}{\tilde{u}_k - \tilde{u}_j - i\dfrac{\tanh\beta_I}{2\tanh(\beta_I/2)}\Big(1 + 4\tilde{u}_k\tilde{u}_j \tanh^2(\beta_I/2)\Big)}$$

where

$$e^{-i\beta_R M} \prod_{k=1}^{M} \frac{\tilde{u}_k + i/2}{\tilde{u}_k - i/2} = 1 \tag{5.4.11}$$

and

$$E = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \tilde{\varepsilon}_j \quad \text{with} \quad \tilde{\varepsilon}_j = \frac{1}{\tilde{u}_k^2 + 1/4} + 2(\cosh\beta_I - 1)\,. \tag{5.4.12}$$

These equations give the corrections to the one-particle states, the two-particle S-matrix and the general n-particle S-matrix which can be found as a product of two-particle S-matrices. The results calculated using form factor perturbation theory will be expressed in terms of rapidities and momenta of the undeformed theory. These can be related to the deformed rapidities using the relation

$$\frac{\tilde{u} + i/2}{\tilde{u} - i/2} = e^{i\beta_R} \frac{u + i/2}{u - i/2} \tag{5.4.13}$$

or to leading order in β_R, $\tilde{u} = u - (u^2 + 1/4)\beta_R$. Hence we find the correction to the energy $\tilde{\varepsilon} = \varepsilon + \delta\varepsilon$, with

$$\delta\varepsilon(u) = \beta_R \frac{2u}{u^2 + 1/4} + \beta_I^2 \tag{5.4.14}$$

and to the S-matrix

$$\delta S(u_1, u_2) = 2i\beta_R \frac{u_2^2 - u_1^2}{(u_1 - u_2 + i)^2} + i\beta_I^2 \frac{(u_1 - u_2)(1 - 4u_1u_2)}{2(u_1 - u_2 + i)^2} . \tag{5.4.15}$$

We can take the low-energy limit of these results to compare with those calculated in the LL-model. The modification of the periodicity condition can be accounted for by shifting the relation between the rapidity and particle momentum

$$\tilde{u}(p) = \frac{1}{2} \cot \frac{p + \beta_R}{2} . \tag{5.4.16}$$

In order to take the low-energy LL limit, we assume the momentum and β_R to scale as κ for $\kappa \to 0$, so that we have

$$\tilde{u}(p) = \frac{1}{p + \beta_R} \simeq \frac{1}{p} - \frac{\beta_R}{p^2} + \mathcal{O}(\beta_R^2) . \tag{5.4.17}$$

The corresponding equation for the change in the energy is

$$\delta\varepsilon = 2\beta_R \, p + \beta_I^2 \tag{5.4.18}$$

and for the change in the S-matrix

$$\delta S(p_1, p_2) = \frac{2i\beta_R(p_1 + p_2)}{(p_1 - p_2)\left(1 - ip_1p_2/p_1 - p_2\right)^2} + \frac{2i\beta_I^2(1 + \gamma\, p_1p_2)}{(p_1 - p_2)\left(1 - ip_1p_2/p_1 - p_2\right)^2} \tag{5.4.19}$$

Here we have introduced a parameter γ in the β_I^2 deformations; in taking the LL-limit previously, Sec. 5.3.4, we have kept sub-leading terms of the form $1/(u_1 - u_2)$ but dropped those of the form $1/(u_1u_2)$ which corresponds to setting $\gamma = 0$.

5.4.3 Deformed LL from Form Factor Perturbations

We can use our previous perturbative calculations of the LL form factors and the general expression (5.4.6) to calculate the corrections to the S-matrix elements to linear order in the deformations.

Vacuum Energy As none of the operators Φ_i have non-vanishing zero-particle form factors the only correction to the vacuum energy comes from the coefficient of the identity operator, namely Λ^2. Using the condition that the vacuum normalization remains unchanged fixes the counterterm coefficient

$$\delta\mathcal{E}_{\text{vac}} = \frac{\Lambda^2}{2} \,. \tag{5.4.20}$$

One-particle states More interestingly, the quadratic operators Φ_1 and Φ_2 give rise to corrections to the one-particle state normalizations

$$\delta\mathcal{E}^{(1)} = (\beta_I^2 - \Lambda^2)\,, \quad \delta\mathcal{E}^{(2)} = -2i\beta_R \,. \tag{5.4.21}$$

These deformations correspond to corrections to the dispersion relation

$$\omega(p) = p^2 + 2\beta_R\, p + (\beta_I^2 - \Lambda^2) \tag{5.4.22}$$

and to calculate the one-particle energies one should multiply by the factor of $\lambda/8\pi^2$ that arises from the rescaling of the time coordinate. If we consider the case $\Lambda = 0$ we have

$$\varepsilon(p) = |p + \beta_{\mathbb{C}}|^2 \tag{5.4.23}$$

which strictly speaking should only be trusted to $\mathcal{O}(\beta_R)$ in our calculations, where $\beta_{\mathbb{C}} = \beta_R + i\beta_I$ and this result can be seen to agree with that previously calculated in the deformed Landau-Lifshitz model [97].

Two-particle states For the two-particle form factors for the operators Φ_3 and Φ_4 we have the result

$$\begin{aligned} \delta S(p_1, p_2) &= \frac{-i}{2(p_1 - p_2)}\left[(\Lambda^2 - \beta_I^2)F^{\Phi_3} + i\beta_R F^{\Phi_4}\right] \\ &= \frac{2i(p_1 + p_2)\beta_R}{(p_1 - p_2)\,(1 - ip_1p_2/p_1 - p_2)^2} + \frac{2i(\beta_I^2 - \Lambda^2)}{(p_1 - p_2)\,(1 - ip_1p_2/p_1 - p_2)^2} \,. \end{aligned} \tag{5.4.24}$$

were we have taken into account the Jacobian, $-i/2(p_1 - p_2)$, relating the usual energy-momentum δ-function and the momentum δ-functions in front of the S-matrix (in addition to the factor of $-i$ from (5.4.6)). To compare with the Bethe Ansatz (5.4.19) results we simply set $\Lambda = 0$ and the results can be seen to match. For the β_R term this deformation essentially follows from the shift of the rapidities. The same deformed S-matrix could be found by taking the fast-string limit of the string world-sheet theory in the β_R-deformed geometry. A perturbative calculation [98] of the world-sheet S-matrix in the near-BMN limit of the deformed theory [99] has been previously carried out and is consistent with the above result. For the β_I^2 term we see that we find the $\gamma = 0$ result. In order to reproduce the $\gamma = 1$ result we would to have to add appropriate derivative corrections to the deformation operator.

In principle there should be additional corrections to the S-matrix from Φ_1 and Φ_2 which have non-vanishing two-particle diagonal form factors:

$$\begin{aligned}\delta S(p_1, p_2) &= \frac{-i}{2(p_1 - p_2)}\left[(\beta_I^2 - \Lambda^2)F^{\Phi_1} - i\beta_R F^{\Phi_2}\right] \\ &= \frac{2i}{(p_1 - p_2)^3\,(1 - ip_1p_2/(p_1 - p_2))} \times \\ &\qquad \times \left[(\beta_I^2 - \Lambda^2)(p_1^2 + p_2^2) - 2\beta_R\, p_1p_2(p_1 + p_2)\right], \end{aligned} \tag{5.4.25}$$

where it is important to note that we use the symmetric prescription to evaluate the diagonal form factors. These corrections correspond to the changes in the S-matrix as a result of changes in the dispersion relation. The relativistic analogue of this was discussed in [20], where as the invariant

$$s = 2m^2(1 + \cosh\theta) \tag{5.4.26}$$

is held constant under the deformation of a parameter, which we call δg, the resulting change in the particle mass, δm, necessarily causes a shift of the rapidity

$$\delta\theta = -2\frac{\delta m}{m}\coth\frac{\theta}{2} \tag{5.4.27}$$

and so the change in the S-matrix has two components

$$\delta S(\theta) = \frac{\partial S(\theta)}{\partial\theta}\delta\theta + \left.\frac{\partial S(\theta, g)}{\partial g}\right|_{g=0}\delta g\,. \tag{5.4.28}$$

The generalized LL-model, being non-relativistic, does not satisfy the same relation but we can define an analogous variation due to changes in the particle momenta

$$\delta S(p_1, p_2) = \frac{\partial S(p_1,p_2)}{\partial p_1}\delta p_1 + \frac{\partial S(p_1,p_2)}{\partial p_2}\delta p_2 + \sum_i \frac{\partial S(p_1,p_2,g_i)}{\partial g_i}\bigg|_{g_i=0}\delta g_i \quad (5.4.29)$$

where in this case we are considering $g_i \in \{\beta_R, \beta_I^2, \Lambda^2\}$. The variations w.r.t. the couplings give the terms calculated previously (5.4.24) while the first two terms should correspond to (5.4.25). This is clearest for the β_I^2, Λ^2 deformations where if we demand that total incoming momentum and energy are unchanged by the deformation, i.e. $\delta\varepsilon_1 + \delta\varepsilon_2 = 0$, we have that

$$2\delta p_1 p_1 + 2\delta p_2 p_2 = -2(\beta_I^2 - \Lambda^2)\,, \quad \delta p_1 + \delta p_2 = 0\,. \quad (5.4.30)$$

and solving for δp_1 and δp_2 and substituting into the first two terms of (5.4.29) we find the corresponding terms in (5.4.25). To reproduce the β_R terms we must modify the variation conditions such that

$$2\delta p_1 p_1 + 2\delta p_2 p_2 = -4(p_1+p_2)\beta_R - 2(\beta_I^2 - \Lambda^2)\,, \quad \delta p_1 + \delta p_2 = -2\beta_R\,. \quad (5.4.31)$$

Integrable form factors As the spin-chain form factors have been computed via the algebraic Bethe ansatz, our results are in fact generalizable to that theory without the need to take the LL low-energy limit. While the perturbative approach we have taken can be used to find the deformed S-matrix for low numbers of external particles, such integrable methods potentially give a method to completely determine the n-particle S-matrix. To leading order in the deformations, we can write the Hamiltonian as

$$H^D = H^{\text{XXX}} + \frac{\lambda}{8\pi}\mathcal{O}^D\,, \quad (5.4.32)$$

where

$$\mathcal{O}^D = \sum_{\ell=1}^{L}\left[i\beta_R\left(o_2^2(\ell) - o_2^3(\ell)\right) + (\Lambda^2 - \beta_I^2)\left(o_2^1(\ell) - o^1(\ell)\right) + \frac{1}{2}\Lambda^2\right]. \quad (5.4.33)$$

A proposal for the n-particle diagonal form factors (including $n > 2$) of this deformation can be given by simply taking linear combinations of the results found in [16], multiplying by the appropriate factor of the *undeformed* S-matrix, and including factors of particle energies to correct the state normalizations:

$$F^{\mathcal{O}_D}(\emptyset) = \frac{1}{2}\Lambda^2\,, \qquad F^{\mathcal{O}_D}(u_1) = \frac{1}{\epsilon_1} f^{\mathcal{O}_D}(u_1)\,, \tag{5.4.34}$$

and for $n \geq 2$

$$F^{\mathcal{O}_D}(u_1,\ldots,u_n) = \frac{\prod_{i\neq j}^{n} S(u_i,u_j)}{\epsilon_1\ldots\epsilon_n} f^{\mathcal{O}^D}(u_1,\ldots,u_n)\,, \tag{5.4.35}$$

where the spin-chain form factors $f^{\mathcal{O}^D}$ are given by (5.3.60) with

$$\begin{aligned} \mathfrak{f}^{\mathcal{O}^D} &= i\beta_R(\mathfrak{f}^{o_2^2} - \mathfrak{f}^{o_2^3}) + (\Lambda^2 - \beta_I^2)(\mathfrak{f}^{o_2^1} - 1) \\ &= -2\beta_R \frac{u}{u^2 + 1/4} + (\Lambda^2 - \beta_I^2) \end{aligned} \tag{5.4.36}$$

and

$$\psi_{ij}^{\mathcal{O}^D} = (u_i - u_j)\Big[\beta_R(u_i + u_j) - (\Lambda^2 - \beta_I^2)(u_i u_j - \tfrac{1}{4})\Big]\phi_{ij}\,. \tag{5.4.37}$$

These can now be used to compute the corrections to the spin-chain S-matrix. It can be seen by comparison with (5.4.14) that the factor $\mathfrak{f}^{\mathcal{O}^D}$, which only contributes to the f_E term in (5.3.60), gives the (negative of) the corrections to the magnon energies which is consistent as it is the sole contribution to the $n = 1$ form factor in the absence of the counterterms. Similarly by comparison with (5.4.15) we can see that the deformation of the two-particle S-matrix is reproduced entirely by the f_S part of the two-particle form factor from (5.3.60). Thus we have

$$\delta S = -iS(u_1,u_2)\psi_{12}\,, \tag{5.4.38}$$

with the additional terms appearing in $F^{\mathcal{O}_D}(u_1,u_2)$ being cancelled by the Jacobian from the energy-momentum δ-functions. Importantly here we are not taking the low-energy LL-limit and the results are valid for arbitrary momenta in the infinite volume limit and in particular we capture the factor of $1/4$ in $\psi_{12}^{\mathcal{O}^D}$ that is missed in the LL limit. Additionally, there is a contribution from the f_E part of the two-particle form factor; as in the LL theory these should be related to the change in the S-matrix due to the change in the definition of the rapidity.

Conclusions

The focus of this thesis has been on the computation of form factors in the world-sheet theory describing strings in $AdS_5 \times S^5$ and in the related generalized Landau–Lifshitz model. We started with an introduction to the AdS/CFT correspondence. We then presented in detail the world-sheet theory describing strings in $AdS_5 \times S^5$ in the uniform light-cone gauge, discussing its quantization and symmetries in the large-tension limit. We turned to the gauge side, introducing $\mathcal{N} = 4$ super Yang–Mills and explaining the connection with the spin chain and the emergence of integrability in this context. The goal was to provide a concise and almost self-contained presentation of the notions needed in the rest of the thesis, where we discussed the form factors for the world-sheet string and the generalized Landau-Lifshitz model.

In Chapter 4, we defined the form factors, with particular attention to the case of the world-sheet string. We mentioned the world-sheet form factors axioms [12], which would allow to extend the S-matrix bootstrap approach to off-shell quantities if a general solution is found. We then studied form factors perturbatively. We extended the calculations done in the closed $\mathfrak{su}(2)$ sector in [12, 13] to other fields of the theory, obtaining the full tree-level three-particle form factor, which is one of the main results of the thesis.

Chapter 5 was dedicated to the Landau-Lifshitz model, an integrable non-relativistic two-dimensional theory, which can be derived as a thermodynamic limit of the Heisenberg $XXX_{1/2}$ spin chain. Thus from the calculation of the conformal dimensions in $\mathcal{N} = 4$ SYM, which can be obtained as the eigenvalues of the Hamiltonian of a spin chain, we have a connection to a generalized LL model through the thermodynamic limit. We also discussed how the LL model has been derived as a double limit of the $AdS_5 \times S^5$ string, which allowed to explore how to match string energies and anomalous dimensions in a simpler setting. We then turned to the perturbative computation of the two- and three-particle diagonal form factors for different operators in the generalized LL model. In particular, we were able to compute the form factors at any loop, writing them as a series and resumming to obtain the all-loop result. More specifically, the other main result of this work is the all-loop computation of the two-particle diagonal form factor for the quadratic operator φ^2. To better understand the answers, we compared the tree-level and one-loop part to the spin-chain calculations done in [16] at leading order in λ, reproducing part of the expected $\mathrm{XXX}_{1/2}$ spin chain form factor, for small momenta. Finally, we mentioned how the form factors can be used to study perturbations of integrable models, introducing FFPT [20].

While there are a number of different directions to pursue, for example other form factors at higher-orders in λ, different deformations, and deformations in larger sectors of the theory, they all ultimately require the exact calculation of the form factors for the AdS string world-sheet theory. Such quantities would provide an alternative method for computing planar gauge-theory structure constants, or equivalently the string vertex operator [100] which satisfies a similar set of axioms, and would also provide a means for computing the world-sheet S-matrix for deformed theories to all orders in λ. One approach to the computation of form factors is the free field representation developed by Lukyanov [101] (see also [102]) which has been successfully applied to a range of models, for example the SU(2) Thirring and sine-Gordon models [101], the O(3) non-linear sigma-model [103], the SU(N) Gross-Neveu models [104] and, of particular relevance to the string world-sheet theory, the principal chiral model with a product group structure [105].

A semi-classical approach to studying deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry, being valid at large g, would be complementary to the methods considered here. The classical world-sheet theory in deformed backgrounds, for example the marginal deformations discussed above but also black-hole geometries, will no longer be integrable but in those cases where there is a parameter that can be taken small one may attempt to use techniques based upon the inverse scattering transform or related methods, previously used for nearly integrable systems [106], to construct classical solutions and compute their charges. Given the relation between deformations and FFPT such methods may also be useful for studying world-sheet form factors semi-classically [14, 15].

Appendix A

Introduction to the S-matrix

A.1 The definition

The S-matrix $\mathbb{S}$ is a unitary operator relating free (asymptotic) states called "in"-states and "out"-states, as we will explain below. Let us consider first a generic theory with creation/annihilation operators $a_i^\dagger$, a_i, where i labels the flavor of the particles, and from them define the operators $a_{\text{in},i}^\dagger$, $a_{\text{in},i}$ and $a_{\text{out},i}^\dagger$, $a_{\text{out},i}$ as operators acting on the same Hilbert space which satisfy

$$\begin{aligned} a(p,\tau) &= \mathbb{U}_{\text{in}}^\dagger(\tau)\cdot a_{\text{in}}(p,\tau)\cdot \mathbb{U}_{\text{in}}(\tau)\,, \\ a(p,\tau) &= \mathbb{U}_{\text{out}}(\tau)\cdot a_{\text{out}}(p,\tau)\cdot \mathbb{U}_{\text{out}}^\dagger(\tau)\,, \end{aligned} \tag{A.1}$$

where the $\mathbb{U}$ are unitary operator, so that the "in" and "out" operators (separately) satisfy the canonical commutation relations. The $\mathbb{U}$ are determined up to a constant unitary transformation, which can be fixed (up to an overall phase) by imposing the boundary conditions

$$\mathbb{U}_{\text{in}}(-\infty) = 1\,, \qquad \mathbb{U}_{\text{out}}(+\infty) = 1\,. \tag{A.2}$$

which imply

$$a_{\text{in},i}^\dagger|_{-\infty} = a_i^\dagger|_{-\infty}\,, \quad a_{\text{in},i}|_{-\infty} = a_i|_{-\infty}\,, \quad a_{\text{out},i}^\dagger|_{+\infty} = a_i^\dagger|_{+\infty}\,, \quad a_{\text{out},i}|_{+\infty} = a_i|_{+\infty}\,,$$

We call "in"-states and "out"-states respectively the states obtained by acting with the "in" and "out" creation operators on the vacuum of the Hilbert space $|0\rangle$, i.e.

$a_{i,in/out}\,|0\rangle = 0\ \forall i,$

$$|p_1,\ldots,p_n\rangle^{(\mathrm{in})}_{i_1,\ldots,i_n} = a^\dagger_{\mathrm{in},i_1}(p_1)\,a^\dagger_{\mathrm{in},i_2}(p_2)\,\cdots\,a^\dagger_{\mathrm{in},i_n}(p_n)\,|0\rangle\ , \tag{A.3}$$

$$|p_1,\ldots,p_n\rangle^{(\mathrm{out})}_{i_1,\ldots,i_n} = a^\dagger_{\mathrm{out},i_1}(p_1)\,a^\dagger_{\mathrm{out},i_2}(p_2)\,\cdots\,a^\dagger_{\mathrm{out},i_n}(p_n)\,|0\rangle\ , \tag{A.4}$$

where p_l and i_l are respectively the momentum and flavor of the l-th particle.

The S-matrix is defined as a unitary operator which maps out-states to in-states (or vice versa depending on conventions)

$$|p_1,p_2,\ldots,p_n\rangle^{\mathrm{in}}_{i_1,i_2,\ldots,i_n} = \mathbb{S}\,|p_1,p_2,\ldots,p_n\rangle^{\mathrm{out}}_{i_1,i_2,\ldots,i_n}\ , \tag{A.5}$$

or in terms of the creation/annihilation operators

$$a_{\mathrm{in}}(p,\tau) = \mathbb{S}\cdot a_{\mathrm{out}}(p,\tau)\cdot\mathbb{S}^\dagger\,, \qquad \mathbb{S}\,|0\rangle = |0\rangle\ . \tag{A.6}$$

In terms of creation and annihilation operators $a, a^\dagger$ the free Hamiltonian $\mathbb{H}_f$ takes the form

$$\mathbb{H}_f = \int \mathrm{d}p\,\sum_i \varepsilon^{(i)}_p\,a^\dagger_i(p,\tau)a_i(p,\tau)\,, \tag{A.7}$$

where the full Hamiltonian is $\mathbb{H} = \mathbb{H}_f + \mathbb{V}$. The operators $a, a^\dagger$ (and $\mathbb{H}_f$) are interacting Heisenberg fields obeying the equations of motion

$$\frac{\partial}{\partial\tau}a_i(p,\tau) = i\,[\mathbb{H}\,,\,a_i(p,\tau)\,] = -i\,\varepsilon^{(i)}\,a_i(p,\tau) + i\,[\mathbb{V}\,,\,a_i(p,\tau)\,]\,, \tag{A.8}$$

where $\mathbb{V} = \mathbb{V}(a^\dagger\,,\,a)$ is a function of $a^\dagger_k$ and a^k.

From the definitions of the "in" and "out" creation/annihilation operators (A.1) and the S-matrix (A.6), it follows

$$\mathbb{S} = \mathbb{U}_{\mathrm{in}}(\tau)\cdot\mathbb{U}_{\mathrm{out}}(\tau)\,. \tag{A.9}$$

Since we have the same time-evolution in the Heisenberg picture for "in" and "out" operators by definition

$$\begin{aligned} \frac{\partial}{\partial\tau} a^{\dagger}_{\text{in},i}(p,\tau) &= i\left[H_2(a^{\dagger}_{\text{in}},a_{\text{in}}),a^{\dagger}_{\text{in},i}(p,\tau)\right] , \\ \frac{\partial}{\partial\tau} a^{\dagger}_{\text{out},i}(p,\tau) &= i\left[H_2(a^{\dagger}_{\text{out}},a_{\text{out}}),a^{\dagger}_{\text{out},i}(p,\tau)\right] , \end{aligned} \tag{A.10}$$

the time dependence in (A.6) and (A.9) factors out and the S-matrix is actually time-independent, thus we can fix τ to any convenient value in the definition of $\mathbb{S}$.

$$\mathbb{S} = \mathbb{U}_{\text{in}}(+\infty) = \mathbb{U}_{\text{out}}(-\infty) . \tag{A.11}$$

We can write $\mathbb{U}_{in/out}$ explicitly using (A.1) derived w.r.t. τ and the equations of motion above to find

$$\dot{\mathbb{U}}_{\text{in}}\mathbb{U}^{\dagger}_{\text{in}} + i\,\mathbb{V}(a^{\dagger}_{\text{in}},a_{\text{in}}) = 0 , \qquad \dot{\mathbb{U}}_{\text{out}}\mathbb{U}^{\dagger}_{\text{out}} - i\,\mathbb{V}(a^{\dagger}_{\text{out}},a_{\text{out}}) = 0 \tag{A.12}$$

These equations for $\mathbb{U}_{in/out}$ with boundary conditions (A.2) have the unique solution

$$\mathbb{U}_{\text{in}}(\tau) = \text{Texp}\left(-i\int_{-\infty}^{\tau} \mathrm{d}\tau'\ \mathbb{V}\left(a^{\dagger}_{\text{in}}(\tau'),a_{\text{in}}(\tau')\right)\right) , \tag{A.13}$$

$$\mathbb{U}_{\text{out}}(\tau) = \text{Texp}\left(-i\int_{\tau}^{+\infty} \mathrm{d}\tau'\ \mathbb{V}\left(a^{\dagger}_{\text{out}}(\tau'),a_{\text{out}}(\tau')\right)\right) , \tag{A.14}$$

where Texp is the time-ordered exponential. Then (A.11) becomes

$$\mathbb{U}_{\text{in}}(\tau) = \text{Texp}\left(-i\int_{-\infty}^{\tau} \mathrm{d}\tau'\ \mathbb{V}\left(a^{\dagger}_{\text{in}}(\tau'),a_{\text{in}}(\tau')\right)\right) , \tag{A.15}$$

$$\mathbb{U}_{\text{out}}(\tau) = \text{Texp}\left(-i\int_{\tau}^{+\infty} \mathrm{d}\tau'\ \mathbb{V}\left(a^{\dagger}_{\text{out}}(\tau'),a_{\text{out}}(\tau')\right)\right) . \tag{A.16}$$

We can expand perturbatively the S-matrix (A.9) to obtain

$$\mathbb{S} = \mathbb{I} + i\,\frac{1}{g}\,\mathbb{T} , \qquad \mathbb{T} = -g\int_{-\infty}^{\infty} \mathrm{d}\tau\,\mathbb{V}(\tau) + \cdots , \tag{A.17}$$

where $\mathbb{T}$ is called T-matrix. For our world-sheet string, we have $\mathbb{H}_f = H_2$ and $\mathbb{V} = H_4$.

A.2 S-matrix bootstrap

In a two-dimensional (1+1) integrable QFT, the bootstrap program allows to determine the S-matrix exactly from scattering data, together with the symmetries of the theory [8]. More precisely, the existence of local conserved charges implies that (*i*) there is no particle production, (*ii*) the final (out) state has the same particles' momenta than the initial one[1], and (*iii*) the $n \to n$ S-matrix factorizes into a product of $2 \to 2$ S-matrices. Asking that the S-matrix satisfies (*i*) and (*ii*) is enough to determine the $2 \to 2$ scattering amplitude, and then the $n \to n$ one can be found using (*iii*).

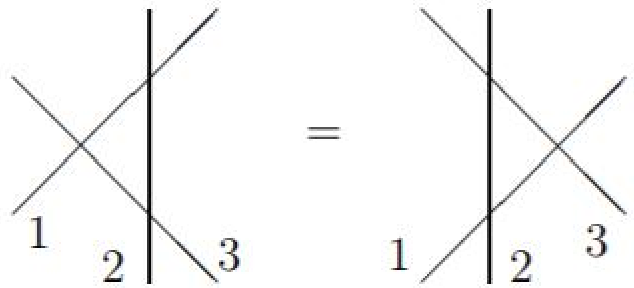


FIGURE A.1: Graphical representation of the Yang–Baxter equation.

Moreover, the two-particle S-matrix must satisfy the Yang–Baxter equation

$$S(p_2, p_3)S(p_1, p_3)S(p_1, p_2) = S(p_1, p_2)S(p_1, p_3)S(p_2, p_3)\,,$$

represented in Fig. A.1, which acts as a consistency condition for the S-matrix. The YBE is related to associativity of the symmetry algebra of the theory, and it can be used as a starting point to find the S-matrix, together with additional conditions, though depending on the theory there may be more convenient ways to complete the bootstrap program, i.e. to conjecture the structure of the S-matrix. See e.g. [107] for a review of the topic, or [9] for a pedagogical introduction.

[1] Though not necessarily in the same order.

Appendix B

Global Symmetry Currents

While the complete expressions for the gauge fixed global currents are quite involved, the explicit expressions at the quadratic level are relatively straightforward and can be found in the literature [58, 42] (however we will here follow the notations and conventions of [51]). We reproduce them here for completeness and consider their transformation properties under parity and time-reversal.

The world-sheet fields $Y_{a\dot{b}}, Z_{\alpha\dot{\beta}}$ and $\Psi_{a\dot{\alpha}}, \Upsilon_{\alpha\dot{a}}$ transform under the parity transformation, $\mathcal{P} : \sigma \to -\sigma$, as

$$\mathcal{P} : \{Y, Z, \Psi, \Upsilon\}(\sigma) \to \{Y, Z, i\Psi, i\Upsilon\}(-\sigma) \,, \tag{B.1}$$

where in particular the fermionic fields acquire a factor of i [42]. Time-reversal $\mathcal{T}$: $\tau \to -\tau$ is implemented by an anti-unitary operator, U_τ under which

$$U_\tau\{Y, Z, \Psi, \Upsilon\}(\tau)U_\tau^{-1} = \{Y, Z, \Psi, \Upsilon\}(-\tau) \,. \tag{B.2}$$

We will also define the operation $\mathcal{R}$ which acts on charges by reversing the contour of integration defining the charge. This reversal can be taken in two different senses (by a clockwise or counterclockwise rotation) and so we define $R_{\pm\pi} = \mathcal{R}_{\pm\pi}\mathcal{P}\mathcal{T}$. As the bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetry charges are purely local, their behaviour under parity and time-reversal is straightforward and follows from the corresponding action on the fundamental fields. That is from

$$\begin{aligned}
\mathbb{L}_a{}^b &= \int d\sigma \left[\frac{i}{2} \left(P^{b\dot{c}} Y_{a\dot{c}} - P_{a\dot{c}} Y^{b\dot{c}} \right) , + \frac{1}{2} \left(\Psi^*_{a\dot{\gamma}} \Psi^{b\dot{\gamma}} - \Psi^{*b\dot{\gamma}} \Psi_{a\dot{\gamma}} \right) \right] \\
\mathbb{R}_\alpha{}^\beta &= \int d\sigma \left[\frac{i}{2} \left(P^{\beta\dot{\gamma}} Z_{\delta\dot{\gamma}} - P_{\alpha\dot{\gamma}} Z^{\beta\dot{\gamma}} \right) + \frac{1}{2} \left(\Upsilon^*_{\alpha\dot{c}} \Upsilon^{\beta\dot{c}} - \Upsilon^{*\beta\dot{c}} \Upsilon_{\alpha\dot{c}} \right) \right] .
\end{aligned} \tag{B.3}$$

and making use of the antiunitarity of the time-reversal it is straightforward to see that

$$R_{\pm\pi} : \mathbb{L}_a{}^b \to -\mathbb{L}_a{}^b\,, \qquad R_{\pm\pi} : \mathbb{R}_\alpha{}^\beta \to -\mathbb{R}_\alpha{}^\beta\,. \tag{B.4}$$

Correspondingly, for the Hamiltonian and the world-sheet momentum

$$\begin{aligned}
\mathbb{H} = \int d\sigma\, \Big[&\frac{1}{2}P_{a\dot{a}}P^{*a\dot{a}} + \frac{1}{2}\acute{Y}^*_{a\dot{a}}\acute{Y}^{a\dot{a}} + \frac{1}{2}Y^*_{a\dot{a}}Y^{a\dot{a}} + \frac{1}{2}P_{\alpha\dot{\alpha}}P^{*\alpha\dot{\alpha}} + \frac{1}{2}\acute{Z}^*_{\alpha\dot{\alpha}}\acute{Z}^{\alpha\dot{\alpha}} + \frac{1}{2}Z^*_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} \\
&+ \frac{i}{2}\left(\Psi^*_{a\dot{\alpha}}\acute{\Psi}^{*a\dot{\alpha}} + \Psi_{a\dot{\alpha}}\acute{\Psi}^{a\dot{\alpha}}\right) + \Psi^*_{a\dot{\alpha}}\Psi^{*a\dot{\alpha}} + \frac{i}{2}\left(\Upsilon^*_{\alpha\dot{a}}\acute{\Upsilon}^{*\alpha\dot{a}} + \Upsilon_{\alpha\dot{a}}\acute{\Upsilon}^{\alpha\dot{a}}\right) + \Upsilon^*_{\alpha\dot{a}}\Upsilon^{*\alpha\dot{a}}\Big]\,, \\
\mathbb{P} = -\frac{1}{g}\int d\sigma\, &\left(P_{a\dot{a}}\acute{Y}^{a\dot{a}} + P_{\alpha\dot{\alpha}}\acute{Z}^{\alpha\dot{\alpha}} + i\Psi^*_{a\dot{\alpha}}\acute{\Psi}^{a\dot{\alpha}} + i\Upsilon^*_{\alpha\dot{a}}\acute{\Upsilon}^{\alpha\dot{a}}\right)\,,
\end{aligned} \tag{B.5}$$

we naturally have $R_{\pm\pi} : \{\mathbb{H}, \mathbb{P}\} \to \{-\mathbb{H}, -\mathbb{P}\}$.

The odd generators are defined by

$$\begin{aligned}
\mathbb{Q}_\alpha{}^b &= \frac{i}{\sqrt{2}}\int d\sigma\, e^{iX^-_C/2}\Big[P^{b\dot{c}}\Upsilon^*_{\alpha\dot{c}} - iY^{b\dot{c}}\Upsilon^*_{\alpha\dot{c}} - \acute{Y}^{b\dot{c}}\acute{\Upsilon}_{\alpha\dot{c}} + P^*_{\alpha\dot{\alpha}}\Psi^{b\dot{\alpha}} + iZ_{\alpha\dot{\alpha}}\Psi^{b\dot{\alpha}} - \acute{Z}_{\alpha\dot{\alpha}}\acute{\Psi}^{*b\dot{\alpha}}\Big]\,, \\
\mathbb{Q}^{\dagger\alpha}_b &= \frac{-i}{\sqrt{2}}\int d\sigma\, e^{-iX^-_C/2}\Big[P^*_{b\dot{c}}\Upsilon^{\alpha\dot{c}} + iY^*_{b\dot{c}}\Upsilon^{\alpha\dot{c}} - \acute{Y}^*_{b\dot{c}}\acute{\Upsilon}^{*\alpha\dot{c}} + P^{\alpha\dot{\alpha}}\Psi^*_{b\dot{\alpha}} \\
&\qquad\qquad - iZ^{*\alpha\dot{\alpha}}\Psi^*_{b\dot{\alpha}} - \acute{Z}^{*\alpha\dot{\alpha}}\acute{\Psi}_{b\dot{\alpha}}\Big]\,.
\end{aligned} \tag{B.6}$$

Due to the non-locality described by the contour C defining X^-_C the transformation is slightly more non-trivial:

$$\mathcal{PT} : X^-_C(\sigma) \to -X^-_{\tilde{C}}(-\sigma) = \frac{1}{g}\int_\infty^{-\sigma} d\sigma' P_M \acute{X}^M(\sigma') + \ldots \tag{B.7}$$

where $X^-_{\tilde{C}}$ is defined by a contour $\tilde{C}$ which starts at $\sigma = \infty$ rather than $\sigma = -\infty$. If we write this as $\int_{\tilde{C}} = \int_\infty^{-\infty} + \int_C$ we see that

$$-X^-_{\tilde{C}}(-\sigma) = \mathbb{P} - X^-_C(-\sigma) \tag{B.8}$$

so that $\mathcal{PT} : \mathbb{Q}_\alpha{}^b \to e^{-\frac{i}{2}\mathbb{P}}\mathbb{Q}_\alpha{}^b$, and hence

$$R_{\pm\pi} : \mathbb{Q}_\alpha{}^b \to -e^{-\frac{i}{2}\mathbb{P}}\mathbb{Q}_\alpha{}^b\,. \tag{B.9}$$

Similarly for the central charges we use the expressions in the literature [58, 42]

$$\mathbb{C} = -\frac{g}{2}\int_{-\infty}^{\infty} d\sigma\, \acute{X}^{-} e^{iX_{\mathbb{C}}^{-}}\ , \qquad \mathbb{C}^{\dagger} = -\frac{g}{2}\int_{-\infty}^{\infty} d\sigma\, \acute{X}^{-} e^{-iX_{\mathbb{C}}^{-}} \tag{B.10}$$

to find $\mathcal{PT} : \mathbb{C} \to -\frac{g}{2}\int_{-\infty}^{\infty} d\sigma\, \acute{X}^{-}(-\sigma)e^{i(-\mathbb{P}+X_{\mathbb{C}}^{-}(-\sigma))}$ and again when we include the reversal of integration orientation we find an additional sign. To summarize, we find using the notation of (2.2.11)

$$R_{\pm\pi} : \{Q_A{}^B, \mathbb{H}, \mathbb{C}, \mathbb{C}^{\dagger}\} \to \{-e^{-i\epsilon_{AB}\mathbb{P}} Q_A{}^B, -\mathbb{H}, -e^{-i\mathbb{P}}\mathbb{C}, -e^{i\mathbb{P}}\mathbb{C}^{\dagger}\}\ , \tag{B.11}$$

and we note the $e^{-i\alpha\mathbb{P}}$ are simply related to the inverse of the global braiding factors Θ^{I}_{J}.

Appendix C

Asymptotic Fields

We collect here the explicit formulae for the free on-shell fields, which are needed to define our asymptotic states. We will introduce the rapidity parameter θ in terms of which we can write the particle energy, momenta and fermionic wave functions

$$\epsilon = \cosh\theta\,, \quad p = \sinh\theta\,, \quad u(p) = \cosh\tfrac{\theta}{2}\,, \quad v(p) = \sinh\tfrac{\theta}{2}\,. \tag{C.1}$$

For the bosonic fields with commutation relations

$$[Y_{a\dot{a}}(\sigma), P^{b\dot{b}}(\sigma')] = i\delta(\sigma-\sigma')\delta_a^b\delta_{\dot{a}}^{\dot{b}}\,, \quad [Z_{\alpha\dot{\alpha}}(\sigma), P^{\beta\dot{\beta}}(\sigma')] = i\delta(\sigma-\sigma')\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}}\,, \tag{C.2}$$

we have the mode expansions (see (C.3))

$$\begin{aligned}
Y_{a\dot{a}}(\sigma) &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\epsilon}}\left[a_{a\dot{a}}(p)e^{-ip\sigma} + a^\dagger_{a\dot{a}}(p)e^{ip\sigma}\right], \\
P^{a\dot{a}}(\sigma) &= -i\int \frac{dp}{2\pi}\sqrt{\frac{\epsilon}{2}}\left[a^{a\dot{a}}(p)e^{-ip\sigma} - a^{\dagger a\dot{a}}(p)e^{ip\sigma}\right], \\
Z_{\alpha\dot{\alpha}}(\sigma) &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\epsilon}}\left[a_{\alpha\dot{\alpha}}(p)e^{-ip\sigma} + a^\dagger_{\alpha\dot{\alpha}}(p)e^{ip\sigma}\right], \\
P^{\alpha\dot{\alpha}}(\sigma) &= -i\int \frac{dp}{2\pi}\sqrt{\frac{\epsilon}{2}}\left[a^{\alpha\dot{\alpha}}(p)e^{-ip\sigma} - a^{\dagger\alpha\dot{\alpha}}(p)e^{ip\sigma}\right].
\end{aligned}$$

Additionally due to the reality condition for the bosonic fields, i.e. $Y_{a\dot{a}}(\sigma) = Y^*_{a\dot{a}}(\sigma)$ and $P_{a\dot{a}}(\sigma) = P^*_{a\dot{a}}(\sigma)$, the oscillators satisfy $(a_{a\dot{a}})^* = a^{\dagger a\dot{a}}$. For the fermionic fields with anti-commutators

$$\{\Upsilon^{*\alpha\dot{a}}(\sigma), \Upsilon_{\beta\dot{b}}(\sigma')\} = \delta(\sigma-\sigma')\delta^\alpha_\beta\delta^{\dot{a}}_{\dot{b}}\,, \quad \{\Psi^{*a\dot{\alpha}}(\sigma), \Psi_{b\dot{\beta}}(\sigma')\} = \delta(\sigma-\sigma')\delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{C.3}$$

we have the mode expansions

$$
\begin{aligned}
\Upsilon_{\alpha\dot{a}} &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{\epsilon}}\left(u(p)b_{\alpha\dot{a}}(p)e^{-ip\sigma} + v(p)b^{\dagger}_{\alpha\dot{a}}(p)e^{+ip\sigma}\right), \\
\Upsilon^{*\alpha\dot{a}} &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{\epsilon}}\left(v(p)b^{\alpha\dot{a}}(p)e^{-ip\sigma} + u(p)b^{\dagger\alpha\dot{a}}(p)e^{+ip\sigma}\right), \\
\Psi_{a\dot{\alpha}} &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{\epsilon}}\left(u(p)b_{a\dot{\alpha}}(p)e^{-ip\sigma} + v(p)b^{\dagger}_{a\dot{\alpha}}(p)e^{+ip\sigma}\right), \\
\Psi^{*a\dot{\alpha}} &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{\epsilon}}\left(v(p)b^{a\dot{\alpha}}(p)e^{-ip\sigma} + u(p)b^{\dagger a\dot{\alpha}}(p)e^{+ip\sigma}\right).
\end{aligned}
\tag{C.4}
$$

Appendix D

Hopf Algebra Consistency Conditions

We will present here the consistency conditions of the graded Hopf algebra, expressed in terms of the coefficients $m_{\mathrm{ab}}^{\mathrm{c}}$, $\mu_{\mathrm{a}}^{\mathrm{bc}}$, $s_{\mathrm{a}}^{\mathrm{b}}$, ϵ_{a} defined by

$$e_{\mathrm{a}}e_{\mathrm{b}} = m_{\mathrm{ab}}^{\mathrm{c}}e_{\mathrm{c}}\,, \qquad \Delta(e_{\mathrm{a}}) = \mu_{\mathrm{a}}^{\mathrm{bc}}e_{\mathrm{b}} \otimes e_{\mathrm{b}}\,,$$
$$s(e_{\mathrm{a}}) = s_{\mathrm{a}}^{\mathrm{b}}e_{\mathrm{b}}\,, \qquad \epsilon(e_{\mathrm{a}}) = \epsilon_{\mathrm{a}}\,, \tag{D.1}$$

and $\epsilon^{\mathrm{a}}e_{\mathrm{a}} = \mathbb{1}$. To be consistent with our grading we define $\epsilon(e_{\mathrm{a}}) = 0$ for $[\mathrm{a}] = 1$. This is in keeping with our physical interpretation of the co-unit as the vacuum expectation value of the generators. We can express the consistency conditions as

$$m_{\mathrm{ab}}^{\mathrm{c}}\epsilon^{\mathrm{a}} = m_{\mathrm{ba}}^{\mathrm{c}}\epsilon^{\mathrm{a}} = \delta_{\mathrm{b}}^{\mathrm{c}}\,, \qquad \mu_{\mathrm{a}}^{\mathrm{bc}}\epsilon_{\mathrm{c}} = \mu_{\mathrm{a}}^{\mathrm{cb}}\epsilon_{\mathrm{c}} = \delta_{\mathrm{a}}^{\mathrm{b}}$$
$$m_{\mathrm{ab}}^{\mathrm{c}}m_{\mathrm{de}}^{\mathrm{b}} = m_{\mathrm{ad}}^{\mathrm{b}}m_{\mathrm{be}}^{\mathrm{c}}\,, \qquad \mu_{\mathrm{a}}^{\mathrm{bc}}\mu_{\mathrm{b}}^{\mathrm{de}} = \mu_{\mathrm{a}}^{\mathrm{db}}\mu_{\mathrm{b}}^{\mathrm{ec}}\,,$$
$$m_{\mathrm{ij}}^{\mathrm{b}}s_{\mathrm{k}}^{\mathrm{j}}\mu_{\mathrm{a}}^{\mathrm{ik}} = m_{\mathrm{ji}}^{\mathrm{b}}s_{\mathrm{k}}^{\mathrm{j}}\mu_{\mathrm{a}}^{\mathrm{ki}} = \epsilon_{\mathrm{a}}\epsilon^{\mathrm{b}}\,,$$

along with $m_{\mathrm{ab}}^{\mathrm{c}}\epsilon_{\mathrm{c}} = \epsilon_{\mathrm{a}}\epsilon_{\mathrm{b}}$ and

$$m_{\mathrm{ab}}^{\mathrm{i}}s_{\mathrm{i}}^{\mathrm{c}} = (-1)^{[\mathrm{a}][\mathrm{b}]}s_{b}^{\mathrm{j}}s_{\mathrm{a}}^{\mathrm{k}}m_{\mathrm{jk}}^{\mathrm{c}}\,, \qquad \mu_{\mathrm{i}}^{\mathrm{ab}}s_{\mathrm{c}}^{\mathrm{i}} = (-1)^{[\mathrm{a}][\mathrm{b}]}s_{\mathrm{j}}^{\mathrm{a}}s_{\mathrm{k}}^{\mathrm{b}}\mu_{\mathrm{c}}^{\mathrm{kj}}\,,$$
$$(-1)^{[k][j]}\mu_{\mathrm{a}}^{\mathrm{ij}}\mu_{\mathrm{b}}^{\mathrm{kl}}m_{\mathrm{ik}}^{\mathrm{c}}m_{\mathrm{jl}}^{\mathrm{d}} = m_{\mathrm{ab}}^{\mathrm{i}}\mu_{\mathrm{i}}^{\mathrm{cd}}\,.$$

The skew-coproduct can be defined by

$$\Delta^{\mathrm{op}}(e_{\mathrm{a}}) = (-1)^{[\mathrm{b}][\mathrm{c}]}\mu_{\mathrm{a}}^{\mathrm{bc}}e_{\mathrm{c}} \otimes e_{\mathrm{b}}\,, \tag{D.2}$$

and the skew-antipode, s', can be then consistently defined if

$$(-1)^{[\mathrm{b}][\mathrm{c}]}\mu_{\mathrm{a}}^{\mathrm{bc}}(s')_{\mathrm{c}}^{\mathrm{d}}m_{\mathrm{db}}^{\mathrm{e}} = (-1)^{[\mathrm{b}][\mathrm{c}]}\mu_{\mathrm{a}}^{\mathrm{bc}}(s')_{\mathrm{b}}^{\mathrm{d}}m_{\mathrm{cd}}^{\mathrm{e}} = \epsilon_{\mathrm{a}}\epsilon^{\mathrm{e}} \; . \tag{D.3}$$

Appendix E

AdS strings/gauge theory duality

In this appendix we will introduce briefly the dictionary between the world-sheet string in $AdS_5 \times S^5$ and the corresponding sector of the SYM operators. The first quantities to take into consideration are the string energy $E = i\partial_t$ and angular momentum $J = -i\partial_\psi$. The former corresponds to the conformal dimension Δ of the SYM operator, while J in the gauge theory is the $U(1)$ R-charge rotating ϕ_5 into ϕ_6 for example.

In order to understand the connection between these quantities, let us examine the implications of the BMN limit in the dual theory. Remembering the definition of the light-cone coordinates, we have the following relation between the conjugated variables

$$2\,p^- = i\partial_{x^+} = \mu\, i(\partial_t + \partial_\psi) = \mu\,(E - J) \tag{E.1}$$

$$2\,p^+ = i\partial_{x^-} = \frac{1}{\mu\,R^2} i(\partial_t - \partial_\psi) = \frac{E+J}{\mu\,R^2}\,, \tag{E.2}$$

and we also define $2\,p^- = \mathcal{H}$, the new light-cone Hamiltonian. Then, in the limit $R \to \infty$, the momentum p^+ vanishes unless $J \sim R^2$ but we see from (E.2) that we need also $E \sim J$. Now we can translate the r.h.s. of (E.1) in the gauge theory language to obtain the key relation

$$\frac{E_{lc}}{\mu} \longleftrightarrow \Delta - J\,. \tag{E.3}$$

Returning to the discussion of $R \to \infty$ with $J \sim R^2$, we need the AdS/CFT relation $R^4 = \alpha'^2 g_{YM}^2 N$ to see that the corresponding limit in the gauge theory is $N \to \infty$ with $J \sim \sqrt{N}$ and g_{YM} fixed. Moreover, because of the condition $E \sim J$, the limit selects operators with $\Delta \sim J$, and these are the protected ones, e.g. $\mathrm{Tr}[Z^J]$ or $\mathrm{Tr}[\phi^i Z^J]$.

Let us consider now a small number of insertions of operators with $\Delta - J = 1$, the so-called "impurities", which violate the protectedness of the BMN operator. Although the usual loop-counting parameter λ is divergent, we can write the quantum corrections as an expansion in g_{YM}^2.

Now we have the tools to identify string states and SYM operators: we need to compare those that satisfy (E.3). Thus the operator dual to the ground state $|0,p^+\rangle$ ($E_{lc} = 0$) is Tr Z^J, which has the protected dimension $\Delta = J$. For the zero-mode excitations we have the following relations with the chiral primaries :

$$\begin{array}{llcll} E_{\rm lc} = 0 & |0,p^+\rangle & \longleftrightarrow & {\rm Tr}\; Z^J & \Delta - J = 0 \\ E_{\rm lc} = \mu & \alpha_0^{\dagger\, I}\,|0,p^+\rangle & \longleftrightarrow & {\rm Tr}\,(\phi^I\, Z^J) & \Delta - J = 1 \\ E_{\rm lc} = 2\mu & \alpha_0^{\dagger\, I}\,\alpha_0^{\dagger\, K}\,|0,p^+\rangle & \longleftrightarrow & \sum_{j=0}^{J} {\rm Tr}\,(\phi^I\, Z^j\, \phi^K\, Z^{J-j}) & \Delta - J = 2 \\ & & \ldots & & \end{array}$$

The other zero-modes in each level correspond to the descendants of the chiral primaries, e.g. at first level ($E_{\rm lc} = \mu$) to describe the rest of the 8 bosonic excitations we have $\alpha_{0\,\mu}^{\dagger}\,|0,p^+\rangle$, which is dual to Tr $(D_\mu Z\, Z^{J-1})$, and similarly for the 8 fermionic ones. For non-zero modes building the dictionary is not as straightforward: we see from (E.3) that they cannot be dual to protected operators, "impurities" must be added to have the right dimension Δ according to (E.3). To clarify this statement, let us look at the first mode:

$$E_{\rm lc} = 2\mu\,\sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}} \qquad\qquad \alpha_{-n}^{I}\,\tilde{\alpha}_{-n}^{K}\,|0,p^+\rangle\,. \tag{E.4}$$

The dual operator is

$$\mathcal{O}_n^{ij} \sim \sum_{l=0}^{J} {\rm Tr}\,(\phi_i\, Z^l\, \phi_j\, Z^{J-l})\, e^{2\,\pi\, i\, n\, l/J} \qquad\qquad \Delta_{pl} = J + 2 + \frac{g_{YM}^2\, N}{J^2}\, n^2 + \mathcal{O}(g_{YM}^2)$$

and using the relations above it can be shown that they satisfy (E.3) at first order in g_{YM}^2.

Appendix F

Higher-Order Potential Terms

We record here the quartic and sextic terms of the potential to $\mathcal{O}(g^4)$ which are used to compute the Feynman rules for the generalized LL model in Section 5.2.3.

Quartic terms up to order g^4:

$$\begin{aligned}
V_{\text{quartic}} &= \frac{b_0}{2}(\varphi^{*2}(\partial_x\varphi)^2 + \varphi^2(\partial_x\varphi^*)^2) + \frac{g^2}{2}\Big[b_1\left(\varphi^2(\partial_x^2\varphi^*)^2\right.\\
&+ \left.\varphi^{*2}(\partial_x^2\varphi)^2 + 8\partial_x\varphi\partial_x\varphi^*(\varphi^*\partial_x^2\varphi + \varphi\partial_x^2\varphi^*)\right) + 4(3b_1+2b_2)(\partial_x\varphi)^2(\partial_x\varphi^*)^2\Big]\\
&- g^4\Big[b_3\left(\varphi^{*2}(\partial_x^3\varphi)^2 + (\partial_x^3\varphi^*)^2\varphi^2 + 18(\partial_x\varphi^*)^2(\partial_x^2\varphi)^2 + 18(\partial_x^2\varphi^*)^2(\partial_x\varphi)^2\right.\\
&\qquad +6(\varphi\partial_x\varphi^* - \varphi^*\partial_x\varphi)\left(\partial_x^3\varphi^*\partial_x^2\varphi - \partial_x^3\varphi\partial_x^2\varphi^*\right)\\
&\qquad +6\partial_x^3\varphi^*\varphi\partial_x^2\varphi^*\partial_x\varphi + 6\varphi^*\partial_x^3\varphi\partial_x\varphi^*\partial_x^2\varphi\\
&\qquad \left.+6\partial_x\varphi\partial_x\varphi^*\left(6\partial_x^2\varphi^*\partial_x^2\varphi - \partial_x^3\varphi^*\partial_x\varphi - \partial_x^3\varphi\partial_x\varphi^*\right)\right)\\
&\qquad +8b_4\,\partial_x\varphi^*\partial_x^2\varphi^*\partial_x\varphi\partial_x^2\varphi + 2b_5\left(\partial_x\varphi^*\partial_x^2\varphi + \partial_x^2\varphi^*\partial_x\varphi\right)^2\Big] + O(g^6)\,.
\end{aligned}$$

Sextic terms up to order g^4:

$$
\begin{aligned}
V_{\text{sextic}} \quad = \quad & -\frac{b_0}{4}\varphi\varphi^*(\varphi^*\partial_x\varphi+\varphi\partial_x\varphi^*)^2+\frac{g^2}{2}\Big[b_1\Big(8|\varphi|^2|\partial_x\varphi|^4 \\
& \quad +\varphi^*\partial_x^2\varphi(2|\varphi|^2|\partial_x\varphi|^2+\tfrac{1}{2}|\varphi|^2\varphi^*\partial_x^2\varphi-3\varphi^{*2}(\partial_x\varphi)^2 \\
& \quad +\varphi\partial_x^2\varphi^*(2|\varphi|^2|\partial_x\varphi|^2+\tfrac{1}{2}|\varphi|^2\varphi\partial_x^2\varphi^*-3\varphi^2(\partial_x\varphi^*)^2\Big) \\
& +8b_2(\varphi^{*2}(\partial_x\varphi)^2+\varphi^2(\partial_x\varphi^*)^2)\Big] \\
& +\frac{g^4}{2}\Big[-b_3\,(2\varphi^{*2}\partial_x^3\varphi^*\partial_x^3\varphi\varphi^2-3\partial_x^3\varphi\varphi^2(\partial_x\varphi^*)^3+3\varphi^*\partial_x^3\varphi \\
& +\varphi^2\partial_x\varphi^*\partial_x^2\varphi^*+\varphi^{*3}\partial_x^3\varphi^2\varphi+36\varphi^*\varphi(\partial_x\varphi^*)^2(\partial_x^2\varphi)^2 \\
& +9\varphi^*\partial_x^3\varphi^*\varphi^2\partial_x\varphi^*\partial_x^2\varphi+36\varphi^2(\partial_x\varphi^*)^2\partial_x^2\varphi^*\partial_x^2\varphi-3\varphi^{*2}\partial_x^3\varphi^*(\partial_x\varphi)^3 \\
& +36\varphi^*\varphi(\partial_x^2\varphi^*)^2(\partial_x\varphi)^2+36(\partial_x\varphi^*)^3(\partial_x\varphi)^3+9\varphi^*\partial_x^3\varphi^*\varphi^2\partial_x^2\varphi^*\partial_x\varphi \\
& -3\partial_x^3\varphi^*\varphi^2(\partial_x\varphi^*)^2\partial_x\varphi-6\varphi^*\partial_x^3\varphi^*\varphi\partial_x\varphi^*(\partial_x\varphi)^2+36\varphi^2\partial_x\varphi^*(\partial_x^2\varphi^*)^2\partial_x\varphi \\
& +72\varphi(\partial_x\varphi^*)^2\partial_x^2\varphi^*(\partial_x\varphi)^2+36\varphi^*\partial_x\varphi^*\partial_x^2\varphi^*(\partial_x\varphi)^3+9\varphi^{*2}\partial_x^3\varphi\varphi\partial_x\varphi^*\partial_x^2\varphi \\
& +9\varphi^{*2}\partial_x^3\varphi\varphi\partial_x^2\varphi^*\partial_x\varphi-6\varphi^*\partial_x^3\varphi\varphi(\partial_x\varphi^*)^2\partial_x\varphi-3\varphi^{*2}\partial_x^3\varphi\partial_x\varphi^*(\partial_x\varphi)^2 \\
& +3\varphi^{*2}\partial_x^3\varphi^*\varphi\partial_x\varphi\partial_x^2\varphi+36\varphi^{*2}\partial_x^2\varphi^*(\partial_x\varphi)^2\partial_x^2\varphi+36\varphi(\partial_x\varphi^*)^3\partial_x\varphi\partial_x^2\varphi \\
& +36\varphi^{*2}\partial_x\varphi^*\partial_x\varphi(\partial_x^2\varphi)^2+72\varphi^*(\partial_x\varphi^*)^2(\partial_x\varphi)^2\partial_x^2\varphi+\varphi^*(\partial_x^3\varphi^*)^2\varphi^3 \\
& +108\varphi^*\varphi\partial_x\varphi^*\partial_x^2\varphi^*\partial_x\varphi\partial_x^2\varphi+3\varphi^{*3}\partial_x^3\varphi\partial_x\varphi\partial_x^2\varphi+3\partial_x^3\varphi^*\varphi^3\partial_x\varphi^*\partial_x^2\varphi^*) \\
& \quad -b_4\,(8\partial_x^2\varphi^*\partial_x^2\varphi\,(\varphi^{*2}(\partial_x\varphi)^2+\varphi^2(\partial_x\varphi^*)^2) \\
& +8\partial_x\varphi^*\partial_x\varphi\,(\varphi^2(\partial_x^2\varphi^*)^2+\varphi^{*2}(\partial_x^2\varphi)^2-2\varphi(\partial_x\varphi^*)^2\partial_x^2\varphi-2\varphi^*\partial_x^2\varphi^*(\partial_x\varphi)^2 \\
& \qquad\qquad +4\varphi\partial_x\varphi^*\partial_x^2\varphi^*\partial_x\varphi+4\varphi^*\partial_x\varphi^*\partial_x\varphi\partial_x^2\varphi+8(\partial_x\varphi^*)^2(\partial_x\varphi)^2)) \\
& \quad -8b_5\,(\partial_x\varphi^*\partial_x^2\varphi+\partial_x^2\varphi^*\partial_x\varphi)\,(\varphi(\partial_x\varphi^*)^2\partial_x\varphi+\varphi^*\partial_x\varphi^*(\partial_x\varphi)^2 \\
& \qquad\qquad\qquad +(\varphi^*)^2\partial_x\varphi\partial_x^2\varphi+\varphi^2\partial_x\varphi^*\partial_x^2\varphi^*) \\
& \quad -64b_6(\partial_x\varphi^*)^3(\partial_x\varphi)^3\Big]+O(g^6)\,.
\end{aligned}
$$

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