

Fibrations of Predicates and Bicategories of Relations

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Declaration

I hereby declare:

- that this work has not been submitted as an exercise for a degree at this or any other University;
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Summary

We reconcile the two different category-theoretic semantics of regular theories in predicate logic. A 2-category of *regular fibrations* is constructed, as well as a 2-category of *regular proarrow equipments*, and it is shown that the two are equivalent. A regular equipment is a *cartesian equipment* satisfying certain axioms, and a cartesian equipment is a slight generalization of a cartesian bicategory.

This is done by defining a tricategory 2-Prof whose objects are bicategories and whose morphisms are category-valued profunctors, and then defining an *equipment* to be a pseudo-monad in this tricategory. The resulting notion of equipment is compared to several existing ones. Most importantly, this involves showing that every pseudo-monad in 2-Prof has a *Kleisli object*. A strict 2-category of equipments, over locally discrete base bicategories, is identified, and cartesian equipments are defined to be the cartesian objects in this 2-category. Thus cartesian equipments themselves form a 2-category, and this is shown to admit a 2-fully-faithful functor from the 2-category of regular fibrations. The cartesian equipments in the image of this functor are characterized as those satisfying certain axioms, and hence a 2-category of *regular equipments* is identified that is equivalent to that of regular fibrations.

It is then shown that a regular fibration admits *comprehension* for predicates if and only if its corresponding regular equipment admits *tabulation* for morphisms, and further that the presence of tabulations for morphisms is equivalent to the existence of *Eilenberg–Moore objects* for co-monads. We conclude with a brief examination of the two different constructions of the effective topos, via triposes and via assemblies, in the light of the foregoing.

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Chapter 1

Introduction

1.1 Background

This work is intended primarily as a contribution to the category-theoretic understanding of predicate logic, with an eye to clarifying the relationship between the two different constructions of realizability toposes. The following section gives more details on the motivation behind this work, the next explains its development and major results, and the last gives a detailed outline of the remaining chapters.

1.1.1 Motivation

For our purposes, a *logic* specifies, given a collection of types, and terms that map from one type to another, and of predicates, each of which lives over some type, and derivations or proofs that map from one predicate to another, a set of admissible ways to build new types, terms, predicates and derivations from existing ones. A theory T over a logic is then given by a collection of basic types and terms and of (equational) *axioms* (equations between terms), and a collection of basic predicates and derivations and of (propositional) axioms (equations between derivations). Traditionally, one did not distinguish between different derivations of the same entailment, so that a collection of derivations and propositional axioms is determined by a collection of statements that one predicate entails another. But we will take the view that it is useful to keep different proofs distinct — one might say that we are doing type theory, rather than logic as traditionally understood.

Category theory formalizes this situation in one of the following two ways (see e.g. [Law69, Jac99] and [FŠ90, CW87] respectively):

1. The types and terms form a category \mathbf{B}_T , with equality on its morphisms

generated by the equational axioms of T . The predicates over each type X form a category $\mathbf{E}_T(X)$, whose morphisms $P \rightarrow Q$ are given by (proofs of) entailments $P(x) \vdash Q(x)$, and the propositional axioms furnish an equality relation on these. The terms $t: X \rightarrow Y$ act on these categories by substitution, so as to make $\mathbf{E}_T(-)$ a pseudo-functor $\mathbf{B}_T^{\text{op}} \rightarrow \mathcal{C}at$, or a fibration over \mathbf{B}_T .

2. A bicategory $\text{Rel}(T)$ is formed, whose objects are the types, and in which a morphism $X \multimap Y$ is a *relation* from X to Y , that is, a predicate on $X \times Y$. The composite of $R(x, y)$ and $S(y, z)$ is the relation $\exists y. R(x, y) \wedge S(y, z)$, and the 2-cells are morphisms of predicates as above. Each term $t: X \rightarrow Y$ gives rise to a relation $t_\bullet: X \multimap Y$, given by $tx = y$ and called the *graph* of t . The equational axioms of T determine propositional equations between graphs.

Notice that the first (*fibrational*) approach requires very little structure to be present in the theory T . On the other hand, the second (*relational* or *bicategorical*) approach requires that (the logic underlying) T have at least finite conjunctions and the existential quantifier; that is, that T be a *regular theory*.

By the usual ‘yoga’ (to use Grothendieck’s term) of categorical logic, syntactic models such as these carry structure determined by the logic underlying the theory T ; more general models are structures of the same kind, and an interpretation of the theory in a model is a homomorphism. In the above two cases, the most common kinds of ‘model’ are the subobject fibrations and bicategories of relations of regular categories. But these two kinds of structure also arise in the two distinct recipes for constructing *realizability toposes*: one approach [Hyl82] goes via fibrations, and the other [CFŠ88] via bicategories. The initial motivation for the research described here was to understand the relationship between these two constructions.

1.2 Outline

1.2.1 Development and results

We will show that the two ways given above of describing regular theories and their models are equivalent. That is, there is a kind of fibration called a *regular fibration*, and a kind of bicategory, or rather proarrow equipment [Woo82], that we call a *regular equipment*, and the bicategories of which these are the objects are equivalent. In particular, the syntactic examples above correspond to each other, and we describe also how the two constructions of the effective topos fit into this framework.

A regular fibration is a bifibration with fibred finite products, satisfying the Frobenius condition and the Beck–Chevalley conditions for certain (*product-absolute*) pullback squares. Structures like these have been studied before, although except for in [Pav96] this has usually been restricted to those fibrations whose fibres are preorders. In the syntactic case described above, these are the term models that record only the existence of a proof of one proposition from another. Our results apply in full generality.

Similarly, the locally preordered versions of the bicategories described in (2) above are well known as *allegories* [FŠ90]. The allegories that arise in the ‘regular’ context carry certain extra structure, making them *unitary* and *pre-tabular*. We show that such allegories are the same thing as *bicategories of relations* [CW87]. These are locally ordered *cartesian bicategories* [CKWW08] satisfying some extra axioms.

In order to construct an equivalence between regular fibrations and cartesian bicategories, it is necessary to equip the latter with distinguished subcategories of morphisms with right adjoints, making them into proarrow equipments. Intuitively, this lets a cartesian bicategory remember the difference, which fibrations account for, between *functions* or *terms* on the one hand, and *functional relations* on the other. So we are looking for a notion of *cartesian equipment*.

There are several definitions of equipments in the literature, namely Wood’s original one [Woo82], Shulman’s *framed bicategories* [Shu08], and the (strictly more general) equipments of Carboni et. al. [CKVW98]. We give an abstract definition, involving the tricategory whose objects are bicategories and whose morphisms are category-valued profunctors, that subsumes those of Wood and of Shulman, whose relation to that of Carboni et. al. is clear, and that is quite similar to Verity’s notion of *double bicategory* [Ver92]. We also show that our equipments form a category that is equivalent to the ordinary category underlying Shulman’s strict 2-category of framed bicategories, and so we may take 2-cells between equipment-morphisms to be transformations between the associated framed functors, yielding a 2-category of equipments.

A cartesian equipment is then defined to be a cartesian object in this last 2-category, and a *regular equipment* to be a cartesian one satisfying some well-known axioms; we show that cartesian bicategories are a special case of cartesian equipments, and that the 2-category of regular equipments is equivalent to that of regular fibrations, as expected. We can then show that (suitable notions of) *comprehension* in a fibration and *tabulation* in an equipment correspond to each other, and that completion with respect to these is equivalent, in the preordered case, to one of the steps in the construction of the effective topos.

1.2.2 Detailed outline

Chapter 2 begins with basic background definitions, before going on to describe the syntax of regular logic and its semantics in regular fibrations. Comprehension in regular fibrations is also discussed. Section 2.1.4 shows that a regular theory gives rise to a ‘syntactic’ or classifying regular fibration, a result that we will not make essential further use of but that it is worth giving in the context of section 2.1 as a whole. The next section defines allegories and the structures on them that we want, and describes idempotents and the construction of the universal allegory in which a class of them splits. The last section of the chapter defines bicategories of relations and proves that they are equivalent to unitary pre-tabular allegories.

Chapter 3 introduces more new ideas and results than the preceding one. It starts with definitions of adjunctions and mates and of monads and modules in a bicategory. This material is of course very well known, but we present the theory of monads and modules in what seems to be a somewhat original way. Section 3.1 concludes with definitions of monoidal bicategories and pseudo-monads, which will be used in chapter 4. As mentioned above, we want to define equipments to be pseudo-monads in the tricategory of bicategories and category-valued profunctors; in order to define this tricategory we mimic the definition of the usual bicategory of profunctors as consisting of presheaf categories and cocontinuous functors. So we spend section 3.2, the remainder of chapter 3, defining and exploring the properties of bicategorical colimits. In particular, our description of 2-dimensional (co)ends appears to be new, as do the results of section 3.2.4 on computing bicategorical colimits in *Cat*.

Chapter 4 is the core of this work. In it we define the tricategory 2-Prof as promised, and show that it admits the construction of Kleisli objects for pseudo-monads. This is what enables us to go on and show, in section 4.1.3, that to give a pseudo-monad in 2-Prof , satisfying certain properties, is precisely to give a proarrow equipment in the sense of Wood [Woo82]. The remainder of that section compares our notion of equipment to Shulman’s notion [Shu08] of framed bicategory, showing that together with their morphisms (equipment-morphisms having been defined) they make up equivalent categories. Even though our abstract approach to equipments via pseudo-monads works well for 0- and 1-cells, it does not quite go through when it comes to 2-cells. Section 5.1.2 discusses how we might rectify this, and it is certainly work that ought to be done, but for our purposes here we can get away with simply defining equipment 2-cells to be transformations between corresponding functors between framed bicategories.

Section 4.2 is where our earlier work begins to bear fruit. Section 4.2.1 defines

what it is for an equipment to be *cartesian*, and gives equivalent descriptions of this structure in both equipments and framed bicategories. In section 4.2.2 it is shown that Shulman’s construction [Shu08, theorem 14.2] of a monoidal equipment from a regular fibration extends to a functor from the bicategory of regular fibrations to that of cartesian equipments, and further that this functor is fully faithful. The construction of a would-be right inverse to its action on objects shows that a regular fibration will only result if two additional axioms are assumed to hold in a given cartesian bicategory. One of these is well-known, and the second is a Beck–Chevalley-type condition that automatically holds in the locally ordered case when a simpler *Frobenius* axiom holds, as well as in the cases of bicategories of spans and of relations, which may explain why it has not previously been considered in the bicategorical context. With this done, we have an equivalence of bicategories between regular fibrations and these *regular equipments*. The last part of section 4.2 compares comprehension in regular fibrations to tabulation in regular equipments, showing that they are equivalent modulo the equivalence of bicategories just noted. The existence of tabulation is also shown to be equivalent to the existence of *Eilenberg–Moore objects* for co-monads. Chapter 4 ends with an application to the original motivation for our work: a discussion of the effective topos and the relationship between its two constructions, through the lens what we have already done.

Finally, chapter 5 reviews the results of the preceding three chapters, noting some links with existing work. We conclude with some prospects for future work, and some ideas on how to go about doing it: further elaboration of the abstract approach to equipments in section 4.1, and an attempt to generalize the equipment side of the correspondence we have established in order to go beyond the regular context. There is also reason to hope that the latter may help to connect our work with some other abstract approaches to realizability.

Chapter 2

Categories, fibrations and allegories

This chapter serves as background on the structures that will be used in those to come. After giving some very basic definitions, we define what is meant by *regular logic*, and then discuss the fibrations in which regular theories find their models, namely *regular fibrations*. We show that any regular theory gives rise to a syntactic model. Then the definition of *allegory* is recalled and the splitting of idempotents described, material that will be used later to connect our work with one of the constructions of the effective topos. Because the structures we will go on to use are a slightly generalized version of cartesian bicategories, we show that certain locally ordered cartesian bicategories, namely *bicategories of relations*, are the same as certain allegories, namely the *unitary pre-tabular* ones.

It is assumed that the reader is familiar with elementary category theory, as expounded in e.g. [Mac98], as well as the theory of enriched categories, for which see e.g. [Kel82], and with ‘formal category theory’, i.e. those parts of ordinary category theory, such as the theory of adjunctions, monads and Kan extensions, that can be replicated in 2-categories other than $\mathcal{C}at$.

Everything we talk about will be assumed to be ‘weak’ or ‘pseudo’ by default — if something is strict or lax we will say so. A ‘2-category’ is therefore a bicategory, a ‘functor’ is a pseudofunctor, and so on. On the other hand, we will make broad use of coherence and strictification theorems in order to simplify definitions and calculations. For example, monoidal categories and bicategories will be (mostly) silently assumed to have been strictified.

Ordinary (possibly monoidal) categories are written in bold face: **Cat**, **Gray**, 2-categories in ‘calligraphic’: \mathcal{K} , $\mathcal{C}at$, and 3-categories with ‘blackboard

bold’: 2-Cat , 2-Prof . Transformations and other 2-cells are written with a double arrow: $\alpha: F \Rightarrow G$, extranaturals (section 3.2.2) with a dotted arrow \Rightarrow . Modifications and other 3-cells are written with a triple arrow: $m: \alpha \Rrightarrow \beta$.

Identities are called 1 and terminal objects are called $\mathbf{1}$.

2.1 Regular fibrations and regular logic

2.1.1 Basic definitions

We give some elementary definitions in order to fix terminology and notation.

2.1.1 Definition. The *image* of a morphism $f: A \rightarrow B$ is a factorisation

$$f = A \xrightarrow{e} \text{im}(f) \xrightarrow{m} B$$

in which m is a monomorphism, and such that in any other such factorisation $f = m'e'$, $m' \subseteq m$ as subobjects of B .

2.1.2 Definition. A *regular category* is a category with finite limits in which every morphism has an image, and in which images are pullback-stable; that is, if $f: A \rightarrow B$ and $g: C \rightarrow B$, then $g^* \text{im}(f) \cong \text{im}(g^* f)$.

We assume familiarity with the notions of fibrations and of indexed categories, and of the equivalence between the two. In fact, we will rarely distinguish between them, and will mostly use the term ‘fibration’ to denote either concept. A *bifibration* is of course a functor that is both a fibration and an opfibration. We will write f^* etc. for the pullback functors of fibrations and either $f_!$ or \exists_f for the pushforwards of opfibrations.

The 2-category \mathcal{Fib} of fibrations can then be thought of as the ‘2-category of elements’ (def. 3.2.14) of either of two equivalent functors

$$\mathcal{Fib}(-) \sim [-^{\text{op}}, \mathcal{Cat}]: \mathcal{Cat}^{\text{co op}} \rightarrow 2\text{-Cat}$$

2.1.3 Definition. The 2-category \mathcal{Fib} is defined as follows:

- an object is a pair of a category \mathbf{B} and a fibration \mathbf{E} over \mathbf{B} ;
- a morphism $(\mathbf{B}_1, \mathbf{E}_1) \rightarrow (\mathbf{B}_2, \mathbf{E}_2)$ is a functor $F: \mathbf{B}_1 \rightarrow \mathbf{B}_2$ and a morphism of fibrations $\phi: \mathbf{E}_1 \rightarrow F^* \mathbf{E}_2$, i.e. either a natural transformation $\phi_X: \mathbf{E}_1(X) \rightarrow \mathbf{E}_2(FX)$ or a cartesian-morphism-preserving functor

$\mathbf{E}_1 \rightarrow \mathbf{E}_2$ between total categories that fits into a commuting square

$$\begin{array}{ccc} \mathbf{E}_1 & \xrightarrow{\phi} & \mathbf{E}_2 \\ \downarrow & & \downarrow \\ \mathbf{B}_1 & \xrightarrow{F} & \mathbf{B}_2 \end{array}$$

- a 2-cell $(F, \phi) \rightarrow (G, \gamma)$ is a transformation $\alpha: F \Rightarrow G$ such that

$$\begin{array}{ccc} \mathbf{E}_1 & \xrightarrow{\phi} & F^* \mathbf{E}_2 \\ & \searrow \gamma & \downarrow \alpha^* \mathbf{E}_2 \\ & & G^* \mathbf{E}_2 \end{array}$$

commutes.

Fib then has a locally full sub-2-category $\mathcal{B}iFib$ consisting of bifibrations, opcartesian-morphism-preserving fibration morphisms and all fibration 2-cells.

2.1.4 Definition. A *monoidal (bi)fibration* is given by a pair of monoidal categories together with a functor between them that is both (strong) monoidal and a (bi)fibration.

2.1.5 Proposition ([Shu08, theorem 12.7]). *If \mathbf{B} is a cartesian monoidal category, then the category of monoidal fibrations over \mathbf{B} is equivalent (via the usual Grothendieck construction and its inverse) to the category of (pseudo)functors from \mathbf{B}^{op} to the 2-category of monoidal categories.*

2.1.6 Definition ([Str81, 2.8]). Let \mathbf{C} and \mathbf{B} be categories. A *two-sided fibration* from \mathbf{B} to \mathbf{C} is given by a span $(p, q): \mathbf{E} \rightarrow \mathbf{C} \times \mathbf{B}$ such that

- p is a fibration whose chosen cartesian lifts are q -vertical (i.e. they are inverted by q);
- q is an opfibration whose chosen opcartesian lifts are p -vertical;
- for any composable cartesian-opcartesian pair $i^*x \rightarrow x \rightarrow j_!x$ in \mathbf{E} , the canonical morphism $j_!i^*x \rightarrow i^*j_!x$ is invertible.

We will say that a two-sided fibration the opposite of whose underlying span is also such is a *two-sided bifibration*.

2.1.7 Definition. An adjoint pair $F \dashv G$ of *colax* monoidal functors between symmetric monoidal categories satisfies *Frobenius reciprocity* [Law70] if the canonical morphism

$$F(A \otimes GB) \longrightarrow FA \otimes FGB \longrightarrow FA \otimes B$$

is invertible. (Such an adjoint pair is also called a *Hopf adjunction* [BLV11]).

2.1.2 Regular logic

Regular logic is the fragment of first-order predicate logic that uses only the connectives \top for truth, \wedge for conjunction and \exists for existential quantification. We will mostly follow [See83].

2.1.8 Definition. A (regular) *signature* S is given by a collection X, Y, \dots of sorts, together with a collection of typed predicate and function symbols. A *type* is a finite sequence X_1, X_2, \dots of sorts, and types will also be denoted X, Y, \dots . If P is a predicate of type X we may write $P: X$, and similarly $f: X \rightarrow Y$ indicates the type of f . Every signature contains at least the equality predicate $=_X: X, X$.

We assume given an inexhaustible supply of free variables $x, x', y, y' \dots$ and bound variables $\xi, \xi', v, v' \dots$ of each sort, with the notation extended to types so that a variable of type X, Y is the same as a pair x, y of variables of sorts X and Y .

2.1.9 Definition. A *context* is a finite list $x: X, y: Y, \dots$ of sorted variables, or equivalently a single variable $z: X, Y, \dots$. A *term* is either a variable, a tuple of terms or a function symbol f applied to a term, all with the obvious well-typedness constraints. Every term lives in a context, which is assumed to contain every variable in the term, perhaps together with ‘dummy’ variables that don’t. We write $t[x]$ to indicate that x is the context of t , and $t[s]$ to denote the substitution of the term s for the variable(s) x in t .

2.1.10 Definition. A (regular) *formula* is either the constant \top , a predicate symbol $P(t)$ applied to a term, the conjunction $\phi \wedge \psi$ of two formulas, a quantified formula $\exists \xi. \phi$ or the substitution $\phi[t]$ of the term t into the formula ϕ , defined in the usual way. Every formula lives in a context, which we assume contains (perhaps strictly) all of its free variables, and we write $\phi[x]$ for this.

2.1.11 Definition. The inference rules of regular logic are as follows: conjunction is governed by

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \qquad \frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

truth by

$$\frac{\phi}{\top}$$

existentials by

$$\frac{\phi[t]}{\exists \xi. \phi[\xi]} \qquad \frac{\begin{array}{c} \overline{\phi[x]} \\ \vdots \\ \psi \end{array}}{\exists \xi. \phi[\xi]} \qquad \psi$$

where on the right x is not free in ψ , and equality by

$$\frac{}{t = t} \qquad \frac{t = s \quad \phi[t]}{\phi[s]}$$

The notion of context is easily extended to derivations. Observe that the rules for \exists are the only rules that do not preserve the contexts of formulas.

Derivations using these rules may be composed:

$$\begin{array}{ccc} & & \phi \\ \phi & \psi & \vdots \\ \vdots & , & \vdots \quad \mapsto \quad \psi \\ \psi & \chi & \vdots \\ & & \chi \end{array}$$

as long as both derivations have the same context, and this composition is clearly associative, with units the identity derivations ϕ . We may write $p: \phi \xrightarrow{x} \psi$ to indicate that p is a derivation of ψ from the assumption ϕ with context x , arriving at the rules

$$\frac{}{1_\phi: \phi \xrightarrow{x} \phi} \qquad \frac{p: \phi \xrightarrow{x} \psi \quad q: \psi \xrightarrow{x} \chi}{q \circ p: \phi \xrightarrow{x} \chi}$$

and thus at a category of derivations in any given context x .

The substitution $p[t]$ of a term $t: Y \rightarrow X$ into a derivation $p[x]$ with x free is defined in the obvious way, and an induction over the structure of derivations shows that the ‘substitute t ’ mapping t^* is a functor from the category of derivations in the context x to derivations in the context y that commutes with the finite-product structure given by the following.

If $p_i: \phi \xrightarrow{x} \psi_i$ for $i = 1, 2$, then we may use the \wedge -introduction rule to form a derivation $\langle p_1, p_2 \rangle: \phi \xrightarrow{x} \psi_1 \wedge \psi_2$, and conversely given a derivation p of the latter type the elimination rules give $\pi_i \circ p: \phi \xrightarrow{x} \psi_i$. Imposing the (β - and η -)equalities

$$\pi_i \langle p_1, p_2 \rangle = p_i \qquad \langle \pi_1 p, \pi_2 p \rangle = p$$

then gives a ‘bijective’ rule

$$\frac{p_1: \phi \xrightarrow{x} \psi_1 \quad p_2: \phi \xrightarrow{x} \psi_2}{\langle p_1, p_2 \rangle: \phi \xrightarrow{x} \psi_1 \wedge \psi_2}$$

where to move from bottom to top we compose with π_i , and this gives binary products in each category of derivations. As for \top , we will say that any derivation $p: \phi \xrightarrow{x} \top$ is equal to the canonical $!_\phi: \phi \xrightarrow{x} \top$, making \top the terminal object in each category of derivations.

Similarly, there is a β rule for equality:

$$\frac{\overline{t = t} \quad \begin{array}{c} \vdots \\ \phi[t] \end{array}}{\phi[t]} = \begin{array}{c} \vdots \\ \phi[t] \end{array}$$

and an η rule:

$$\begin{array}{c} p: \\ t = t' \\ q[t, t']: \\ \phi[t, t'] \end{array} = \frac{\overline{t = t} \quad \begin{array}{c} p: \\ q[t, t]: \\ t = t' \\ \phi[t, t] \end{array}}{\phi[t, t']}$$

and these set up a bijection

$$\frac{\phi, x = x' \xrightarrow{x, x'} \psi[x, x']}{\phi \xrightarrow{x} \psi[x, x]} \quad (2.1.1)$$

between derivations of the indicated types [Jac99]. There is also a ‘coherence’ rule

$$\frac{\begin{array}{c} \vdots \\ \frac{t = t}{\top} \\ t = t \end{array}}{t = t} = \begin{array}{c} \vdots \\ t = t \end{array}$$

which makes sure that $\top_X \Leftrightarrow x = x$, so that $x = x$ is the terminal object in the category of derivations in the context x .

2.1.12 Definition. A (regular) theory T over a signature S is given by a collection of axioms (derivation constants, perhaps including purely equational axioms $t = t'$) together with a collection of equations between derivations built from those axioms and the above rules.

The terms of a signature, together with the equational axioms $t = t'$ of a theory over that signature, give rise to a category \mathbf{B}_T with finite products — the ‘multisorted Lawvere theory’ associated to the theory. In this category an object is a type X_1, X_2, \dots, X_n , and a morphism from X_1, X_2, \dots, X_n to Y_1, Y_2, \dots, Y_m is given by an m -tuple $\langle t_1, t_2, \dots, t_m \rangle$ of terms, where each $t_i: X_1, X_2, \dots, X_n \rightarrow Y_i$. Thus a theory T gives rise to a pseudofunctor $\mathbf{E}_T(-): \mathbf{B}_T^{\text{op}} \rightarrow \mathcal{C}at$, which takes a type X to the finite-product category $\mathbf{E}_T(X)$ of formulas and derivations

whose context is of type X , and takes a term $t: X \rightarrow Y$ to the substitution functor $t^*: \mathbf{E}_T(Y) \rightarrow \mathbf{E}_T(X)$.

2.1.3 Regular fibrations

In this section we define the structures that serve as fibrational models of regular theories. We also recall and discuss Lawvere's notion of comprehension in a fibration.

2.1.13 Definition. Let \mathbf{B} be a category with finite products. The following squares are pullbacks in \mathbf{B} ([See83], cf. [Law70, p. 9]) for any morphisms t, t' .

$$\begin{array}{ccc} X & \xrightarrow{\langle X, t \rangle} & X \times Y \\ \downarrow t & \lrcorner & \downarrow t \times Y \\ Y & \xrightarrow{d} & Y \times Y \end{array} \quad \text{(A)} \qquad \begin{array}{ccc} X & \xrightarrow{X} & X \\ \downarrow X & \lrcorner & \downarrow d \\ X & \xrightarrow{d} & X \times X \end{array} \quad \text{(B)}$$

and

$$\begin{array}{ccc} X' \times X & \xrightarrow{X' \times t} & X' \times Y \\ t' \times X \downarrow & \lrcorner & \downarrow t' \times Y \\ Y' \times X & \xrightarrow{Y' \times t} & Y' \times Y \end{array} \quad \text{(C)}$$

Also, if $tu = sv$ is a pullback, then so is its product with any object:

$$\begin{array}{ccc} P \times Z & \xrightarrow{u \times Z} & X \times Z \\ v \times Z \downarrow & \lrcorner & \downarrow t \times Z \\ X' \times Z & \xrightarrow{s \times Z} & Y \times Z \end{array} \quad \text{(D)}$$

and similarly for products on the right.

The squares (A), (B) and (C), and those built from them using (D) and pasting side-by-side, are called *product-absolute* pullbacks [WW08], because they are preserved by any functor that preserves products.

2.1.14 Remark. The coassociativity square for the diagonal d is product-absolute:

$$\begin{array}{ccc} X & \xrightarrow{d} & X^2 \\ \downarrow d & & \downarrow 1 \times d \\ X^2 & \xrightarrow{d \times 1} & X^3 \end{array} = \begin{array}{ccc} X & \xrightarrow{(1, d)} & X^3 & \xrightarrow{X \times p_2} & X^2 \\ \downarrow d & & \downarrow d \times X^2 & & \downarrow d \times X \\ X^2 & \xrightarrow{d} & X^4 & \xrightarrow{X^2 \times p_2} & X^3 \end{array}$$

See the example after definition 5 at [Tri13].

2.1.15 Definition. A *regular fibration* is a fibration $\mathbf{E}: \mathbf{B}^{\text{op}} \rightarrow \mathcal{C}at$, such that

1. \mathbf{B} , and $\mathbf{E}X$ for each object X of \mathbf{B} , have finite products (the product in \mathbf{B} is denoted $(\times, \mathbf{1})$ and that in each $\mathbf{E}X$ as (\cap, \top));
2. $f^* = \mathbf{E}f$, for each morphism f of \mathbf{B} , has a left adjoint, denoted \exists_f or $f_!$, and (hence) preserves finite products;
3. these adjoints satisfy Frobenius reciprocity (def. 2.1.7) and the Beck–Chevalley (def. 3.1.2) conditions with respect to product-absolute pull-backs (def. 2.1.13) in \mathbf{B} .

A morphism of regular fibrations is a product-preserving morphism of bifibrations, and a transformation of such is simply a transformation of fibration-morphisms. These make up the 2-category $\mathcal{R}egFib$.

Our regular fibrations are (nearly) those of [Pav96]. A similar definition is given in [Jac99], the only difference being that the latter sort of regular fibration is required to have all fibres preordered. These we call *ordered* regular fibrations. They form a full sub-2-category $\mathcal{O}rdRegFib \hookrightarrow \mathcal{R}egFib$.

2.1.16 Remark. In logical terms, the point of the Frobenius condition is that together with the Beck–Chevalley conditions it ensures that $\exists_t \phi$ is equivalent to $\exists \xi. t[\xi] = y \wedge \phi[\xi]$. See [Law70, Theorem, p. 8].

2.1.17 Definition. The *internal language* of a regular fibration $\mathbf{E} \rightarrow \mathbf{B}$ is the regular theory defined as follows:

- The sorts and terms are those of the Lawvere theory \mathbf{B} , so that a sort is a finite list of objects of \mathbf{B} , with products $X \times Y$ identified with lists X, Y , and a term is either a variable (product projection) or the application (composition) of a function symbol (morphism of \mathbf{B}) to a tuple of terms.
- The predicates and derivations of sort X are given by the objects and morphisms of $\mathbf{E}(X)$. That is, a predicate P of sort X_1, \dots, X_n is an object $\llbracket P \rrbracket$ of \mathbf{E} in the fibre over $X_1 \times \dots \times X_n$, conjunction \wedge and quantification \exists are given by the regular structure of p , and a derivation is a p -vertical morphism of \mathbf{E} .

2.1.18 Definition. The *soundness theorem* [vO08, theorem 2.1.6] says that if $\mathbf{E} \rightarrow \mathbf{B}$ is a regular fibration, then to each proof of a sequent

$$P_1, \dots, P_n \xRightarrow{x} Q$$

where \vec{x} contains the free variables of the P_i, Q , there corresponds a vertical morphism $[[P_1]] \times \cdots \times [[P_n]] \rightarrow [[Q]]$ in \mathbf{E} over the type of x .

We therefore say that a fibration *satisfies* a sequent if such a vertical morphism exists.

2.1.19 Proposition ([See83, Theorem, §8]). *If a hyperdoctrine satisfies the Beck–Chevalley condition (def. 3.1.2) for the product-absolute pullbacks of def. 2.1.13, then it satisfies the condition for an arbitrary pullback $tu = sv$ if and only if it satisfies*

$$t[m] = s[m'] \implies \exists \xi. (u[\xi] = m \wedge v[\xi] = m')$$

and

$$u[p] = u[p'], v[p] = v[p'] \implies p = p'$$

that is, if the hyperdoctrine ‘knows’ that the diagram is a pullback.

Seely’s proof of prop. 2.1.19 goes through unchanged for a regular fibration.

The connection with regular categories (def. 2.1.2) is as follows.

2.1.20 Proposition. *A category \mathbf{C} is regular if and only if its subobject fibration $\text{Sub}(\mathbf{C}) \rightarrow \mathbf{C}$ that sends $S \hookrightarrow X$ to X is a (necessarily ordered) regular fibration.*

Proof. If \mathbf{C} is a regular category, then the adjunctions $\exists_f \dashv f^*$ come from pullbacks and images in \mathbf{C} [Joh02, lemma A1.3.1] as does the Frobenius property [op. cit., lemma A1.3.3]. The terminal object of $\text{Sub}(X) = \text{Sub}(\mathbf{C})_X$ is the identity 1_X on X , and binary products in the fibres $\text{Sub}(X)$ are given by pullback. These products are preserved by reindexing functors f^* because the f^* are right adjoints, and the projection to \mathbf{C} clearly preserves them too. The Beck–Chevalley condition follows from pullback-stability of images in \mathbf{C} .

Conversely, suppose $\text{Sub}(\mathbf{C}) \rightarrow \mathbf{C}$ is a regular fibration. We need to show that \mathbf{C} has equalizers (to get finite limits) and pullback-stable images. But the equalizer of $f, g: X \rightrightarrows Y$ is $(f, g)^*d$. For images, let $\text{im } f = \exists_f \top_X$ as in [Joh02, lemma A1.3.1]. Pullback-stability follows from the Beck–Chevalley condition, together with the fact that $\text{Sub}(\mathbf{C}) \rightarrow \mathbf{C}$ ‘knows’, in the sense of prop. 2.1.19, that any pullback is indeed a pullback. \square

We will write $\text{Arr}(\mathbf{C}) \rightarrow \mathbf{C}$ for the codomain projection out of the category $[\mathbf{2}, \mathbf{C}]$ of morphisms of a category \mathbf{C} . It is well known that this is a regular fibration if and only if \mathbf{C} has finite limits; the non-trivial parts of the proof are essentially as above.

Images $\text{im } t = \exists_t \top_X = t_! \top_X$ as above make sense in any regular fibration, and can be made functorial: for a morphism

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ & \searrow t & \swarrow t' \\ & X & \end{array}$$

in \mathbf{B}/X , the morphism $\text{im } g: \text{im } t \rightarrow \text{im } t'$ is the composite

$$t_! \top_X \xrightarrow{\sim} t'_! g_! \top_X \longrightarrow t'_! \top_Z$$

where the second morphism is $t'_!$ applied to the unique $g_! \top_X \rightarrow \top_Z$; if g is the identity then the composite is the identity, by the coherence laws for the pseudofunctor $t \mapsto t_!$ together with uniqueness of maps into a terminal object. For a composable pair g, g' of morphisms over X we get

$$\begin{array}{ccccc} t_! \top_X & & t'_! \top_Z & & t''_! \top_W \\ \sim \downarrow & \nearrow & \sim \downarrow & \nearrow & \nearrow \\ t'_! g_! \top_X & & t''_! g'_! \top_Z & & \\ \sim \downarrow & \nearrow & & \nearrow & \\ t''_! g'_! g_! \top_X & & & & \end{array}$$

where the rectangular cell commutes by naturality and the other by functoriality of $t'_!$ and uniqueness of maps into terminals again. So image is a functor $\text{im}: \mathbf{B}/X \rightarrow \mathbf{E}X$, for each $X \in \mathbf{B}$.

2.1.21 Definition ([Law70]). A regular fibration \mathbf{E} over \mathbf{B} has *comprehension* or *is comprehensive* if for each $X \in \mathbf{B}$ the functor $\text{im}: \mathbf{B}/X \rightarrow \mathbf{E}X$ has a right adjoint $P \mapsto \{P\}: \mathbf{E}X \rightarrow \mathbf{B}/X$, called *extension*. \mathbf{E} has *full comprehension* if each such functor is fully faithful.

This means that for each $P \in \mathbf{E}X$ there is a morphism $i_P: \{P\} \rightarrow X$ such that for each $t: Y \rightarrow X$ there is a bijection between factorizations

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \{P\} \\ & \searrow t & \swarrow i_P \\ & X & \end{array}$$

in \mathbf{B} and morphisms

$$t_! \top_Y \longrightarrow P$$

in $\mathbf{E}X$. Notice that these are the same as morphisms $\top_Y \rightarrow t^*P$ in $\mathbf{E}Y$, and hence correspond to maps $\top_Y \rightarrow P$ over t in the total category \mathbf{E} (cf. the definition of ‘subset types’, i.e. comprehension, in [Jac99, def. 4.6.1]).

For an object X in the base of a regular fibration, the equality predicate over X is given by the image $d_! \top_X$ of the diagonal morphism $d: X \rightarrow X \times X$. The Beck–Chevalley condition for squares of type (B) in def. 2.1.13 requires that the unit $1 \Rightarrow d^* d_!$ of the adjunction $d_! \dashv d^*$ be invertible. Using Frobenius reciprocity we can show

$$\begin{aligned} d_! \top \cap d_! \top &\cong d_! (\top \cap d^* d_! \top) \\ &\cong d_! (\top \cap \top) \\ &\cong d_! \top \end{aligned}$$

But this isomorphism means that the following square is a pullback:

$$\begin{array}{ccc} d_! \top & \xrightarrow{1} & d_! \top \\ \downarrow 1 & \lrcorner & \downarrow \\ d_! \top & \longrightarrow & \mathbf{1} \end{array}$$

and that is equally to say that $d_! \top$ is subterminal in $\mathbf{E}(X \times X)$, so that there can be at most one proof of any equality.

2.1.22 Definition. Equality in a regular fibration \mathbf{E} over \mathbf{B} is *extensional* (cf. the ‘very strong equality’ of [Jac99, 3.4.2]) if two parallel morphisms $f, g: Y \rightrightarrows X$ in \mathbf{B} are equal whenever the (then necessarily unique) morphism

$$\top_Y \longrightarrow \llbracket fy = gy \rrbracket = (f, g)^* d_! \top_X$$

in $\mathbf{E}Y$ exists.

For the following result compare [Law70, Theorem, p. 13] and [Jac99, exercise 4.6.6].

2.1.23 Proposition. *A comprehensive regular fibration over \mathbf{B} has extensional equality if and only if, for any parallel pair $f, g: Y \rightrightarrows X$, the map $i: \{\llbracket fy = gy \rrbracket\} \rightarrow Y$ exhibits its domain as the equalizer of f and g .*

Proof. For any $t: Z \rightarrow Y$ in \mathbf{B} , there is a bijection between morphisms $\top_Z \rightarrow \llbracket ftz = gtz \rrbracket$ in $\mathbf{E}Z$ and morphisms $t \rightarrow i$ in \mathbf{B}/Y , there being therefore at most one of the latter. If equality is extensional, then t factors through i if and only if $ft = gt$, making i the equalizer of f and g . Conversely, taking $t = 1$, there is a morphism $1 \rightarrow i$ if and only if $f = g$, but this then corresponds to a morphism $\top = \text{im } 1 \rightarrow \llbracket fy = gy \rrbracket$. \square

The adjunction $\text{im} \dashv \{-\}$ gives, for each $t: Y \rightarrow X$, a unit

$$\begin{array}{ccc} Y & \xrightarrow{e_t} & \{\text{im } t\} \\ & \searrow t & \swarrow i_t \\ & & Z \end{array}$$

We will say that t is an injection if this e_t is invertible. Notice that injections in an ordered fibration must be monomorphisms. Conversely, if comprehension is full then $\text{hom}(Q, P) \cong \text{hom}(\text{im } i_Q, P) \cong \text{hom}(i_Q, i_P)$, so that if i_P is a monomorphism then there is at most one morphism into P from any object in the same fibre, and so if every injection is a monomorphism then \mathbf{E} is ordered.

The following is proved in [Jac99, prop. 4.9.3] in a somewhat more general context than ours, but only for ordered fibrations.

2.1.24 Proposition. *A regular fibration has extensional equality if and only if each diagonal $d: X \rightarrow X \times X$ is an injection (supposing $\{\text{im } d\}$ to exist).*

Proof. For any parallel pair f, g , there are bijections

$$\frac{\frac{\top \longrightarrow (f, g)^* d \dashv \top}{\text{im}(f, g) \longrightarrow \text{im } d}}{Y \xrightarrow{\quad} \{\text{im } d\}} \quad \begin{array}{ccc} & & \\ & \searrow (f, g) & \swarrow i_d \\ & & X \times X \end{array}$$

If d is an injection, then factorizations of the last form are in bijection with factorizations of (f, g) through d , but since d is monic, there can be at most one such, which exists precisely when $f = g$. Conversely, to say that equality is extensional is to say that morphisms $\top \rightarrow \llbracket fy = gy \rrbracket$ are in bijection with factorizations of (f, g) through d , but by the correspondence above the former are also in bijection with maps $(f, g) \rightarrow i_d$, naturally in (f, g) . Taking $f = g = 1$ shows that there is exactly one morphism $d = (1, 1) \rightarrow i_d$, which must be e_d , and because the induced map $\text{hom}(-, d) \cong \text{hom}(-, i_d)$ is an isomorphism e_d must be invertible. \square

The following proposition does not seem to have been published before in this particular form, but it is a generalization of a very well-known fact. Although the fibration that sends a category \mathbf{C} to $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ is not regular, as noted already by Lawvere [Law70], it does have full comprehension, with the extension of a presheaf given by its category of elements. In that case the proposition reduces to the fact [MLM92, exercise III.8(a)] that for any presheaf P on \mathbf{C} there is an equivalence $[\mathbf{C}^{\text{op}}, \mathbf{Set}]/P \simeq [(f P)^{\text{op}}, \mathbf{Set}]$.

2.1.25 Proposition. *Let P be a predicate over X in a regular fibration \mathbf{E} with full comprehension. There is then an equivalence*

$$\mathbf{E}\{P\} \simeq \mathbf{E}X/P \quad (2.1.2)$$

Proof. Write $(\mathbf{B}/X)_i$ for the full subcategory of \mathbf{B}/X on the injections. Because the extension functor is fully faithful, it restricts to equivalences

$$\begin{aligned} \mathbf{E}\{P\} &\simeq (\mathbf{B}/\{P\})_i \\ \mathbf{E}X/P &\simeq ((\mathbf{B}/X)/i_P)_i \end{aligned}$$

But the right-hand sides are themselves equivalent, by a standard argument on slices of slices. \square

2.1.4 The classifying fibration of a regular theory

As something of an aside, we will construct in this section the ‘syntactic model’ of a regular theory. Most of this material is at least sketched in [See83] for the hyperdoctrine corresponding to a first-order theory, but an explicit presentation of what remains for the regular case is useful and illuminating.

We want to show firstly that a regular theory T gives rise to a bifibration $\mathbf{E}_T \rightarrow \mathbf{B}_T$.

2.1.26 Proposition. *Let T be a regular theory. For each term $t: X \rightarrow Y$, the functor $t^*: \mathbf{E}_T(Y) \rightarrow \mathbf{E}_T(X)$ has a left adjoint \exists_t .*

Proof. Define \exists_t on formulas as

$$\exists_t \phi = \exists \xi. (t[\xi] = y \wedge \phi[\xi])$$

It suffices to show that for any $\phi[x]$ of type X there is a universal $\eta_\phi^t: \phi \xrightarrow{x} t^* \exists_t \phi$; that is, for any equivalence class of proofs $p: \phi \xrightarrow{x} t^* \psi$, there is a unique $\hat{p}: \exists_t \phi \xrightarrow{y} \psi$ such that $t^* \hat{p} \circ \eta_\phi^t$ is equal to p . The derivation η_ϕ^t is obtained by forming the derivation

$$\frac{\frac{x = x' \quad \overline{t[x] = t[x]}}{t[x'] = t[x]} \quad \frac{x = x' \quad \phi[x]}{\phi[x']}}{t[x'] = t[x] \wedge \phi[x']}}{\exists \xi. (t[\xi] = t[x] \wedge \phi[\xi])} \quad (2.1.3)$$

of type $\phi[x], x = x' \xrightarrow{x, x'} t^* \exists_t \phi$ and using the bijection (2.1.1) above to get rid of the hypothesis $x = x'$. Given $p: \phi \xrightarrow{x} t^* \psi$, let \hat{p} be

$$\frac{\frac{\frac{\frac{\overline{t[x] = y \wedge \phi[x]}}{\phi[x]}}{\vdots}}{t[x] = y}}{\psi[t[x]]}}{\psi[y]}}{\frac{\exists \xi.(t[\xi] = y \wedge \phi[\xi])}{\psi[y]}}$$

The β and η equalities given above show that the composite $t^* \hat{p} \circ \eta_\phi^t$ is equal to p , and uniqueness of \hat{p} follows from the normal form theorem for natural deduction [Pra06]. So we have another bijection

$$\frac{\frac{\exists_t \phi \xrightarrow{y} \psi}{\phi \xrightarrow{x} t^* \psi}}$$

□

In particular, we have the usual rewriting rules, as given in [See83]:

$$\frac{\frac{p \vdots \quad \overline{\phi[x]}}{\phi[t]} \quad \vdots q[x]}{\frac{\exists \xi.\phi[\xi]}{\psi}} \quad \psi = \frac{p \vdots \quad \phi[t]}{q[t] \vdots \quad \psi}$$

and

$$\frac{p \vdots \quad \exists \xi.\phi[\xi]}{q \vdots \quad \psi} = \frac{\frac{\overline{\phi[x]}}{\exists \xi.\phi[\xi]}}{p \vdots \quad q \vdots \quad \psi}$$

For $\mathbf{E}_T \rightarrow \mathbf{B}_T$ to be a regular fibration, it must satisfy the Frobenius and Beck–Chevalley conditions. The former means that for any term t the canonical map $\exists_t(\phi \wedge t^* \psi) \xrightarrow{y} (\exists_t \phi) \wedge \psi$ is an isomorphism. This canonical map is given [Joh02, definition D1.3.1(i)] by

$$\frac{\frac{\phi \wedge t^* \psi \xrightarrow{x} t^* \psi}{\exists_t(\phi \wedge t^* \psi) \xrightarrow{y} \psi} \quad \frac{\frac{\phi \wedge t^* \psi \xrightarrow{x} \phi}{\phi \wedge t^* \psi \xrightarrow{x} t^* \exists_t \phi}}{\exists_t(\phi \wedge t^* \psi) \xrightarrow{x} \exists_t \phi}}{\exists_t(\phi \wedge t^* \psi) \xrightarrow{x} (\exists_t \phi) \wedge \psi}$$

So we must insist that in \mathbf{E}_T the above proof, call it p , have a formal inverse $p^{-1}: (\exists_t \phi) \wedge \psi \xrightarrow{x} \exists_t(\phi \wedge t^* \psi)$, adding to the equations above $p^{-1}p = 1$ and $pp^{-1} = 1$.

The Beck–Chevalley conditions for the product-absolute pullbacks (A), (C) and (D) in def. 2.1.13 are shown as in [See83, §4].

2.1.27 Proposition. *The Beck–Chevalley condition for 2.1.13(B) holds; that is, η^d , as defined by (2.1.3), is invertible.*

Proof. An inverse is given by

$$\frac{\frac{\frac{\exists \xi. d[\xi] = d[x] \wedge \phi[\xi]}{\phi[x]} \quad \frac{\frac{(x', x') = (x, x) \wedge \phi[x']}{(x', x') = (x, x)}{\phi[x']}}{\phi[x]}}{\phi[x]}}$$

That this derivation is a left inverse for η_ϕ^d follows from the β -reductions given above, and conversely that it is a right inverse follows from the η -reductions for \wedge, \exists and $=$. \square

We can now perform the usual rites of categorical logic: a model of a regular theory T in a regular fibration $\mathbf{E} \rightarrow \mathbf{B}$ is a morphism of regular fibrations from $\mathbf{E}_T \rightarrow \mathbf{B}_T$ to $\mathbf{E} \rightarrow \mathbf{B}$, and it is easy to see that this is equivalent to the traditional notion. Completeness is automatic, because if a sequent is true in every model then it is true in the syntactic model and thence provable.

2.2 Allegories and bicategories of relations

In this section we recall the structures used to give the bicategorical or relational semantics of regular theories, in the locally ordered case. First *allegories* and then *bicategories of relations* are defined, and the latter are shown to be the same as certain allegories. Later on it will become clear that they are also a special case of the *regular equipments* that we will define.

2.2.1 Allegories and their completions

Here we define allegories and the idempotent splitting construction. Nothing in this section is original.

2.2.1 Definition ([FŠ90, Joh02]). An *allegory* \mathcal{A} is a strict 2-category whose each hom-category $\mathcal{A}(X, Y)$ is a poset with binary meets \cap , and that comes

equipped with a strict involution $(-)^{\circ}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{A}$ that is the identity on objects and satisfies the *modular law* for all suitably-typed morphisms r, s, t :

$$sr \cap t \leq (s \cap tr^{\circ})r \quad (2.2.1)$$

An allegory functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is a 2-functor that preserves \cap and $(-)^{\circ}$. A transformation $\alpha: F \Rightarrow G$ is an oplax transformation (i.e. $\alpha_Y \circ Fs \leq Gs \circ \alpha_X$ for any $s \in \mathcal{A}(X, Y)$) whose components α_X have right adjoints (they are *maps*). There is then a 2-category *All* of allegories, functors and transformations.

Note that hom-posets are not required to have top elements, and that composition is not required to preserve local meets (although it must preserve the local ordering). Note also that the modular law as above is equivalent to the dual form

$$sr \cap t \leq s(r \cap s^{\circ}t) \quad (2.2.2)$$

Morphisms are written as e.g. $r: X \multimap Y$. A morphism is called a *map* if it has a right adjoint. Maps are written as $f: X \dashrightarrow Y$; the right adjoint of f is f^{\bullet} .

We recall some basic facts about allegories.

2.2.2 Lemma ([Joh02, lemma A3.2.3]). *If $f: X \dashrightarrow Y$ is a map, then its right adjoint is f° . Further, the ordering on maps is discrete: if $f \leq g$ then $f = g$. Hence the evident sub-2-category $\text{Map}(\mathcal{A})$ is a category.*

2.2.3 Remark. It follows that the componentwise ordering on transformations between allegory functors is also discrete, so that the 2-category *All* is just a 2-category.

2.2.4 Lemma. *If $r^{\circ}r \leq 1$ (e.g. if $r = f^{\bullet}$ is a right adjoint), then the modular law (2.2.1) is an identity, and if $ss^{\circ} \leq 1$ (e.g. if $s = f$ is a map) then the dual modular law (2.2.2) is an identity.*

Proof. $(s \cap tr)r^{\circ} \leq (sr \cap t)r^{\circ}r \leq sr \cap t$, and dually. \square

2.2.5 Lemma ([Joh02, corollary A3.1.6]). *The distributivity laws hold:*

$$\begin{aligned} (r \cap s)t &\leq rt \cap st \\ t(r \cap s) &\leq tr \cap ts \end{aligned}$$

If $tt^{\circ} \leq 1$ then the first is an identity, and dually if $t^{\circ}t \leq 1$ then the second is an identity.

2.2.6 Definition. A *tabulation* of $r: X \multimap Y$ is a span of maps $f: Z \dashrightarrow X$, $g: Z \dashrightarrow Y$ such that $r = gf^{\circ}$ and $f^{\circ}f \cap g^{\circ}g = 1$. An allegory is *tabular* if every

morphism has a tabulation; it is *pre-tabular* if every morphism is contained in one that has a tabulation. Clearly, an allegory whose every hom-poset has a top element is pre-tabular if and only if each such element has a tabulation.

2.2.7 Definition. A *unit* in an allegory \mathcal{A} is an object U such that 1_U is the top element of $\mathcal{A}(U, U)$ and for any X there exists a morphism $p: X \multimap U$ satisfying $p^\circ p \geq 1$. An allegory is called *unitary* if it has a unit.

2.2.8 Lemma ([Joh02, lemmas A3.2.8, A3.2.9]).

1. If \mathcal{A} has a unit then its hom-posets have top elements.
2. If U is a unit in \mathcal{A} , then it is the terminal object of $\text{Map}(\mathcal{A})$.

2.2.9 Lemma ([Joh02, lemma A3.2.4]). Suppose that $r: X \multimap Y$ has a tabulation (f, g) , and that $i: W \dashrightarrow X$, $j: W \dashrightarrow Y$ are maps. Then $ji^\circ \leq r$ if and only if there exists a map h such that $i = fh$ and $j = gh$. Such a h is necessarily unique.

2.2.10 Corollary. If the top element of $\mathcal{A}(X, Y)$ has a tabulation (f, g) , then that span is a product cone. Hence, if \mathcal{A} is a unitary pre-tabular allegory, then $\text{Map}(\mathcal{A})$ has finite products.

2.2.11 Proposition ([Joh02, theorem A3.2.10]). An allegory \mathcal{A} is unitary and tabular if and only if $\text{Map}(\mathcal{A})$ is a regular category. In that case $\mathcal{A} \simeq \text{Rel}(\text{Map}(\mathcal{A}))$. If \mathbf{C} is a regular category, then $\text{Rel}(\mathbf{C})$ is a unitary tabular allegory, and $\mathbf{C} \simeq \text{Map}(\text{Rel}(\mathbf{C}))$.

Next we recall the theory of idempotents in allegories and the construction of the universal allegory in which a given class of idempotents splits.

2.2.12 Definition. An endomorphism $r: X \multimap X$ in an allegory is called

- *reflexive* if $1 \leq r$;
- *transitive* if $rr \leq r$;
- *symmetric* if $r^\circ = r$;
- *coreflexive* if $r \leq 1$;
- *idempotent* if $rr = r$.

A morphism that is reflexive, transitive and symmetric is called an *equivalence*.

2.2.13 Lemma ([Joh02, lemma A3.3.2]). A symmetric transitive morphism is idempotent. A coreflexive morphism is symmetric and idempotent.

2.2.14 Definition. An idempotent $r: X \looparrowright X$ *splits* if there is an object X_s and a pair of morphisms $s: X_s \looparrowright X$, $s': X \looparrowright X_s$ such that $ss' = r$ and $s's = 1$.

2.2.15 Lemma ([Joh02, lemma A3.3.3]). *If a symmetric idempotent $r: X \looparrowright X$ splits as $r = ss'$, then $s' = s^\circ$. If r is reflexive then s' is a map; if it is coreflexive then s is a map.*

2.2.16 Definition. An allegory is *effective* if all of its equivalences split.

2.2.17 Proposition ([Joh02, prop. A3.3.6]). *A unitary tabular allegory \mathcal{A} is effective if and only if $\text{Map}(\mathcal{A})$ is (Barr) exact. A regular category \mathbf{C} is exact if and only if $\mathcal{R}el(\mathbf{C})$ is effective.*

2.2.18 Definition. Let \mathcal{A} be an allegory and S a class of symmetric idempotents in \mathcal{A} that includes the identities. Then the *splitting* of S is the allegory $\mathcal{A}[\check{S}]$ with objects the elements $s: X \looparrowright X$ of S and morphisms $s \looparrowright s'$ given by morphisms $m: X \looparrowright X'$ such that $ms = m = s'm$.

2.2.19 Proposition ([Joh02, theorem A3.2.10]). *$\mathcal{A}[\check{S}]$ is an allegory, and there is a functor $\mathcal{A} \rightarrow \mathcal{A}[\check{S}]$, which preserves the unit, if \mathcal{A} has one.*

2.2.20 Remark. The functor $\mathcal{A} \rightarrow \mathcal{A}[\check{S}]$ sends an object X to 1_X and a morphism $X \looparrowright Y$ to the same morphism considered as a morphism $1_X \looparrowright 1_Y$ of idempotents. It is thus fully faithful.

2.2.21 Proposition ([Joh02, prop. A3.3.6]). *An allegory is (unitary and) tabular if and only if it is (unitary and) pre-tabular and all of its coreflexives split. If \mathcal{A} is (unitary and) pre-tabular and crf is the class of coreflexives in \mathcal{A} , then the functor $\mathcal{A} \rightarrow \mathcal{A}[\check{\text{crf}}]$ is universal from \mathcal{A} to (unitary and) tabular allegories.*

2.2.22 Proposition ([Joh02, prop. A3.3.9]). *If \mathcal{A} is any allegory, and eqv is its class of equivalences, then $\mathcal{A}[\check{\text{eqv}}]$ is effective, and tabular if \mathcal{A} is, and the functor $\mathcal{A} \rightarrow \mathcal{A}[\check{\text{eqv}}]$ is universal from \mathcal{A} to effective allegories.*

2.2.23 Definition. If \mathbf{C} is a finitely complete category, let $\text{Span}(\mathbf{C})$ denote the 2-category of spans in \mathbf{C} . The allegory $\text{Span}'(\mathbf{C})$ is the local poset reflection of $\text{Span}(\mathbf{C})$. It is unitary and pre-tabular [Joh02, example 3.3.8].

2.2.24 Corollary (of props 2.2.11, 2.2.17, 2.2.21 and 2.2.22).

1. *If \mathbf{C} is a finitely complete category, then its regular completion is given by*

$$\mathbf{C}_{\text{reg/lex}} \simeq \text{Map}(\text{Span}'(\mathbf{C})[\check{\text{crf}}])$$

2. *If \mathbf{C} is a regular category, then its exact completion is given by*

$$\mathbf{C}_{\text{ex/reg}} \simeq \text{Map}(\mathcal{R}el(\mathbf{C})[\check{\text{eqv}}])$$

2.2.2 Bicategories of relations

2.2.25 Definition ([CW87]). A (locally ordered) *cartesian bicategory* is a locally partially ordered 2-category \mathcal{C} satisfying the following:

1. \mathcal{C} is symmetric monoidal: there is a pseudofunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ together with natural isomorphisms α, λ, ρ and σ satisfying the usual coherence conditions;
2. every object of \mathcal{C} is a commutative comonoid, that is, comes equipped with maps

$$d_X: X \dashrightarrow X \otimes X \quad e_X: X \dashrightarrow I$$

whose right adjoints we write d_X^\bullet, e_X^\bullet , where I is the tensor unit, satisfying the obvious associativity, symmetry and unitality axioms, and this is the only such comonoid structure on X ;

3. every morphism $r: X \multimap Y$ is a lax comonoid morphism:

$$d_Y \circ r \leq (r \otimes r) \circ d_X \quad e_Y \circ r \leq e_X$$

A *cartesian functor* between cartesian bicategories is a (strong) monoidal 2-functor, and a cartesian transformation is an oplax transformation whose components are maps, as for allegories.

2.2.26 Proposition ([CW87, theorem 1.6]). *A 2-category \mathcal{C} is a cartesian bicategory if and only if the following hold:*

1. $\text{Map}(\mathcal{C})$ has finite 2-products (given by \otimes and I).
2. The hom-posets of \mathcal{C} have finite products \cap, \top , and 1_I is the terminal object of $\mathcal{C}(I, I)$.
3. The tensor product defined as

$$r \otimes s = (p_1^\bullet r p_1) \cap (p_2^\bullet s p_2)$$

where the p_i are the product projections, is functorial.

2.2.27 Remark. This definition clearly makes sense even if \mathcal{C} is not locally ordered, and indeed is the one given in [CKWW08, defs 3.1, 4.1], but we will stick to the locally ordered ones until section 4.2.1. The local finite products referred to are given by: $r \cap s = d^\bullet(r \otimes s)d$ and $\top = e^\bullet e$.

2.2.28 Definition ([CW87, def. 2.1]). An object X in a cartesian bicategory is called *Frobenius* (Carboni–Walters say *discrete*) if it satisfies

$$d \circ d^\bullet = (d^\bullet \otimes 1) \circ (1 \otimes d) \quad (2.2.3)$$

or, in other words, if the (in fact, either, see [WW08, lemma 3.2]) Beck–Chevalley condition holds for d_X 's associativity square $(1 \otimes d)d = (d \otimes 1)d$.

A *bicategory of relations* is a cartesian bicategory in which every object is Frobenius.

2.2.29 Remark. By [CW87, remark 2.2] the unit I is always Frobenius, and $X \otimes Y$ is Frobenius if X and Y are. So a full sub-2-category of a bicategory of relations that contains I and is closed under \otimes is again a bicategory of relations.

2.2.30 Proposition ([CW87, theorem 2.4]). A *bicategory of relations* \mathcal{B} is *compact closed*, that is, there is an *identity-on-objects involution* $(-)^{\circ} : \mathcal{B}^{\text{op}} \rightarrow \mathcal{B}$ and a *natural isomorphism*

$$\mathcal{B}(X \otimes Y, Z) \cong \mathcal{B}(X, Z \otimes Y)$$

In addition, two dual forms of the modular law hold:

$$(r \otimes 1)d \leq (1 \otimes r^\circ)dr \quad (2.2.4)$$

$$d^\bullet(r \otimes 1) \leq rd^\bullet(1 \otimes r^\circ) \quad (2.2.5)$$

with equality in the first if $r^\circ r \leq 1$ (e.g. if $r = f^\bullet$ is a right adjoint) and in the second if $rr^\circ \leq 1$ (e.g. if $r = f$ is a map) (cf. lemma 2.2.4).

Sketch of proof. The bijection is given by composition with $1 \otimes \eta_Y$ in one direction and $1 \otimes \zeta_Y$ in the other, where

$$\eta_Y = I \xrightarrow{e_Y^\bullet} Y \xrightarrow{d_Y} Y \otimes Y$$

$$\zeta_Y = Y \otimes Y \xrightarrow{d_Y^\bullet} Y \xrightarrow{e_Y} I$$

One then shows that these are the unit and counit for a duality $Y \dashv Y$. The bijection above is natural in X and Z and ‘extranatural’ in Y , meaning that the correspondence

$$\frac{X \otimes Y' \xrightarrow{1 \otimes s} X \otimes Y \xrightarrow{r} Z}{X \xrightarrow{r'} Z \otimes Y \xrightarrow{1 \otimes s^\circ} Z \otimes Y'}$$

holds, where $r \mapsto r'$ is transposition and $(-)^{\circ}$ is given by composition with $1 \otimes \eta$ on one side and $\zeta \otimes 1$ on the other. \square

2.2.31 Lemma ([CW87, corollary 2.6], cf. lemma 2.2.2). *In a bicategory of relations, if f is a map then $f^{\bullet} = f^{\circ}$, and if f and g are maps and $f \leq g$ then $f = g$.*

2.2.32 Lemma. *If \mathbf{C} is a regular category, then $\mathcal{R}el(\mathbf{C})$ is a bicategory of relations.*

Proof. Conditions 1 and 2 of prop. 2.2.26 clearly hold. For the third, we may reason in the internal language of \mathbf{C} . Clearly

$$1 \otimes 1 = \llbracket x = x \wedge x' = x' \rrbracket = 1$$

Suppose $X \overset{r}{\dashv} Y \overset{s}{\dashv} Z$ and $X' \overset{r'}{\dashv} Y' \overset{s'}{\dashv} Z'$. Then $sr \otimes s'r'$ is the meaning of

$$\begin{aligned} & (\exists v. r(x, v) \wedge s(v, z)) \wedge (\exists v'. r'(x', v') \wedge s'(v', z')) \\ & \Leftrightarrow \exists v. (r(x, v) \wedge s(v, z) \wedge v^* \exists v'. r'(x', v') \wedge s'(v', z')) \\ & \Leftrightarrow \exists v. (r(x, v) \wedge s(v, z) \wedge \exists v'. v^* (r'(x', v') \wedge s'(v', z'))) \\ & \Leftrightarrow \exists v, v'. r(x, v) \wedge s(v, z) \wedge r'(x', v') \wedge s'(v', z') \end{aligned}$$

by two uses of Frobenius reciprocity and one of Beck–Chevalley (for a product-absolute pullback), and this last is the meaning of $(s \otimes s')(r \otimes r')$. Finally, the Frobenius law is

$$\begin{aligned} \exists \xi'. (x_1, x_2) = (\xi', \xi') &= (x_3, x_4) \\ \Leftrightarrow \exists \xi'. (x_1, \xi') = (x_3, x_3) \wedge (x_2, x_2) &= (x_4, \xi') \end{aligned}$$

which follows simply from transitivity and symmetry of $=$. \square

2.2.33 Proposition. *A bicategory of relations is the same thing as a unitary tabular allegory.*

Proof. Suppose \mathcal{B} is a bicategory of relations. It is thus a locally partially ordered 2-category equipped with an identity-on-objects involution. It satisfies the modular law by [CW87, remark 2.9(ii)] and so is an allegory. The tensor unit I , the terminal object of $\text{Map}(\mathcal{B})$, is a unit (def. 2.2.7): there is a unique map $X \rightarrow I$ for any X , and 1_I is the top element of $\mathcal{B}(I, I)$ by prop. 2.2.26. By corollary 2.2.10 the product projections tabulate the top elements, so \mathcal{B} is pre-tabular.

Conversely¹, let \mathcal{A} be a unitary pre-tabular allegory. By remark 2.2.20, \mathcal{A} embeds faithfully into $\mathcal{A}[\check{\text{crf}}]$, which is unitary and tabular by prop. 2.2.21, hence equivalent by prop. 2.2.11 to $\mathcal{R}el(\mathbf{C})$ for \mathbf{C} the regular category $\text{Map}(\mathcal{A})$, hence a bicategory of relations by lemma 2.2.32. So by remark 2.2.29 it suffices to show that \mathcal{A} is closed under \times in $\mathcal{A}[\check{\text{crf}}]$. Any allegory functor must preserve tabulations (because it preserves \cap and $(-)^{\circ}$), while the inclusion $\mathcal{A} \rightarrow \mathcal{A}[\check{\text{crf}}]$ preserves the unit and the property of being a map, and thus preserves top morphisms. So the tabulation

$$X \leftarrow X \times Y \rightarrow Y$$

of \top_{XY} in \mathcal{A} is a tabulation of $\top_{1_X 1_Y}$ in $\mathcal{A}[\check{\text{crf}}]$, and therefore $1_{X \times Y} \cong 1_X \times 1_Y$. \square

2.2.34 Theorem. *The 2-category $\mathcal{B}iRel$ of bicategories of relations and cartesian functors and transformations is equivalent to the locally full sub-2-category $\mathcal{U}PtAll$ of All on the unitary pre-tabular allegories and unit-preserving functors.*

Proof. It suffices to show that a 2-functor is a cartesian functor if and only if it is a unit-preserving allegory functor. But a strong monoidal functor must preserve products in categories of maps, hence d and t and their right adjoints, hence \cap and $(-)^{\circ}$, and also the unit object. Conversely, a functor that preserves \cap and $(-)^{\circ}$ must preserve (tabulations and thus) products, and so preserve the tensor product. \square

¹This part of the proof was suggested by Mike Shulman. A direct proof is possible, but essentially amounts to translating the proof of lemma 2.2.32 through the equivalence $\mathcal{A}[\check{\text{crf}}] \simeq \mathcal{R}el(\mathbf{C})$.

Chapter 3

2- and 3-categories

This chapter prepares the ground for the next by recalling some existing definitions and facts regarding higher categories, and developing some new ones that will be needed later.

The next section reviews some notions of formal category theory in a 2-category, and defines monoidal 2-categories and functors between them, along with a few other 3-dimensional notions. The subsequent section is where our original work begins: we want to define a 3-category of ‘2-profunctors’ while avoiding the long and tedious calculations that would be needed to prove that it is a 3-category. Instead we will mimic the definition of $\mathcal{P}rof$ as the 2-category of presheaf categories and cocontinuous functors. This is clearly a 2-category, because it is a sub-2-category of Cat . So in section 3.2 we review the relevant facts about 2-dimensional limits and colimits, and define 2-dimensional ends and coends and show how they may be computed in Cat . In the next chapter we will define 2-Prof as the locally full sub-3-category of 2-Cat on the ‘presheaf 2-categories’ and the colimit-preserving functors.

3.1 Adjunctions and monads

3.1.1 Adjunctions

We take as known the notion of adjoint morphisms in a 2-category. In that setting there is a useful generalization of adjoint transposition.

3.1.1 Definition ([KS74]). Given adjunctions $f \dashv u$ and $f' \dashv u'$ in a 2-category

\mathcal{K} , the *mate* of a 2-cell

$$\begin{array}{ccc} x & \longrightarrow & x' \\ f \downarrow & \Downarrow & \downarrow f' \\ y & \longrightarrow & y' \end{array}$$

is the 2-cell

$$\begin{array}{ccc} x & \longrightarrow & x' \\ u \uparrow & \Downarrow & \uparrow u' \\ y & \longrightarrow & y' \end{array} = \begin{array}{ccccc} & y & \xrightarrow{u} & x & \longrightarrow & x' \\ & \searrow & \Downarrow & \downarrow f & \Downarrow & \downarrow f' \\ & & & y & \longrightarrow & y' \\ & & & & \xrightarrow{u'} & x' \end{array}$$

given by pasting with the counit of $f \dashv u$ and the unit of $f' \dashv u'$. Dually, the mate of a square with opposite sides u and u' is given by pasting with the unit of the first adjunction and the counit of the other. This correspondence is bijective, by the triangle equalities.

The mate of an invertible cell is not in general invertible. On the other hand, given a square each of whose sides has a right adjoint, its mate, defined as above, has a further mate with respect to the other pair of opposite sides. It then follows from the triangle equalities that this ‘double mate’ is invertible if the original 2-cell was.

3.1.2 Definition. Given a bifibration \mathbf{E} over a category \mathbf{B} , a commuting square in \mathbf{B} is sent by \mathbf{E} to a square in $\mathcal{C}at$ that is filled by an isomorphism and each of whose sides has a left adjoint.

$$\begin{array}{ccc} Y & \longrightarrow & W \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \mapsto \begin{array}{ccc} \mathbf{E}Y & \longleftarrow & \mathbf{E}W \\ \uparrow & \cong & \uparrow \\ \mathbf{E}X & \longleftarrow & \mathbf{E}Z \end{array}$$

Because this 2-cell is invertible, there are two possible mates that could be taken: we say that *the Beck–Chevalley condition holds* for the square in \mathbf{B} if *both* of the mates of its image are invertible.

3.1.2 Monads and modules

We review the notions of monads and modules in a 2-category. For background, etc. see [KS74, Str72]. This material is classical, but the presentation of modules in terms of a canonical distributive law seems to be new. It gives a pleasing characterization of the Kleisli and Eilenberg–Moore completions of a 2-category \mathcal{K} as being locally full in the 2-category $\text{Mod}(\mathcal{K})$ on the left- or right-free modules.

3.1.3 Definition. A *monad* in a 2-category \mathcal{K} is given by a morphism $t: x \rightarrow x$ together with 2-cells $\mu: tt \Rightarrow t$ and $\eta: 1 \Rightarrow t$ that make t a monoid in the monoidal category $\mathcal{K}(x, x)$. A *comonad* in \mathcal{K} is a monad in \mathcal{K}^{co} .

Given a monad $t: x \rightarrow x$ in \mathcal{K} and any other object $y \in \mathcal{K}$, there is a monad $t^* = \mathcal{K}(t, y): \mathcal{K}(x, y) \rightarrow \mathcal{K}(x, y)$ in $\mathcal{C}at$ given by pre-composition with t , and of course there is also a post-composition monad $t_* = \mathcal{K}(z, t)$ on $\mathcal{K}(z, x)$ for any object z .

3.1.4 Definition. If $t: x \rightarrow x$ is a monad in \mathcal{K} , then a *left t -module* is an algebra in the usual sense for (one of) the monad(s) t_* : it is given by an object z , a morphism $a: z \rightarrow x$ and a 2-cell $\alpha: ta \Rightarrow a$ satisfying the appropriate identities. A *right t -module* is a t^* -algebra. The category $\text{LMod}(t, z)$ is the category of algebras for the monad $\mathcal{K}(z, t)$; the category $\text{RMod}(t, y)$ is the category of $\mathcal{K}(t, y)$ -algebras.

Left and right *comodules* for a comonad are defined analogously.

Given monads t and s on x and y respectively, the associator of \mathcal{K} gives rise to an invertible distributive law $t^*s_* \cong s_*t^*$, so that both of these composites are themselves monads on $\mathcal{K}(x, y)$ with equivalent categories of algebras. The category $\text{Mod}(t, s)$ of *bimodules from t to s* is the category of algebras for this composite monad, which we will call $\mathcal{K}(t, s)$. Objects of this category will be written thus: $m: t \leftrightarrow s$.

Standard facts about distributive laws [Bec69] then show that there is a commuting square of monadic functors:

$$\begin{array}{ccc}
 & \text{Mod}(t, s) & \\
 & \simeq \mathcal{K}(t, s)\text{-Alg} & \\
 U_{s_*} \swarrow & & \searrow U_{t^*} \\
 \text{RMod}(t, y) & & \text{LMod}(s, x) \\
 \simeq \mathcal{K}(t, y)\text{-Alg} & & \simeq \mathcal{K}(x, s)\text{-Alg} \\
 U_{t^*} \searrow & & \swarrow U_{s_*} \\
 & \mathcal{K}(x, y) &
 \end{array}$$

and that each of $\mathcal{K}(t, y)$ and $\mathcal{K}(x, s)$ canonically induces a monad on the other's category of algebras called \hat{t}^* and \hat{s}_* respectively.

3.1.5 Definition. Given a monad $t: x \rightarrow x$ in a 2-category \mathcal{K} , the *Eilenberg–Moore object* x^t of t is, if it exists, the universal left t -module; equivalently, it is ‘the’ representation of the functor $y \mapsto \text{LMod}(t, y)$. In more concrete terms, the EM object comes equipped with the structure of a left t -module $u^t: x^t \rightarrow x$, composition with which sets up an equivalence $\mathcal{K}(y, x^t) \simeq \text{LMod}(t, y)$.

The *Kleisli object* x_t of t is the universal right t -module: it comes with a right t -module structure $f_t: x \rightarrow x_t$ that mediates an equivalence $\mathcal{K}(x_t, z) \simeq \text{RMod}(t, z)$. Equivalently, it is the EM object of t considered as a monad in \mathcal{K}^{op} .

The *co-Kleisli* and *co-Eilenberg–Moore* objects of a comonad in \mathcal{K} are its Kleisli and EM objects in \mathcal{K}^{co} . (The co- prefix will be omitted where it is unnecessary.)

Eilenberg–Moore objects are weighted limits [Str76], and so Kleisli objects are colimits. The theory of completions under colimits is well understood, and leads in this case to the following.

3.1.6 Definition. The *Kleisli completion* $\text{Kl}(\mathcal{K})$ [LS02] of a 2-category \mathcal{K} is the full sub-2-category of $[\mathcal{K}^{\text{op}}, \text{Cat}]$ on those functors that are Kleisli objects of monads on representable functors. It is convenient to take the objects of $\text{Kl}(\mathcal{K})$ to be the monads in \mathcal{K} themselves.

The *Eilenberg–Moore completion* $\text{EM}(\mathcal{K})$ of \mathcal{K} is $\text{Kl}(\mathcal{K}^{\text{op}})^{\text{op}}$.

The *co-Kleisli* and *co-Eilenberg–Moore* completions of \mathcal{K} are then $\text{Kl}^{\text{co}}(\mathcal{K}) = \text{Kl}(\mathcal{K}^{\text{co}})^{\text{co}}$ and $\text{EM}^{\text{co}}(\mathcal{K}) = \text{EM}(\mathcal{K}^{\text{co}})^{\text{co}}$.

We may follow [LS02] and give a more hands-on description of the Kleisli completion: if $t: x \rightarrow x$ and $s: y \rightarrow y$ are monads in \mathcal{K} as above, then morphisms $t \rightarrow s$ in $\text{Kl}(\mathcal{K})$ are transformations $\mathcal{K}(1, x)_{\bar{t}} \Rightarrow \mathcal{K}(1, y)_{\bar{s}}$ in $[\mathcal{K}^{\text{op}}, \text{Cat}]$ between the Kleisli objects of $\bar{t} = \mathcal{K}(1, t)$ and $\bar{s} = \mathcal{K}(1, s)$. The universal property of the domain makes the category of these equivalent to $\text{RMod}(\bar{t}, \mathcal{K}(1, y)_{\bar{s}})$. The Yoneda lemma then shows that this category is in turn equivalent to $\text{RMod}(\hat{t}^*, \mathcal{K}(x, y)_{s_*})$ — that is, $\text{Kl}(\mathcal{K})(t, s)$ is the category of algebras for the monad induced by t on the Kleisli category of the monad $\mathcal{K}(x, s)$. This can be expanded in two ways, corresponding to the two different constructions of the Kleisli category: the first takes $\mathcal{K}(x, y)_{s_*}$ to be the full subcategory of $\mathcal{K}(x, y)^{s_*} = \text{LMod}(s, x)$ on the free s -modules, and the monad induced by t to be simply precomposition with t . Note that this makes $\text{Kl}(\mathcal{K})(t, s)$ equivalent to the full subcategory of $\text{Mod}(t, s)$ on the ‘left-free bimodules’ from t to s (cf. [Woo85, p. 166]). The second description of $\text{Kl}(\mathcal{K})(t, s)$ uses the direct presentation of the Kleisli category [Mac98, VI.5]: $\mathcal{K}(x, y)_{s_*}$ has objects the morphisms $x \rightarrow y$ in \mathcal{K} , and as morphisms $a \Rightarrow b: x \rightarrow y$ the 2-cells $a \Rightarrow sb$ in \mathcal{K} , with identities and composition given by η^s and the usual Kleisli composition. Then the monad induced by t is given by pre-(Kleisli-)composition with $\eta^s t$, but the monad axioms make this the same as precomposing the underlying \mathcal{K} -morphism with t . Working everything out as in [LS02], we see that a Kleisli morphism from t to s is given by a morphism

$a: x \rightarrow y$ in \mathcal{K} together with a 2-cell $\alpha: at \Rightarrow sa$ that satisfies:

$$\begin{array}{ccc} att & \xrightarrow{\alpha t} & sat & \xrightarrow{s\alpha} & ssa \\ a\mu \downarrow & & & & \downarrow \mu a \\ at & \xrightarrow{\alpha} & sa & & \end{array} \quad \begin{array}{ccc} a & \xrightarrow{a\eta} & at \\ \eta a \searrow & & \downarrow \alpha \\ & & sa \end{array}$$

and a 2-cell $a \Rightarrow b$ is given by a 2-cell $\phi: a \Rightarrow sb$ of \mathcal{K} satisfying:

$$\begin{array}{ccccc} at & \xrightarrow{\alpha} & sa & \xrightarrow{s\phi} & ssb \\ \phi t \downarrow & & & & \downarrow \mu b \\ sbt & \xrightarrow{s\beta} & ssb & \xrightarrow{\mu b} & sb \end{array}$$

Notice that a Kleisli morphism $t \rightarrow s$ is precisely a ‘monad op-functor’ from t to s in the sense of [Str72]. Such a morphism determines and is determined by [LS02, section 2.1] an essentially-commuting square

$$\begin{array}{ccc} \mathcal{K}(1, x) & \longrightarrow & \mathcal{K}(1, x)_{\mathcal{K}(1, t)} \\ \downarrow & \cong & \downarrow \\ \mathcal{K}(1, y) & \longrightarrow & \mathcal{K}(1, y)_{\mathcal{K}(1, s)} \end{array}$$

Similarly, a ‘monad opfunctor transformation’ is precisely a ‘free’ Kleisli 2-cell, i.e. one of the form $\eta \circ \phi': a \Rightarrow b \Rightarrow sb$, hence a commuting cylinder of the following form:

$$\begin{array}{ccc} \mathcal{K}(1, x) & \longrightarrow & \mathcal{K}(1, x)_{\mathcal{K}(1, t)} \\ \left(\begin{array}{c} \leftarrow \\ \downarrow \\ \leftarrow \end{array} \right) & & \left(\begin{array}{c} \leftarrow \\ \downarrow \\ \leftarrow \end{array} \right) \\ \mathcal{K}(1, y) & \longrightarrow & \mathcal{K}(1, y)_{\mathcal{K}(1, s)} \end{array}$$

3.1.7 Definition. Given modules $m: t \rightarrow s$ and $n: s \rightarrow r$, where t, s, r are monads on x, y, z respectively, the *composite* $n \circ m$ is given by the following coequalizer in $\mathcal{K}(x, z)$ [Woo85, p. 165], [CKW87, 4.1]:

$$nsm \rightrightarrows nm \longrightarrow n \circ m$$

where the parallel morphisms are the actions of s on n and m and their codomain is the composite nm in \mathcal{K} . This is a reflexive coequalizer, with section $n\eta m$. If it is preserved by $\mathcal{K}(t, r)$ then it is reflected by the monadic functor $\text{Mod}(\mathcal{K})(t, r) \rightarrow \mathcal{K}(x, z)$ [Bor94, prop. 4.3.2] and so $n \circ m$ is a module from t to r .

If the hom-categories of \mathcal{K} admit all such coequalizers, and if these are preserved by composition on either side, then this formula defines the composition

operation of a 2-category $\text{Mod}(\mathcal{K})$. The identity on a monad t is t equipped with its left and right self-actions, and the 2-category axioms follow from the universal property of the coequalizer above. (Note that $\text{Mod}(\mathcal{K})$ will almost never be strict even if \mathcal{K} is, because composition depends on a choice of colimit.)

Observe that if m in the above is a left-free module (i.e. a morphism in $\text{Kl}(\mathcal{K})$) then we may write $m = sm'$, and the pair to be coequalized is then

$$nssm' \begin{array}{c} \xrightarrow{\nu sm'} \\ \xrightarrow{n\mu m'} \end{array} nsm'$$

where ν is the action of s on n and μ is the multiplication of s . But because ν is an algebra for $\mathcal{K}(s, z)$, it is the coequalizing map [Bor94, lemma 4.3.3] in

$$nss \begin{array}{c} \xrightarrow{\nu s} \\ \xrightarrow{n\mu} \end{array} ns \xrightarrow{\nu} n$$

Moreover, this is a split, hence absolute, coequalizer, so that whiskering by m' yields the composite $n \circ sm' \simeq nm'$, with $\nu m'$ as the coequalizing map. This fact ensures that $\text{Kl}(\mathcal{K})$ always exists, even if $\text{Mod}(\mathcal{K})$ does not. It is also not hard to see that the former will be strict if \mathcal{K} is, because composition of left-free modules can be taken to be just composition in \mathcal{K} .

Thus there are inclusions

$$\text{Mnd}^{\text{op}}(\mathcal{K}) \rightarrow \text{Kl}(\mathcal{K}) \rightarrow \text{Mod}(\mathcal{K}) \leftarrow \text{EM}(\mathcal{K}) \leftarrow \text{Mnd}(\mathcal{K})$$

which are all the identity on objects. Here $\text{Mnd}(\mathcal{K})$ is the 2-category described by [Str72], $\text{Mnd}^{\text{op}}(\mathcal{K})$ is $\text{Mnd}(\mathcal{K}^{\text{op}})^{\text{op}}$, and $\text{EM}(\mathcal{K})$ is the Eilenberg–Moore completion of [LS02], i.e. $\text{Kl}(\mathcal{K}^{\text{op}})^{\text{op}}$.

In chapter 4 we will consider structures called (proarrow) *equipments*, which have several definitions that we will try to relate. For the purposes of the following definition, we may take an equipment to be given by a pair of 2-categories with the same objects and a locally fully faithful identity-on-objects functor between them [Woo82].

3.1.8 Definition ([GS13]). Given an equipment $\mathcal{K} \rightarrow \mathcal{M}$ such that $\text{Mod}(\mathcal{M})$ exists as a 2-category, its *Kleisli completion* is given by the functor

$$\text{Kl}(\mathcal{K} \rightarrow \mathcal{M}) = \text{Kl}_{\mathcal{K}}(\mathcal{M}) \rightarrow \text{Mod}(\mathcal{M})$$

whose domain is the locally full sub-2-category of $\text{Kl}(\mathcal{M})$ on the morphisms whose underlying 1-cell in \mathcal{M} is in the image of the functor $\mathcal{K} \rightarrow \mathcal{M}$.

3.1.3 Pseudo-monads and monoidal bicategories

Recall [Gur07] that **Gray** is the symmetric monoidal closed category of strict 2-categories and strict 2-functors, with the ‘pseudo’ Gray tensor product as \otimes . The hom is the right adjoint $\text{Ps}(\mathcal{C}, -)$ to $- \otimes \mathcal{C}$, where $\text{Ps}(\mathcal{C}, \mathcal{D})$ is the 2-category of strict 2-functors, *pseudonatural* transformations and modifications. Recall next that **Gray-Cat** is the usual category of categories enriched in **Gray**, and that every tricategory is equivalent to a **Gray**-category. These are almost strict 3-categories, except that the interchange law holds only up to coherent isomorphism. So we may pretend that our 3-categories are almost-strict in this sense.

3.1.9 Definition. A *monoidal 2-category* \mathcal{B} is given by a 3-category $\Sigma\mathcal{B}$ with a single object, which by the coherence theorem is essentially the same thing as a (strict) monoid object in **Gray**.

3.1.10 Definition ([DS97]). A *monoidal functor* $(\mathcal{B}, \otimes, i) \rightarrow (\mathcal{B}', \boxtimes, i')$ is given by an ordinary functor $F: \mathcal{B} \rightarrow \mathcal{B}'$ together with transformations μ, η , with components

$$\mu_{xy}: Fx \boxtimes Fy \rightarrow F(x \otimes y) \quad \eta_*: i' \rightarrow Fi$$

and modifications a, l, r , with components (omitting tensor symbols)

$$\begin{array}{ccc} F(x)F(y)F(z) \xrightarrow{\mu F(z)} F(xy)F(z) & & Fx \xrightarrow{F(x)\eta} F(x)F(i) \xleftarrow{\eta F(x)} Fx \\ F(x)\mu \downarrow & a & \downarrow \mu \\ F(x)F(yz) \xrightarrow{\mu} F(xyz) & & \begin{array}{ccc} & & \\ & \downarrow \mu & \\ & Fx & \end{array} \end{array} \quad \begin{array}{ccc} & & \\ & \downarrow \mu & \\ & Fx & \end{array}$$

that satisfy

$$\begin{array}{ccc}
F(x)F(y)F(z)F(w) & \xrightarrow{\mu F(z)F(w)} & F(xy)F(z)F(w) \\
\downarrow F(x)F(y)\mu & \searrow F(x)\mu F(w) & \searrow \mu F(w) \\
F(x)F(y)F(zw) & \xrightarrow{F(x)a} & F(x)F(yz)F(w) \xrightarrow{\mu F(w)} F(xyz)F(w) \\
& \searrow F(x)\mu & \downarrow F(x)\mu \quad \downarrow \mu \\
& & F(x)F(yzw) \xrightarrow{\mu} F(xyzw)
\end{array}
=
\begin{array}{ccc}
F(x)F(y)F(z)F(w) & \xrightarrow{\mu F(z)F(w)} & F(xy)F(z)F(w) \\
\downarrow F(x)F(y)\mu & \searrow \mu\mu & \downarrow F(xy)\mu \quad \searrow \mu F(w) \\
F(x)F(y)F(zw) & \xrightarrow{\mu F(zw)} & F(xy)F(zw) \xrightarrow{a} F(xyz)F(w) \\
& \searrow F(x)\mu & \downarrow \mu \\
& & F(x)F(zyw) \xrightarrow{\mu} F(xyzw)
\end{array}
\tag{3.1.1}$$

where $\mu\mu$ is the relevant interchange isomorphism, and

$$\begin{array}{ccc}
F(x)F(y) & \xrightarrow{F(x)\eta F(y)} & F(x)F(i)F(y) \xrightarrow{\mu F(y)} F(x)F(y) \\
& & \downarrow F(x)\mu \quad \downarrow \mu \\
& & F(x)F(y) \xrightarrow{\mu} F(xy)
\end{array}
=
\begin{array}{ccc}
F(x)F(y) & \xrightarrow{F(x)\eta F(y)} & F(x)F(i)F(y) \\
\downarrow F(x)\eta F(y) & \searrow F(x)l & \downarrow \mu F(y) \\
F(x)F(i)F(y) & \xrightarrow{F(x)\mu} & F(x)F(y) \xrightarrow{\mu} F(xy)
\end{array}
\tag{3.1.2}$$

3.1.11 Definition. A (pseudo-)monoid in a monoidal 2-category \mathcal{B} is a monoidal functor $\mathbf{1} \rightarrow \mathcal{B}$, and the 2-category $\text{PsMon}(\mathcal{B})$ is defined as in [McC00, DS97] to be the 2-category of monoidal functors, transformations and modifications from $\mathbf{1}$ to \mathcal{B} . If $F: \mathcal{B} \rightarrow \mathcal{B}'$ is a monoidal 2-functor then $\text{PsMon}(F): \text{PsMon}(\mathcal{B}) \rightarrow$

$\text{PsMon}(\mathcal{B}')$ is a 2-functor [McC00, section 2].

A (pseudo-)monad [Mar99] in a 3-category \mathbb{T} is given by an object $x \in \mathbb{T}$ and a pseudomonoid in $\mathbb{T}(x, x)$.

3.1.12 Definition ([Mar99]). Given a pseudomonad T on a 2-category \mathcal{K} , its 2-category $T\text{-Alg}$ of *algebras* is given by the following:

- An object is an object x of \mathcal{K} and a morphism $a: Tx \rightarrow x$ together with invertible 2-cells

$$\begin{array}{ccc} T^2x & \xrightarrow{\mu x} & Tx \\ Ta \downarrow & \alpha & \downarrow a \\ Tx & \xrightarrow{a} & x \end{array} \quad \begin{array}{ccc} x & \xrightarrow{\eta x} & Tx \\ & v \searrow & \downarrow a \\ & 1 & Fx \end{array}$$

that satisfy equations (6) and (7) of [Mar99] (these are much the same as equations (3.1.1) and (3.1.2) above, altered in the only way that makes sense).

- A morphism $(x, a) \rightarrow (y, b)$ is given by a morphism $f: x \rightarrow y$ and an invertible 2-cell

$$\begin{array}{ccc} Tx & \xrightarrow{Tf} & Ty \\ a \downarrow & \phi & \downarrow b \\ x & \xrightarrow{f} & y \end{array}$$

that satisfies equations (9) and (10) of [Mar99].

- A 2-cell between algebra morphisms is given by a 2-cell between the underlying morphisms in \mathcal{K} that makes the evident ‘cylinder’ commute.

Now if $t: x \rightarrow x$ is a monad in a 3-category \mathbb{T} , we can define the 2-categories of left and right t -modules as

$$\text{LMod}(t, y) = \mathbb{T}(y, t)\text{-Alg} \quad \text{RMod}(t, z) = \mathbb{T}(t, z)\text{-Alg}$$

These assignments are functorial in the objects y and z [op. cit.], and we may define Eilenberg–Moore and Kleisli objects as representations of these functors just as before. Similarly, the Kleisli completion of a 3-category \mathbb{T} can be defined as the full sub-3-category of $[\mathbb{T}^{\text{op}}, 2\text{-Cat}]$ on the Kleisli objects of representable monads. From this we could, following the reasoning of the previous section, define monad morphisms, EM/Kleisli 2- and 3-cells, and so on. We won’t do that fully here, but we will touch on the matter again in section 4.1.3.

3.1.13 Remark (cf. [KL97, section 2]). If T is a pseudo-monad on and A an object of \mathcal{K} , and if evaluation at A has a right adjoint $\langle -, A \rangle: \mathcal{K} \rightarrow [\mathcal{K}, \mathcal{K}]$, then in the equivalence

$$\frac{TA \longrightarrow A}{T \longrightarrow \langle A, A \rangle}$$

a morphism above is a T -algebra if and only if its transpose below is a morphism of pseudo-monoids. The proof is simply a matter of unwinding the definitions and using the T -algebra and adjunction axioms.

3.2 Limits and colimits

In this section we treat the material on colimits in $\mathcal{C}at$ that will be needed in order to define the 3-category of 2-profunctors in the next chapter. Everything from section 3.2.2 onwards is original, except where noted.

3.2.1 Representables and colimits

Some notation:

3.2.1 Definition. If $H: \mathcal{L}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{C}at$ is a (pro)functor, and $F: \mathcal{K}' \rightarrow \mathcal{K}$ and $G: \mathcal{L}' \rightarrow \mathcal{L}$, then we write

$$H(G, F) = \mathcal{L}'^{\text{op}} \times \mathcal{K}' \xrightarrow{G^{\text{op}} \times F} \mathcal{L}^{\text{op}} \times \mathcal{K} \xrightarrow{H} \mathcal{C}at$$

A profunctor of the form $H(1, F)$ or $H(G, 1)$ is called *representable* or *corepresentable*.

3.2.2 Definition. If $H: \mathcal{L}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{C}at$ is a (pro)functor, then we write an object $h \in H(\ell, k)$ as $h: \ell \rightrightarrows k$, and call it a *heteromorphism* from ℓ to k . We also write the action of morphisms of \mathcal{K} and \mathcal{L} on h as e.g. $k' \xrightarrow{f} k \xrightarrow{h} \ell \xrightarrow{g} \ell'$.

The usual generalities hold, up to the expected level of weakness, for representable 2-functors. In particular, there is a 2-categorical Yoneda lemma.

3.2.3 Proposition ([Str80, 1.9, 1.11]). *For a 2-category \mathcal{K} and a functor $F: \mathcal{K} \rightarrow \mathcal{C}at$, there is an equivalence*

$$[\mathcal{K}, \mathcal{C}at](\mathcal{K}(k, -), F) \simeq Fk$$

If $G: \mathcal{K} \rightarrow \mathcal{C}at$ is representable as $G \simeq \mathcal{K}(k, -)$ then there is an object $x \in Gk$ such that

1. *For any $y \in Gj$, there is a morphism $f: k \rightarrow j$ in \mathcal{K} and an isomorphism $f_*x = Gf(x) \cong y$.*

2. For any $g, h: k \rightarrow j$ in \mathcal{K} and $a: g_*k \rightarrow h_*k$ in Gj , there is a unique 2-cell $q: g \Rightarrow h$ in \mathcal{K} such that $q_*k = a$.

3.2.4 Corollary. *The Yoneda embedding $\mathcal{K} \rightarrow [\mathcal{K}^{\text{op}}, \mathcal{C}at]$ is 2-fully-faithful, that is, locally fully faithful and essentially surjective on morphisms.*

3.2.5 Remark. One consequence of this is the following: suppose $H: \mathcal{L}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{C}at$ is a functor, and that for each $\ell \in \mathcal{L}$ there is an object $\bar{H}\ell \in \mathcal{K}$ and a representation $H(\ell, -) \simeq \mathcal{K}(\bar{H}\ell, -)$, i.e that the corresponding functor $\mathcal{L}^{\text{op}} \rightarrow [\mathcal{K}, \mathcal{C}at]$ takes its values in representables. Then there is an essentially unique way to make \bar{H} into a functor $\mathcal{L} \rightarrow \mathcal{K}$, and H is then equivalent to $\mathcal{K}(\bar{H}, 1)$: if $g: \ell \rightarrow m$ is a morphism of \mathcal{L} , then by Yoneda there are universal objects $\bar{\ell} \in H(\ell, \bar{H}\ell)$ and $\bar{m} \in H(m, \bar{H}m)$, so that $g^*\bar{m}$ induces, by property 1 of the proposition, a morphism $\bar{H}g: \bar{H}\ell \rightarrow \bar{H}m$ such that $(\bar{H}g)_*\bar{\ell} \cong g^*\bar{m}$. The comparison maps of \bar{H} arise from the 2-cells given by property 2 in the proposition above, and their uniqueness implies their coherence.

Of course, the dual property also holds, and we may sum up the two as follows:

3.2.6 Corollary. *A pointwise (co)representable profunctor is (co)representable.*

A 2-category, strict or not, may have colimits of varied strictness. We will be mainly concerned with the weakest sort.

3.2.7 Definition ([Str80, 1.12]). Let $F: \mathcal{J} \rightarrow \mathcal{K}$ and $W: \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}at$ be functors. If the functor $[\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \mathcal{K}(F, 1))$ is representable as

$$\mathcal{K}(W \star F, 1) \simeq [\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \mathcal{K}(F, 1))$$

then we call the representing object $W \star F$ the *2-colimit* (or just the *colimit*) of F *weighted by* W . (This is known in much of the literature as a *bicolimit*.) The *conical* colimit $\text{colim } F$ of F is $\Delta \mathbf{1} \star F$, where $\Delta \mathbf{1}$ is the constant functor at the terminal category.

3.2.8 Remark. From this we may immediately derive two dual forms of the Yoneda lemma:

$$\{\mathcal{J}(j, 1), F\} \simeq Fj \quad \mathcal{J}(1, j) \star F \simeq Fj \quad (3.2.1)$$

(Cf. [Kel82, (3.10)].)

Stricter kinds of colimit are useful in constructing the above sort. If \mathcal{K} is a strict 2-category, then the *pseudo-colimit* [Str80, 1.14] of the functors in the

definition is representable via an *isomorphism*

$$\mathcal{K}(W \star_{\text{ps}} F, k) \cong [\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \mathcal{K}(F, k))$$

If both \mathcal{K} and \mathcal{J} are strict 2-categories, and the functors F and W are strict too, then we can consider the *strict pseudo-colimit* [Lac10, 6.10], which satisfies the property of the pseudocolimit with the 2-category $[\mathcal{J}^{\text{op}}, \mathcal{C}at]$ replaced by the 2-category $\text{Ps}(\mathcal{J}^{\text{op}}, \mathcal{C}at)$ of strict functors, pseudonatural transformations and modifications. The *strict colimit* is the same, except that now the functor 2-category involved is that of strict functors, strict transformations and modifications. This last is the **Cat**-colimit, in the usual enriched sense.

Pseudo-colimits, strict ones in particular, are thus *a fortiori* 2-colimits, and moreover strict pseudo-colimits are strict colimits whose weights are suitably ‘cofibrant’ [Lac10, 6.10]. Further, if \mathcal{K} is a strict 2-category, then for any 2-category \mathcal{J} there is a strict \mathcal{J}' such that $[\mathcal{J}, \mathcal{K}] \sim \text{Ps}(\mathcal{J}', \mathcal{K})$ and so for a diagram $F: \mathcal{J} \rightarrow \mathcal{K}$ and a weight $W: \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}at$, there are strict functors $F': \mathcal{J}' \rightarrow \mathcal{K}$ and $W': \mathcal{J}'^{\text{op}} \rightarrow \mathcal{C}at$ such that

$$[\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \mathcal{K}(F, 1)) \simeq \text{Ps}(\mathcal{J}'^{\text{op}}, \mathcal{C}at)(W', \mathcal{K}(F', 1))$$

That is, the strict pseudo-colimit of the strictified functors is equivalent to the 2-colimit of the originals. So a strict 2-category that has all strict (i.e. **Cat**-weighted) colimits also has all strict pseudo-colimits and hence all 2-colimits. In particular, $\mathcal{C}at$ is strictly 2-cocomplete and so is 2-cocomplete.

3.2.2 2-extranaturality

We will need to talk about 2-categorical ends and coends.

3.2.9 Definition. Let $T: \mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{L}$ be a functor and ℓ be an object of \mathcal{L} . A family $\beta_k: \ell \rightarrow T(k, k)$ is *extranatural* (in k) if for each $f: k \rightarrow j$ in \mathcal{K} there is an invertible 2-cell β_f

$$\begin{array}{ccc} \ell & \xrightarrow{\beta_k} & T(k, k) \\ \beta_j \downarrow & \cong \beta_f & \downarrow T(k, f) \\ T(j, j) & \xrightarrow{T(f, j)} & T(k, j) \end{array}$$

satisfying the following (fairly obvious) axioms:

1. β_f is natural in f : for $p: f \Rightarrow g$

$$\begin{array}{ccc}
\ell & \xrightarrow{\beta_k} & T(k, k) \\
\beta_j \downarrow & \begin{array}{c} T(k, g) \\ \beta_g \cong \end{array} & \begin{array}{c} T(k, p) \\ \leftarrow \quad \rightarrow \\ T(k, f) \end{array} \\
T(j, j) & \xrightarrow{T(g, j)} & T(k, j)
\end{array}
=
\begin{array}{ccc}
\ell & \xrightarrow{\beta_k} & T(k, k) \\
\beta_j \downarrow & \cong \beta_f & \downarrow T(k, f) \\
T(j, j) & \xrightarrow{T(f, j)} & T(k, j) \\
& \begin{array}{c} \parallel \\ T(p, j) \\ \downarrow \\ T(g, j) \end{array} & \uparrow \\
& & T(g, j)
\end{array}$$

2. $\beta_{1_k} = 1_{\beta_k}$, modulo the unitors of T and \mathcal{L} :

$$\begin{array}{ccc}
\ell & \xrightarrow{\beta_k} & T(k, k) \\
\beta_k \downarrow & \begin{array}{c} T(k, 1) \\ \cong \beta_{1_k} \end{array} & \downarrow \cong \\
T(k, k) & \xrightarrow{T(1, k)} & T(k, k) \\
& \cong & \uparrow \\
& 1 &
\end{array}
=
\begin{array}{ccc}
\ell & \xrightarrow{\beta_k} & T(k, k) \\
\beta_k \downarrow & = 1_{\beta_k} & \downarrow 1 \\
T(k, k) & \xrightarrow{1} & T(k, k)
\end{array}$$

3. β respects composition: for $gf: k \rightarrow j \rightarrow i$

$$\begin{array}{ccc}
\ell & \xrightarrow{\beta_k} & T(k, k) \\
\beta_i \downarrow & \cong \beta_{gf} & \downarrow T(k, gf) \\
T(i, i) & \xrightarrow{T(gf, j)} & T(k, i)
\end{array}
=
\begin{array}{ccc}
\ell & \xrightarrow{\beta_k} & T(k, k) \\
\beta_i \downarrow & \beta_j & \downarrow T(k, f) \\
& T(j, j) & \xrightarrow{T(f, j)} T(k, j) \\
& \begin{array}{c} \downarrow T(j, g) \\ \cong \beta_g \end{array} & \downarrow T(f, g) \\
T(i, i) & \xrightarrow{T(g, i)} T(j, i) & \xrightarrow{T(f, i)} T(k, i)
\end{array}$$

where the triangles at the lower right contain the obvious compositors of T and (T applied to) the unitors of $\mathcal{K}^{\text{op}} \times \mathcal{K}$, and the compositors of T on the boundaries are left implicit.

There is an obvious notion of modification between two extranatural transformations, so we get a category $\text{Exnat}(\ell, T)$.

3.2.10 Lemma. *If $T, S: \mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{L}$ are functors, $\alpha: T \Rightarrow S$ is a transformation and the family $\beta_k: \ell \rightarrow T(k, k)$ is extranatural in k , then the family $\alpha_{k, k} \beta_k$ is again extranatural.*

Proof. For $f: k \rightarrow j$ as before, the structure 2-cell $(\alpha\beta)_f$ is

$$\begin{array}{ccccc}
 \ell & \xrightarrow{\beta_k} & T(k, k) & \xrightarrow{\alpha_{k, k}} & S(k, k) \\
 \beta_j \downarrow & & \cong \beta_f \downarrow & & \cong \alpha_{k, f} \downarrow \\
 T(j, j) & \xrightarrow{T(f, j)} & T(k, j) & & S(k, j) \\
 \alpha_{j, j} \downarrow & & \cong \alpha_{f, j} \downarrow & \searrow \alpha_{k, j} & \downarrow S(k, f) \\
 S(j, j) & \xrightarrow{S(f, j)} & S(k, j) & & S(k, j)
 \end{array}$$

The naturality and unit axioms follow more or less obviously from this and the corresponding properties of β and α .

Using the definition above, and the composition axioms for β and α , we may expand $(\alpha\beta)_{gf}$ to

$$\begin{array}{ccccccc}
 \ell & \xrightarrow{\quad} & T(k, k) & \xrightarrow{\quad} & S(k, k) & & \\
 \downarrow & \searrow & \downarrow \beta_f & \downarrow \alpha_{k, f} & \downarrow & & \\
 & & T(j, j) & \xrightarrow{\quad} & T(k, j) & \xrightarrow{\quad} & S(k, j) \\
 & & \downarrow \beta_g & \downarrow \alpha_{k, g} & \downarrow & & \\
 T(i, i) & \xrightarrow{\quad} & T(j, i) & \xrightarrow{\quad} & T(k, i) & & \\
 \downarrow \alpha_{g, i} & & \downarrow \alpha_{f, i} & & \downarrow & & \\
 S(i, i) & \xrightarrow{\quad} & S(j, i) & \xrightarrow{\quad} & S(k, i) & &
 \end{array}$$

which is to equal

$$\begin{array}{ccccccc}
 \ell & \xrightarrow{\quad} & T(k, k) & \xrightarrow{\quad} & S(k, k) & & \\
 \downarrow & \searrow & \downarrow \beta_f & \downarrow \alpha_{k, f} & \downarrow & & \\
 & & T(j, j) & \xrightarrow{\quad} & T(k, j) & & \\
 & & \downarrow \beta_g & \downarrow \alpha_{f, j} & \downarrow & & \\
 T(i, i) & \xrightarrow{\quad} & T(j, i) & \xrightarrow{\alpha_{j, g}} & S(j, j) & \xrightarrow{\quad} & S(k, j) \\
 \downarrow \alpha_{g, i} & & \downarrow & \downarrow & \downarrow & & \\
 S(i, i) & \xrightarrow{\quad} & S(j, i) & \xrightarrow{\quad} & S(k, i) & &
 \end{array}$$

So it suffices to show that the composite 2-cells, γ and δ , say, from

$$T(j, j) \rightarrow T(k, j) \rightarrow S(k, j) \rightarrow S(k, i)$$

to

$$T(j, j) \rightarrow T(j, i) \rightarrow S(j, i) \rightarrow S(k, i)$$

in the lower right-hand corners are equal. The diagonals of these are the boundaries of $\alpha_{f,g}$, and by gluing γ and δ together along this and their boundaries we get two ‘cones’ of 2-cells, one relating $\alpha_{f,g}$ to $\alpha_{j,g}$ and $\alpha_{f,i}$ and the other relating it to $\alpha_{f,j}$ and $\alpha_{k,g}$, whose commutativity implies that γ and δ are equal. Because $(f, i)(j, g) \cong (f, g) \cong (k, g)(f, j)$ in $\mathcal{K}^{\text{op}} \times \mathcal{K}$, the naturality and composition axioms for $\alpha_{f,g}$ show that these cones do indeed commute, so that $\alpha\beta$ satisfies the composition axiom and hence is extranatural. \square

If \mathcal{K} is a 2-category, then the family $1_k: \mathbf{1} \rightarrow \mathcal{K}(k, k)$ is extranatural in k , and in fact this is the universal extranatural transformation out of $\mathbf{1}$, in the following sense.

3.2.11 Proposition (Extranatural Yoneda). *Let $H: \mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{C}at$ be a functor. Then there is an equivalence of categories*

$$\text{Exnat}(\mathbf{1}, H) \simeq \text{Nat}(\text{hom}_{\mathcal{K}}, H)$$

given from right to left by composition with the extranatural $1: \mathbf{1} \rightrightarrows \mathcal{K}$.

Proof. Given an extranatural $\beta: \mathbf{1} \rightrightarrows H$ we get a natural $\hat{\beta}$ with components $\hat{\beta}_{k,j}: f \mapsto H(k, f)(\beta_k)$ (we could equally choose the isomorphic $H(f, j)(\beta_j)$). For morphisms g, h , the mediating 2-cell $\hat{\beta}_{g,h}$ comes from the unitors of H , the compositors of \mathcal{K} , and β_g . These are all suitably natural and thus so is $\hat{\beta}$.

If $\alpha: \mathcal{K} \rightrightarrows H$ is natural, then the isomorphisms $(\alpha_{k,f})_{1_k}$ provide the components of an invertible modification $\hat{\alpha}1 \rightrightarrows \alpha$, which in fact is natural in α . The equivalence is completed by the fact that $\beta_k \cong H(k, 1_k)(\beta_k)$. \square

In enriched category theory, (co)ends are hom-weighted (co)limits [Kel82, section 3.10]. Because in our setting $\text{Nat}(F, G)$ is equivalent to the F -weighted limit $\{F, G\}$ of G [Str80, prop. 1.15], we have shown that $\mathcal{C}at$ admits ‘2-ends’

$$\int_k H(k, k) \simeq \text{Exnat}(\mathbf{1}, H).$$

If $T: \mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{L}$, we find that

$$\text{Exnat}(\ell, T) \simeq \text{Exnat}(\mathbf{1}, \mathcal{L}(\ell, T)) \simeq \{\mathcal{L}, \mathcal{L}(\ell, T)\}$$

where we write \mathcal{L} for $\text{hom}_{\mathcal{L}}$. If any of these is representable as a functor of ℓ we may call the representing object the end $\int_k T(k, k)$ of T .

The following result is immediate.

3.2.12 Proposition. *If $F, G: \mathcal{K} \rightarrow \mathcal{L}$ are functors, then*

$$\text{Nat}(F, G) \cong \text{Exnat}(\mathbf{1}, \mathcal{L}(F-, G-))$$

and hence

$$\text{Nat}(F, G) \simeq \int_k \mathcal{L}(Fk, Gk).$$

For strictly enriched categories this is a definition rather than a theorem [Kel82, section 2.2], but here it shows that our definition of 2-ends is the right one.

Coends are dual: if $S: \mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{L}$ we have that

$$\text{Exnat}(S, \ell) \simeq \text{Exnat}(\mathbf{1}, \mathcal{L}(S, \ell)) \simeq \{\mathcal{L}, \mathcal{L}(S, \ell)\}$$

and a representation of any of these may be called the coend $\int^k S(k, k)$ of S .

3.2.3 The free cocompletion of a 2-category

We want to show now that if \mathcal{K} is a 2-category then its *free 2-cocompletion* is given by $\mathbb{P}\mathcal{K} = [\mathcal{K}^{\text{op}}, \mathcal{C}at]$. We know that $\mathcal{C}at$ is cocomplete, and suspect that colimits in $\mathbb{P}\mathcal{K}$ will be calculated pointwise: let $F: \mathcal{J} \rightarrow \mathbb{P}\mathcal{K}$ and $W: \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}at$ and set $(W \star F)k = W \star F(-, k)$. Then, using prop. 3.2.12,

$$\begin{aligned} \mathbb{P}\mathcal{K}(W \star F, G) &\simeq \int_k \mathcal{C}at(W \star F(-, k), Gk) \\ &\simeq \int_k [\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \mathcal{C}at(F(-, k), Gk)) \\ &\simeq [\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \int_k \mathcal{C}at(F(-, k), Gk)) \\ &\simeq [\mathcal{J}^{\text{op}}, \mathcal{C}at](W, \mathbb{P}\mathcal{K}(F-, G)) \end{aligned}$$

As a corollary to this and prop. 3.2.3, one easily verifies the *co-Yoneda lemma*: if $Y: \mathcal{K} \rightarrow \mathbb{P}\mathcal{K}$ is the Yoneda embedding, and $W: \mathcal{K}^{\text{op}} \rightarrow \mathcal{C}at$ is a weight, then

$$W \star Y \simeq W$$

3.2.13 Proposition. *The (strict) 2-category $\mathbb{P}\mathcal{K} = [\mathcal{K}^{\text{op}}, \mathcal{C}at]$ is the free cocompletion of \mathcal{K} , in that for a cocomplete 2-category \mathcal{L} there is a 2-equivalence*

$$[\mathcal{K}, \mathcal{L}] \sim \text{Cocont}(\mathbb{P}\mathcal{K}, \mathcal{L})$$

given from right to left by composition with the Yoneda embedding $\mathcal{K} \rightarrow \mathbb{P}\mathcal{K}$.

Proof. The inverse to $- \circ Y$ sends $F: \mathcal{K} \rightarrow \mathcal{L}$ to $\bar{F}: W \mapsto W \star F$. Applying

corollary 3.2.6, and the cocompleteness of \mathcal{L} , to the functor

$$(W, F) \mapsto [\mathcal{K}^{\text{op}}, \mathcal{C}at](W, \mathcal{K}(F, 1))$$

shows that $(W, F) \mapsto W \star F$ extends to a functor $[\mathcal{K}^{\text{op}}, \mathcal{C}at] \times [\mathcal{K}, \mathcal{L}] \rightarrow \mathcal{L}$, and hence fixing F does yield a functor $(- \star F): \mathbb{P}\mathcal{K} \rightarrow \mathcal{L}$, which is cocontinuous essentially because representables are continuous (cf. [Kel82, section 3.3]).

We want to show that this inverse is a 2-equivalence. The co-Yoneda lemma above implies that if $H: \mathbb{P}\mathcal{K} \rightarrow \mathcal{L}$ is cocontinuous then $H(W) \simeq H(W \star Y) \simeq W \star HY$, showing that $F \mapsto \overline{F} = (- \star F)$ is essentially surjective. That it is 2-fully-faithful again follows easily from the Yoneda lemma. \square

3.2.4 Computing colimits

An explicit description of conical colimits in $\mathcal{C}at$ is not too difficult to find, thanks in part to a classical result due to Grothendieck and Verdier. First recall the following:

3.2.14 Definition ([Str80, 1.10]). If \mathcal{J} is a 2-category and $D: \mathcal{J} \rightarrow \mathcal{C}at$ is a functor then the *2-category of elements* $\int D$ of D is given as follows:

- an object is an object $j \in \mathcal{J}$ together with an object $x \in Dj$;
- a morphism from $x \in Dj$ to $y \in Di$ is given by a morphism $m: j \rightarrow i$ in \mathcal{J} together with a morphism $m_*x \rightarrow y$ in Di (where $m_*x = (Dm)(x)$);
- a 2-cell from $(m, m_*x \rightarrow y)$ to $(n, n_*x \rightarrow y)$ is given by a 2-cell $\alpha: m \Rightarrow n$ such that $\alpha_*x: m_*x \rightarrow n_*x$ fits into a commuting triangle over y .

The projection $\int D \rightarrow \mathcal{J}$ is a strict functor, and a morphism $(m, f: m_*x \rightarrow y)$ in $\int D$ is called *opcartesian* when f is invertible. Of course, if \mathcal{J} is an ordinary category then so is $\int D$.

3.2.15 Proposition ([GVSD72, exposé VI, def. 6.3]). *If \mathbf{C} is an ordinary category and $D: \mathbf{C} \rightarrow \mathcal{C}at$ is a pseudofunctor, then the 2-colimit of D is obtained by taking the category of elements $\int D$ of D and formally inverting the opcartesian morphisms.*

In more detail, one verifies that $\int D$ is the *lax conical colimit*¹ of D , i.e. that there is an equivalence

$$\text{Lax}(D, \Delta \mathbf{B}) \simeq [\int D, \mathbf{B}]$$

¹Note that lax transformations $D \rightarrow \Delta \mathbf{B}$ correspond to *oplax* transformations $\Delta \mathbf{1} \rightarrow \mathcal{C}at(D, \mathbf{B})$, so that a lax conical colimit in this sense is actually an oplax weighted colimit.

where the left-hand side is the category of lax transformations and modifications from D to the constant functor at the category \mathbf{B} . Furthermore, the pseudonatural transformations on the left correspond to the functors on the right that invert the opcartesian morphisms of $\int D$, so that

$$\mathrm{Ps}(D, \Delta \mathbf{B}) \simeq [\int D, \mathbf{B}]_{S^{-1}} \simeq [\int D[S^{-1}], \mathbf{B}]$$

where S is the class of opcartesian morphisms of $\int D$ and the notation $[-, -]_{S^{-1}}$ denotes the full subcategory of the functor category on those functors that invert the elements of S . The objects of the category of fractions $\int D[S^{-1}]$ are those of $\int D$, and its morphisms are zig-zags of morphisms in $\int D$ in which the backwards-pointing components are in S .

To extend this result to the case of diagrams indexed by 2-categories, we first recall that there is a monoidal adjunction $\pi \dashv d$, in which $\pi = \pi_0: \mathbf{Cat} \rightarrow \mathbf{Set}$ is the ‘connected components’ functor and $d: \mathbf{Set} \rightarrow \mathbf{Cat}$ the ‘discrete category’ functor. Then a suitably ‘weak’ version of the usual change-of-enrichment arguments, or simply direct calculation, verify the following.

3.2.16 Proposition. *There is an adjunction*

$$[\pi_* \mathcal{K}, \mathbf{B}] \simeq [\mathcal{K}, d_* \mathbf{B}]$$

in which the functors $\pi_* \dashv d_*$ apply π or d hom-wise. Moreover, this adjunction descends to the case of functors that invert a class S of morphisms of \mathcal{K} :

$$[\pi_* \mathcal{K}, \mathbf{B}]_{S^{-1}} \simeq [\mathcal{K}, d_* \mathbf{B}]_{S^{-1}}$$

Proof. A functor $\mathcal{K} \rightarrow d_* \mathbf{B}$ must take any 2-cell of \mathcal{K} to an identity, and therefore identify any pair of connected morphisms, which defines an essentially unique functor out of $\pi_* \mathcal{K}$. The former inverts a specified morphism if and only if the latter inverts its equivalence class, because their image in \mathbf{B} is the same and invertibility in $d_* \mathbf{B}$ is precisely invertibility in \mathbf{B} .

A transformation between two such functors takes objects of \mathcal{K} to morphisms of \mathbf{B} , and morphisms to strictly commuting squares. Naturality of these with respect to 2-cells means that any pair of connected morphisms must be assigned the same square, and this specifies a unique transformation out of $\pi_* \mathcal{K}$. \square

3.2.17 Proposition. *If $D: \mathcal{J} \rightarrow \mathbf{Cat}$, then*

$$\mathrm{Lax}(D, \Delta \mathbf{B}) \simeq [\int D, d_* \mathbf{B}] \simeq [\pi_* \int D, \mathbf{B}]$$

and

$$\text{Ps}(D, \Delta \mathbf{B}) \simeq [\int D, d_* \mathbf{B}]_{S^{-1}} \simeq [\pi_* \int D, \mathbf{B}]_{S^{-1}} \simeq [(\pi_* \int D)[S^{-1}], \mathbf{B}]$$

so that the lax colimit of D is $\pi_* \int D$, and the colimit of D is got by inverting the images of the opcartesian morphisms of $\int D$ in this category.

Proof. (In light of the previous proposition, only the first in each chain of equivalences requires proof.) Let $A: \int D \rightarrow d_* \mathbf{B}$ be a functor. The inclusions $\delta_i: Di \rightarrow \int D$ of the fibres of D give a family of functors $(A\delta_i: Di \rightarrow d_* \mathbf{B})_i$. For each $m: i \rightarrow j$ and $x \in Di$, the canonical opcartesian $(m, 1): x \rightarrow m_* x$ gives a family $\delta_m x: \delta_i x \rightarrow \delta_j m_* x$, and $A\delta_m x$ is then natural in x , because given $k: x \rightarrow x'$ both sides of the naturality square are equal to $A(m, m_* k)$. It is easy to check then that $m \mapsto A\delta_m$ is functorial — the two morphisms $x \rightarrow (nm)_* x$ in $\int D$ that are required to be equal for δ to be a lax transformation are instead just isomorphic, but A turns this 2-cell into an identity. Similarly, a 2-cell $\phi: m \Rightarrow n$ gives rise to a 2-cell between the two morphisms $x \rightarrow n_* x$ that naturality would require to be equal, but applying A ensures that their images are equal in $d_* \mathbf{B}$. So $A\delta$ is a lax transformation, and because the components of δ are opcartesian, $A\delta$ will be pseudo-natural if A inverts them. Moreover, if $\mu: A \Rightarrow B$ is a transformation, then so is $\mu\delta_i$, and these form a modification $A\delta \Rightarrow B\delta$ by virtue of the interchange law for $\mathcal{C}at$, which applies because $d_* \mathbf{B}$ is locally discrete, and this assignment is clearly functorial.

Conversely, suppose given a lax transformation $\alpha: D \Rightarrow \Delta \mathbf{B}$ and consider a morphism

$$(m: i \rightarrow j, f: m_* x \rightarrow y): (i, x) \rightarrow (j, y)$$

in $\int D$. We get a configuration like this:

$$\begin{array}{ccc}
 & Di & \\
 x \nearrow & \downarrow & \searrow \alpha_i \\
 \mathbf{1} & f \Downarrow Dm & \Downarrow \alpha_m \mathbf{B} \\
 y \searrow & \downarrow & \nearrow \alpha_j \\
 & Dj &
 \end{array}$$

which defines a morphism $\alpha_i x \rightarrow \alpha_j y$ in \mathbf{B} . This assignment is functorial because of the coherence of the α_m with respect to identities and composition in \mathcal{J} , and it takes 2-cells $\phi: f \Rightarrow g$ to identities in \mathbf{B} because of the naturality condition on α . If α is pseudo, moreover, then this functor $\hat{\alpha}$ clearly inverts any morphism (m, f) in $\int D$ for which f is invertible. If $p: \alpha \Rightarrow \beta$ is a modification, then the morphisms $p_i x: \alpha_i x \rightarrow \beta_i x$ assemble, by the modification axiom and

naturality of each p_i , into a natural transformation $\hat{\alpha} \Rightarrow \hat{\beta}$.

Applying this recipe to the transformation $A\delta$ arising from a functor $A: \int D \rightarrow d_*\mathbf{B}$ gives a functor whose value at a morphism (m, f) as above is

$$Ax \xrightarrow{A(m,1)} Am_*x \xrightarrow{A(1,f)} Ay$$

which is the A -image of the opcartesian–vertical factorization of (m, f) . Now the latter is only isomorphic in $\int D$ to (m, f) itself, but as before applying A makes this an equality. So the functor arising from $A\delta$ is equal to A . In the other direction, the functor $\hat{\alpha}: \int D \rightarrow d_*\mathbf{B}$ corresponding to a lax transformation α is equal to α_i on each fibre Di , and applying it to δ_m produces exactly α_m . Finally, a simple calculation shows that this correspondence is also bijective on morphisms. \square

3.2.5 Coends again

As in ordinary category theory, there are useful relationships between (co)ends and (co)limits. If the W -weighted colimit $W \star F$ of F (def. 3.2.7) exists in \mathcal{K} we find that

$$\begin{aligned} \mathcal{K}(W \star F, k) &\simeq \{W, \mathcal{K}(F-, k)\} \\ &\simeq \int_j \mathcal{C}at(Wj, \mathcal{K}(Fj, k)) \\ &\simeq \int_j \mathcal{K}(Wj \otimes Fj, k) \end{aligned}$$

if the tensors $Wj \otimes Fj$ exist, so that

$$W \star F \simeq \int_j Wj \otimes Fj \tag{3.2.2}$$

as usual [Kel82, section 3.10]. Dually, of course, we may, by only a slight generalization of prop. 3.2.12, write

$$\{W, G\} \simeq \int_j Wj \pitchfork Gj$$

as long as the necessary cotensors exist.

3.2.18 Definition. The category **Dsc** is the subcategory of the (unaugmented) simplex category Δ generated by the following diagram:

$$\mathbf{3} \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{2} \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{1}$$

A *codescent diagram* in \mathcal{K} is a functor $\mathbf{Dsc}^{\text{op}} \rightarrow \mathcal{K}$. The conical colimit of such a diagram is called its *codescent object*.

Coends may be expressed as codescent objects (cf. the usual presentation of 1-dimensional coends as coequalizers, as in [Kel82, (2.2)]): for any 2-category \mathcal{K} , the co-Yoneda lemma lets us write $\mathcal{K}(k, \ell) \simeq \mathcal{K}(k, -) \star \mathcal{K}(-, \ell)$, and the (dual of the) construction of weighted 2-limits in [Str87] yields $\mathcal{K}(k, \ell)$ as the following codescent object, where we adopt a tensor-style notation as in $\mathcal{K}_j^k \mathcal{K}_\ell^j = \coprod_j \mathcal{K}(k, j) \times \mathcal{K}(j, \ell)$, etc., with the obvious summation convention for repeated indices:

$$\mathcal{K}_j^k \mathcal{K}_i^j \mathcal{K}_h^i \mathcal{K}_\ell^h \begin{array}{c} \rightrightarrows \\ \rightleftarrows \\ \rightleftarrows \\ \rightleftarrows \end{array} \mathcal{K}_j^k \mathcal{K}_i^j \mathcal{K}_\ell^i \begin{array}{c} \leftleftarrows \\ \leftleftarrows \\ \leftleftarrows \\ \leftleftarrows \end{array} \mathcal{K}_j^k \mathcal{K}_\ell^j \longrightarrow \mathcal{K}_\ell^k \quad (3.2.3)$$

The morphisms in the diagram are the actions of \mathcal{K} on itself given by composition and the insertion of identities. The whole diagram is functorial in (k, ℓ) and so presents $\text{hom}_{\mathcal{K}}$ as a codescent object in the functor category. (The diagram is also the ‘canonical presentation’ [LCMV02] of the algebra $\text{hom}_{\mathcal{K}}$ for the 2-monad on $[\text{ob } \mathcal{K}, [\mathcal{K}, \mathcal{C}at]]$ whose algebras are functors $\mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{C}at$.)

Now if $T: \mathcal{K}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{L}$ is a functor, then applying the functor $(- \star T)$ (which is cocontinuous, as observed in the proof of prop. 3.2.13) to the codescent diagram (3.2.3) for $\text{hom}_{\mathcal{K}^{\text{op}}}$ yields the following codescent object in \mathcal{L} :

$$T_j^k \mathcal{K}_i^j \mathcal{K}_k^i \begin{array}{c} \rightrightarrows \\ \rightleftarrows \\ \rightleftarrows \\ \rightleftarrows \end{array} T_j^k \mathcal{K}_k^j \begin{array}{c} \leftleftarrows \\ \leftleftarrows \\ \leftleftarrows \\ \leftleftarrows \end{array} T_k^k \longrightarrow \int^k T(k, k) \quad (3.2.4)$$

In more detail, $(- \star T)$ preserves codescent objects and coproducts, so we get e.g.

$$\begin{aligned} \left(\coprod_{j,i} \mathcal{K}(-, j) \times \mathcal{K}(j, i) \times \mathcal{K}(i, -) \right) \star T &\simeq \coprod_{j,i} \left(\mathcal{K}(-, j) \times \mathcal{K}(j, i) \times \mathcal{K}(i, -) \right) \star T \\ &\simeq \coprod_{j,i} \left((\mathcal{K}^{\text{op}} \times \mathcal{K})(-, (i, j)) \star T \right) \times \mathcal{K}(j, i) \\ &\simeq T_j^i \mathcal{K}_i^j \end{aligned}$$

where in the second line we use the fact that product with $\mathcal{K}(j, i)$ is the tensor in the functor category and so is preserved by $(- \star T)$, and in the last we use the Yoneda equivalence (3.2.1). It follows that the rightward arrows in (3.2.4) are given by composition in \mathcal{K} and the left and right actions of \mathcal{K} on T . The results of the previous section now yield an explicit recipe for computing coends in $\mathcal{C}at$.

Codescent objects also figure in the 2-categorical version of Beck’s theorem, which we shall need for theorem 4.1.3.

3.2.19 Proposition (2-monadicity theorem, [LCMV02, theorem 3.6]). *Let*

$U: \mathcal{K} \rightarrow \mathcal{L}$ be a 2-functor with left adjoint F . The canonical functor $\mathcal{K} \rightarrow \mathcal{L}^{UF}$ into the 2-category of (pseudo) UF -algebras is an equivalence if and only if U reflects adjoint equivalences and \mathcal{K} has and U preserves colimits of codescent diagrams whose U -image has an absolute colimit.

Chapter 4

Equipments

This chapter contains our main results. The first section constructs the 3-category of 2-profunctors as promised, and shows that it has well-behaved Kleisli objects for pseudo-monads. This then gives a correspondence between pseudo-monads on a 2-category \mathcal{K} and identity-on-objects functors $\mathcal{K} \rightarrow \mathcal{M}$, which is the basis for our comparison of notions of equipment in section 4.1.3. We stop short of trying to construct a 2- or 3-category of equipments directly from the monad definition: this could certainly be done, and section 5.2.1 discusses how one might go about it, but it is more than we need. The correspondence on objects and morphisms that we give is enough to get the results that we want, so it is a natural stopping point.

Section 4.2 then defines cartesian equipments as cartesian objects in the 2-category of equipments constructed in the previous section, and shows that there is a fully faithful functor from the 2-category of regular fibrations into that of cartesian equipments. Axioms are given that ensure that a given cartesian equipment is in the image of this functor; these we call *regular* equipments, and the full sub-2-category of cartesian equipments on the regular ones is therefore equivalent to the 2-category of regular fibrations. The last subsection compares comprehension for predicates in a regular fibration (def. 2.1.21) with tabulation for morphisms and with Eilenberg–Moore objects for co-monads in a regular equipment.

Finally, section 4.3 examines the two constructions of the effective topos through the lens of the preceding material, showing how the equivalence we have given between regular fibrations and equipments can be used to relate them.

4.1 2-profunctors and equipments

A 2-profunctor $H: \mathcal{K} \rightarrow \mathcal{L}$, as we have said, will be a functor $\mathcal{L}^{\text{op}} \times \mathcal{K} \rightarrow \mathcal{C}at$. By the results of section 3.2.3, this is essentially the same thing as a cocontinuous functor $\mathbb{P}\mathcal{K} \rightarrow \mathbb{P}\mathcal{L}$. Because 2-categories of the form $\mathbb{P}\mathcal{K}$ are strict, composition of such functors is associative and unital on the nose. So one would hope that 2-Prof would turn out to be a **Gray**-category, or even a strict 3-category, but it is neither: whiskering a transformation by a functor fails to be strictly functorial.

4.1.1 The tricategory 2-Prof

4.1.1 Definition. 2-Prof is the tricategory whose objects are 2-categories $\mathcal{K}, \mathcal{L}, \dots$ and whose homs are given by $\text{hom}(\mathcal{K}, \mathcal{L}) = \text{Cocont}(\mathbb{P}\mathcal{K}, \mathbb{P}\mathcal{L})$.

By this definition, all of the structure of 2-Prof bar the objects is imported directly from 2-Cat . Because 2-Cat is known to be a tricategory [Gur07, section 6.3], then, so is 2-Prof .

Suppose we are given profunctors $H: \mathcal{K} \rightarrow \mathcal{L}$ and $G: \mathcal{L} \rightarrow \mathcal{M}$, corresponding to $\hat{H}: \mathbb{P}\mathcal{K} \rightarrow \mathbb{P}\mathcal{L}$ and $\hat{G}: \mathbb{P}\mathcal{L} \rightarrow \mathbb{P}\mathcal{M}$. Then we can compose H with G by composing \hat{H} directly with \hat{G} and passing back across the equivalence of prop. 3.2.13 to get a profunctor $GH: \mathcal{K} \rightarrow \mathcal{M}$:

$$GH(m, k) \simeq \hat{G}(\hat{H}(Yk))(m) \simeq H(-, k) \star G(m, -)$$

By the coend formula (3.2.2) for weighted limits, this gives:

$$G \circ H \simeq \int^{\ell} G(-, \ell) \times H(\ell, -) \quad (4.1.1)$$

just as for ordinary profunctors. So we may switch freely between profunctors considered as cocontinuous functors between 2-presheaf categories, composed as ordinary functors, and profunctors considered as $\mathcal{C}at$ -valued functors composed as above.

It follows immediately from this and the cartesian closedness of $\mathcal{C}at$ that

4.1.2 Proposition. 2-Prof has stable local colimits; that is, the colimits in $2\text{-Prof}(\mathcal{K}, \mathcal{L}) \simeq [\mathcal{L}^{\text{op}} \times \mathcal{K}, \mathcal{C}at]$ are preserved by composition with a profunctor on either side.

Sending a functor $F: \mathcal{K} \rightarrow \mathcal{L}$ to the profunctor $\mathcal{L}(1, F): \mathcal{K} \rightarrow \mathcal{L}$ gives a mapping from 2-Cat to 2-Prof that is the identity on objects and locally fully faithful by the Yoneda lemma (corollary 3.2.4). The co-Yoneda lemma shows that it is functorial, i.e. that $\mathcal{M}(1, G) \circ \mathcal{L}(1, F) \simeq \mathcal{M}(1, GF)$. (Indeed, more is

true: by the same lemma, we have, for functors $F: \mathcal{K} \rightarrow \mathcal{L}$ and $G: \mathcal{J} \rightarrow \mathcal{M}$, and a profunctor $H: \mathcal{L} \rightarrow \mathcal{M}$, that

$$\mathcal{M}(G, 1) \circ H \circ \mathcal{L}(1, F) \simeq H(G, F)$$

as for the analogous functor $\mathcal{C}at \rightarrow \mathcal{P}rof$ [Woo82, Shu08].)

The functor $\widehat{\mathcal{L}(1, F)}: \mathbb{P}\mathcal{K} \rightarrow \mathbb{P}\mathcal{L}$ (sometimes called the *Yoneda extension* of F) takes a weight W and an object $\ell \in \mathcal{L}$ to $W \star \mathcal{L}(\ell, F-)$. As a functor of ℓ , this is the pointwise colimit $W \star \mathcal{L}(1, F)$, where $\mathcal{L}(1, F)$ is taken as a functor $\mathcal{K} \rightarrow \mathbb{P}\mathcal{L}$. Dually, $\widehat{\mathcal{L}(F, 1)}$ takes $V \in \mathbb{P}\mathcal{L}$ and $k \in \mathcal{K}$ to $V \star \mathcal{L}(Fk, -)$; but by the co-Yoneda lemma this is just VFk , so that $\widehat{\mathcal{L}(F, 1)}$ is the pullback-along- F functor F^* . Using this, we may calculate

$$\begin{aligned} \mathbb{P}\mathcal{L}(\widehat{\mathcal{L}(1, F)}W, V) &\simeq \mathbb{P}\mathcal{K}(W, \mathbb{P}\mathcal{L}(\mathcal{L}(1, F), V)) \\ &\simeq \mathbb{P}\mathcal{K}(W, VF) && \text{by Yoneda} \\ &\simeq \mathbb{P}\mathcal{K}(W, \widehat{\mathcal{L}(F, 1)}V) \end{aligned}$$

Thus $\widehat{\mathcal{L}(1, F)}$ is left adjoint to $\widehat{\mathcal{L}(F, 1)}$, and so we have

$$\mathcal{L}(1, F) \dashv \mathcal{L}(F, 1) \tag{4.1.2}$$

in 2-Prof . The functor $2\text{-Cat} \rightarrow 2\text{-Prof}$ is therefore a proarrow equipment, in a suitable 3-categorical sense.

4.1.2 Kleisli objects in 2-Prof

The Kleisli object of a monad $H: \mathbf{C} \rightarrow \mathbf{C}$ in $\mathcal{P}rof$ is given by the category whose objects are those of \mathbf{C} and whose homs $\mathbf{C}_H(a, b)$ are given by $H(a, b)$. Identities and composition are defined using the unit and multiplication of H . So a monad on \mathbf{C} in $\mathcal{P}rof$ is essentially the same thing as a functor $\mathbf{C} \rightarrow \mathbf{D}$ that is bijective on objects. Things are much the same in our 2-categorical setting.

4.1.3 Theorem. *2-Prof has tight Kleisli objects: if $T: \mathcal{K} \rightarrow \mathcal{K}$ is a monad in 2-Prof , then there is a 2-category \mathcal{K}_T and a functor $F_T: \mathcal{K} \rightarrow \mathcal{K}_T$ such that composition with the right T -module $\mathcal{K}_T(1, F_T)$ gives rise to an equivalence $2\text{-Prof}(\mathcal{K}_T, -) \sim \text{RMod}(T, -)$.*

Proof. A 2-category \mathcal{K} is the same thing as a pseudo double category [GP99, section 1.9] whose category of objects is discrete, and this in turn means that \mathcal{K} is a monad in $\mathcal{S}pan(\mathcal{C}at)$ on the discrete category $\mathcal{K}_0 = \text{ob}\mathcal{K}$. We will show

first that the forgetful functor

$$2\text{-Prof}(\mathcal{K}, \mathcal{K}) = [\mathcal{K}^{\text{op}} \times \mathcal{K}, \mathcal{C}at] \longrightarrow [\mathcal{K}_0 \times \mathcal{K}_0, \mathcal{C}at] = \mathit{Span}(\mathcal{C}at)(\mathcal{K}_0, \mathcal{K}_0)$$

is monoidal (def. 3.1.10) and so takes the monad T to a monad in $\mathit{Span}(\mathcal{C}at)$ (def. 3.1.11).

The monoidal structure on $[\mathcal{K}^{\text{op}} \times \mathcal{K}, \mathcal{C}at]$ is given by profunctor composition (4.1.1). Writing the image of a profunctor H under the restriction functor above as \tilde{H} , we find that the composite $\tilde{G}\tilde{H}$ in $[\mathcal{K}_0 \times \mathcal{K}_0, \mathcal{C}at]$ is given by

$$\tilde{G}\tilde{H}(k, \ell) = G_j^k H_\ell^j = \coprod_j G(k, j) \times H(j, \ell)$$

The identity for this composition is the equality predicate on \mathcal{K}_0 , which sends k, ℓ to the terminal category $\mathbf{1}$ if $k = \ell$, or to the empty category \emptyset otherwise. Equivalently, it is the identity span on \mathcal{K}_0 .

The required comparison morphisms are then given by the codescent morphism (3.2.4)

$$G_j^k H_\ell^j = \coprod_j G(k, j) \times H(j, \ell) \longrightarrow \int^j G(k, j) \times H(j, \ell) = (GH)_\ell^k$$

and the identity-assigning functor $1_- : \mathbf{1} \rightarrow \mathcal{K}(k, k)$. We must show that these satisfy the conditions of def. 3.1.10.

For any three composable profunctors, there is a diagram $\mathbf{Dsc}^{\text{op}} \times \mathbf{Dsc}^{\text{op}} \rightarrow [\mathcal{K}^{\text{op}} \times \mathcal{K}, \mathcal{C}at]$ (where \mathbf{Dsc} is as in def. 3.2.18) whose colimit is their composite. By universality, this may be calculated directly, or as the colimit of the colimit of either of the two adjoints $\mathbf{Dsc}^{\text{op}} \rightarrow [\mathbf{Dsc}^{\text{op}}, [\mathcal{K}^{\text{op}} \times \mathcal{K}, \mathcal{C}at]]$ of the diagram. This gives the injections

$$\begin{array}{ccc} G_j^k H_i^j K_\ell^i & \longrightarrow & G_j^k (HK)_\ell^j \\ \downarrow & & \downarrow \\ (GH)_i^k K_\ell^i & \longrightarrow & (GHK)_\ell^k \end{array}$$

and the canonical isomorphism filling this square is the required associator a . The commutative cube formed from the six different such squares associated to a fourfold composite is exactly the coherence condition (3.1.1) for a .

To express the coherence condition on the unitors, we first note that the required l and r for the composite of G and H arise from the left unitor of G

and the right unitor of $H: l$ is the 2-cell in

$$\begin{array}{ccc}
 G_j^k H_\ell^j & \xrightarrow{1} & G_j^k H_\ell^j \\
 & \searrow & \nearrow \\
 & \lambda & \\
 & G_j^k \mathcal{K}_i^j H_\ell^i &
 \end{array}$$

whose components are the induced isomorphisms $(g, h) \cong (g1, h)$, where $g1$ is the action by G on g of the appropriate identity morphism of \mathcal{K} , and similarly for r . The coherence condition itself then requires that the morphism in $(GH)_\ell^k$:

$$(g, h) \leftarrow (g1, h) \leftarrow (g, 1, h) \rightarrow (g, 1h) \rightarrow (g, h) \quad (4.1.3)$$

be the identity. This is equal to the composite of (the formal inverse of)

$$(g, h) \rightarrow (g, 1, h) \rightarrow (g1, h) \rightarrow (g, h)$$

with

$$(g, h) \rightarrow (g, 1, h) \rightarrow (g, 1h) \rightarrow (g, h)$$

where in both cases the factor $(g, h) \rightarrow (g, 1, h)$ is given by the action of the splitting in the codescent diagram. In each of these the underlying morphism of \mathbf{Dsc} is the identity, and the morphism $1^*(g, h) \rightarrow (g, h)$ is the composite $(g, h) \rightarrow (g1, h) \rightarrow (g, h)$, respectively $(g, h) \rightarrow (g, 1h) \rightarrow (g, h)$, of inverses. Both morphisms are thus identities, and so their formal composite in the coend $(GH)_\ell^k$ is also the identity. Hence the functor $[\mathcal{K}^{\text{op}} \times \mathcal{K}, \mathcal{Cat}] \rightarrow [\mathcal{K}_0 \times \mathcal{K}_0, \mathcal{Cat}]$ is indeed monoidal.

The monad T in 2-Prof is sent by this functor to a 2-category \mathcal{K}_T , with objects those of \mathcal{K} and hom-categories $\mathcal{K}_T(k, \ell)$ the values $T(k, \ell)$ of T . Identities in \mathcal{K}_T are the η -images of identities in \mathcal{K} , and composition is given by the action of μ . The unit η of T is a morphism of monoids in $2\text{-Prof}(\mathcal{K}, \mathcal{K})$ and thus is sent to a functor $F_T: \mathcal{K} \rightarrow \mathcal{K}_T$, which of course is the identity on objects.

It remains to show that the profunctor $\mathcal{K}_T(1, F_T)$ is the universal right T -module. The adjunction $\mathcal{K}_T(1, F_T) \dashv \mathcal{K}_T(F_T, 1)$ (4.1.2) gives rise to an adjunction

$$2\text{-Prof}(\mathcal{K}, \mathcal{L}) \begin{array}{c} \xrightarrow{-\circ \mathcal{K}_T(F_T, 1)} \\ \perp \\ \xleftarrow{-\circ \mathcal{K}_T(1, F_T)} \end{array} 2\text{-Prof}(\mathcal{K}_T, \mathcal{L})$$

The unitors of T supply an equivalence $T \simeq \mathcal{K}_T(F_T, F_T)$ (whose components are identities), which respects their monad structures essentially by definition — the unit and counit of the adjunction above are given by the unit and multiplication of T . Thus $\text{RMod}(T, \mathcal{L})$ is equivalent to the category of algebras for the monad

induced by the adjunction above, and so there is a canonical comparison functor $2\text{-Prof}(\mathcal{K}_T, \mathcal{L}) \rightarrow \text{RMod}(T, \mathcal{L})$ given by composition with the module $\mathcal{K}_T(1, F_T)$. To show that this functor is an equivalence, then, and hence that this module is the universal one, it suffices to show that the right adjoint $-\circ\mathcal{K}_T(1, F_T): H \mapsto H(1, F_T)$ above is monadic, in the sense of prop. 3.2.19. We already know (prop. 4.1.2) that 2-Prof has stable local colimits, so that $2\text{-Prof}(\mathcal{K}_T, \mathcal{L})$ has, and $-\circ\mathcal{K}_T(1, F_T)$ preserves, the required codescent objects. It remains only to show that this functor reflects adjoint equivalences: if $\alpha: G \Rightarrow H$ is a transformation such that $\alpha \circ \mathcal{K}_T(1, F_T): G(1, F_T) \Rightarrow H(1, F_T)$ is an equivalence, then because F_T is the identity on objects, the components of α are precisely the components of $\alpha \circ \mathcal{K}_T(1, F_T)$, and hence if the latter are all equivalences then so are the former. \square

4.1.4 Corollary. *The corepresentable profunctor $\mathcal{K}_T(F_T, 1)$ exhibits \mathcal{K}_T as the Eilenberg–Moore object of T .*

Proof. Apply the argument above to the adjunction

$$2\text{-Prof}(\mathcal{L}, \mathcal{K}) \begin{array}{c} \xrightarrow{\mathcal{K}_T(1, F_T) \circ -} \\ \perp \\ \xleftarrow{\mathcal{K}_T(F_T, 1) \circ -} \end{array} 2\text{-Prof}(\mathcal{L}, \mathcal{K}_T)$$

to show that $2\text{-Prof}(\mathcal{L}, \mathcal{K}_T) \sim \text{LMod}(T, \mathcal{L})$. \square

4.1.5 Corollary. *Precomposition with $\mathcal{K}_T(1, F_T)$ preserves and detects representables.*

Proof. If $H: \mathcal{K}_T \rightarrow \mathcal{L}$ is representable then clearly so is $H(1, F_T)$. Suppose conversely that $H(1, F_T)$ is representable, as $\mathcal{L}(1, G)$, say. Then, because F_T is the identity on objects, each $H(-, k) = H(-, F_T k)$ is a representable presheaf $\mathcal{L}(-, Gk)$, and so by remark 3.2.5 H is representable. \square

This means in particular that representable right modules correspond to representable profunctors out of the Kleisli object. It also means that if $T = \mathcal{K}(1, T')$ is a representable monad, then the Kleisli object \mathcal{K}_T of T in 2-Prof is also the Kleisli object of T' in 2-Cat ; because T is representable, the right adjoint $\mathcal{K}_T(F_T, 1)$ of the Kleisli morphism, arising as it does from the right T -module structure of T itself, is representable, as $\mathcal{K}(1, U_T)$, say. But the second-to-last corollary shows that this is not necessarily the case for Eilenberg–Moore objects.

As an aside, we can say something similar about coproducts in 2-Prof (cf. axioms 4 and 5 of [Woo85]):

4.1.6 Proposition. *2-Prof has representable coproducts: if $\{\mathcal{K}_i\}_i$ is a small family of 2-categories, then its coproduct $\coprod_i \mathcal{K}_i$ in 2-Cat is also its coproduct*

in 2-Prof . As before, the injections are representable and preserve and jointly detect representables, and their adjoints together exhibit the coproduct of the \mathcal{K}_i as their product.

Proof. Let $\{\iota_i: \mathcal{K}_i \rightarrow \coprod_i \mathcal{K}_i\}_i$ be the obvious injections, and assume given a family $\{\mathcal{K}_i \rightarrow \mathcal{L}\}_i$. Then there are equivalences

$$\frac{\frac{\{\mathcal{K}_i \rightarrow \mathcal{L}\}_i}{\mathcal{K}_i \rightarrow [\mathcal{L}^{\text{op}}, \mathcal{C}at]_i}}{\coprod \mathcal{K}_i \rightarrow [\mathcal{L}^{\text{op}}, \mathcal{C}at]} \quad \frac{\quad}{\coprod \mathcal{K}_i \rightarrow \mathcal{L}}$$

showing that $\coprod \mathcal{K}_i$ is again a coproduct in 2-Prof . The equivalence (corollary 3.2.6) between representability and pointwise representability shows that the profunctor in the bottom line is representable if and only if the family in the top line is so. Finally, much the same argument (together with the fact that $(\coprod \mathcal{K}_i)^{\text{op}} = \coprod \mathcal{K}_i^{\text{op}}$) shows that the corepresentables $\mathcal{K}_i(\iota_i, 1)$ mediate an equivalence between profunctors $\mathcal{L} \rightarrow \coprod \mathcal{K}_i$ and families $\{\mathcal{L} \rightarrow \mathcal{K}_i\}_i$. \square

4.1.3 Equipments and their morphisms

We now want to argue that the various notions of proarrow equipment in the literature are either subsumed by or at least clearly related to the notion of pseudo-monad in 2-Prof , or, what is essentially the same thing, the Kleisli object of one such. We will see, however, that even though monads and monad morphisms capture the right notion of equipments and functors between them, the situation is more subtle when it comes to transformations. Here we will treat only the case of equipments over (i.e. monads on) 2-categories that are locally discrete, because that is the important one, but we will touch in the general case again in chapter 5.

4.1.7 Definition. An *equipment* is, equivalently, a monad in 2-Prof on a 1-category \mathbf{K} or an identity-on-objects functor $\mathbf{K} \rightarrow \mathcal{M}$.

We will follow [LS12] in calling \mathbf{K} the category of *tight* morphisms of the equipment, a morphism in \mathcal{M} being called tight if it is the image of a morphism of \mathbf{K} .

4.1.8 Definition ([Woo82]). An *equipment in the sense of Wood* is given by 2-categories \mathcal{K} and \mathcal{M} with the same objects, where \mathcal{K} is strict, and a 2-functor $(-)_\bullet: \mathcal{K} \rightarrow \mathcal{M}$ that is the identity on objects and locally fully faithful, and such that the image f_\bullet of every morphism f of \mathcal{K} has a right adjoint f^\bullet in \mathcal{M} (i.e. the functor $(-)_\bullet$ factors through $\text{Map}(\mathcal{M})$).

Leaving out the condition on the existence of right adjoints, it is clear that an identity-on-objects functor $\mathcal{K} \rightarrow \mathcal{M}$ (out of a strict 2-category) that is locally fully faithful is the same thing as an identity-on-objects functor $\mathbf{K} \rightarrow \mathcal{M}$ out of a locally discrete 2-category. By the results of the previous section, this is the same thing as the Kleisli object of an essentially unique monad in 2-Prof on \mathbf{K} . An equipment that satisfies the condition that tight morphisms have right adjoints we will call a *map-equipment*.

4.1.9 Definition ([CKVW98]). An *equipment in the sense of Carboni et. al.* is given by a category \mathbf{K} together with a 2-functor $M: \mathbf{K}^{\text{op}} \times \mathbf{K} \rightarrow \text{Cat}$. A *pointed equipment* in their sense is given by an equipment together with a transformation $\text{hom}_{\mathbf{K}} \Rightarrow M$.

Any monad in 2-Prof has a canonical underlying pointed equipment in this sense. Conversely, to give the structure of such a monad on a pointed equipment M is precisely to specify how heteromorphisms in the putative Kleisli 2-category \mathbf{K}_M are to be composed.

4.1.10 Definition ([Shu08]). An *equipment in the sense of Shulman* (or a *framed bicategory*) is given by a pseudo double category whose underlying span $\mathbf{K} \leftarrow \mathbf{M} \rightarrow \mathbf{K}$ in Cat is a two-sided bifibration (def. 2.1.6).

One half of this property is equivalent to requiring that every vertical morphism have a *horizontal companion* in the sense of [GP04]: the companion of $f: x \rightarrow y$ is a horizontal morphism $f_{\bullet}: x \rightarrowtail y$ equipped with cells

$$\begin{array}{ccc} x & \xrightarrow{f_{\bullet}} & y \\ f \downarrow & \Downarrow & \downarrow 1 \\ y & \xrightarrow{1} & y \end{array} \quad \begin{array}{ccc} x & \xrightarrow{1} & x \\ 1 \downarrow & \Downarrow & \downarrow f \\ x & \xrightarrow{f_{\bullet}} & y \end{array} \quad (4.1.4)$$

that compose vertically and horizontally to the identities on f and f_{\bullet} . Similarly, the other half of the bifibration property requires every vertical morphism f to have a *horizontal adjoint* f^{\bullet} , which is then right adjoint to f_{\bullet} in the horizontal 2-category of (\mathbf{K}, \mathbf{M}) , i.e. the 2-category of cells with identity vertical boundaries (we will call such cells *globular* and write \mathbf{M}_{gl} for this 2-category).

In [Shu08, appendix C] it is shown that every (map-)equipment in the sense of Wood gives rise to a framed bicategory (as long as the former's 2-category \mathcal{K} of tight maps is strict), and vice versa, and it is stated that these constructions are inverses up to isomorphism. In more detail, from a framed bicategory as above we get an identity-on-objects functor from \mathbf{K} to the horizontal 2-category \mathcal{M} of (\mathbf{K}, \mathbf{M}) , which sends a vertical map $f: x \rightarrow y$ to its companion $f_{\bullet}: x \rightarrowtail y$.

This is then a map-equipment $\mathbf{K} \rightarrow \mathcal{M}$. In the other direction, given a map-equipment $\mathbf{K} \rightarrow \mathcal{M}$ over a locally discrete 2-category, there is a pseudo double category $(\mathbf{K}, \text{Sq}_{\mathbf{K}}(\mathcal{M}))$ with the same objects, with \mathbf{K} as vertical category, the morphisms of \mathcal{M} as horizontal morphisms and cells

$$\begin{array}{ccc} x & \xrightarrow{m} & y \\ f \downarrow & & \downarrow g \\ z & \xrightarrow{n} & w \end{array} \quad (4.1.5)$$

the 2-cells $g_{\bullet}M \Rightarrow Nf_{\bullet}$ in \mathcal{M} . (If we write $F_T: \mathbf{K} \rightarrow \mathcal{M}$ for the inclusion, then the category of cells $\text{Sq}_{\mathbf{K}}(\mathcal{M})$ is the category of elements $\int \mathcal{M}(F_T, F_T)$.) By [Shu08, prop. C.3] this is a framed bicategory. Clearly, these constructions are inverses up to an isomorphism that is the identity on \mathbf{K} :

$$\begin{array}{ccc} & \mathbf{K} & \\ & \swarrow & \searrow \\ \mathcal{M} & \xrightarrow{\sim} & (\text{Sq}_{\mathbf{K}}\mathcal{M})_{\text{gl}} \\ & & \end{array} \quad \begin{array}{ccc} \mathbf{M} & \xrightarrow{\sim} & \text{Sq}_{\mathbf{K}}(\mathbf{M}_{\text{gl}}) \\ & \searrow & \swarrow \\ & \mathbf{K} \times \mathbf{K} & \end{array} \quad (4.1.6)$$

that on the right arising from the bijection between globular cells $g_{\bullet}M \Rightarrow Nf_{\bullet}$ and cells of the form (4.1.5) in (\mathbf{K}, \mathbf{M}) .

4.1.11 Definition. An *equipment profunctor* from (\mathbf{K}, T) to (\mathbf{L}, S) is a profunctor $H: \mathbf{K} \rightarrow \mathbf{L}$ that underlies a monad (op-)morphism $HT \rightarrow SH$, that is, an algebra for the monad \hat{T}^* given by precomposition with T on the Kleisli 2-category $2\text{-Prof}(\mathbf{K}, \mathbf{L})_{S^*}$ of the monad given by postcomposition with S (cf. section 3.1.2).

An *equipment morphism* is a representable equipment profunctor.

We will now compare this definition to the others.

Wood [Woo85] defines an equipment morphism to be a functor $F: \mathbf{K} \rightarrow \mathbf{L}$ that fits into a square

$$\begin{array}{ccc} \mathbf{K} & \longrightarrow & \mathbf{K}_T \\ F \downarrow & \simeq & \downarrow \tilde{F} \\ \mathbf{L} & \longrightarrow & \mathbf{L}_S \end{array} \quad (4.1.7)$$

To give such a lift \tilde{F} of F is equivalently to give a right T -module structure on $F_S F$, by the universal property of \mathbf{K}_T (\tilde{F} will be representable if F is, by corollary 4.1.5). The equivalence in the square is also essentially unique, given F and \tilde{F} , for the same reason.

Recall that the injection $F_S: \mathbf{L} \rightarrow \mathbf{L}_S$ satisfies $S \simeq \mathbf{L}_S(F_S, F_S)$, so that

$$S_* \simeq \mathbf{L}_S(F_S, F_S)_* \simeq \mathbf{L}_S(F_S, 1)_* \circ \mathbf{L}_S(1, F_S)_*$$

This latter monad is a representable profunctor in any tricategory $2\text{-Prof}'$ of 2-categories and profunctors large enough to contain $2\text{-Prof}(\mathbf{K}, \mathbf{L})$ as an object, and as such its Kleisli 2-category may be constructed by the above recipe. Thus by the adjunction $\mathbf{L}_S(1, F_S) \dashv \mathbf{L}_S(F_S, 1)$ (4.1.2) we have

$$2\text{-Prof}(\mathbf{K}, \mathbf{L})(1, S_*) \simeq 2\text{-Prof}(\mathbf{K}, \mathbf{L})(\mathbf{L}_S(1, F_S)_*, \mathbf{L}_S(1, F_S)_*)$$

and the Kleisli 2-category of the latter is simply the full image of the functor $\mathbf{L}_S(1, F_S)_*$ — its objects are profunctors $H: \mathbf{K} \rightarrow \mathbf{L}$ and the hom-object from H to H' is

$$2\text{-Prof}(\mathbf{K}, \mathbf{L})(\mathbf{L}_S(1, F_S) \circ H, \mathbf{L}_S(1, F_S) \circ H') \simeq 2\text{-Prof}(\mathbf{K}, \mathbf{L})(H, SH')$$

Precomposition T^* with T is a monad on this 2-category, and its algebras are the monad op-morphisms (thus equipment profunctors) $T \rightarrow S$, or equivalently the right T -modules whose underlying morphism is of the form $\mathbf{L}_S(1, F_S) \circ H$. But as noted above, the latter are precisely the right T -modules $\mathbf{L}_S(1, F_S) \circ HT \rightarrow \mathbf{L}_S(1, F_S) \circ H$ that arise from squares of the form (4.1.7) above, with representable modules corresponding to representable profunctors $H = \mathbf{L}(1, F)$. So equipment morphisms in the sense of def. 4.1.11 are equivalent to morphisms (4.1.7) in the sense of Wood.

Suppose $G: \mathbf{K} \rightarrow \mathbf{L}$ is a functor. An action of T on G has a mate

$$\frac{\mathbf{L}(1, G) \circ T \longrightarrow \mathbf{L}(1, G)}{T \longrightarrow \mathbf{L}(G, G)}$$

and the first underlies a right T -module if and only if the second is a morphism of pseudo-monoids, by remark 3.1.13. The morphism $T \rightarrow \mathbf{L}(G, G)$ corresponding to a right action is then sent by the construction of theorem 4.1.3 to a functor $\mathbf{K}_T \rightarrow \mathbf{K}_{\mathbf{L}(G, G)}$ into the full image of G that is the identity on objects and whose action on hom-categories is given by the components of the monoid morphism. This then composes with the fully faithful $\mathbf{K}_{\mathbf{L}(G, G)} \rightarrow \mathbf{L}$ given by the action of G on objects to give a functor $\mathbf{K}_T \rightarrow \mathbf{L}$. The unit axiom for a morphism of monoids then shows that the composite of this functor with the canonical $\mathbf{K} \rightarrow \mathbf{K}_T$ is equivalent to G . This shows how to compute the functor $\mathbf{K}_T \rightarrow \mathbf{L}$ corresponding to a representable right T -module, a recipe that is difficult to extract from the proof of theorem 4.1.3.

Now suppose given an equipment morphism $(F, \tilde{F}): T \rightarrow S$ of the form (4.1.7). The functor \tilde{F} gives rise to a right T -module by composition with the

canonical one, and this has a transpose $T \rightarrow \mathbf{L}_S(\tilde{F}F_T, \tilde{F}F_T)$, which naturality of transposition shows is the composite

$$T \xrightarrow{\sim} \mathbf{K}_T(F_T, F_T) \longrightarrow \mathbf{L}_S(\tilde{F}F_T, \tilde{F}F_T)$$

Here the right-hand morphism arises from the unit $1 \rightarrow \mathbf{L}_S(\tilde{F}, \tilde{F})$, which is the effect on hom-categories of \tilde{F} . The codomain of this is equivalent (as a monad, because $\tilde{F}F_T$ and FF_S are equivalent T -modules) to $S(F, F)$, so that the Kleisli objects of the two are equivalent under \mathbf{K}_T . We thus get an equivalence of factorizations of F_T

$$\begin{array}{ccc} \mathbf{K}_T & \longrightarrow & \mathbf{K}_{S(F,F)} \\ \downarrow & \simeq & \swarrow \\ \mathbf{K}_{\mathbf{L}_S(\tilde{F}F_T, \tilde{F}F_T)} & \longrightarrow & \mathbf{L}_S \end{array}$$

where the functors out of \mathbf{K}_T and the diagonal one are the identity on objects and the others are fully faithful — that on the right acts as F on objects and that on the bottom as \tilde{F} . The two composites $\mathbf{K}_T \rightarrow \mathbf{L}_S$ are canonically equivalent to \tilde{F} , because they give rise to equivalent modules; the left-and-bottom factorization is essentially F_T itself, but the top-and-right factorization takes the values of F on objects. This shows that in an equipment morphism of the form (4.1.7), we can always, up to canonical equivalence, take \tilde{F} to coincide strictly with F on objects.

A morphism $(\mathbf{K}, \mathbf{M}) \rightarrow (\mathbf{L}, \mathbf{N})$ of framed bicategories is defined [Shu08, def. 6.5] to be a pseudo-functor between their underlying double categories, i.e. a pair of functors between their vertical and horizontal categories that commute with the projections, together with invertible globular cells witnessing functoriality. This data immediately gives rise to a morphism of equipments

$$\begin{array}{ccc} \mathbf{K} & \longrightarrow & \mathbf{M}_{\text{gl}} \\ \downarrow & & \downarrow \\ \mathbf{L} & \longrightarrow & \mathbf{N}_{\text{gl}} \end{array}$$

Conversely, given a morphism $(F, \tilde{F}): (\mathbf{K}, T) \rightarrow (\mathbf{L}, S)$ of equipments, its tight part F is a functor between the vertical parts of their corresponding double categories. As noted above, \tilde{F} can be taken to coincide with F on objects, and it acts on a general cell $g_\bullet M \Rightarrow Nf_\bullet$ in $\text{Sq}_{\mathbf{K}}(\mathbf{K}_T)$ to form

$$F(g)_\bullet \tilde{F}(M) \xrightarrow{\sim} \tilde{F}(g_\bullet) \tilde{F}(M) \xrightarrow{\sim} \tilde{F}(g_\bullet M) \rightarrow \tilde{F}(Nf_\bullet) \xrightarrow{\sim} \tilde{F}(N) \tilde{F}(f_\bullet) \xrightarrow{\sim} F(N)F(f)_\bullet$$

Vertical functoriality follows from the naturality and associativity of \tilde{F} 's compositor and the pseudo-naturality of the equivalence $\tilde{F}F_T \simeq F_S F$, so that we get a morphism of spans $(\mathbf{K}, \text{Sq}_{\mathbf{K}}(\mathbf{K}_T)) \rightarrow (\mathbf{L}, \text{Sq}_{\mathbf{L}}(\mathbf{L}_S))$, and the 2-functoriality of \tilde{F} makes this into a morphism of double categories. This assignment is in fact strictly functorial, because of how the comparison cells of a composite functor are defined.

When we come to define equipment 2-cells, however, we run into a problem. A transformation between double functors assigns a vertical morphism of the target double category to each object of the source, and a cell of the target to each horizontal morphism of the source (subject to some axioms). So for functors arising from equipment morphisms $F, G: (\mathbf{K}, T) \rightarrow (\mathbf{L}, S)$, a double transformation would send $k \in \mathbf{K}$ to $\alpha_k: Fk \rightarrow Gk$ in \mathbf{L} , and $m: k \rightarrow k'$ to some $\alpha_m: (\alpha_{k'}) \bullet Fm \Rightarrow Gm(\alpha_k) \bullet$, such that horizontal identities are sent to identity cells, and the cell assigned to a composite is the horizontal composite of the cells assigned to the components. This amounts precisely to an *oplax* transformation $\tilde{F} \Rightarrow \tilde{G}$, with tight components, but we have no recipe for producing these from our abstract monad machinery. By the above discussion, a Kleisli 2-cell in the sense of section 3.1.2 between monad op-morphisms would amount to a morphism of right T -modules from $\mathbf{L}_S(1, F_S F)$ to $\mathbf{L}_S(1, F_S G)$, which corresponds to a *pseudonatural* transformation $\tilde{F} \Rightarrow \tilde{G}$, whose components are not required to be tight. A ‘free’ Kleisli 2-cell would be one that fits into a cylinder

$$\begin{array}{ccc} \mathbf{K} & \longrightarrow & \mathbf{K}_T \\ \left(\begin{array}{c} \Downarrow \\ \Leftarrow \\ \Downarrow \end{array} \right) & & \left(\begin{array}{c} \Downarrow \\ \Leftarrow \\ \Downarrow \end{array} \right) \\ \mathbf{L} & \longrightarrow & \mathbf{L}_S \end{array}$$

thus amounting to a transformation $\tilde{F} \rightarrow \tilde{G}$ with tight components, that is however still required to be pseudonatural.

We will take the easy way out by noting that, just as 2-categories and pseudofunctors form a strict 2-category, so do equipments and their morphisms: in a 3-fold composite of squares (4.1.7), the 1-cells are uniquely determined, and the equivalence filling the composite square is determined up to unique isomorphism, as noted above. This then forms a category that we will call $\mathcal{E}qt_1$, and the preceding discussion supplies a functor $\text{Sq}: \mathcal{E}qt_1 \rightarrow \mathcal{F}r\text{Bicat}_1$ into the category (underlying the strict 2-category) of framed bicategories and their morphisms that we have already seen to be essentially surjective on objects. To show that it is surjective on morphisms, let $G: \text{Sq}_{\mathbf{K}}(\mathcal{M}) \rightarrow \text{Sq}_{\mathbf{L}}(\mathcal{N})$ be a double functor. To show that G is equal to the ‘conjugate’ of $\text{Sq}(G_{\text{gl}})$ by the relevant isomorphisms (4.1.6) it suffices to show that G commutes with the process of passing between

squares of the following form:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{m} & \bullet \\
 f \downarrow & \Downarrow & \downarrow g \\
 \bullet & \xrightarrow{n} & \bullet
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \bullet & \xrightarrow{m} & \bullet & \xrightarrow{g_\bullet} & \bullet \\
 1 \downarrow & & \Downarrow & & \downarrow 1 \\
 \bullet & \xrightarrow{f_\bullet} & \bullet & \xrightarrow{n} & \bullet
 \end{array}$$

But this process is given by pasting with the universal squares (4.1.4), and these are preserved by double functors by [Shu08, prop. 6.4]. As for injectivity, given two functors $F, G: (\mathbf{K}, \mathcal{M}) \rightarrow (\mathbf{L}, \mathcal{N})$ in $\mathcal{E}qt_1$, the definition of $\text{Sq}(F)$ and $\text{Sq}(G)$ uses all of the structure of the two, namely their action on objects, morphisms and 2-cells and their functoriality constraints, and if they differ in any of these then so will their images under $\text{Sq}(-)$. So this functor is an equivalence. We can now simply define an equipment 2-cell to be a double transformation between the appropriate double functors.

4.2 Equipments and fibrations

In this section we define cartesian equipments (section 4.2.1), and show that the 2-category of them receives an ‘equipment-of-matrices’ functor from that of regular fibrations that moreover is fully faithful (section 4.2.2). In that same section we give axioms on a cartesian equipment that ensures it is in the image of this functor. Section 4.2.3 then shows that a regular equipment has comprehension in the sense of def. 2.1.21 if and only if the corresponding equipment has *tabulations* in a sense that we will define, and that this holds if and only if every co-monad in the equipment has an Eilenberg–Moore object.

Henceforth we will use the term ‘equipment’ to mean a map-equipment over a locally discrete 2-category. It follows from the discussion of the previous section that equipments and equipment functors and transformations between them form a 2-category $\mathcal{E}qt$ equivalent to Shulman’s strict 2-category of framed bicategories [Shu08, prop. 6.8]. It carries a monoidal structure given by the cartesian product of equipments.

4.2.1 Definition. A cell in a map-equipment $\mathcal{K} \rightarrow \mathcal{M}$

$$\begin{array}{ccc}
 X & \dashrightarrow & X' \\
 \downarrow & & \downarrow \\
 Y & \dashrightarrow & Y'
 \end{array}$$

is *exact* if its mate in \mathcal{M} is invertible.

A commuting square of tight maps gives rise to two distinct vertically in-

vertible cells, so that there are two senses in which it can be said to be exact.

4.2.1 Cartesian equipments

4.2.2 Definition. Let \mathcal{M} be a cartesian monoidal 2-category. A *cartesian object* in \mathcal{M} is a pseudomonoid M in \mathcal{M} whose multiplication map is right adjoint to the diagonal at M , and whose unit map is right adjoint to the map to the terminal object:

$$M \times M \begin{array}{c} \xleftarrow{d_M} \\ \perp \\ \xrightarrow{\otimes} \end{array} M \qquad M \begin{array}{c} \xleftarrow{I} \\ \top \\ \xrightarrow{!} \end{array} \mathbf{1}$$

Clearly, such objects form a full sub-2-category of $\text{PsMon}(\mathcal{M})$ (def. 3.1.11).

A *cartesian equipment* is a cartesian object in $\mathcal{E}qt$. The full sub-2-category of the latter on the former will be called $\mathcal{C}artEq$.

To give a right adjoint $G: (\mathbf{L}, \mathbf{N}) \rightarrow (\mathbf{K}, \mathbf{M})$ to a morphism F of framed bicategories is, by (the dual of) [Shu08, prop. 8.4], to give the following:

1. for each object $\ell \in \mathbf{L}$, a universal morphism $e_\ell: FG\ell \rightarrow \ell$;
2. for each horizontal morphism $n: \ell \rightarrow \ell'$ a cell ϵ_n

$$\begin{array}{ccc} FG\ell & \xrightarrow{FGn} & FG\ell' \\ e_k \downarrow & \Downarrow & \downarrow e_{\ell'} \\ \ell & \xrightarrow{n} & \ell' \end{array}$$

such that any cell as on the left below factors as on the right:

$$\begin{array}{ccc} Fk & \xrightarrow{Fm} & Fk' \\ f \downarrow & \Downarrow & \downarrow f' \\ \ell & \xrightarrow{n} & \ell' \end{array} = \begin{array}{ccc} Fk & \xrightarrow{Fm} & Fk' \\ Fg \downarrow & \Downarrow & \downarrow Fg' \\ FG\ell & \xrightarrow{FGn} & FG\ell' \\ e_k \downarrow & \Downarrow & \downarrow e_{\ell'} \\ \ell & \xrightarrow{n} & \ell' \end{array}$$

where the upper square on the right is the F -image of a unique square in (\mathbf{K}, \mathbf{M}) ;

3. such that horizontal composites of universal cells and identity cells on universal vertical morphisms are again universal.

The first condition supplies a right adjoint G_0 for F_0 , the second a right adjoint G_1 for F_1 that makes (G_0, G_1) a morphism of spans, and the third ensures that this is a double functor. This shows that to give a framed bicategory (\mathbf{K}, \mathbf{M}) the structure of a cartesian object is to give finite products in both \mathbf{K} and \mathbf{M} that are preserved by the projections and by the composition and identity functors.

We can transfer these conditions across the equivalence $\mathcal{E}qt \simeq FrBicat$. A family of universal vertical morphisms e_ℓ in $Sq_{\mathbf{L}}(\mathcal{N})$ gives, trivially, a family of universal tight morphisms in the equipment $(\mathbf{L}, \mathcal{N})$. Suppose given also a family of universal cells ϵ_n : for each functor $\tilde{F}_{kk'}: \mathcal{M}(k, k') \rightarrow \mathcal{N}(Fk, Fk')$ define a map on objects in the opposite direction by $G'_{kk'}n = h_{k'}^\bullet(Gn)h_{k\bullet}$, where the h_k are the units of the adjunction $F_0 \dashv G_0$. Then the mate of the universal η_m , unit of the adjunction $F_1 \dashv G_1$, is a globular cell $m \Rightarrow G'\tilde{F}m$, and it is universal from m to G' by the universal property of η_m . Hence F has *local* right adjoints $\tilde{F}_{kk'} \dashv G'_{kk'}$ for each pair k, k' of objects, the G' being functorial because G is so locally.

Conversely, suppose given a morphism $(F, \tilde{F}): (\mathbf{K}, \mathcal{M}) \rightarrow (\mathbf{L}, \mathcal{N})$ of equipments, together with objects $G\ell$ and universal 1-cells $e_\ell: FG\ell \rightarrow \ell$ in \mathbf{L} , and universal 2-cells $\epsilon'_n: \tilde{F}\ell\ell'G'\ell\ell'n \Rightarrow n$ supplying adjunctions $\tilde{F}_{kk'} \dashv G'_{kk'}$ as above. Clearly the universal 1-cells are also universal vertical morphisms in $Sq_{\mathbf{L}}(\mathcal{N})$. Define $\tilde{G}n = G'_{GkGk'}(e_{\ell'}^\bullet ne_{\ell\bullet})$. Then the transpose of $\epsilon'_{e_{\ell'}^\bullet ne_{\ell\bullet}}$, as on the left below

$$\begin{array}{ccc} FG\ell & \xrightarrow{\tilde{F}\tilde{G}n} & FG\ell' \\ e_\ell \downarrow & \Downarrow & \downarrow e_{\ell'} \\ \ell & \xrightarrow{\tilde{n}} & \ell' \end{array} \qquad \begin{array}{ccc} Fk & \xrightarrow{\tilde{F}m} & Fk' \\ f \downarrow & \Downarrow & \downarrow f' \\ \ell & \xrightarrow{\tilde{n}} & \ell' \end{array}$$

is universal in $Sq_{\mathbf{L}}(\mathcal{N})$: in a cell as on the right, the vertical maps are $f = e_\ell \circ Gg$ for a unique $g: k \rightarrow G\ell$, and similarly for f' , and so the cell corresponds to a unique 2-cell in \mathcal{N}

$$\tilde{F}(g'_\bullet \circ m \circ g_\bullet) \xrightarrow{\sim} Fg'_\bullet \circ \tilde{F}m \circ Fg_\bullet \longrightarrow e_{\ell'}^\bullet ne_{\ell\bullet}$$

which we can chase through the following bijections:

$$\frac{\tilde{F}(g'_\bullet \circ m \circ g_\bullet) \longrightarrow e_{\ell'}^\bullet ne_{\ell\bullet}}{g'_\bullet \circ m \circ g_\bullet \longrightarrow \tilde{G}n} \\ \frac{k \xrightarrow{\tilde{m}} k'}{g \downarrow \quad \Downarrow \quad \downarrow g'} \\ Gl \xrightarrow{\tilde{G}n} Gl'$$

So we get a family of universal cells in $\text{Sq}_{\mathbf{L}}(\mathcal{N})$, as required. In short, a right adjoint to an equipment functor (F, \tilde{F}) is a right adjoint G of F together with local right adjoints $\tilde{F}_{kk'} \dashv G'_{kk'}$ for \tilde{F} , such that the resulting \tilde{G} is functorial. (But note carefully that \tilde{G} is not itself right adjoint to \tilde{F} .)

The above gives rise straightforwardly to a description of cartesian objects in $\mathcal{E}qt$ that extends the characterizations of cartesian bicategories in [CW87, thm. 1.6] and [CKWW08, def. 4.1, prop. 4.2].

4.2.3 Proposition. *To give cartesian structure on an object $\mathbf{K} \rightarrow \mathcal{M}$ of $\mathcal{E}qt$ is to give either:*

1. *equipment morphisms as follows, that give \mathbf{K} finite products:*

$$\begin{array}{ccccc} \mathbf{1} & \xrightarrow{\mathbf{1}} & \mathbf{K} & \xleftarrow{\times} & \mathbf{K} \times \mathbf{K} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{1} & \xrightarrow{\mathbf{1}} & \mathcal{M} & \xleftarrow{\otimes} & \mathcal{M} \times \mathcal{M} \end{array}$$

such that $m \wedge m' = d^\bullet(m \otimes m')d_\bullet$ and $\top = e^\bullet \mathbf{1}_\bullet e_\bullet$ provide finite products in the hom-categories of \mathcal{M} ; or

2. *finite products in \mathbf{K} and the hom-categories of \mathcal{M} , such that $(\mathbf{1}_\mathbf{1})_\bullet \cong \top_{\mathbf{1},\mathbf{1}}$ and*

$$m \otimes m' = p^\bullet m p_\bullet \wedge q^\bullet m' q_\bullet$$

is functorial, or equivalently such that the universal cells in $\text{Sq}_{\mathbf{K}}(\mathcal{M})$ derived as above from the local products in \mathcal{M} satisfy the coherence conditions of (3) above.

Of course, the corresponding framed bicategory then has products as described above; in particular, $\text{Sq}(\otimes)$ is the cartesian product on $\text{Sq}_{\mathbf{K}}(\mathcal{M})$.

By comparing this with prop. 2.2.26 and remark 2.2.27 we can immediately conclude the following.

4.2.4 Proposition. *A chordate cartesian equipment is the same thing as a cartesian bicategory.*

This will enable us to make use, in what follows, of results from e.g. [WW08] and [LWW10] that are proved there for cartesian bicategories, as long as we are careful to distinguish between maps and tight maps.

Clearly, a monoid morphism between cartesian framed bicategories is a double functor whose components are (strong) monoidal, and likewise a monoid

2-cell is a pair of monoidal transformations that underlie a double transformation, meaning that their components commute with the monoidal constraints of the functors involved.

An equipment morphism that preserves the products \times in the base category of its domain preserves the global tensor \otimes if and only if it preserves the local products \wedge , and this is what a monoid morphism in $\mathcal{E}qt$ between cartesian objects amounts to. On the other hand, even though a ‘local’ description of monoid 2-cells could probably be derived, we will have no use for one, and will stick with the ‘global’ description provided by the framed-bicategory perspective.

4.2.2 Comparison with regular fibrations

The following proposition constructs from a regular fibration \mathbf{E} a cartesian equipment of ‘matrices’ in \mathbf{E} . The next result then shows that this construction is part of a fully-faithful functor $\text{Matr}(-): \mathcal{R}egFib \rightarrow \mathcal{C}artEq$. The image $\mathcal{R}egEq$ of this functor is then characterized, so that we get an equivalence between $\mathcal{R}egFib$ and $\mathcal{R}egEq$.

4.2.5 Proposition. *If $\mathbf{E} \rightarrow \mathbf{B}$ is a regular fibration, then there is a cartesian equipment $\text{Matr}(\mathbf{E})$, with objects and tight maps the objects and morphisms of \mathbf{B} , and hom categories $\text{Matr}(\mathbf{E})(X, Y)$ the fibres $\mathbf{E}(X \times Y)$.*

Proof. A regular fibration \mathbf{E} over \mathbf{B} is, in particular, a symmetric monoidal bifibration with cartesian base, and [Shu08, theorem 14.2] shows that the $\text{Matr}(-)$ construction applied to one such yields a symmetric monoidal framed bicategory, which is a symmetric monoid in $\mathcal{E}qt$.¹ For reference, here are the essential details: composites are given by ‘relational composition’:

$$S \circ R = p_{Y!}(p_Z^* R \cap p_X^* S) = \llbracket \exists v. R(x, v) \wedge S(v, z) \rrbracket$$

and identities by ‘identity relations’:

$$1_X = d_1 \top_X = \llbracket \exists \xi. (x, x') = (\xi, \xi) \rrbracket = \llbracket x = x' \rrbracket$$

while a morphism $f: X \rightarrow X'$ of \mathbf{B} becomes a tight map like so:

$$\begin{aligned} f_\bullet &= (f \times 1)^* 1_{X'} \\ f^\bullet &= (1 \times f)^* 1_{X'} \end{aligned}$$

¹That result is stated for fibrations satisfying Beck–Chevalley for either all pullback squares or for a restricted class as long as the fibration satisfies Frobenius, but inspection of the proof shows that the conditions are only applied for product-absolute pullback squares.

For a cell in $\text{Matr}(\mathbf{E})$ of the form

$$\begin{array}{ccc} X & \xrightarrow{R} & Y \\ f \downarrow & \Downarrow & \downarrow g \\ X' & \xrightarrow{S} & Y' \end{array}$$

we have by [Shu08, (10, 11)] that

$$\begin{aligned} g_{\bullet}R &\cong (1 \times g)_!R \\ Sf_{\bullet} &\cong (f \times 1)^*S \end{aligned}$$

and hence a morphism $g_{\bullet}R \rightarrow Sf_{\bullet}$ like that above has mates

$$\frac{\begin{array}{c} R \longrightarrow (f \times g)^*S \cong g^{\bullet}Sf_{\bullet} \\ \hline g_{\bullet}R \cong (1 \times g)_!R \longrightarrow (f \times 1)^*S \cong Sf_{\bullet} \\ \hline g_{\bullet}Rf^{\bullet} \cong (f \times g)_!R \longrightarrow S \end{array}}{\quad}$$

The category \mathbf{B} we already know to be cartesian, and the tensor product on $\text{Matr}(\mathbf{E})$ is defined as

$$R \otimes R' = p_{X'Y'}^*R \cap p_{XY}^*R' : X \times X' \looparrowright Y \times Y'$$

which implies that

$$d^{\bullet}(R \otimes R')d_{\bullet} \cong (d_X \times d_Y)^*(p_{X'Y'}^*R \cap p_{XY}^*R') \cong R \cap R'$$

and so we have local binary products. Because $d_{\mathbf{1}}$ is an isomorphism, the associated pull–push adjunction is an equivalence, and so

$$e^{\bullet}(\mathbf{1}_{\mathbf{1}})e_{\bullet} \cong (e_X \times e_Y)^*d_{\mathbf{1}}\top_{\mathbf{1}} \cong (e_X \times e_Y)^*\top_{\mathbf{1} \times \mathbf{1}} \cong \top_{X \times Y}$$

Hence in fact $\text{Matr}(\mathbf{E})$ has local finite products given by the formulas in (1) of prop. 4.2.3, and so it is a cartesian equipment. \square

Proving the following is a simple matter of unwinding definitions.

4.2.6 Corollary. *A commuting square in the base of a regular fibration \mathbf{E} satisfies the Beck–Chevalley condition (def. 3.1.2) if and only if it is exact (def. 4.2.1) in both senses in $\text{Matr}(\mathbf{E})$.*

4.2.7 Theorem. *The $\text{Matr}(-)$ construction of prop. 4.2.5 extends to a fully faithful functor $\mathcal{R}egFib \rightarrow \mathcal{C}artEq$.*

Proof. Functoriality follows from [Shu08, theorem 14.9]. In brief, a morphism $(F, \phi): (\mathbf{B}, \mathbf{E}) \rightarrow (\mathbf{B}', \mathbf{E}')$ of regular fibrations preserves all of the structure used to define $\text{Matr}(\mathbf{E})$, so that F , together with the functors

$$\tilde{F}_{XY}: \mathbf{E}(X \times Y) \xrightarrow{\phi_{X \times Y}} \mathbf{E}'(F(X \times Y)) \xrightarrow{\sim} \mathbf{E}'(FX \times FY)$$

which sends $m: X \rightarrow Y$ to the pushforward of ϕm along the relevant coherence map of F , gives rise to a functor $\tilde{F}: \text{Matr}(\mathbf{E}) \rightarrow \text{Matr}(\mathbf{E}')$, with local monoidal constraints obtained similarly by pushforward. The functoriality isomorphisms of $\text{Matr}(-)$ come from the pseudonaturality cells of the ϕ, γ , etc., and the former are coherent because the latter are.

Conversely, if $(F, \tilde{F}): \text{Matr}(\mathbf{E}) \rightarrow \text{Matr}(\mathbf{E}')$ is a map of cartesian equipments, then $F: \mathbf{B} \rightarrow \mathbf{B}'$ preserves products, and the required transformation ϕ from $\mathbf{E} \Rightarrow F^* \mathbf{E}'$ is given by

$$\phi_X = \mathbf{E}X \cong \text{Matr}(\mathbf{E})(X, \mathbf{1}) \xrightarrow{\tilde{F}_{X, \mathbf{1}}} \text{Matr}(\mathbf{E}')(FX, F\mathbf{1}) \cong \mathbf{E}'(FX)$$

which is natural in X because its components are, and it preserves products because \tilde{F} does so locally. Naturality of $\tilde{F}_{X(Y \times \mathbf{1})}$ with respect to $Y \times \mathbf{1} \cong Y$ shows that $\text{Matr}(-)$ applied to this gives an equipment morphism isomorphic to (F, \tilde{F}) , so that $\text{Matr}(-)$ is essentially surjective on morphisms.

Thinking of a transformation $(F, \phi) \Rightarrow (G, \gamma)$ as a ‘cylinder’ $(\alpha, \bar{\alpha})$, where $\alpha: F \Rightarrow G$ and $\bar{\alpha}: \bar{F} \Rightarrow \bar{G}$, \bar{F} and \bar{G} being the functors between total categories corresponding to ϕ and γ , we get for each $m: X \rightarrow Y$ a morphism $\bar{\alpha}_m: \bar{F}m \rightarrow \bar{G}m$ over $\alpha_{X \times Y}$, and composing this with the (op)cartesian morphisms indicated we get

$$\tilde{F}m \cong \bar{F}m \xrightarrow{\bar{\alpha}_m} \bar{G}m \cong \tilde{G}m$$

over

$$FX \times FY \cong F(X \times Y) \xrightarrow{\alpha_{X \times Y}} G(X \times Y) \cong GX \times GY$$

The latter is $\alpha_X \times \alpha_Y$, so the former corresponds to a unique cell $\tilde{\alpha}_m: \alpha_{Y \bullet} \circ Fm \Rightarrow Gm \circ \alpha_{X \bullet}$, this assignment being natural in m . It also respects horizontal composition and identities, because $\bar{\alpha}$ is a monoidal transformation, and hence commutes with the monoidal constraints of \bar{F} and \bar{G} , therefore with those of \tilde{F} and \tilde{G} . It respects cartesian products in the category of cells for the same reason. The map $\bar{\alpha} \mapsto \tilde{\alpha}$ is itself clearly functorial.

Conversely, let $(\beta, \tilde{\beta}): \text{Matr}(F, \bar{F}) \Rightarrow \text{Matr}(G, \bar{G})$ be a monoidal equipment-transformation. An object n over X in the domain \mathbf{E} of \bar{F} and \bar{G} gives a morphism $r_1^{-1}n: X \rightarrow \mathbf{1}$ in $\text{Matr}(\mathbf{E})$, where $r: X \times \mathbf{1} \cong X$, and this in turn gives $\tilde{\beta}_{r_1^{-1}n}: \tilde{F}(r_1^{-1}n) \rightarrow \tilde{G}(r_1^{-1}n)$ over $\beta_X \times \beta_{\mathbf{1}}$. By the above, this is the composite

with the evident isomorphisms of some morphism $\bar{F}(r_1^{-1}n) \rightarrow \bar{G}(r_1^{-1}n)$ over $\beta_{X \times 1}$, and composing this with the \bar{F} - and \bar{G} -images of the isomorphism $n \cong r_1^{-1}n$ gives a morphism $\bar{\beta}_n: \bar{F}n \rightarrow \bar{G}n$ over β_X , which is natural in n and monoidal because $\tilde{\beta}$ is. It is easy to see then that $\text{Matr}(\bar{\beta}) = \tilde{\beta}$, by cancelling inverses and using naturality, and the same in the other direction. So $\text{Matr}(-)$ is locally fully faithful, hence locally an equivalence, hence fully faithful as a 2-functor. \square

4.2.8 Definition (cf. def. 2.1.13). An object A in a cartesian equipment is *separable* [LWW10, def. 3.2] if the pullback square that expresses the monicity of d_A :

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow d_A \\ A & \xrightarrow{d_A} & A \times A \end{array} \quad (4.2.1)$$

is exact. The object A is *Frobenius* if the coassociativity square

$$\begin{array}{ccc} A & \xrightarrow{d_A} & A \times A \\ d_A \downarrow & & \downarrow A \times d_A \\ A \times A & \xrightarrow{d_A \times A} & A^3 \end{array} \quad (4.2.2)$$

is exact, in both senses. (In fact, [WW08, lemma 3.2] shows that either condition implies the other.)

The separability and Frobenius conditions hold in an equipment of the form $\text{Matr}(\mathbf{E})$: they follow from the Beck–Chevalley conditions (def. 2.1.13 type (A), and remark 2.1.14).

If $\mathbf{B} \rightarrow \mathcal{B}$ is a cartesian equipment, then

$$\text{Pred}(\mathcal{B}) = \mathcal{B}((-)\bullet, \mathbf{1}): \mathbf{B}^{\text{op}} \rightarrow \mathcal{C}at$$

is a bifibration, because if f is a tight map then the pullback functor $f^* = \text{Pred}(\mathcal{B})(f\bullet)$ has a left adjoint $f_! = \text{Pred}(\mathcal{B})(f\bullet)$. It clearly also has fibred finite products. The Beck–Chevalley condition for squares $df = (f \otimes 1)\langle 1, f \rangle$ of type (A) in def. 2.1.13 requires invertibility of

$$\begin{aligned} f_\bullet d^\bullet(1 \times f)^\bullet &\longrightarrow d^\bullet(f \times 1)_\bullet \\ (1 \times f)_\bullet d_\bullet f^\bullet &\longrightarrow (f \times 1)^\bullet d_\bullet \end{aligned}$$

these being the mates of the isomorphism exhibiting f as a d -homomorphism (compare the inequalities 2.2.4, 2.2.5). They are clearly dual. Frobenius reci-

procuity requires invertibility of

$$(R \wedge Sf_{\bullet})f^{\bullet} \longrightarrow Rf^{\bullet} \wedge Sf_{\bullet}f^{\bullet} \longrightarrow Rf^{\bullet} \wedge S$$

but this is the whiskering of the second Beck–Chevalley morphism above by $d^{\bullet}(R \otimes S)$, and hence the former is invertible if the latter is. The condition for type-(B) squares is precisely separability (4.2.1). The condition for squares of type (C) is a special case of the functoriality of \otimes , as is that for type (D), and side-by-side pastings always preserve the condition (cf. [See83, p. 512]).

So if every object in \mathcal{B} is separable, then the only thing keeping $\text{Pred}(\mathcal{B})$ from being a regular fibration is the type-(A) Beck–Chevalley condition. Unfortunately, despite a strong suspicion that the Frobenius condition implies it, I have been unable to find a proof (note that the converse implication holds by remark 2.1.14). So we must assume it as an axiom, in the most general cases, in order to get a regular fibration out of the $\text{Pred}(-)$ construction. Therefore we define a *regular equipment* to be a cartesian equipment satisfying these two conditions. There is thus a 2-category \mathcal{RegEq} of regular equipments, whose 1-cells are monoidal equipment functors and whose 2-cells are equipment transformations. It is easy to see that if $\text{Pred}(\mathcal{B})$ exists then $\text{Matr}(\text{Pred}(\mathcal{B})) \simeq \mathcal{B}$, so that \mathcal{RegEq} is equivalent to \mathcal{RegFib} .

Note, however, that the troublesome condition does follow from the Frobenius condition in the locally ordered context: by [WW08, prop. 3.6] each object in a regular equipment is self-dual, giving an identity-on-objects contravariant involution $(-)^{\circ}$. In the locally ordered case the results of [CW87, theorem 2.4] follow (modulo the caveats above regarding the difference between maps and tight maps), showing that the dual of $(df)_{\bullet} = ((f \times f)d)_{\bullet}$ is an equality of precisely the type required, by naturality of duality and by the fact that the dual of d^{\bullet} is d_{\bullet} . This isomorphism does still exist in the non-locally-ordered setting, but there seems to be no good reason why it should be the inverse of the Beck–Chevalley morphism.

4.2.3 Tabulation and comprehension

In this section we examine notions of tabulation for morphisms in a regular equipment, and compare them to comprehension in the corresponding regular fibration.

4.2.9 Definition. Let $R: X \rightleftarrows Y$ be a morphism in an equipment. A *tabulation*

of R is an object $\{R\}$ together with a universal cell

$$\begin{array}{ccc} \{R\} & \dashrightarrow & \{R\} \\ i \downarrow & \Downarrow & \downarrow j \\ X & \dashrightarrow_R & Y \end{array}$$

that is, such that a 2-cell $g_{\bullet} \rightarrow Rf_{\bullet}$ is given by composing the above cell with (the identity cell on) a unique tight $Z \rightarrow \{R\}$.

An equipment *has tabulation* if every morphism has such a tabulation, and we say that these tabulations are *full* if the mate $j_{\bullet}i^{\bullet} \rightarrow R$ of the universal cell is invertible for each R .

The following result explains why we use the same notation for tabulations as for comprehension (def. 2.1.21).

4.2.10 Proposition. *A regular fibration \mathbf{E} has (full) comprehension if and only if $\text{Matr}(\mathbf{E})$ has (full) tabulation.*

Proof. The following sequence of bijections shows that the extension $\{R\}$ of an object R of $\mathbf{E}(X \times Y)$ is also the tabulation of R considered as a morphism $X \multimap Y$ in $\text{Matr}(\mathbf{E})$:

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & \{R\} \\ & \searrow & \swarrow \\ & (f,g) & \\ & X \times Y & \end{array} \quad \begin{array}{c} \hline (f,g)! \top_Z \cong (f \times g)! 1_Z \longrightarrow R \\ \hline \end{array}$$

$$\begin{array}{ccc} Z & \dashrightarrow & Z \\ f \downarrow & \Downarrow & \downarrow g \\ X & \dashrightarrow_R & Y \end{array}$$

Setting $Y = \mathbf{1}$, the same sequence read backwards shows that the extension of a predicate $P \in \mathbf{E}X$ is given by the tabulation of $P: X \multimap \mathbf{1}$. It also shows that the morphisms required to be invertible by the two forms of fullness are in fact the same. \square

Proposition 3.4 of [LWW10] shows that the separability axiom for an object X of a cartesian bicategory is equivalent to the identity 1_X 's being subterminal, so that an endomorphism G of X can admit at most one 'copoint' $G \rightarrow 1_X$. Their lemma 3.15 then shows that if it does then there exists a unique 2-cell $\gamma: G \rightarrow G^2$ making G into a comonad.

We can go further: the proposition referred to also shows that separability is equivalent to the statement that for any $\epsilon: G \rightarrow 1_X$, the span $G \leftarrow G \rightarrow 1_X$ is a product. In that case there is a unique morphism $G \rightarrow G \wedge 1_X$, necessarily given by $(1_G, \epsilon)$, which is invertible and natural in G (i.e. with respect to morphisms between copointed endomorphisms of X). A useful consequence of this is an isomorphism

$$\begin{aligned}
d^\bullet(G \otimes 1) &\cong d^\bullet((G \wedge 1) \otimes 1) \\
&\cong d^\bullet(d^\bullet \otimes 1)(G \otimes 1 \otimes 1)(d \otimes 1) \\
&\cong d^\bullet(1 \otimes d^\bullet)(G \otimes 1 \otimes 1)(d \otimes 1) \\
&\cong d^\bullet(G \otimes 1)(1 \otimes d^\bullet)(d \otimes 1) \\
&\cong d^\bullet(G \otimes 1)dd^\bullet \\
&\cong Gd^\bullet
\end{aligned} \tag{4.2.3}$$

using coassociativity of d , Frobenius, and separability. From this in turn we see that, for example,

$$G(M \wedge N) \cong GM \wedge N \cong M \wedge GN \tag{4.2.4}$$

and in particular that $G \wedge G \cong G(1 \wedge G) \cong GG$. Indeed, the 2-cell $\gamma: G \rightarrow GG$ given by [LWW10, lemma 3.15] is equivalently the composite

$$G \xrightarrow{\delta} G \wedge G \xrightarrow{\sim} GG$$

of this isomorphism with the local diagonal at G , because the latter begins with

$$G \xrightarrow{\delta} G \wedge G \xrightarrow{\sim} G \wedge 1 \wedge G \longrightarrow \dots$$

and the former with

$$G \xrightarrow{\delta_3} G \wedge G \wedge G \xrightarrow{G \wedge \epsilon \wedge G} G \wedge 1 \wedge G \longrightarrow \dots$$

and the composites of the first two morphisms in each are clearly equal, while both continue identically. The same lemma then shows that the product projections $GG \rightarrow G$ are given by ϵG and $G\epsilon$.

There is a not-too-dissimilar result for comodules, which allows us to describe categories of comodules in a neat and useful way.

4.2.11 Proposition. *Let G be a comonad on X in the regular equipment*

$\mathbf{Matr}(\mathbf{E})$, and let $M: Z \looparrowright X$ be a morphism. The isomorphism

$$GM \cong G(M \wedge \top) \cong M \wedge G\top$$

whose second factor is (4.2.4) is natural in M and exhibits the endofunctor $M \mapsto M \wedge G\top$ as a comonad isomorphic as such to $G_*: M \mapsto GM$. There is then an equivalence of categories

$$\mathbf{LComod}(G, Z) \simeq \mathbf{E}(Z \times X)/G\top$$

Proof. The isomorphism is natural in M because its components are, so that $G_* \cong (- \wedge G\top)$, and hence the latter acquires the structure of a comonad. For any object A in a monoidal category, comonad structures on $(- \otimes A)$ are in bijection with comonoid structures on A . But because $\mathbf{E}(Z \times X)$ is cartesian, there is one and only one comonoid structure on $G\top$, given by projection and diagonal, and hence the resulting comonad structure on $(- \wedge G\top)$ must be identical with that transferred from G_* .

It follows that the categories of coalgebras of $(- \wedge G\top)$ and of G_* are equivalent. But the category of coalgebras of the latter is the category of left G -comodules, while it is a generality that the category of coalgebras for a comonad of the form $(- \times A)$ on a cartesian category \mathbf{C} is just \mathbf{C}/A . Hence

$$\mathbf{LComod}(G, Z) \simeq \mathbf{Coalg}(G_*) \simeq \mathbf{E}(Z \times X)/G\top$$

□

Now we want to compare the presence of tabulation for arbitrary morphisms with the existence of Eilenberg–Moore objects for comonads.

4.2.12 Definition. An Eilenberg–Moore object for a comonad G on an object X in an equipment is, as in def. 3.1.5, an object X^G that represents (left) G -comodules, in that $\mathbf{hom}(Z, X^G) \simeq \mathbf{LComod}(G, Z)$, naturally in Z , but with this equivalence also holding for the restriction of each side to tight maps. That is equivalently to say that the universal $X \looparrowright X^G$ is a tight map, and that composition with it preserves and detects tight maps (cf. [GS13]).

We will say that an object in an equipment is an EM object *with respect to tight maps* if only the second part of this property holds. In general, of course, such an object is not necessarily a genuine EM object.

4.2.13 Lemma. *If G is a comonad in a regular equipment, f and g are tight maps and $g_\bullet \rightarrow Gf_\bullet$ is a 2-cell, then there is a unique isomorphism $f_\bullet \cong g_\bullet$, modulo which the given 2-cell makes f a G -comodule.*

Proof. Composing the given 2-cell with the counit $G \rightarrow 1$ gives a 2-cell $f_\bullet \rightarrow g_\bullet$, which by [LWW10, theorem 3.14] is unique and invertible. Then 2-cells $g_\bullet \rightarrow Gf_\bullet$ are in natural canonical bijection with 2-cells $f_\bullet \rightarrow Gf_\bullet$. To say that a 2-cell of the latter form is a G -comodule is the same as to say that its mate $f_\bullet f^\bullet \rightarrow G$ is a morphism of comonoids, but by [*op. cit.*, theorem 4.2(ii)] every such morphism is so. \square

4.2.14 Proposition. *A regular equipment has tabulation if and only if it has EM objects with respect to tight maps. Moreover, the latter are genuine EM objects if and only if the corresponding tabulations are full.*

Proof. By [LWW10, theorem 4.3], a morphism $R: X \multimap Y$ gives rise to a comonad G_R

$$X \times Y \xrightarrow{d \otimes Y} X \times X \times Y \xrightarrow{X \otimes R \otimes Y} X \times Y \times Y \xrightarrow{X \otimes d^\bullet} X \times Y$$

that comes equipped with a cell (given by projecting out R)

$$\begin{array}{ccc} X \times Y & \xrightarrow{G_R} & X \times Y \\ p_1 \downarrow & \Downarrow \mu & \downarrow p_2 \\ X & \xrightarrow{R} & Y \end{array}$$

whose mate $p_2 G_R p_1^\bullet \rightarrow R$ is invertible. To give a tight comodule for G_R is to give a morphism $(f, g): Z \rightarrow X \times Y$ and a cell

$$\begin{array}{ccc} Z & \xrightarrow{1} & Z \\ (f, f, g) \downarrow & \Downarrow & \downarrow (f, g, g) \\ X \times X \times Y & \xrightarrow{X \otimes R \otimes Y} & X \times Y \times Y \end{array}$$

Because the lower morphism is a three-fold product in the category of cells (prop. 4.2.3), to give such a cell is to give three cells of the following form:

$$\begin{array}{ccc} Z \xrightarrow{1} Z & Z \xrightarrow{1} Z & Z \xrightarrow{1} Z \\ f \downarrow \Downarrow \downarrow f & f \downarrow \Downarrow \downarrow g & g \downarrow \Downarrow \downarrow g \\ X \xrightarrow{1} X & X \xrightarrow{R} Y & Y \xrightarrow{1} Y \end{array}$$

But the outer two of these are always unique (as in the proof of the lemma above), so that to give a square of the central form is precisely to give a tight G_R -comodule, and if one is universal then so is the other. Such a universal cell into R is then, as in [LWW10, theorem 4.7], the composite of the EM comodule

with μ above. But the mate of the structure 2-cell of a genuine EM object is always invertible, as is that of μ , so that the mate of the cell exhibiting the tabulation of R is invertible too (being the composite of cells with invertible mates) and hence in this case the tabulation is full.

In the other direction, if G is a comonad, then its tabulation comes together with a universal 2-cell $j_\bullet \rightarrow Gi_\bullet$. By lemma 4.2.13 above, i is then a G -comodule, and the universal property of the tabulation is (again by the lemma) precisely the universal property of an EM object wrt tight maps. By prop. 4.2.11 above, a comodule $j_\bullet \rightarrow Gj_\bullet$ is the same thing as a morphism $j_\bullet \rightarrow G\top$, which is the same thing as a cell

$$\begin{array}{ccc} Z & \xrightarrow{\quad \dagger \quad} & Z \\ t \downarrow & & \Downarrow \\ \mathbf{1} & \xrightarrow[\dagger_\bullet]{} X & \xrightarrow[G]{} X \\ & & \downarrow j \end{array}$$

That means that if i is the EM object with respect to tight maps of G , then i tabulates $Gt^\bullet \cong p_{X!}G$, where $p_X: X \times X \rightarrow X$ is the projection onto the first component, and hence $\{G\} \cong \{p_{X!}G\}$. If these tabulations are full, then, propositions 4.2.11 and 2.1.25 give

$$\begin{aligned} \text{hom}(Z, \{G\}) &\simeq \mathbf{E}(Z \times \{G\}) \\ &\simeq \mathbf{E}(Z \times \{p_{X!}G\}) \\ &\simeq \mathbf{E}\{p_Z^*p_{X!}G\} \\ &\simeq \mathbf{E}(Z \times X)/p_Z^*p_{X!}G \\ &\simeq \mathbf{E}(Z \times X)/G\top \\ &\simeq \text{LComod}(Z, G) \end{aligned}$$

and so full tabulations give rise to genuine EM objects. □

4.3 The effective topos

A *realizability topos* [vO08] is, roughly, a topos built out of some collection of computable objects (a *partial combinatory algebra*, or pca). In particular, the *effective topos* \mathbf{Eff} is constructed relative to the partial recursive functions $\mathbb{N} \rightarrow \mathbb{N}$ on the natural numbers. The connection with realizability in the traditional sense is that the canonical interpretation of higher-order Heyting arithmetic in \mathbf{Eff} yields precisely Kleene's realizability interpretation [Kle45, Tro98] of intuitionistic arithmetic.

There are two ostensibly quite different ways to build a realizability topos starting from a given pca, and here we will use the results of the preceding

sections to explain (to a certain extent, at least) how they are related.

4.3.1 The two constructions

The first definition of the effective topos was Hyland's [Hyl82]. We start by considering sets $S \subseteq \mathbb{N}$ as non-standard truth values, so that the set $[X, \mathcal{PN}]$ of functions $X \rightarrow \mathcal{PN}$ is thought of as the set of non-standard predicates, called \mathcal{PN} -sets, on the set X . This set carries the structure of a category: if $\phi, \psi: X \rightarrow \mathcal{PN}$, then a morphism $\phi \rightarrow \psi$ is given by a partial recursive function Φ that satisfies the following condition: for any $x \in X$ and any $n \in \phi x$, Φn is defined and $\Phi n \in \psi x$. Moreover, as a category, the set $[X, \mathcal{PN}]$ has finite products: the terminal object is given by $x \mapsto \mathbb{N}$, while the product $\phi \times \psi$ is $x \mapsto \{(n, m) \mid n \in \phi x \text{ and } m \in \psi x\}$. (Recall that pairing $\langle -, - \rangle: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ can be chosen to be a total bijection).

The usual construction of the effective topos uses the *preorder reflection* of this category structure on $[X, \mathcal{PN}]$: it is the preorder where $\phi \leq \psi$ if there is a morphism $\phi \rightarrow \psi$, that is, if there exists a partial recursive Φ that satisfies the condition above. This preorder $[X, \mathcal{PN}]$ is (equivalent to) a Heyting algebra, but the finite meets given by the finite products defined above are enough for our purposes.

4.3.1 Definition ([Hyl82]). The *effective tripos* $\mathbf{ET}(-): \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Heyt}$ is the functor that sends a set X to the Heyting algebra $[X, \mathcal{PN}]$ and a function $f: X \rightarrow Y$ to the Heyting algebra homomorphism $f^*: [Y, \mathcal{PN}] \rightarrow [X, \mathcal{PN}]$ given by precomposition with f .

The total category $\int \mathbf{ET}$ is the category of \mathcal{PN} -sets.

The following is proved in [vO08, p. 53].

4.3.2 Proposition. *\mathbf{ET} is an ordered regular fibration.*

4.3.3 Definition. A *partial equivalence relation* (per) on a type X is a symmetric transitive relation on X ; that is, a binary relation $R(x, x')$ of type (X, X) such that

$$\begin{aligned} R(x_1, x_2) &\implies R(x_2, x_1) \\ R(x_1, x_2), R(x_2, x_3) &\implies R(x_1, x_3) \end{aligned}$$

If $\mathbf{E} \rightarrow \mathbf{B}$ is an ordered regular fibration and $X \in \mathbf{B}$, then a per on X is thus an object $r = \llbracket R \rrbracket$ over $X \times X$ satisfying $r \leq r^\circ = \sigma^* r$ (where σ is the symmetry map of $X \times X$) and $r \cap r \leq r$.

A morphism $(R, X) \rightarrow (S, Y)$ of pers is a relation F of type (X, Y) satisfying

$$\begin{aligned}
F(x, y) &\implies R(x, x) \wedge S(y, y) && \text{(strict)} \\
F(x, y) \wedge R(x, x') \wedge S(y, y') &\implies F(x', y') && \text{(relational)} \\
F(x, y) \wedge F(x, y') &\implies S(y, y') && \text{(single-valued)} \\
R(x, x) &\implies \exists y. F(x, y) && \text{(total)}
\end{aligned}$$

4.3.4 Definition. The *effective topos* is the category of pers in the effective tripos **ET**.

That this category is indeed a topos is proved in e.g. [vO08, theorem 2.2.1].

The second approach to constructing the effective topos is due to Carboni, Freyd and Ščedrov [CFŠ88].

4.3.5 Definition. An *assembly* A over a set X is given by an \mathbb{N} -indexed sequence $\{A_i \subseteq X\}_{i \in \mathbb{N}}$. The sets A_i are the *caucuses* and the set $|A| = \bigcup_i A_i$ the *carrier* of A . A morphism $A \rightarrow B$ of assemblies is given by a function $f: |A| \rightarrow |B|$ such that there exists a partial recursive Φ_f satisfying the following condition: for any i and any $a \in A_i$, $\Phi_f i$ is defined and $f a \in B_{\Phi_f i}$.

4.3.6 Remark. An assembly $\{A_i \subseteq X\}_i$ is essentially the same thing as a function $X \rightarrow \mathcal{P}\mathbb{N}$, because $[X, \mathcal{P}\mathbb{N}] \cong [\mathbb{N}, \mathcal{P}X]$ as sets. Moreover, the ordering on assemblies over X induced by morphisms whose underlying function is (a restriction of) the identity on X coincides with the ordering on $\mathcal{P}\mathbb{N}$ -sets defined above.

However, morphisms ‘between the fibres’ are not the same: an assembly morphism takes no account of elements that are not contained in any caucus. In particular, assemblies with exactly the same caucuses (even ones over different sets) must be isomorphic in **Asm**, but need not be so in (the total category of) **ET**.

It does, however, follow from this that every assembly A over X , say, is isomorphic to an assembly over its carrier $|A|$, and this is clearly the same thing as a $\mathcal{P}\mathbb{N}$ -set ϕ such that each ϕx is non-empty. Taking that point of view, a morphism of assemblies is then precisely a morphism of $\mathcal{P}\mathbb{N}$ -sets, and so **Asm** is equivalent to a full subcategory of $\int \mathbf{ET}$.

4.3.7 Proposition ([CFŠ88, Proposition 1]). *The category **Asm** of assemblies and assembly morphisms is regular.*

4.3.8 Definition. The *effective topos* **Eff** is the exact completion $\mathbf{Asm}_{\text{ex/reg}}$ of **Asm**.

By corollary 2.2.24, the exact completion of \mathbf{Asm} is the category of maps in the splitting (def. 2.2.18) of the equivalences in $\mathcal{R}el(\mathbf{Asm})$. An equivalence in an allegory is the same thing as a monad s that is symmetric (i.e. $s^\circ = s$), and a morphism of idempotents between two such is precisely a (bi)module (def. 3.1.4), because in this locally ordered context a module $m: s \rightarrow s'$ is indeed simply a morphism such that $ms = m = s'm$.

4.3.2 Relating the two

4.3.9 Definition. We will denote by $\mathcal{R}el(\mathbf{Asm})_{|\mathbf{Set}}$ the full sub-2-category of $\mathcal{R}el(\mathbf{Asm})$ on the constant assemblies, which can be identified with the locally ordered 2-category whose objects are sets X, Y, \dots and in which a morphism $X \rightarrow Y$ is given by an assembly $\{A_i \subseteq A\}_i$ together with a jointly monic span of functions $X \leftarrow |A| \rightarrow Y$. The ordering is induced in the obvious way by (necessarily unique) assembly morphisms.

4.3.10 Remark. Because assemblies with the same caucuses are isomorphic in \mathbf{Asm} and in $\mathcal{R}el(\mathbf{Asm})_{|\mathbf{Set}}$ (remark 4.3.6), we may assume without loss of generality that a morphism in the latter from X to Y is given by an assembly $\{A_i \subseteq X \times Y\}_i$.

4.3.11 Lemma. $\mathcal{R}el(\mathbf{Asm})_{|\mathbf{Set}}$ is equivalent to the underlying 2-category of $\mathbf{Matr}(\mathbf{ET})$.

Proof. By definition, the two have the same objects. Isomorphism on hom posets follows essentially from remark 4.3.6. In detail, the equivalence sends $r: X \times Y \rightarrow \mathcal{P}\mathbb{N}$ to the assembly $\bar{r} = \{\bar{r}_i \subseteq X \times Y\}_i$, where $(x, y) \in \bar{r}_i$ if $i \in r(x, y)$, together with the projections to X and Y . It follows from remark 4.3.10 that this assignment is a bijection.

If Φ tracks $r \leq s$ in $\mathbf{ET}(X \times Y)$, then it also tracks $\bar{r} \leq \bar{s}$ and conversely, because $\Phi i \in s(x, y)$ if and only if $(x, y) \in \bar{s}_{\Phi i}$.

It is a simple exercise in set theory to show that this correspondence preserves identities and composites, and so we have a 2-functor that is the identity on objects and locally an isomorphism, hence an equivalence. \square

4.3.12 Lemma ([CFŠ88]). $\mathcal{R}el(\mathbf{Asm})$ is equivalent to $(\mathcal{R}el(\mathbf{Asm})_{|\mathbf{Set}})_{[\text{crf}]}$.

By lemma 4.3.11, the functor $\mathbf{Set} \rightarrow \mathcal{R}el(\mathbf{Asm})_{|\mathbf{Set}}$ is a regular equipment in the image of $\mathbf{Matr}(-)$, so that the results of section 4.2.3 apply: a coreflexive morphism is precisely a comonad, and the splitting of these is precisely the category of comodules. So by the last result $\mathcal{R}el(\mathbf{Asm})$ is $\mathbf{Mod}^{\text{co}}(\mathbf{Matr}(\mathbf{ET}))$, and in particular a comonad in $\mathbf{Matr}(\mathbf{ET})$ is an assembly. One might then

wonder whether \mathbf{Asm} itself could turn out to be the co-Eilenberg–Moore completion $\mathbf{EM}^{\text{co}}(\mathbf{Matr}(\mathbf{ET}))$, but it is not: a coreflexive $h: X \rightarrow X$ is a \mathcal{PN} -set such that $hxx' \subseteq \llbracket x = x' \rrbracket$, and so $\hat{h}x = hxx$ is a \mathcal{PN} -set, or indeed an assembly, over X . A morphism $h \rightarrow g$ of coreflexives is a function $f: X \rightarrow Y$ such that $g(x, x') \leq h(fx, fx')$, or equivalently such that there exists a recursive Φ such that if $n \in \hat{h}$ then $\Phi n \in \hat{h}fx$. In other words, $\mathbf{EM}^{\text{co}}(\mathbf{Matr}(\mathbf{ET}))$ is precisely the category $\int \mathbf{ET}$ of \mathcal{PN} -sets. This shouldn't be too surprising, since we are dealing with the co-Eilenberg–Moore completion of an equipment, which we know corresponds to the comprehensive completion of a fibration, and the base category of the latter is the total category of the original fibration [MR12, theorem 3.1].

4.3.13 Proposition. *Let $\mathbf{E} \rightarrow \mathbf{B}$ be an ordered regular fibration. The category of pers (def. 4.3.3) in \mathbf{E} is equivalent to $\text{Map}(\mathbf{Matr}(\mathbf{E})[\text{sym}])$, where sym is the class of symmetric idempotents in $\mathbf{Matr}(\mathbf{E})$ (def. 2.2.12).*

Proof. (Cf. [Joh02, corollary A3.3.13(ii) et seq.]) It is obvious that a symmetric idempotent in $\mathbf{Matr}(\mathbf{E})$ is the same thing as a per in \mathbf{E} .

Suppose $f: r \rightarrow s$ is a morphism of symmetric idempotents that has a right adjoint f^\bullet (which is necessarily equal to f°).

The axioms (strict) and (relational) are equivalent to f 's being a morphism of idempotents, i.e. its satisfying $fr = f$ and $sf = f$ (and consequently $f = sfr$). In one direction, we have that $f(x, y)$ is equivalent to $f(x', y) \wedge r(x, x')$, and by symmetry and transitivity $r(x, x')$ implies $r(x, x)$. The same works for s and so (strict) follows. The condition $f = sfr$ easily implies (relational). Conversely, (strict) and (relational) together imply that the three conditions

$$\begin{aligned} sfr &= \exists \xi, v. f(\xi, v) \wedge r(x, \xi) \wedge s(y, v) \\ fr &= \exists \xi. f(\xi, y) \wedge r(x, \xi) \\ sf &= \exists v. f(x, v) \wedge s(y, v) \end{aligned}$$

are equivalent. If (strict) holds then $f(x, y)$ implies $r(x, x)$ and so implies $\exists \xi. f(\xi, y) \wedge r(x, \xi)$, while (relational) yields that sfr as above implies $f(x, y)$, so that $f = sfr = fr = sf$.

The axioms (total) and (single-valued) correspond to the adjunction $f \dashv f^\circ$; that is, to $r \leq f^\circ \circ f$ and $f \circ f^\circ \leq s$. The latter gives

$$f \circ f^\circ \leq s \quad \text{iff} \quad \exists \xi. f(\xi, y) \wedge f(\xi, y') \Rightarrow s(y, y')$$

which by adjointness of \exists and weakening is equivalent to (single-valued). Finally, we have

$$r \leq f^\circ \circ f \quad \text{iff} \quad r(x, x') \Rightarrow \exists v. f(x, v) \wedge f(x', v)$$

which yields (total) when $x = x'$. Conversely, $r(x, x')$ yields $r(x, x)$ and $r(x', x')$, and from these we get $\exists v, v'. f(x, v) \wedge f(x', v')$; (relational) gives $f(x, y) \wedge f(x, y')$ from this and $r(x, x')$, (single-valued) gives $s(y, y')$ and (relational) again yields $f(x, y) \wedge f(x', y)$. \square

So the two different constructions of **Eff** are linked as shown by the ‘map’ in figure 4.1 (where \bullet denotes a category that we don’t really care about): we may start with the effective tripos **ET**, move to the corresponding framed bicategory of relations, take the co-Eilenberg–Moore and then the Kleisli completions, functionally complete the result and pass back to a regular fibration (which is allowed because everything is locally ordered), and the effective topoi will be the base category of the result. The construction that starts with the category of assemblies merges with this one at the stage indicated, modulo the slight mismatch noted above between **Asm** and $\text{EM}^{\text{co}}(\text{Matr}(\mathbf{ET}))$.

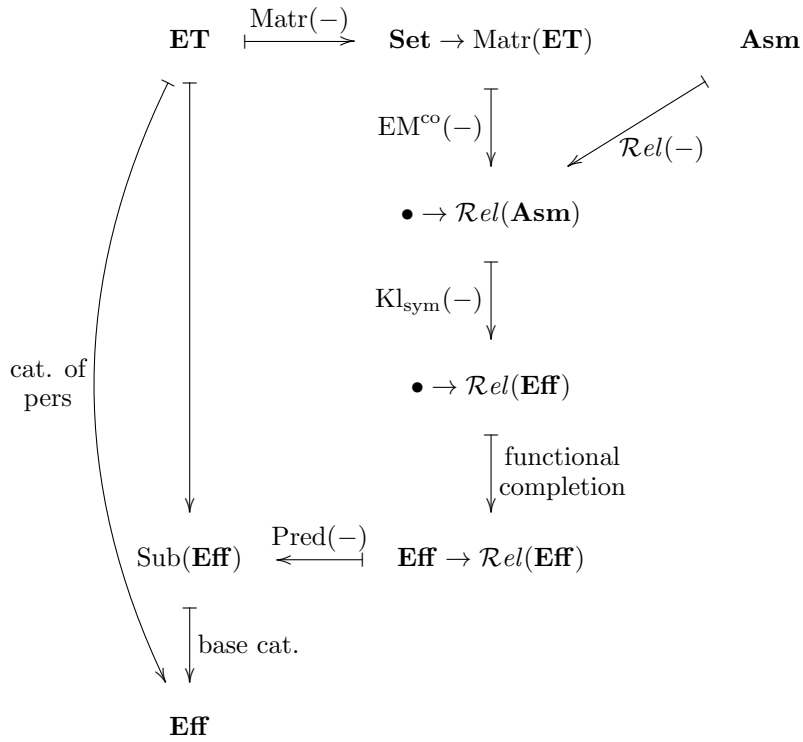


Figure 4.1: Constructions of the effective topoi

Chapter 5

Conclusions and future work

5.1 Recapitulation and comparison with existing work

Our main concrete results have been as follows:

- For regular theories, the fibrational and bicategorical semantics outlined in section 1.1.1 are equivalent once the latter is slightly augmented, and definitions and constructions may be translated back and forth across this equivalence in interesting and useful ways. (And this is true not just in the (locally/fibrewise) ordered context of logic in the traditional sense, but also in the ‘proof-relevant’ realm of type theory and category theory.)
- In particular, one may translate the category-of-pers construction into the world of equipments, where it naturally decomposes into a sequence of constructions that each has a universal property, namely, the co-Eilenberg–Moore completion, followed by the Kleisli completion with respect to symmetric monads, followed by the functional completion.

Re-translating this back into the world of fibrations exhibits the category of pers as the category of definable functions of the effective completion of the comprehensive completion. This decomposition illuminates, to a certain extent, the relationship between the two ostensibly quite different constructions of the effective topos, showing that they ‘converge’ sooner than one might expect.

It is also worth commenting on the techniques we have used to obtain these results, and on the auxiliary results we have got along the way:

- Proposition 2.2.33, that a bicategory of relations is the same thing as a unitary tabular allegory, does not seem to have been published before, although it is hardly a surprising result. It is what connects our work to the construction of the effective topos by taking the exact completion of \mathbf{Asm} , i.e. by splitting idempotents in the allegory $\mathcal{R}el(\mathbf{Asm})$.
- Section 3.2.2 defined bicategorical ends and coends in $\mathcal{C}at$, showing that the former behaved exactly as one would expect. Section 3.2.5 then showed how to compute coends as weighted and as conical colimits, the previous section having shown how to construct the latter in $\mathcal{C}at$. All of these results are new, as far as I can tell, although it would be useful to compare our construction of colimits with that of ‘2-filtered’ ones given in [DS06].
- These results then meant that we could define $2\text{-}\mathbb{P}rof$ as a full sub-3-category of $2\text{-}\mathcal{C}at$, thereby avoiding a lot of calculation, but also that we could treat its morphisms as category-valued functors composed using coends in the usual way.

In the following sections we discuss in a little more detail how our work on regular fibrations and regular equipments, and on equipments in general, is related to some existing work.

5.1.1 Comprehension and tabulation

Suppose given a regular fibration \mathbf{E} over \mathbf{B} that has full comprehension. If equality in \mathbf{E} is extensional (def. 2.1.22), then \mathbf{B} has all pullbacks, which satisfy the Beck–Chevalley condition in \mathbf{E} (cf. [LWW10, theorem 4.8]): for a cospan $(f, g): X \rightarrow Z \leftarrow Y$, put $P(f, g) = (f \times g)^* d_i \top z$. Then $\{P(f, g)\}$ is the pullback of f along g , by the following bijections:

$$\begin{array}{ccc}
 W & \xrightarrow{\quad} & \{P(f, g)\} \\
 & \searrow^{(h, j)} & \swarrow \\
 & & X \times Y \\
 \hline
 & \text{im}(h, j) \xrightarrow{\quad} & P(f, g) \\
 \hline
 & \text{im}(fh, gj) \xrightarrow{\quad} & \text{im } d \\
 \hline
 W & \xrightarrow{\quad} & \{\text{im } d\} \\
 & \searrow^{(fh, gj)} & \swarrow \\
 & & Z \times Z
 \end{array}$$

By prop. 2.1.24 extensionality means that each diagonal $d: X \rightarrow X \times X$ is an injection, so that morphisms of the last form are the same as factorizations

of (fh, gj) through d , of which there is at most one, which exists precisely when $fh = gj$. (The Beck–Chevalley condition then follows from the fullness of tabulations in $\text{Matr}(\mathbf{E})$.) So \mathbf{B} has finite limits, and hence $X \mapsto \mathbf{B}/X$ is a regular fibration. Note that this also means that the type-(A) Beck–Chevalley condition holds in cartesian equipments that satisfy the separability condition and that admit full tabulations.

The adjunctions $\text{im} \dashv \{-\}$ exhibit each fibre $\mathbf{E}X$ as a reflective subcategory of \mathbf{B}/X . If injections are closed under composition, then this is equivalent [CJKP97, 2.12] to giving a factorization system on \mathbf{B} , whose right class consists of the injections. Then the image functors preserve pullbacks (they always preserve pushforwards) if and only if this factorization system is pullback-stable. Consider the pullback square defined above: by the Beck–Chevalley condition, we have $f^*g_! \cong (f^*g)_!(g^*f)^*$, but $(g^*f)^*$ preserves the terminal object, so that $f^*(\text{im } g) \cong \text{im } f^*g$. Hence full comprehension implies that image preserves pullbacks. Therefore, from the definition of a regular category from def. 2.1.2, and the fact that in the presence of full comprehension, orderedness of a fibration is equivalent to every injection’s being a monomorphism, we have the following (cf. [Jac99, theorem 4.9.4]).

5.1.1 Proposition. *A regular fibration \mathbf{E} over \mathbf{B} is equivalent to $\text{Sub}(\mathbf{B})$ if and only if \mathbf{E} is locally ordered and has full comprehension, such that every monomorphism in \mathbf{B} is an injection.*

We also have the following result, an evident consequence of the definition of injections.

5.1.2 Proposition. *A regular fibration over \mathbf{B} is equivalent to $\text{Arr}(\mathbf{B})$ if and only if it has full comprehension and every morphism in \mathbf{B} is an injection.*

We clearly have

$$\begin{aligned} \text{Matr}(\text{Sub}(\mathbf{C})) &\simeq \mathbf{C} \rightarrow \mathcal{R}el(\mathbf{C}) \\ \text{Matr}(\text{Arr}(\mathbf{C})) &\simeq \mathbf{C} \rightarrow \mathcal{S}pan(\mathbf{C}) \end{aligned}$$

and so the previous two results translate to characterizations of equipments of relations and of spans. Saying that a span $(f, g): X \rightarrow Y \times Z$ tabulates itself if the following diagram is a tabulation

$$\begin{array}{ccccc} X & \xrightarrow{\quad \dagger \quad} & X & & \\ f \downarrow & & \downarrow & & \downarrow g \\ Y & \xrightarrow{\quad \dagger \bullet \quad} & X & \xrightarrow{\quad \bullet \quad} & Z \end{array}$$

then we have

5.1.3 Proposition. *A regular equipment $\mathbf{B} \rightarrow \mathcal{B}$ with co-Eilenberg–Moore objects is*

- *the equipment of relations in \mathbf{B} if it is locally ordered and if every relation in \mathbf{B} tabulates itself in \mathcal{B} ; or*
- *the equipment of spans in \mathbf{B} if every span in \mathbf{B} tabulates itself in \mathcal{B} .*

This is clearly very similar to the characterization in [CKS84, theorems 4, 7] of 2-categories of relations and of spans, although they do not require even cartesianness of the underlying bicategory \mathcal{B} but instead that $\text{Map}(\mathcal{B})$ be locally discrete.

In [LWW10] another characterization is given of 2-categories of spans: they are those that are cartesian and admit Eilenberg–Moore objects for comonads, and in which every map is comonadic. Expressed in our language, taking a cartesian bicategory to be a chordate cartesian equipment as in prop. 4.2.4, comonadicity means that if $f: X \rightarrow Y$ is a map then the cell

$$\begin{array}{ccc} X & \xrightarrow{\quad \dagger \quad} & X \\ f \downarrow & & \Downarrow \\ Y & \xrightarrow{\quad f_{\bullet} \quad} & X \xrightarrow{\quad f_{\bullet} \quad} Y \end{array}$$

exhibits X as the EM object of $f_{\bullet} f^{\bullet}$. By prop. 4.2.14 this is the same as saying that

$$\begin{array}{ccc} X & \xrightarrow{\quad \dagger \quad} & X \\ t \downarrow & & \Downarrow \\ \mathbf{1} & \xrightarrow{\quad t_{\bullet} \quad} & Y \xrightarrow{\quad f_{\bullet} f^{\bullet} \quad} Y \end{array}$$

is a tabulation. But $f_{\bullet} f^{\bullet} t^{\bullet}$ is canonically isomorphic to $f_{\bullet} t^{\bullet}$, which is precisely $\text{im } f$, and the tabulation above is the extension $\{\text{im } f\}$. So to say that a tight map is comonadic in \mathcal{B} is precisely to say that it is an injection with respect to the fibration $\text{Pred}(\mathcal{B})$. Again, this is very similar, though not identical, to the result above. In fact, the only real difference here is that [LWW10] derives the separability and Frobenius conditions from the comonadicity axiom rather than postulating them. It would be interesting to see whether a similar but restricted condition would suffice to axiomatize regular equipments, with or without the type-(A) Beck–Chevalley condition.

5.1.2 Equipments

Apart from the concrete results listed above, probably the most significant thing we have done is to define the 3-category 2-Prof and to identify equipments, in a suitably general sense, as pseudo-monads in it. The important step here was the construction of Kleisli objects in 2-Prof in theorem 4.1.3. We have seen that these monads, when taken over locally discrete 2-categories, are essentially the same as both Wood’s and Shulman’s notions when these are taken not to require right adjoints for tight morphisms, and their relationship with the equipments of Carboni et. al. is clear.

One kind of equipment that we have not compared with ours is Verity’s notion of a *double bicategory* [Ver92]. Such a thing is given by a pair of 2-categories with the same objects, thought of as ‘vertical’ and ‘horizontal’, and the 2-cells of these act in a functorial way on a set of ‘squares’, whose boundaries are vertical and horizontal 1-cells, as in a double category. There is also an operation of horizontal composition on squares, that commutes suitably with the action of vertical and horizontal 2-cells. We won’t work out the details here, but one would expect our equipments, in the most general sense, to be to double bicategories as Wood’s equipments (etc.) are to double categories, that is, to be (equivalent to) double bicategories whose squares are uniquely determined by certain horizontal 2-cells. Indeed, a monad $T: \mathcal{K} \rightarrow \mathcal{K}$ in 2-Prof has an underlying vertical 2-category, namely \mathcal{K} , a horizontal one, namely \mathcal{K}_T , and a set of squares given by the objects of the 2-category of elements $\int T$. The squares are acted on by the 2-cells of \mathcal{K} and \mathcal{K}_T and inherit a horizontal composition operation from the multiplication of T . (This structure is what [Ver92, def. 1.2.4] would call $\text{Sq}(\mathcal{K}_T, \mathcal{K}, F_T)$.) After defining double bicategories, Verity goes on to use them to discuss morphisms of equipments more general than the ones we have defined. We will suggest some ways of doing this in our context in section 5.2.1 below.

There has been work done before on constructing Kleisli objects in 2-Cat . In [CHP04], it is shown that for a pseudo-monad $T: \mathcal{K} \rightarrow \mathcal{K}$ in 2-Cat , the objects of \mathcal{K} and the hom-categories $\mathcal{K}(k, Tk')$ form a 2-category, and their theorem 4.3 then says that this 2-category represents right T -modules (which they call ‘co-cones’) in 2-Cat . Our theorem 4.1.3 strictly generalizes this result, because (as noted after corollary 4.1.5) the Kleisli object in 2-Prof of a representable monad is also its Kleisli object as a monad in 2-Cat . The idea behind the construction in *op. cit.* is, in our language, that for an equipment $\mathcal{K} \rightarrow \mathcal{M}$ that is the Kleisli object of a monad T on \mathcal{K} in 2-Cat , together with another monad S on \mathcal{K} , to lift the latter to a pseudo-monad on the equipment is precisely to give a distributive law [Mar99] of S over T . Now the point of our theorem 4.1.3 is that

in fact every equipment is the Kleisli object of some monad in 2-Prof , just not necessarily a representable one. So the problem of lifting monads in the above sense is contained in the problem of constructing distributive laws in 2-Prof .

5.2 Future directions

In this last section we give some ideas and prospects for future work based on what we have already done.

5.2.1 More on equipments

We have seen, in section 4.1.3, that while equipments in the most general sense can be viewed as monads in 2-Prof and equipment morphisms are then monad morphisms in a straightforward way, this doesn't quite work for 2-cells. The right notion of equipment 2-cell would reduce to a vertical transformation between double categories in the case of a locally discrete base bicategory, but neither monad 2-cells nor Kleisli 2-cells fit the bill. As it turned out, we were able to define an ordinary category of equipments, which, together with double transformations between associated double functors, was enough to get the results of section 4.2.2.

Section 4.1.3 showed that an equipment morphism from $\mathcal{K} \rightarrow \mathcal{K}_T$ to $\mathcal{L} \rightarrow \mathcal{L}_S$ is given by a functor $F: \mathcal{K} \rightarrow \mathcal{L}$ and a morphism $T \rightarrow S(F, F)$ of pseudo-monoids, the latter giving the effect on hom-categories of a functor $\tilde{F}: \mathcal{K}_T \rightarrow \mathcal{L}_S$. Along the same lines, an equipment 2-cell will be a transformation $\alpha: F \Rightarrow G$ together with a coherent isomorphism between the two evident morphisms $T \rightarrow S(F, G)$, which gives the morphism-components required to make α a pseudonatural transformation $\tilde{F} \Rightarrow \tilde{G}$. Relaxing the condition that this latter morphism be invertible should then give the right notion of 2-cell, and, just as an equipment 2-cell between morphisms \tilde{F} and \tilde{G} from T to S is a (pseudo) morphism of T^* -algebras, where T^* is the precomposition-with- T monad on the Kleisli 2-category of S_* , a lax equipment 2-cell ought to be a lax algebra morphism. This raises the possibility of using the theory of lax morphism classifiers [Lac02] to reduce the lax case to the pseudo case.

A similar possibility suggests itself when it comes to defining lax morphisms of equipments. One may define lax algebras for pseudo-monads just as in def. 3.1.12, except that the 2-cells α and ν are not required to be invertible. Then a lax T^* -algebra structure on F in the Kleisli 2-category of S_* should correspond to a 'lax monoid morphism' $\phi: T \rightarrow S(F, F)$, i.e. one that comes equipped with coherent morphisms $\phi \cdot \eta^T \rightarrow \eta^{S(F, F)}$ and $\mu^{S(F, F)} \cdot (\phi \circ \phi) \rightarrow \phi \cdot \mu^T$, which will give a lax functor $\tilde{F}: \mathcal{K}_T \rightarrow \mathcal{L}_S$ together with a transformation $\tilde{F} \circ F_T \Rightarrow F_S \circ F$,

not invertible in general. Again, it may be possible to use or generalize existing work on 2-dimensional monads to reduce the lax case to the pseudo: for a strict 2-monad T on a strict 2-category, it is possible, under certain conditions, to construct a new monad T' such that lax T -algebras are precisely strict T' -algebras. That 2-Prof has well-behaved local colimits suggests that it might be possible to do something similar in this context, in order to construct lax morphism classifiers in some 3-category of equipments. This would be useful for studying the kind of change-of-base questions that Verity [Ver92] considers, as well as liftings of monads to equipments, as discussed in the last section, but where the lift is a lax monad [Bun74] rather than pseudo. A classic example of the latter situation is the fact that the ultrafilter monad on \mathbf{Set} lifts to a lax monad on $\mathcal{R}el$, whose lax algebras are topological spaces [Bar70], but there are many other contexts in which such constructions arise [CHT04].

5.2.2 ‘Variation through enrichment’

For \mathbf{C} any category, there is a 2-category $\mathcal{S}(\mathbf{C})$ given by the full sub-2-category of $\mathit{Span}([\mathbf{C}^{\text{op}}, \mathbf{Set}])$ on the representables. Then categories enriched in $\mathcal{S}(\mathbf{C})$, in the sense of e.g. [Bén67, (5.5)] or [Wal82], are very nearly the same as fibrations over \mathbf{C} : by [BCSW83] there is an equivalence between the 2-categories of ‘Cauchy-complete’ objects of each sort. In fact, it seems (although I do not know of a published proof) that if $\mathcal{S}(\mathbf{C})$ -enriched functors are defined using the equipment/double-category structure of $\mathcal{S}(\mathbf{C})$, i.e. if they are required to give *vertical* morphisms between extents, then the equivalence includes even the non-complete categories and fibrations.

There are two reasons for considering this as a framework in which to interpret our results. The first is that we would like to be able to pare away at the structure of a regular fibration or equipment to see what the axioms on one side of the equivalence correspond to on the other side. The problem, of course, is that nearly all of the structure of a regular fibration is required in order even to define the functor $\text{Matr}(-)$ as in section 4.2.2. So it would make sense to try to recover this latter construction as a special case of the more general one: that is, we know there is an equivalence $\mathit{Fib}(\mathbf{C}) \rightarrow \mathcal{S}(\mathbf{C})\text{-Cat}$, and we might ask what is required of a fibration over \mathbf{C} in order for this functor to factor through equipments over \mathbf{C} in such a way as to reproduce the results of section 4.2.2, if indeed that is possible at all. What is ‘regular structure’ on a fibration as an object of $\mathit{Fib}(\mathbf{C})$? What does that mean for the corresponding $\mathcal{S}(\mathbf{C})$ -category? Does this structure on an $\mathcal{S}(\mathbf{C})$ -category make it ‘equivalent’ to a regular equipment in some way, in a way that coheres with the $\text{Matr}(-)$ construction?

The second reason for moving to this level of generality is a potential connection with more general forms of realizability. Longley has recently proposed a notion of ‘computability structure’ [Lon13] that encompasses partial combinatory algebras and is similar to the ‘basic combinatorial objects’ of Hofstra [Hof06]. Each of these is clearly trying very hard to be a category enriched in some sort of 2-category, and so one might wonder whether they are examples of a still more general notion of ‘coefficient object’ for realizability that encompasses the two, and whether the passage from a partial combinatory algebra to its associated tripos can be seen in the context of the equivalence between fibrations and categories enriched in certain 2-categories. That is rather a vague idea, of course, but it holds out the possibility of a structural account of realizability: an equivalence of categories between very general collections (of whatever sort) of computable objects and the fibrations they induce would allow a direct comparison between structures borne by the one and by the other; regular structure, tripos structure, and so on. That, after all, was the motivation behind the work reported here in the first place.

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