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A Symmetric Alternative to Geary and John's Expression of Product Changes

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Geary and John in this issue of the *Review* prove the identity $d(x_1 \ldots x_k) = \Sigma f_i dx_i$, where $d(x_1 \ldots x_k) = \pi x_i^2 - \pi x_i^1$, $dx_i = x_i^2 - x_i^1$ and f_i is the symmetric function of the x's other than x_i, viz.:

$$f_i = \left\{ \Sigma^0_{(i)} \Big/ \binom{k-1}{0} + \Sigma^1_{(i)} \Big/ \binom{k-1}{1} + \ldots + \Sigma^{k-1}_{(i)} \Big/ \binom{k-1}{k-1} \right\} \Big/ k.$$

Here, $\Sigma^r_{(i)} = \Sigma x_1^{t_1} \ldots x_{i-1}^{t_{i-1}} \; x_{i+1}^{t_{i+1}} \ldots x_k^{t_k}$ with r of the (k−1) x's having the superscript 1 (i.e., the t's of those x's are set at 1) and the remaining (k−1−r) x's having superscript 2 (t's at 2) and the sum is over the $\binom{k-1}{r}$ distinct ways of placing the superscripts in the k−1 positions available. To illustrate, for k=5,

$$\Sigma^3_{(1)} = x_2^1 x_3^1 x_4^1 x_5^2 + x_2^1 x_3^1 x_4^2 x_5^1 + x_2^1 x_3^2 x_4^1 x_5^1 + x_2^2 x_3^1 x_4^1 x_5^1.$$

A different argument leading directly to the Geary-John identity is sketched below but, for interest, is organised to produce an alternative symmetric expression which is perhaps equally attractive. Using bars to indicate arithmetic means, it follows and is easily shown from the calculus of finite differences that $d(x_1 \ldots x_k) = \bar{x}_k d(x_1 \ldots x_{k-1}) + (\overline{x_1 \ldots x_{k-1}})dx_k$. A similar expression can be written for $d(x_1 \ldots x_{k-1})$, and continuation of the expansion leads to a sum of k terms, the first involving dx_1, the second dx_2, etc.

This process can be carried out for all k! permuted orders of $x_1 \ldots x_k$. Adding and averaging yields

$$d(x_1 \ldots x_k) = \Sigma g_i dx_i \tag{1}$$

where

$$g_1 = \Bigg\{\left[\overline{(x_2 \ldots x_k)}\right] \Big/ \binom{k-1}{0} + \left[\bar{x}_2(\overline{x_3 \ldots x_k}) + \bar{x}_3(\overline{x_2 x_4 \ldots x_k}) + \ldots\right] \Big/ \binom{k-1}{1}$$

$$+ \left[\bar{x}_2 \bar{x}_3 (\overline{x_4 \ldots x_k}) + \bar{x}_2 \bar{x}_4 (\overline{x_3 x_5 \ldots x_k}) + \ldots\right] \Big/ \binom{k-1}{2} + \ldots$$

$$+ \left[\bar{x}_2 \bar{x}_3 \ldots \bar{x}_{k-1}(\bar{x}_k) + \bar{x}_2 \bar{x}_3 \ldots \bar{x}_{k-2} \bar{x}_k (\bar{x}_{k-1}) + \ldots\right] \Big/ \binom{k-1}{k-2}$$

$$+ \left[\bar{x}_2 \bar{x}_3 \ldots \bar{x}_k\right] \Big/ \binom{k-1}{k-1} \Bigg\} \Big/ k \tag{2}$$

with similar expressions for $g_2, \ldots, g_k$. The number of terms in the r'th square bracket is $\binom{k-1}{r-1}$, the number of ways of selecting r-1 variables (whose product is to be averaged) from k-1. Thus, g_i can be seen as an intuitively appealing kind of generalised average of the product of the k-1 x's excluding x_i.

From the Geary and John identity, it follows that $\Sigma g_i dx_i = \Sigma f_i dx_i$. From symmetry it is expected that $g_i = f_i$, all i, which is indeed the case, although term by term equality within g_i and f_i obviously fails. The relationship between the structure of g_i and that of f_i is complex but pleasingly elegant. Showing equality of g_i and f_i depends on showing that

$$H(p,m) + I(p,m) = 1, p = 0, \ldots, m,$$

where

$$H(p,m) = \sum_{r=0}^{m-p} (\tfrac{1}{2})^{p+r+1} \binom{p+r}{p} \text{ and } I(p,m) = \sum_{r=0}^{p} (\tfrac{1}{2})^{m-p+r+1} \binom{m-p+r}{r}$$

By noting that $H(p,m) = H(p-1,m) - (\tfrac{1}{2})^{m+1} \binom{m+1}{p}$, $p = 1, \ldots, m$ and that $I(p,m) = I(p+1,m) - (\tfrac{1}{2})^{m+1} \binom{m+1}{p+1}$, $p = 0, \ldots, m-1$ and using the easily calculated values for $H(0,m)$ and $I(0,m)$, the desired result follows. As an aside, if the index r runs to infinity in the H and I series, both series converge and sum to unity – a fact which allows the generation of further interesting combinatorial identities and yields further insights into the respective structures of g_i and f_i.

If a reader can simplify the above arguments or would wish to see them expressed in fuller detail, he is invited to contact the author.