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On the central Haagerup tensor product and completely bounded mappings of a C^*-algebra

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Abstract

We consider the natural contractive map from the central Haagerup tensor product of a unital C^*-algebra A with itself to the space of completely bounded maps $CB(A)$ on A. We establish the necessity of the known sufficient condition for isometry of the map, namely that all Glimm ideals of A are primal. However, when the map is restricted to tensors with length bounded by a fixed quantity, a weaker necessary and sufficient condition is established.

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0. Introduction

Let A be a unital C^*-algebra, $A \otimes_h A$ the Haagerup tensor product, $CB(A)$ the space of completely bounded maps $T: A \to A$ and $\mathcal{E}\ell(A)$ the subspace of elementary operators on A (those expressible in the form $Tx = \sum_{j=1}^{\ell} a_j x b_j$ with $a_j, b_j \in A$)

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[4, Chapter 5]. There is a natural contraction $\theta: A \otimes_h A \to CB(A)$ (mapping $\sum_{j=1}^n a_j \otimes b_j$ to $T \in \mathcal{E}\ell(A)$ as above). Following the pioneering work of Haagerup in the case of $B(H)$ (see [16], [4, 5.4.7, 5.4.9] and [17]), Chatterjee and Sinclair [9] showed that θ is isometric if A is a separably-acting von Neumann factor. More generally, Mathieu showed that θ is isometric if and only if A is a prime C^*-algebra (see [4, 5.4.11]).

If A is not prime then θ is not even injective, and it is then natural to consider the central Haagerup tensor product $A \otimes_{Z,h} A$ (the quotient of the Haagerup tensor product $A \otimes_h A$ by the closure of the span of elements of the form $az \otimes b - a \otimes zb$, $a, b \in A$, $z \in Z(A)$, where $Z(A)$ is the centre of A). The mapping θ induces a contraction $\theta_Z: A \otimes_{Z,h} A \to CB(A)$. Chatterjee and Smith [10] showed that θ_Z is isometric if A is a von Neumann algebra or if the primitive ideal space $\mathrm{Prim}(A)$ is Hausdorff (see also [11]). More generally, Ara and Mathieu (see [3] and [4, 5.4.26]) showed that θ_Z is isometric if A is a boundedly centrally closed C^*-algebra.

A further generalization was obtained by Somerset [19, Theorem 4], who showed that θ_Z is isometric if every Glimm ideal of A is primal. It was also shown in [19] that θ_Z is injective if and only if every Glimm ideal of A is 2-primal, and that if A has a Glimm ideal which fails to be 3-primal then there is a "pre-derivation" $1 \otimes a - a \otimes 1$ for which θ_Z reduces the norm (see also [18]). In particular, while the primality of every Glimm ideal is sufficient for θ_Z to be an isometry, the 3-primality of every Glimm ideal is necessary. This seemed to suggest that it should be possible to find a necessary and sufficient condition in terms of ideal structure for θ_Z to be an isometry.

In Section 2, we construct an example to show that the 3-primality of all Glimm ideals is not sufficient for θ_Z to be an isometry. Indeed, we explicitly exhibit an element whose norm is reduced by θ_Z. In Section 3, we extend the ideas and computations associated with this example to a general situation. Our first main result (Theorem 7) is that the primality of all Glimm ideals is necessary for θ_Z to be an isometry (this is the converse of [19, Theorem 4]). The proof makes crucial use of a result of Akemann and Pedersen [1, Proposition 2.6] concerning orthogonal lifting from a quotient of a C^*-algebra. At the end of Section 3, we consider the case of a non-unital C^*-algebra A by using the multiplier algebra $M(A)$ in the usual way.

In Section 4, we go on to consider the more difficult question of how the degree of primality of the Glimm ideals is related to the isometric behaviour of θ_Z on (cosets of) tensors $u \in A \otimes A$ of bounded length. For this, we exploit the recent results of Timoney [21] on matrix numerical ranges, together with a corollary of Carathéodory's theorem on convex hulls in $\mathbb{R}^n$. Our second main result (Theorem 17) is that, for fixed $\ell \geqslant 1$, θ_Z is isometric on each $u = \sum_{j=1}^{\ell} a_j \otimes b_j \in A \otimes A$ if and only if every Glimm ideal of A is $(\ell^2 + 1)$-primal.

1. Notation

If A is a unital C^*-algebra and $J \in \mathrm{Max}(Z(A))$ (the maximal ideal space of the centre $Z(A)$), then the *Glimm ideal* of A generated by J is the proper closed two-sided ideal AJ (see [15, §4]). It is closed by Cohen's factorization theorem. Since $AJ \cap Z = J$, the mapping $J \mapsto AJ$ ($J \in \mathrm{Max}(Z(A))$) is a bijection of $\mathrm{Max}(Z(A))$ onto the set $\mathrm{Glimm}(A)$ of all Glimm ideals of A.

A (closed two-sided) ideal I of A is called *n-primal* (for some $n \geqslant 2$) if whenever $J_1, J_2, \dots, J_n$ are ideals of A with product $J_1 J_2 \cdots J_n = \{0\}$, then at least one of the J_i is contained in I. The ideal I is called *primal* if it is n-primal for all $n \geqslant 2$. This concept arose in [6] where it was shown that a state of A is a weak*-limit of factorial states if and only if the kernel of its GNS representation is primal.

In [7, Lemma 1.3], it is shown that an ideal I of A is n-primal if and only if $\bigcap_{i=1}^n P_i$ is primal whenever $P_1, P_2, \dots, P_n$ are primitive ideals of A containing I. Furthermore, the primality of such an intersection $\bigcap_{i=1}^n P_i$ is equivalent to the existence of a net (Q_α) in the primitive ideal space $\mathrm{Prim}(A)$ which converges to each element of $\{P_1, P_2, \dots, P_n\}$ and hence to every element in the closure of this set, namely $\mathrm{Prim}(A/(P_1 \cap P_2 \cap \cdots \cap P_n))$ (see [6, Proposition 3.2]).

In [6, §3] it is shown that for each $n \geqslant 2$ there is a unital C^*-algebra A_n containing an ideal I_n which is n-primal but not $(n+1)$-primal. Note that I_n is not a Glimm ideal because it is non-zero and A_n has trivial centre. In [7, Theorem 2.7], it is shown that, for each odd integer $n \geqslant 3$, there is a 2-step nilpotent Lie group whose (non-unital) C^*-algebra contains a Glimm ideal which is n-primal but not $(n+1)$-primal.

To conclude this notation section, we mention that we denote the norm on $A \otimes_{Z,h} A$ by $\|\cdot\|_{Z,h}$. For convenience, we will often refer to $\|u\|_{Z,h}$ and $\theta_Z(u)$ when $u \in A \otimes A$, where it is to be understood that u is to be replaced by its image in $A \otimes_{Z,h} A$.

2. Basic constructions

We consider in some detail an example of a unital C^*-algebra A in which all Glimm ideals are primitive (and hence primal) except for one particular Glimm ideal G_∞ which is 3-primal but not 4-primal. This example is an elaboration of an example in [5, Example 4.12] which has a Glimm ideal that is 2-primal but not 3-primal, and it is also a prototype for variants which seem to be able to exhibit many of the phenomena that can occur in general.

The basic idea is to build a 4-point compactification of a locally compact Hausdorff space, where in each way of approaching the points at infinity one actually has three limiting values (but not the fourth). This requires four ‘directions’ of approach to infinity. A way to visualise such a space is to consider a disjoint union $\mathcal{T} = \bigcup_{j=1}^4 R_j$ of four semi-infinite closed rays in the plane with four points adjoined as follows. For example $\mathcal{T} = \{(x, y) : xy = 0, x^2 + y^2 \geqslant 1\} \subset \mathbb{R}^2$ with (say) $R_1 = \{(x, 0) : x \geqslant 1\}$. Label the four extra points ω_i ($1 \leqslant i \leqslant 4$). A basis of neighbourhoods of each ω_i is given by the sets

$$\{\omega_i\} \cup \bigcup_{j \neq i} \{t \in R_j : |t| > r\},$$

where $r > 1$. So, for example, the sequence $(n, 0)$ in the ray R_1 of the space $\mathcal{T} \cup \{\omega_1, \omega_2, \omega_3, \omega_4\}$ would have each of ω_2, ω_3 and ω_4 as limits as $n \to \infty$ (but not ω_1).

A 'discrete' version of this space would start with $\mathcal{T} \cap \mathbb{Z}^2$ in place of $\mathcal{T}$. Clearly one can map $\mathcal{T} \cap \mathbb{Z}^2$ to $\mathbb{N}$ by mapping the four directions to equivalence classes in $\mathbb{N}$ modulo 4 (cf. [5, Example 4.12]).

We now construct a C^*-algebra A such that $\mathrm{Prim}(A)$ is (homeomorphic to) $\mathcal{T} \cup \{\omega_1, \omega_2, \omega_3, \omega_4\}$. We consider the C^*-algebra B of bounded continuous functions $x: \mathcal{T} \to M_3(\mathbb{C})$ and we define A to be the C^*-subalgebra of B consisting of all those elements $x \in B$ for which there exist scalars $\lambda_1(x), \lambda_2(x), \lambda_3(x), \lambda_4(x)$ such that

$$\lim_{R_j \ni t \to \infty} x(t) = \mathrm{diag}\,(\lambda_{j+1}(x), \lambda_{j+2}(x), \lambda_{j+3}(x)) \quad (1 \leqslant j \leqslant 4),$$

where we understand the subscripts $j + i$ $(1 \leqslant i \leqslant 3)$ to be reduced modulo 4 to lie in the range $1, 2, 3, 4$. Next, we introduce notation for what we call 'constant' elements of A. Given four scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ we write $c(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for the element $x \in A$ where

$$x(t) = \mathrm{diag}\,(\lambda_{j+1}, \lambda_{j+2}, \lambda_{j+3}) \qquad (t \in R_j,\ j \in \{1, 2, 3, 4\})$$

(where we again understand the subscripts modulo 4). The set A_c of all constant elements of A forms an abelian C^*-subalgebra isomorphic to $\mathbb{C}^4$ and $A = C_0(\mathcal{T}, M_3(\mathbb{C})) + A_c$.

We call this A a '4-spoke' example. The centre $Z(A)$ of A consists of elements x where each $x(t)$ is a multiple of the identity (and hence $\lambda_i(x)$ does not depend on i) and so $Z(A)$ is canonically isomorphic to the algebra of scalar-valued continuous functions on the one-point compactification of $\mathcal{T}$. The space $\mathrm{Glimm}(A)$ can then be identified with this one point compactification, or $\{G_t : t \in \mathcal{T}\} \cup \{G_\infty\}$ where $G_t = \{x \in A : x(t) = 0\}$ and $G_\infty = C_0(\mathcal{T}, M_3(\mathbb{C}))$. As A/G_∞ is abelian, the irreducible representations of A whose kernels contain G_∞ are $x \mapsto \lambda_i(x)$. The remaining irreducible representations of A restrict to irreducible representations of $C_0(\mathcal{T}, M_3(\mathbb{C}))$, and hence have the form $\pi_t: A \to M_3(\mathbb{C})$ where $\pi_t(x) = x(t)$ for $x \in A$ and $t \in \mathcal{T}$. Thus, as $\ker \pi_t = G_t$,

$$\mathrm{Prim}(A) = \{G_t : t \in \mathcal{T}\} \cup \{\ker \lambda_i : 1 \leqslant i \leqslant 4\}.$$

As a topological space $\mathrm{Prim}(A)$ is homeomorphic to $\mathcal{T} \cup \{\omega_i : 1 \leqslant i \leqslant 4\}$. For example, to see that as $t \in R_j$ tends to infinity we have $G_t = \ker \pi_t \to \ker \lambda_i$ for each $i \neq j$, let us fix $i \neq j$ and consider an open neighbourhood U of $\ker \lambda_i$ in $\mathrm{Prim}(A)$. Then there is a closed two-sided ideal J of A with $U = \{I \in \mathrm{Prim}(A) : J \not\subseteq I\}$. Since $\ker \lambda_i \in U$, there is some $x \in J$ with $\lambda_i(x) \neq 0$. Thus there exists $r > 1$ so that if $t \in R_j$ and $|t| > r$ then $x(t) \neq 0$. It follows that $\ker \pi_t \in U$ whenever $t \in R_j$ and $|t| > r$.

The four ideals $J_i = \{x \in A : x(t) = 0 \text{ for all } t \in R_i\}$ have product $\{0\}$ but (for example) J_1 is not contained in G_∞ because $c(1, 0, 0, 0) \in J_1$. Hence G_∞ is not 4-primal. To show that G_∞ is 3-primal, note that there are only four primitive ideals of A which contain G_∞, namely ker λ_i $(i = 1, 2, 3, 4)$. So it suffices to show that, for each i,

$$\ker \lambda_{i+1} \cap \ker \lambda_{i+2} \cap \ker \lambda_{i+3}$$

is primal. But we have just shown that ker π_t converges in $\mathrm{Prim}(A)$ to each of ker λ_{i+1}, ker λ_{i+2} and ker λ_{i+3}, as $t \in R_i$ tends to infinity.

Proposition 1. *For A as above, denote by $\tilde{a}_j, \tilde{b}_j \in A$ for $1 \leqslant j \leqslant 4$ the following elements*:

$$\begin{array}{ll} \tilde{a}_1 = c(0,1,1,1), & \tilde{b}_1 = c(1,0,0,0), \\ \tilde{a}_2 = c(1,0,1,1), & \tilde{b}_2 = c(0,1,0,0), \\ \tilde{a}_3 = c(1,1,0,1), & \tilde{b}_3 = c(0,0,1,0), \\ \tilde{a}_4 = c(1,1,1,0), & \tilde{b}_4 = c(0,0,0,1). \end{array} \tag{1}$$

Then, for $u = \sum_{j=1}^{4} \tilde{a}_j \otimes \tilde{b}_j \in A \otimes A$ and $T = \theta(u) \in \mathcal{E}\ell(A)$,

$$\|u\|_{Z,h} > \|T\|_{cb}.$$

Our verification of the proposition will require an analysis of norms of elementary operators similar to T but acting on M_3 and M_4. We will use e_{ij} for the $n \times n$ matrix with 1 in the (i, j) position and zeros elsewhere (the n will be inferred from the context). We also use δ_{ij} for the Kronecker delta symbol.

Example 2. Consider the (elementary) operator $T_n : M_n \to M_n$ given by $T_n x = \sum_{j=1}^{n} (I_n - e_{jj}) x e_{jj} = x - \sum_{j=1}^{n} e_{jj} x e_{jj}$.

Then $\|T_n\| = \|T_n\|_{cb} = 2(n-1)/n$.

Proof. Note that $T_n e_{ij} = (1 - \delta_{ij}) e_{ij}$ and $T_n I_n = 0$.

We can rewrite $T_n x = ((n-1)/n)x - \sum_{j=1}^{n} (e_{jj} - I_n/n) x (e_{jj} - I_n/n) = ((n-1)/n)x - S_n x$ where S_n is a completely positive operator. Hence $\|S_n\|_{cb} = \|S_n\| = \|S_n(I_n)\| = (n-1)/n$ and thus $\|T_n\| \leqslant \|T_n\|_{cb} \leqslant 2(n-1)/n$.

To show that we have equality in both of these inequalities, we introduce the unit vector $\xi = (1, 1, \ldots, 1)/\sqrt{n} \in \mathbb{C}^n$ and the rank one projection operator $\xi^* \otimes \xi$ of $\mathbb{C}^n$ onto the span of ξ. As a matrix, $\xi^* \otimes \xi$ has $1/n$ in each entry. So we can see that $\langle T_n(\xi^* \otimes \xi)\xi, \xi \rangle = (n-1)/n$. Since $2(\xi^* \otimes \xi) - I_n$ is a norm one operator and $\langle T_n(2(\xi^* \otimes \xi) - I_n)\xi, \xi \rangle = 2\langle T_n(\xi^* \otimes \xi)\xi, \xi \rangle = 2(n-1)/n$ we have $\|T_n\| \geqslant 2(n-1)/n$. $\square$

Proof of Proposition 1. We know (from [4, 5.3.12]) that $\|T\|_{cb} = \sup_{\pi \in \hat{A}} \|T^\pi\|_{cb}$ where the supremum is over all irreducible representations π of A and $T^\pi : \mathcal{B}(H_\pi) \to \mathcal{B}(H_\pi)$ is given by

$$T^\pi(y) = \sum_{j=1}^{4} \pi(\tilde{a}_j) y \pi(\tilde{b}_j).$$

When $\pi = \pi_t$ (for any $t \in \mathcal{T}$) we have $T^\pi = T_3$ and so $\|T^\pi\|_{cb} = 4/3$. For $\pi = \lambda_j$ we have $T^\pi = 0$. Hence $\|T\|_{cb} = 4/3$.

We now claim that $\|u\|_{Z,h} = \|T_4\|_{cb} = 3/2$ for $u = \sum_{j=1}^4 \tilde{a}_j \otimes \tilde{b}_j$. By Somerset [19, Theorem 1], $\|u\|_{Z,h} = \sup_G \|u^G\|_h$ where the sup is over all Glimm ideals of A and $u^G \in (A/G) \otimes_h (A/G)$ is $u^G = \sum_{j=1}^4 (\tilde{a}_j + G) \otimes (\tilde{b}_j + G)$. The case $G = G_\infty$ yields A/G as a four-dimensional commutative algebra. We can identify it as the diagonal in $M_4(\mathbb{C})$. Then the elements $\tilde{b}_j + G$ can be taken to correspond to $e_{jj} \in M_4(\mathbb{C})$ and $\tilde{a}_j + G$ to $I_4 - e_{jj}$. By injectivity of the Haagerup norm (see [14, 9.2.5]) we can compute $\|u^G\|_h$ in $M_4 \otimes M_4$ where, by Haagerup's theorem [4, 5.4.7], it gives $\|T_4\|_{cb}$ which equals 3/2.

Thus $\|u\|_{Z,h} \geqslant 3/2$ (and in fact we could easily show equality as all the other Glimm ideals are primitive, being the kernels of the representations π_t for $t \in \mathcal{T}$). Thus we have $\|u\|_{Z,h} \geqslant 3/2 > 4/3 = \|T\|_{cb} = \|\theta_Z(u)\|_{cb}$. $\square$

Remark 3. The proof can be generalised to produce similar examples where all Glimm ideals are n-primal but not all are $(n+1)$-primal. The elementary operator would have length $n+1$ and the algebra A would be replaced by an '$(n+1)$-spoke' algebra constructed from a $\mathcal{T}$ having $n+1$ rays to infinity R_i $(1 \leqslant i \leqslant n+1)$ and matrices $M_n(\mathbb{C})$. There would be $n+1$ multiplicative linear functionals $x \mapsto \lambda_i(x)$ at 'infinity' with $x(t)$ tending to a diagonal using n of the $n+1$ values $\lambda_i(x)$ as $t \to \infty$ in any R_i. One would obtain $u \in A \otimes A$ such that $\|u\|_{Z,h} = 2n/(n+1)$, $\|\theta_Z(u)\|_{cb} = 2(n-1)/n$ and hence $\|u\|_{Z,h}/\|\theta_Z(u)\|_{cb} = n^2/(n^2-1)$.

In Proposition 1, the (minimal) length of the tensor u is 4, the elementary operator $T = \theta_Z(u)$ is self-adjoint $(T^*(x) = T(x^*)^* = T(x))$ and $\|u\|_{Z,h}/\|T\|_{cb} = 9/8$. For the same algebra A, we now exhibit a tensor u with length 2 on which θ_Z fails to be isometric. In this case, the corresponding elementary operator T is not self-adjoint but $\|u\|_{Z,h}/\|T\|_{cb} = \frac{4}{1+\sqrt{5}} > 9/8$.

Example 4. For A the '4-spoke' C^*-algebra introduced above, take $T\colon A \to A$ to be the generalised derivation given by $Tx = ax - xb$ where

$$a = c(0, \imath, 0, -\imath), \qquad b = c(-1, 0, 1, 0)$$

$(\imath = \sqrt{-1})$. Then $\|T\|_{cb} = 1/2 + \sqrt{5/4} < 2 = \|a \otimes 1 - 1 \otimes b\|_{Z,h}$.

Proof. By a result of [20], the norm of a generalised derivation $S : x \mapsto ax - xb$ on $\mathcal{B}(H)$ (any Hilbert space H, any $a, b \in \mathcal{B}(H)$) is

$$\|S\| = \inf_{\lambda \in \mathbb{C}} \left(\|a - \lambda\| + \|b - \lambda\| \right).$$

For $k = 2, 3, \ldots$, the operator $S^{(k)}$ on $M_k(\mathcal{B}(H))$ given by $S^{(k)}((x_{ij})_{i,j=1}^k) = (Sx_{ij})_{i,j=1}^k$ may be regarded as the generalised derivation on $B(H^k)$ defined by the amplifications of a and b. So, by Stampfli's formula again, $\|S^{(k)}\| = \|S\|$. Hence $\|S\|_{cb} = \|S\|$.

As before we compute $\|T\|$ via the representations π_t $(t \in \mathcal{T})$. When $t \in R_4$ we end up with

$$\pi_t(a) = \mathrm{diag}\,(0, \imath, 0) \qquad \pi_t(b) = \mathrm{diag}\,(-1, 0, 1).$$

One can see geometrically that

$$\|\pi_t(a) - (\imath/2)I_3\| + \|\pi_t(b) - (\imath/2)I_3\| = \frac{1}{2} + \left|1 + \frac{\imath}{2}\right| = \frac{1}{2} + \sqrt{\frac{5}{4}}$$

achieves the minimum in the Stampfli formula, but in any case $\|T^\pi\|$ is bounded above by this number for $\pi = \pi_t$ and $t \in R_4$. A similar analysis applies for all R_i $(1 \leqslant i \leqslant 4)$.

In the one-dimensional irreducible representations $\pi = \lambda_j$ we have $\|T^\pi\| = 1$ and so we end up with $\|T\|_{cb} = \sup_{\pi \in \hat{A}} \|T^\pi\|_{cb} \leqslant 1/2 + \sqrt{5/4}$.

Finally, to show that $\|a \otimes 1 - 1 \otimes b\|_{Z,h} = \|a\| + \|b\| = 2$ we concentrate on the quotient A/G_∞. We then must consider (following the pattern of proof in Example 2) the norm (= cb-norm) of the generalised derivation on M_4 given by

$$x \mapsto \mathrm{diag}\,(0, \imath, 0, -\imath)x - x\mathrm{diag}\,(-1, 0, 1, 0).$$

One may verify that the norm is 2 using Stampfli's formula quoted above. □

3. Solution of the isometry problem for θ_Z

Our aim in this section is to show that if a unital C^*-algebra A has a non-primal Glimm ideal then the mapping θ_Z is not an isometry. In order to utilise the computations of Example 2 in a more general setting, we shall need the following lemma:

Lemma 5. *Let b_j $(1 \leqslant j \leqslant n)$ be orthogonal, positive elements of norm one in a C^*-algebra A (that is, $b_j \geqslant 0$, $\|b_j\| = 1$ and $b_j b_k = 0$ for $j \neq k$, $1 \leqslant j, k \leqslant n$) and let X denote their linear span. Let d_j $(1 \leqslant j \leqslant n)$ be orthogonal positive elements of a C^*-algebra B and let Y denote their linear span. Assume $\|d_j\| \leqslant 1$ for $1 \leqslant j \leqslant n$.*

We can define a linear map $\phi: X \to Y$ by $\phi(b_j) = d_j$ and it has the following properties:

(i) $\|\phi\| \leqslant 1$.
(ii) *The map $\phi \otimes \phi: X \otimes_h X \to Y \otimes_h Y$ (with Haagerup tensor norms in each case) has norm at most one.*
(iii) *If $\|d_j\| = 1$ for each j, then $\phi \otimes \phi$ is an isometry between $X \otimes_h X$ and $Y \otimes_h Y$.*

Proof. Consider the commutative C^*-algebra generated by the b_j $(1 \leqslant j \leqslant n)$. It is isomorphic to an algebra of continuous functions $C_0(K_X)$ on some locally compact Hausdorff space K_X where the b_j must be positive functions that are non-zero on disjoint open sets. It is clear then that the norm of a linear combination $\sum_{j=1}^n \alpha_j b_j$ is $\max_j |\alpha_j| \|b_j\| = \max_j |\alpha_j|$. (In particular the b_j are linearly independent and ϕ is well-defined.) For similar reasons, we may view $Y \subseteq C_0(K_Y)$ for a locally compact Hausdorff space K_Y, and

$$\left\| \sum_{j=1}^n \alpha_j d_j \right\| = \max_j |\alpha_j| \|d_j\| \leqslant \max_j |\alpha_j|.$$

This shows $\|\phi\| \leqslant 1$.

For the second part, note that when we compute the Haagerup tensor norm of $u = \sum_{i=1}^N a_i \otimes c_i \in X \otimes X$, we consider an infimum of expressions

$$\frac{1}{2}\left(\left\| \sum_{i=1}^N a_i a_i^* \right\| + \left\| \sum_{i=1}^N c_i^* c_i \right\| \right)$$

over all representations of u and we can find a representation where this infimum is attained (without going outside representations in $X \otimes X$). We can compute that applying $\phi \otimes \phi$ to this same representation produces a representation of $(\phi \otimes \phi)(u) \in Y \otimes Y$ where the corresponding expression is reduced. For example if we write $a_i = \sum_{j=1}^n a_{ij} b_j \in X$ then, for $\kappa \in K_Y$,

$$\begin{aligned} \left(\sum_{i=1}^N \phi(a_i)\phi(a_i)^* \right)(\kappa) &= \sum_{i=1}^N \left(\sum_{j=1}^n a_{ij} d_j(\kappa) \right) \left(\sum_{j=1}^n \overline{a}_{ij} d_j^*(\kappa) \right) \\ &= \sum_{j=1}^n \sum_{i=1}^N |a_{ij}|^2 |d_j(\kappa)|^2 \end{aligned}$$

because $d_j(\kappa)$ is non-zero for at most one j. The supremum of this latter sum over $\kappa \in K_Y$ is at most

$$\max_j \sum_{i=1}^N |a_{ij}|^2 = \left\| \sum_{i=1}^N a_i a_i^* \right\|.$$

Thus $\|\phi \otimes \phi\| \leqslant 1$.

For the third part, we can apply the second part to the inverse map of ϕ if $\|d_j\| = 1$ for all j. $\square$

Lemma 6. *Let b_j $(1 \leqslant j \leqslant n)$ be positive elements of a C^*-algebra with $b_j b_k = 0$ for $j \neq k$ $(1 \leqslant j, k \leqslant n)$. Let*

$$u = \left(\sum_{j=1}^{n} b_j\right) \otimes \left(\sum_{j=1}^{n} b_j\right) - \sum_{j=1}^{n} b_j \otimes b_j.$$

(i) *If $\|b_j\| \leqslant 1$ for $1 \leqslant j \leqslant n$, then $\|u\|_h \leqslant 2(n-1)/n$.*
(ii) *If $\|b_j\| = 1$ for $1 \leqslant j \leqslant n$, then $\|u\|_h = 2(n-1)/n$.*

Proof. We can deduce this from Lemma 5 and Example 2. We identify $\mathbb{C}^n$ with the diagonals in M_n and consider

$$\phi: \mathbb{C}^n \to Y = \text{span}\{b_j : 1 \leqslant j \leqslant n\}$$

given by $\phi(e_{jj}) = b_j$. Using injectivity of the Haagerup norm, Haagerup's theorem and Example 2, we have

$$\left\| I_n \otimes I_n - \sum_{j=1}^{n} e_{jj} \otimes e_{jj} \right\|_h = \|T_n\|_{cb} = 2(n-1)/n.$$

But the tensor in the left-hand side maps to u under the mapping $\phi \otimes \phi$ of Lemma 5. □

Theorem 7. *Let A be a unital C^*-algebra containing a Glimm ideal G that is not n-primal for some $n \geqslant 2$. Then there exists $u = \sum_{j=1}^{n} a_j \otimes b_j \in A \otimes A$ with*

$$\|u\|_{Z,h} > \|\theta_Z(u)\|_{cb}.$$

Proof. By reducing n if necessary, we may assume that G is $(n-1)$-primal but not n-primal (where we adopt the convention that all closed two-sided ideals in A are 1-primal).

There must exist n primitive ideals P_j of A $(1 \leqslant j \leqslant n)$ with $G \subseteq P_j$ for all j but $J := \bigcap_{j=1}^{n} P_j$ not primal. However

$$R_j := \bigcap_{1 \leqslant k \leqslant n, k \neq j} P_k$$

is primal for each $1 \leqslant j \leqslant n$. Note that since $R_1, R_2, \ldots, R_n$ are primal but J is not, it follows that $P_j \not\supseteq P_k$ for $j \neq k$.

There must exist open neighbourhoods U_j of P_j in $\mathrm{Prim}(A)$ $(1 \leqslant j \leqslant n)$ so that

$$\bigcap_{j=1}^{n} U_j = \emptyset.$$

For, if no such neighbourhoods existed there would be a net $(Q_\alpha)_\alpha$ in $\mathrm{Prim}(A)$ converging to each of the P_j $(1 \leqslant j \leqslant n)$ and hence to every primitive ideal containing J, contradicting the non-primality of J [6, Proposition 3.2]. Now there are closed two-sided ideals J_j in A so that $U_j = \mathrm{Prim}(J_j)$ (hence $U_j = \{Q \in \mathrm{Prim}(A) : J_j \nsubseteq Q\}$).

Let $I_j = J_j R_j$ for $1 \leqslant j \leqslant n$. The ideal I_j cannot be contained in J because then we would have $J_j R_j \subseteq P_j$ and since the primitive ideal P_j is necessarily prime, it would follow that $J_j \subseteq P_j$ or $R_j \subseteq P_j$. Since $P_j \in U_j$, we have $J_j \nsubseteq P_j$. By primeness of P_j, if $R_j \subseteq P_j$, then $P_k \subseteq P_j$ for some $k \neq j$ (again not so).

Let $\Psi: A \to A/J$ denote the quotient map. Let $K_j = \Psi(I_j)$, a non-zero closed ideal of A/J. Note that $K_j K_k = 0$ for $j \neq k$ (as $R_j R_k \subseteq J$).

For $1 \leqslant j \leqslant n$, choose a positive element $d_j \in K_j$ of norm one and $g_j \in I_j$ positive of norm one with $\Psi(g_j) = d_j^{1/3}$. Since $d_j^{1/3} d_k^{1/3} = 0$ for $j \neq k$, we can use [1, Proposition 2.6] to find $c_j \in A$ $(1 \leqslant j \leqslant n)$ with $\Psi(c_j) = \Psi(g_j) = d_j^{1/3}$ and $c_j c_k = 0$ for $j \neq k$. Let $b'_j = c_j d_j c_j \in I_j^+$. Then $b'_j b'_k = 0$ for $j \neq k$ and $\Psi(b'_j) = d_j$ $(1 \leqslant j, k \leqslant n)$.

Let $f: [0, \infty) \to [0, \infty)$ be $f(t) = \min(t, 1)$, a uniform limit on any compact subset of $[0, \infty)$ of polynomials without constant term. Define $b_j = f(b'_j)$ by functional calculus. Then we have $b_j \in I_j^+$, $\Psi(b_j) = d_j$, $\|b_j\| = 1$ and $b_j b_k = 0$ for $j \neq k$.

Consider now

$$u = \left(\sum_{j=1}^{n} b_j\right) \otimes \left(\sum_{j=1}^{n} b_j\right) - \sum_{j=1}^{n} b_j \otimes b_j = \sum_{j=1}^{n} \left(\left(\sum_{k=1}^{n} b_k\right) - b_j\right) \otimes b_j$$

as in Lemma 6. Note that the canonical quotient map from A/G to A/J induces a contraction from $A/G \otimes_h A/G$ to $A/J \otimes_h A/J$. From [19, Theorem 1]), we have $\|u\|_{Z,h} \geqslant \|u^G\|_h$. Applying Lemma 6 to A/J, we deduce

$$\|u\|_{Z,h} \geqslant \|u^G\|_h \geqslant \|u^J\|_h = 2(n-1)/n.$$

On the other hand, by [4, 5.3.12, 5.4.10]

$$\|\theta_Z(u)\|_{cb} = \sup\{\|u^P\|_h : P \in \mathrm{Prim}(A)\}.$$

Let $P \in \mathrm{Prim}(A)$. There exists $j \in \{1, \ldots, n\}$ such that $P \notin U_j$ and hence $b_j \in I_j \subseteq J_j \subseteq P$. Applying Lemma 6 again (this time to A/P with at most $n-1$ non-zero

$b_k + P$), we have

$$\|u^P\| \leqslant 2(n-2)/(n-1).$$

Thus

$$\|\theta_Z(u)\|_{cb} \leqslant 2(n-2)/(n-1) < 2(n-1)/n = \|u\|_{Z,h}. \qquad \square$$

Combining Theorem 7 with [19, Theorem 4], we obtain the following result:

Theorem 8. *Let A be a unital C^*-algebra. The mapping $\theta_Z : A \otimes_{Z,h} A \to CB(A)$ is an isometry if and only if every Glimm ideal of A is primal.*

If A is a non-unital C^*-algebra then it is customary to consider the multiplier algebra $M(A)$. If Z now denotes the centre of $M(A)$, then we have the natural contraction $\theta_Z: M(A) \otimes_{Z,h} M(A) \to CB(M(A))$ for the unital C^*-algebra $M(A)$. But, since A is an ideal in $M(A)$, we obtain a contraction $\Theta_Z: M(A) \otimes_{Z,h} M(A) \to CB(A)$ by defining $\Theta_Z(u) = \theta_Z(u)|_A$ (see [4, 5.4.17]).

Corollary 9. *Let A be a non-unital C^*-algebra. The map $\Theta_Z: M(A) \otimes_{Z,h} M(A) \to CB(A)$ is an isometry if and only if every Glimm ideal of $M(A)$ is primal.*

Proof. Let $u \in M(A) \otimes M(A)$. By taking a faithful non-degenerate representation of A on a Hilbert space H, we may assume the inclusions $A \subseteq M(A) \subseteq A'' \subseteq B(H)$. By tensoring with $M_n(\mathbb{C})$ and using Kaplansky's density theorem, one obtains that $\|\theta_Z(u)\|_{cb} = \|\Theta_Z(u)\|_{cb}$. The result now follows from Theorem 8. $\square$

We can state a necessary condition for Θ_Z to be an isometry in terms of Glimm ideals of A, something that involves an extension of the notion of Glimm ideal to the non-unital case. In a (not necessarily unital) C^*-algebra A, a *Glimm ideal* is the kernel of an equivalence class in $\mathrm{Prim}(A)$, where primitive ideals P and Q are defined to be equivalent if $f(P) = f(Q)$ for all $f \in C^b(\mathrm{Prim}(A))$ [8,12]. By the Dauns–Hofmann theorem, this definition is consistent with the one already given in the unital case.

Lemma 10. *Let A be a* (*non-unital*) *C^*-algebra containing a Glimm ideal G that is not n-primal* (*some $n \geqslant 2$*). *Then $M(A)$ also contains a Glimm ideal that is not n-primal.*

Proof. In this proof, we elaborate an argument in [4, p. 88] and use different notation.

By the Dauns–Hofmann theorem, there is an isomorphism Φ of the algebra C^b $(\mathrm{Prim}(A))$ onto the centre $Z(M(A))$ of $M(A)$ such that for $f \in C^b(\mathrm{Prim}(A))$, $a \in A$ and $P \in \mathrm{Prim}(A)$,

$$(\Phi(f)a) + P = f(P)(a + P)$$

in A/P. Temporarily fix $P \in \mathrm{Prim}(A)$ with $P \supseteq G$ and define a multiplicative linear functional ϕ on $C^b(\mathrm{Prim}(A))$ by $\phi(f) = f(P)$. Clearly ϕ is independent of the choice of $P \supseteq G$. Let $J = \ker(\phi \circ \Phi^{-1})$, a maximal ideal of $Z(M(A))$, and let $H = M(A)J$, a Glimm ideal of $M(A)$.

We have $H \cap A = M(A)JA = AJ$. Let $a \in A$, $z \in J$ and let Q be any primitive ideal of A containing G. In A/Q we have

$$za + Q = (\Phi^{-1}(z))(Q)(a + Q) = \phi(\Phi^{-1}(z))(a + Q) = 0.$$

Hence $AJ \subseteq G$. (In fact $AJ = G$, but we will not need that.)

Suppose that H is n-primal. For any closed ideals $I_1, I_2, \ldots, I_n \subseteq A$, with product $I_1 I_2 \cdots I_n = \{0\}$ we must have $I_i \subseteq H$ (for some $1 \leqslant i \leqslant n$) and so $I_i \subseteq H \cap A = AJ \subseteq G$. Thus G is n-primal, a contradiction showing that H cannot be n-primal. $\square$

From Lemma 10 and Corollary 9, we can make the following assertion:

Corollary 11. *Let A be a (non-unital) C^*-algebra. If the map $\Theta_Z: M(A) \otimes_{Z,h} M(A) \to CB(A)$ is an isometry then every Glimm ideal of A is primal.*

For an odd integer $n \geqslant 3$, let W_n be the simply connected, 2-step nilpotent, Lie group considered in [7] and let $A = C^*(W_n)$. Then A has a Glimm ideal which is not $(n+1)$-primal [7, Theorem 2.7] and so, by Corollary 11, Θ_Z is not an isometry in this case.

The next example, together with Corollary 9, shows that the necessary condition in Corollary 11 is not sufficient for Θ_Z to be an isometry.

Example 12. There is a C^*-algebra A with compact, Hausdorff, primitive ideal space (and hence with every Glimm ideal primal) such that $M(A)$ has a Glimm ideal which is not 2-primal.

Proof. Let X be a non-compact, locally compact Hausdorff space such that the Stone–Čech remainder $\beta X \setminus X$ has at least two distinct points y and z (e.g. we could take $X = \mathbb{N}$ or $X = \mathbb{R}$). Let B be the C^*-algebra $C(\beta X, M_2(\mathbb{C}))$, and let B_1 be the C^*-subalgebra consisting of those functions $f \in B$ for which there exist complex numbers $\lambda_1(f)$, $\lambda_2(f)$, $\lambda_3(f)$ such that $f(y) = \mathrm{diag}(\lambda_1(f), \lambda_2(f))$ and $f(z) = \mathrm{diag}(\lambda_2(f), \lambda_3(f))$.

Let $\pi_x: B_1 \to M_2$ denote the representation $\pi_x(f) = f(x)$. Then

$$\mathrm{Prim}(B_1) = \{\ker \pi_x : x \in \beta X \setminus \{y, z\}\} \cup \{\ker \lambda_1, \ker \lambda_2, \ker \lambda_3\}$$

and, for $G = \ker \lambda_1 \cap \ker \lambda_2 \cap \ker \lambda_3$,

$$\mathrm{Glimm}(B_1) = \{\ker \pi_x : x \in \beta X \setminus \{y, z\}\} \cup \{G\}.$$

The Glimm ideal G is not 2-primal. To see this, let U and V be disjoint neighbourhoods of y and z, respectively, in βX. Let K_U be the closed ideal of B_1 consisting of those functions vanishing off U, and similarly let K_V consist of those $f \in B_1$ vanishing off V, also a closed ideal of B_1. Then $K_U, K_V \nsubseteq G$, but $K_U K_V = \{0\}$.

Let $A = \{f \in B_1 : \lambda_1(f) = \lambda_3(f) = 0\}$, a closed ideal in B_1. We have

$$\mathrm{Prim}(A) = \{\ker \pi_x|_A : x \in \beta X \setminus \{y, z\}\} \cup \{\ker (\lambda_2|_A)\}.$$

Furthermore, $\mathrm{Prim}(A)$ is homeomorphic to the compact Hausdorff space obtained from βX by identifying the points y and z. In particular, therefore, every Glimm ideal of A is primitive and hence primal.

Now let $J = C_0(X, M_2(\mathbb{C}))$. Then $M(J) = C^b(X, M_2(\mathbb{C}))$ by Akemann et al. [2, Corollary 3.4]. Note that the restriction map $f \mapsto f|_X$ is a *-isomorphism between $B = C(\beta X, M_2(\mathbb{C}))$ and $C^b(X, M_2(\mathbb{C}))$. Since J is an essential ideal in A, it is also an essential ideal in $M(A)$ and so we now have $J \subseteq A \subseteq M(A) \subseteq M(J) = B$. Elementary computations show that $M(A) = B_1$. $\square$

4. Length specific results

If every Glimm ideal of a unital C^*-algebra A is 2-primal (so that θ_Z is injective) but not every Glimm ideal is primal, then one may look for a relationship between the degree of primality of the Glimm ideals of A and the length of the shortest tensors $u \in A \otimes A$ on which θ_Z fails to be isometric. We begin by considering the question of whether n-primality of all the Glimm ideals of A is sufficient for θ_Z to be isometric on tensors $u = \sum_{j=1}^{\ell} a_j \otimes b_j \in A \otimes A$, where n and ℓ are related in some way.

We will use results from [21] in the sequel in order to be able to calculate Haagerup norms. By injectivity of the Haagerup norm, we can always make our computation in $\mathcal{B}(H)$ for some H and in this setting we have equality of the Haagerup norm of a tensor $u = \sum_{j=1}^{\ell} a_j \otimes b_j$ and the cb-norm of the elementary operator $T = \theta(u)$ on $\mathcal{B}(H)$ [4, 5.4.9]. The difficulty addressed by [21] is to be able to recognise when a tensor u is represented in an optimal way, meaning a way that gives equality in the infimum

$$\|u\|_h = \inf \frac{1}{2}\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right),$$

where we now adopt the shorthand $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^t$ for the (column) ℓ-tuple of the b_j's and $\mathbf{a} = [a_1, a_2, \ldots, a_\ell]$ for the (row) ℓ-tuple of the a_j's. Recall that $\|\mathbf{a}\|^2 = \left\|\sum_{j=1}^{\ell} a_j a_j^*\right\|$ while $\|\mathbf{b}\|^2 = \left\|\sum_{j=1}^{\ell} b_j^* b_j\right\|$. The infimum for $\|u\|_h$ can also be written using the geometric mean version $\|u\|_h = \inf \|\mathbf{a}\|\|\mathbf{b}\|$ but there is no loss in restricting to representations $u = \sum_{j=1}^{\ell} a_j \otimes b_j$ where $\|\mathbf{a}\| = \|\mathbf{b}\|$ and so the geometric and arithmetic means of $\|\mathbf{a}\|^2$ and $\|\mathbf{b}\|^2$ agree.

The results from [21] use numerical range ideas to characterise the situation where we have equality in

$$\|\theta(u)\| \leqslant \|\theta(u)\|_{cb} \leqslant \frac{1}{2}\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right) \tag{2}$$

and then an extension of this characterisation to amplifications $\theta(u)^{(k)}$ of $\theta(u)$ in order to deal with the equality in the second inequality only.

From [21] we use the notation $W_m(\mathbf{b})$ for the matrix numerical range

$$W_m(\mathbf{b}) = \left\{(\langle b_j^* b_i \xi, \xi\rangle)_{i,j=1}^{\ell} = (\langle b_i \xi, b_j \xi\rangle)_{i,j=1}^{\ell} : \xi \in H, \|\xi\| = 1\right\}$$

associated with a column $\mathbf{b}$. This subset of M_ℓ^+ (the positive semidefinite $\ell \times \ell$ matrices) is in fact the joint spatial numerical range of the ℓ^2 operators $b_j^* b_i$ but it is convenient to consider it as a set of matrices. It is easy to see that each matrix in $W_m(\mathbf{b})$ has trace at most $\|\mathbf{b}\|^2$ and that this is the supremum of the traces. The 'extremal matrix numerical range' $W_{m,e}(\mathbf{b})$ is defined as the subset of the closure of $W_m(\mathbf{b})$ consisting of those matrices with trace equal to $\|\mathbf{b}\|^2$. (In case H is finite dimensional, $W_m(\mathbf{b})$ is already closed and the extremal matrix numerical range corresponds to restricting $\xi \in H$ to be in the eigenspace for the maximum eigenvalue of $\sum_j b_j^* b_j$.) The criterion in [21, Proposition 3.1] for equality in (2) is

$$W_{m,e}(\mathbf{a}^*) \cap W_{m,e}(\mathbf{b}) \neq \emptyset$$

(where $\mathbf{a}^* = [a_1^*, a_2^*, \ldots, a_\ell^*]^t$ is a column).

Let $\operatorname{co}(S)$ denote the convex hull of a set S. Equality in the second inequality of (2) occurs if and only if

$$\operatorname{co}(W_{m,e}(\mathbf{a}^*)) \cap \operatorname{co}(W_{m,e}(\mathbf{b})) \neq \emptyset \tag{3}$$

by Timoney [21, Theorem 3.3]. Given $u \in \mathcal{B}(H) \otimes \mathcal{B}(H)$ of length ℓ, it can be written as $u = \sum_{j=1}^{\ell} a_j \otimes b_j$ so as to get

$$\|u\|_h = \|\mathbf{a}\|^2 = \|\mathbf{b}\|^2 = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}{2} \tag{4}$$

(see [14, Proposition 9.2.6]) with the same ℓ. Via Haagerup's theorem $\|u\|_h = \|\theta(u)\|_{cb}$, we see that (3) and (4) are equivalent for $u \in \mathcal{B}(H) \otimes \mathcal{B}(H)$. We will use this equivalence several times to detect when representations of such u satisfy (4).

Lemma 13. *Consider a Hilbert space H which is a* (*Hilbert space*) *direct sum of Hilbert spaces H_i* ($i \in \mathcal{I} =$ *some index set*). *Let $a_{j,i}, b_{j,i} \in B(H_i)$ for each $i \in \mathcal{I}$*

with $\sup_i \|a_{j,i}\| < \infty$ *and* $\sup_i \|b_{j,i}\| < \infty$ *for* $1 \leqslant j \leqslant \ell$. *Consider* $a_j = (a_{j,i})_{i\in\mathcal{I}}$ *as a 'block diagonal' element in* $\mathcal{B}(H)$, $b_j = (b_{j,i})_{i\in\mathcal{I}}$ *similarly and* $u = \sum_{j=1}^{\ell} a_j \otimes b_j \in \mathcal{B}(H) \otimes \mathcal{B}(H)$.

For a subset $F \subseteq \mathcal{I}$, *let* H_F *be the direct sum of those* H_i *for* $i \in F$ *and let* $a_{j,F} = (a_{j,i})_{i\in F} \in \mathcal{B}(H_F)$, $b_{j,F}$ *similarly defined and* $u_F = \sum_{j=1}^{\ell} a_{j,F} \otimes b_{j,F} \in \mathcal{B}(H_F) \otimes \mathcal{B}(H_F)$.

Then

$$\|u\|_h = \sup\{\|u_F\|_h : F \subseteq \mathcal{I}, F \quad \textit{has at most } \ell^2 + 1 \textit{ elements}\}.$$

Proof. As remarked above, we know that $\|u\|_h = \|\theta(u)\|_{cb}$ for $\theta(u) \in \mathcal{E}\ell(\mathcal{B}(H))$ and similarly for $\|u_F\|_h$.

Let (P_μ) be an increasing net of projections converging in the strong operator topology to the identity operator on H. Since, for the strong operator topology, multiplication is jointly continuous on norm-bounded sets, we have

$$\|\theta(u)\| = \lim_{\mu} \|\theta(u_\mu)\|,$$

where

$$u_\mu = \sum_{j=1}^{n} (P_\mu a_j P_\mu) \otimes (P_\mu b_j P_\mu).$$

Furthermore, for each $k \geqslant 2$, the k-fold amplification of P_μ converges strongly to the identity on H^k and so

$$\|\theta(u)\|_{cb} = \lim_{\mu} \|\theta(u_\mu)\|_{cb}.$$

We may therefore assume that $\mathcal{I}$ is finite.

We assume next that u is written so as to get equality in the Haagerup norm infimum $\|u\|_h = (\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)/2$, hence (3) holds. Since we are in the case where $\mathcal{I}$ is finite,

$$\|\mathbf{a}\|^2 = \max_{i\in\mathcal{I}} \|\mathbf{a}_{\{i\}}\|^2 = \max_{i\in\mathcal{I}} \left\| \sum_{j=1}^{\ell} a_{j,i}(a_{j,i})^* \right\|, \tag{5}$$

where now $\mathbf{a}_{\{i\}} = [a_{1,i}, a_{2,i}, \ldots, a_{\ell,i}]$ relates to the summand i.

A unit vector $\xi \in H = \bigoplus_i H_i$ gives an element of $W_m(\mathbf{a}^*)$ which is a convex combination of elements of $W_m(\mathbf{a}^*_{\{i\}})$ ($i \in \mathcal{I}$). Hence, since closed bounded subsets of M_ℓ are compact and $\mathcal{I}$ is finite,

$$\mathrm{co}\,(\overline{W_m(\mathbf{a}^*)}) = \mathrm{co}\left(\overline{\mathrm{co} \bigcup_{i\in\mathcal{I}} W_m(\mathbf{a}^*_{\{i\}})}\right) = \mathrm{co}\left(\bigcup_{i\in\mathcal{I}} \overline{W_m(\mathbf{a}^*_{\{i\}})}\right).$$

To get elements of the extremal matrix numerical range $W_{m,e}(\mathbf{a}^*)$, we must only use those $i \in \mathcal{I}$ where the maximum in (5) is attained and matrices from $\mathrm{co}\,(W_{m,e}(\mathbf{a}_{\{i\}}))$ in the convex combination. Thus, if $\mathcal{I}_{\mathbf{a}}$ denotes the subset of $i \in \mathcal{I}$ where the maximum in (5) is attained, we have

$$\mathrm{co}\,(W_{m,e}(\mathbf{a}^*)) = \mathrm{co}\left(\bigcup_{i \in \mathcal{I}_{\mathbf{a}}} W_{m,e}(\mathbf{a}^*_{\{i\}})\right). \tag{6}$$

Applying the same argument to $\mathbf{b}$ as applied above to $\mathbf{a}^*$, we obtain a (possibly different) $\mathcal{I}_{\mathbf{b}} \subseteq \mathcal{I}$ so that

$$\mathrm{co}\,(W_{m,e}(\mathbf{b})) = \mathrm{co}\left(\bigcup_{i \in \mathcal{I}_{\mathbf{b}}} W_{m,e}(\mathbf{b}_{\{i\}})\right). \tag{7}$$

We claim that there are non-empty subsets $F_{\mathbf{a}} \subseteq \mathcal{I}_{\mathbf{a}}$ and $F_{\mathbf{b}} \subseteq \mathcal{I}_{\mathbf{b}}$ such that $|F_{\mathbf{a}}| + |F_{\mathbf{b}}| \leqslant (\ell^2 - 1) + 2 = \ell^2 + 1$ and

$$\left(\mathrm{co}\left(\bigcup_{i \in F_{\mathbf{a}}} W_{m,e}(\mathbf{a}^*_{\{i\}})\right)\right) \cap \left(\mathrm{co}\left(\bigcup_{i \in F_{\mathbf{b}}} W_{m,e}(\mathbf{b}_{\{i\}})\right)\right) \neq \emptyset. \tag{8}$$

To see this, note that all the matrices we are considering (in the extremal matrix numerical ranges) are hermitian $\ell \times \ell$ matrices with the same trace $\|\mathbf{a}\|^2 = \|\mathbf{b}\|^2$ and hence they lie in an affine space of real dimension $\ell^2 - 1$ (or affine dimension ℓ^2). By Carathéodory's theorem, any element in the convex hull of a subset S of $\mathbb{R}^n$ can be represented as a convex combination of $n+1$ or fewer elements of S. A slightly less well-known fact is that if the convex hulls of two non-empty sets $S_1, S_2 \subset \mathbb{R}^n$ (or an affine space equivalent to it) intersect, then we can find a convex combination of n_1 elements in S_1 to equal a convex combination of n_2 elements of S_2, where $n_1, n_2 \geqslant 1$ and $n_1 + n_2 \leqslant n + 2$. This follows by applying Carathéodory's theorem to the origin, which belongs to the convex hull of

$$\{(x, 1) : x \in S_1\} \cup \{(-y, -1) : y \in S_2\} \subset \mathbb{R}^{n+1}.$$

We can apply this fact because we have (3) valid, and therefore the subsets $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ exist as claimed.

Let α be in intersection (8) and let $F = F_{\mathbf{a}} \cup F_{\mathbf{b}}$. Let

$$\mathbf{a}_F = [a_{1,F}, a_{2,F}, \ldots, a_{\ell,F}] \text{ and } \mathbf{b}_F = [b_{1,F}, b_{2,F}, \ldots, b_{\ell,F}]^t.$$

Applying (6) and (7) to $\mathbf{a}^*_F$ and $\mathbf{b}_F$, respectively, and noting that $F \cap \mathcal{I}_{\mathbf{a}} \supseteq F_{\mathbf{a}}$ and $F \cap \mathcal{I}_{\mathbf{b}} \supseteq F_{\mathbf{b}}$, we obtain $\|\mathbf{a}_F\| = \|\mathbf{a}\|$, $\|\mathbf{b}_F\| = \|\mathbf{b}\|$ and that

$$\alpha \in \mathrm{co}\,(W_{m,e}(\mathbf{a}^*_F)) \cap \mathrm{co}\,(W_{m,e}(\mathbf{b}_F)).$$

Hence, by criterion (3) we have $\|u_F\|_h = \|\mathbf{a}_F\|^2 = \|\mathbf{b}_F\|^2 = \|\mathbf{a}\|^2 = \|\mathbf{b}\|^2 = \|u\|_h$. Since F has at most ℓ^2+1 elements, the result now follows. □

Proposition 14. *Let A be a unital C^*-algebra and ℓ a positive integer. Suppose that every Glimm ideal in A is (ℓ^2+1)-primal. Let $u = \sum_{j=1}^{\ell} a_j \otimes b_j \in A \otimes A$. Then*

$$\|\theta_Z(u)\|_{cb} = \|u\|_{Z,h}$$

Proof. From [19, Theorem 1 and Proposition 3], we know that

$$\|u\|_{Z,h} = \sup\{\|u^G\|_h : G \in \mathrm{Glimm}(A)\}$$

while

$$\|\theta_Z(u)\|_{cb} = \sup\{\|u^J\|_h : J \text{ minimal primal in } A\}.$$

Let $G \in \mathrm{Glimm}(A)$ and consider $u^G \in (A/G) \otimes_h (A/G)$. In order to compute $\|u^G\|_h$ we embed A/G faithfully as an algebra of operators, and use injectivity of the Haagerup norm [14, Proposition 9.2.5]. We take as our faithful representation the reduced atomic representation

$$\sigma_r : A/G \hookrightarrow \prod_{\pi} \mathcal{B}(H_\pi) \subset \mathcal{B}\left(\bigoplus_{\pi} H_\pi\right)$$

(one irreducible representation π from each equivalence class in $\widehat{A/G}$).

Let $\varepsilon > 0$. By Lemma 13, there exist inequivalent irreducible representations $\pi_1, \ldots, \pi_n$ of A/G such that $n \leqslant \ell^2+1$ and

$$\|u^G\|_h - \varepsilon < \|((\sigma_r \otimes \sigma_r)(u^G))_F\|_h = \|(\sigma \otimes \sigma)(u^G)\|_h,$$

where $H_F = H_{\pi_1} \oplus \cdots \oplus H_{\pi_n}$ and $\sigma = \pi_1 \oplus \cdots \oplus \pi_n$. Let $P_i = \ker \pi_i$ for $1 \leqslant i \leqslant n$ and let $I = \bigcap_{i=1}^n P_i$. By hypothesis, I is a primal ideal of A.

Since σ induces a faithful representation of A/I (given by $a + I \to \sigma(a)$ for $a \in A$), we have $\|(\sigma \otimes \sigma)(u^G)\|_h = \|u^I\|_h$ by injectivity of the Haagerup norm. Now let J be a minimal primal ideal of A contained in I. We have

$$\|u^G\|_h - \varepsilon < \|u^I\|_h \leqslant \|u^J\|_h \leqslant \|\theta_Z(u)\|_{cb}.$$

Since ε and G were arbitrary, $\|u\|_{Z,h} \leqslant \|\theta_Z(u)\|_{cb}$. As θ_Z is a contraction, the result follows. □

Our aim now is to show that the converse of Proposition 14 holds, and for that we need some preparation.

Lemma 15. *Given a positive definite $n \times n$ matrix α of trace 1, there exist n^2 affinely independent rank one (self-adjoint) projections $\rho_i \in M_n$ $(1 \leqslant i \leqslant n^2)$ so that*

$$\alpha = \sum_{i=1}^{n^2} t_i \rho_i$$

is a convex combination of the ρ_i with $t_i > 0$ for each i (and $\sum_{i=1}^{n^2} t_i = 1$).

Proof. Note that positive semidefinite trace 1 matrices $\rho \in M_n$ correspond to states of M_n via $x \mapsto \operatorname{trace}(x\rho)$ and the rank one projections correspond to the pure states. We argue by induction on n. Of course the $n = 1$ case is obvious and so we consider $n > 1$.

Recall that we can write any rank one projection ρ in M_n as $\rho = \xi^* \otimes \xi$ for a unit vector ξ in the range of ρ. We can assume the given matrix α is diagonal with (positive) diagonal entries $\alpha_{11} \geqslant \alpha_{22} \geqslant \cdots \geqslant \alpha_{nn} > 0$ in descending order (by replacing the original α by $u^*\alpha u$ for some suitable unitary $u \in M_n$ and applying $u(\cdot)u^*$ to the rank one projections we find). Since $n > 1$, $\alpha_{11} < 1$. Choose $\delta > 0$ so that $\delta < \alpha_{nn}/\alpha_{11} \leqslant 1$ and $\alpha_{11}(1 + (n-1)\delta^2) < 1$. Let ζ be a primitive mth root of unity with $m = 2n - 1$. Let

$$\xi_i = (1, \delta\zeta^i, \delta\zeta^{2i}, \ldots, \delta\zeta^{(n-1)i})/\sqrt{1 + (n-1)\delta^2} \quad (1 \leqslant i \leqslant m)$$

and observe that

$$\alpha = \sum_{i=1}^{m} \frac{\alpha_{11}\sqrt{1 + (n-1)\delta^2}}{m} (\xi_i^* \otimes \xi_i) + \left(1 - \alpha_{11}\sqrt{1 + (n-1)\delta^2}\right)\alpha',$$

where α' is essentially a positive definite diagonal matrix of trace 1 in M_{n-1}. Strictly speaking, α' is in M_n and has 0 in the $(1, 1)$ entry, but we are able to apply the inductive hypothesis to it. We end up with α as a convex combination of a total of $m + (n-1)^2 = n^2$ rank one projections.

Working with the first row and column (and using a Vandermonde determinant argument), we can check that the projections $\xi_i^* \otimes \xi_i$ are affinely independent among themselves and also when we add in the $(n-1)^2$ projections we get from the inductive step. □

There is a simpler argument which does not quite prove the preceding lemma. The affine dimension of the state space is n^2 and so it is possible to find n^2 affinely independent rank one projections. One can argue that the average β of such a collection of projections has to be positive definite. If not, there is a unit vector $\xi \in \mathbb{C}^n$ with $\langle \beta\xi, \xi \rangle = 0$ and then each of the projections p would necessarily satisfy $\langle p\xi, \xi \rangle = 0$. That is the projections would be restricted to lie in an affine space of dimension strictly less than n^2 (in fact in a face of the state space). So β has to be non-singular.

For us, it is more convenient to be able to express any pre-assigned, positive definite matrix α with $\mathrm{trace}(\alpha) = 1$ as a convex combination of n^2 rank one projections (though we could actually manage with a non-specific β). A variant of the inductive argument above is needed in the next lemma.

Lemma 16. *For $\ell \geqslant 2$ and $(\ell - 1)^2 + 2 \leqslant N \leqslant \ell^2 + 1$ there exists $u = \sum_{j=1}^{\ell} a_j \otimes b_j \in \mathbb{C}^N \otimes \mathbb{C}^N$ such that $\|u\|_h = 1$ (where $\mathbb{C}^N$ is considered as the commutative C^*-algebra of functions on a discrete space with N points) and such that for any non-empty subset $F \subset \{1, 2, \ldots, N\}$ of $N - 1$ points or fewer,*

$$\|u_F\|_h < 1$$

where $u_F = \sum_{j=1}^{\ell} a_{j,F} \otimes b_{j,F}$ and $a_{j,F}, b_{j,F}$ are the restrictions of a_j, b_j to F.

Proof. We will adopt a similar notation to that in Lemma 13 and take $\mathcal{I} = \{1, 2, \ldots, N\}$, $H_i = \mathbb{C}$ (each $i \in \mathcal{I}$) and $H = \bigoplus_{i \in \mathcal{I}} H_i$. Our a_j will be diagonal elements of $\mathcal{B}(H)$ with diagonal entries $(a_{j,i})_{i \in \mathcal{I}}$ and similarly $b_j = (b_{j,i})_{i \in \mathcal{I}}$ (for scalars $a_{j,i}, b_{j,i} \in \mathbb{C}$). We abbreviate $\mathbf{a} = [a_1, a_2, \ldots, a_\ell]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^t$.

Let $m = 2(\ell - 1)$ and $n = N - ((\ell - 1)^2 + 2)$. Our $a_{j,i}$ will be zero for $m < i \leqslant N$ and $b_{j,i}$ will be zero for $1 \leqslant i \leqslant n$. As $0 \leqslant n \leqslant m < N$, we shall be able to arrange that for each $i \in \{1, \ldots, N\}$ there will be a j with $a_{j,i} \neq 0$ or $b_{j,i} \neq 0$ (or both).

We will arrange that

$$\|\mathbf{a}\|^2 = \left\| \sum_{j=1}^{\ell} a_j a_j^* \right\| = \max_{1 \leqslant i \leqslant m} \sum_{j=1}^{\ell} |a_{j,i}|^2 = 1$$

and that the maximum is achieved in each position $1 \leqslant i \leqslant m$ (so that $\sum_{j=1}^{\ell} |a_{j,i}|^2 = 1$ for $1 \leqslant i \leqslant m$). We will also arrange that

$$\|\mathbf{b}\|^2 = \left\| \sum_{j=1}^{\ell} b_j^* b_j \right\| = \max_{n < i \leqslant N} \sum_{j=1}^{\ell} |b_{j,i}|^2 = 1$$

and each $\sum_{j=1}^{\ell} |b_{j,i}|^2 = 1$ for $n < i \leqslant N$. We will use (3) to ensure $\|u\|_h = (\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)/2 = 1$ by ensuring that $\alpha \in \mathrm{co}\,(W_{m,e}(\mathbf{a}^*)) \cap \mathrm{co}\,(W_{m,e}(\mathbf{b}))$ with α the diagonal $\ell \times \ell$ matrix with diagonal entries all equal to $1/\ell$. In fact, α will be the only matrix in the intersection. But we achieve this in such a way that all N summands in H are required and therefore for any choice of F giving $N-1$ or fewer summands we do not satisfy criterion (3) (and hence $\|u_F\|_h$ is strictly less than 1 by Timoney [21, Theorem 3.3]).

For the a_j $(1 \leqslant j \leqslant \ell)$, it is helpful to think of ℓ rows $a_1, \ldots, a_\ell$ which we will specify column by column (where each column has length ℓ). We take a primitive mth root of unity ζ and, for $i \in \{1, \ldots, m\}$, we define

$$(a_{1,i}, a_{2,i}, \ldots, a_{\ell,i}) = (1, \zeta^i, \zeta^{2i}, \ldots, \zeta^{(\ell-1)i})/\sqrt{\ell}.$$

Recall that $a_{j,i}$ is to be zero for $i > m$ and $1 \leqslant j \leqslant \ell$. Any unit vector $\xi \in H$ supported in the summands H_i $(1 \leqslant i \leqslant m)$ gives a matrix in $W_{m,e}(\mathbf{a}^*)$, specifically the matrix

$$\sum_{i=1}^{m} |\xi_i|^2 (\eta_i^* \otimes \eta_i)$$

(a convex combination of the $\eta_i^* \otimes \eta_i$, from which we see that $W_{m,e}(\mathbf{a}^*)$ is convex) where

$$\eta_i = (1, \bar{\zeta}^i, \bar{\zeta}^{2i}, \ldots, \bar{\zeta}^{(\ell-1)i})/\sqrt{\ell}.$$

Taking each $\xi_i = 1/\sqrt{m}$ we get the matrix α. For future reference, notice that $\zeta^{\ell-1} = -1$ and so, for $1 \leqslant i \leqslant m$, the matrix $\eta_i^* \otimes \eta_i$ has the real number $(-1)^i/\ell$ in the $(1, \ell)$ position.

As with the a_j, it is helpful to think of the b_j as ℓ rows which we will specify column by column. The first two non-zero columns (column $n+1$ and column $n+2$) are as follows:

$$\theta_1 = \left(\sqrt{2}, 0, \ldots, 0, \imath\right)/\sqrt{3} \text{ and } \theta_2 = \left(\sqrt{2}, 0, \ldots, 0, -\imath\right)/\sqrt{3},$$

where $\imath = \sqrt{-1}$. We choose the remaining $(\ell-1)^2$ columns by using Lemma 15. According to that lemma, we can find $(\ell-1)^2$ affinely independent rank one projections ρ_k $(1 \leqslant k \leqslant (\ell-1)^2)$ in $M_{\ell-1}$ so that the diagonal $(\ell-1) \times (\ell-1)$ matrix

$$\beta = \begin{pmatrix} \frac{2}{2\ell-3} & 0 & \cdots & 0 \\ 0 & \frac{2}{2\ell-3} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & \frac{1}{2\ell-3} \end{pmatrix} = \sum_{k=1}^{(\ell-1)^2} t_k \rho_k$$

is a convex combination of *all* of the ρ_k (that is, $t_k > 0$ for all k and $\sum_k t_k = 1$). (Note that only the final diagonal entry of β is reduced to the value $1/(2\ell-3)$.) Take unit vectors μ_k $(1\leqslant k\leqslant(\ell-1)^2)$ in $\mathbb{C}^{\ell-1}$ to be in the ranges of ρ_k, and extend them to vectors $(0,\mu_k)=\tilde{\mu}_k\in\mathbb{C}^\ell$. Let

$$(b_{1,i},b_{2,i},\dots,b_{\ell,i})=\tilde{\mu}_k \qquad (i=n+2+k,\ 1\leqslant k\leqslant(\ell-1)^2).$$

We can check that

$$\alpha=\frac{3}{4\ell}(\theta_1^*\otimes\theta_1)+\frac{3}{4\ell}(\theta_2^*\otimes\theta_2)+\sum_{k=1}^{(\ell-1)^2}\frac{2\ell-3}{2\ell}t_k(\tilde{\mu}_k^*\otimes\tilde{\mu}_k),$$

a convex combination. Thus $\alpha\in W_{m,e}(\mathbf{b})$. Since $\alpha\in W_{m,e}(\mathbf{a}^*)$ also, the criterion (3) guarantees that $\|u\|_h=1$.

We show next that α is the unique element of $\operatorname{co}(W_{m,e}(\mathbf{a}^*))\cap\operatorname{co}(W_{m,e}(\mathbf{b}))=W_{m,e}(\mathbf{a}^*)\cap W_{m,e}(\mathbf{b})$. Suppose that

$$\sum_{i=1}^{m}c_i\eta_i^*\otimes\eta_i=r\theta_1^*\otimes\theta_1+s\theta_2^*\otimes\theta_2+\sum_{k=1}^{(\ell-1)^2}t_k\tilde{\mu}_k^*\otimes\tilde{\mu}_k,$$

where $c_i,r,s,t_k\geqslant 0$, $\sum_i c_i=1$ and $r+s+\sum_k t_k=1$, and let the common value be the $\ell\times\ell$ matrix γ. By considering the $(1,1)$-entry of γ, we see that $1/\ell=2(r+s)/3$. On the other hand, recalling that the $(1,\ell)$-entry of γ must be real, we see that $-r+s=0$. Thus $r=s=3/(4\ell)$. By considering the first row of γ and also the entries $\gamma_{\ell-1,1},\gamma_{\ell-2,1},\dots,\gamma_{2,1}$, we obtain that $\mathbf{Vc}=\mathbf{e_1}$ where $\mathbf{V}$ is the $m\times m$ matrix whose (i,j)-entry is $\zeta^{j(i-1)}$, $\mathbf{c}=(c_1,\dots,c_m)^t$ and $\mathbf{e_1}=(1,0,\dots,0)^t$. By inspection, one solution is $c_1=c_2=\cdots=c_m=1/m$ (giving $\gamma=\alpha$), and this solution is unique because the determinant of $\mathbf{V}$ is a non-zero alternant of Vandermonde.

What remains, in order to show that $\|u_F\|_h<1$ for any non-empty proper subset F of $\{1,2,\dots,N\}$, is to show that we cannot find a common element of the convex hulls of the corresponding extremal matrix numerical ranges when we remove any summand H_i (or more than one H_i). However, by the uniqueness established above, the matrix α is the only possible candidate for being such a common element. Removing the summand H_i implies removing one of the η_i if $1\leqslant i\leqslant m$, and one of θ_1,θ_2 or some $\tilde{\mu}_k$ if $n<i\leqslant N$. (If $N<\ell^2+1$, then there will be some i falling into both groups.) But to get α on the $\mathbf{a}_F^*$ side, we need all of the $\eta_i^*\otimes\eta_i$ $(1\leqslant i\leqslant m)$ because they form an affinely independent set (since the equation $\mathbf{Vd}=\mathbf{0}$ has unique solution $\mathbf{d}=\mathbf{0}$). Thus F must contain all i in the range $1\leqslant i\leqslant m$. On the other hand, it is easily checked that the set $\{\theta_1^*\otimes\theta_1,\theta_2^*\otimes\theta_2\}\cup\{\tilde{\mu}_k^*\otimes\tilde{\mu}_k : 1\leqslant k\leqslant(\ell-1)^2\}$ is affinely independent. Hence, to get α on the $\mathbf{b}_F$ side, F must contain all i in the range $n<i\leqslant N$. So if F is a proper subset of $\{1,2,\dots,N\}$, then we cannot satisfy the criterion (3) of [21, Theorem 3.3] and so $\|u_F\|_h<1$. □

Theorem 17. *Let A be a unital C^*-algebra. Fix $\ell \geqslant 1$. Then*

$$\|\theta_Z(u)\|_{cb} = \|u\|_{Z,h}$$

holds for each $u = \sum_{j=1}^{\ell} a_j \otimes b_j \in A \otimes A$ if and only if every Glimm ideal in A is (ℓ^2+1)-primal.

Proof. One direction is already done in Proposition 14 above. For the converse, suppose that A has a Glimm ideal G which is not (ℓ^2+1)-primal. If G is not 2-primal then there exists $u = a \otimes b \in A \otimes A$ such that $\|u\|_{Z,h} \neq 0$ and $\theta_Z(u) = 0$ (see the proof of [19, Lemma 5]). If G is 2-primal (so $\ell > 1$) then there exists $\ell' \in \{2, \dots, \ell\}$ and $N \in \{(\ell'-1)^2+2, \dots, \ell'^2+1\}$ such that G is $(N-1)$-primal but not N-primal. Since a tensor with ℓ' summands may be regarded as a tensor with ℓ summands, by the addition of zeros, we may as well assume (for notational convenience) that $\ell' = \ell$.

As in the proof of Theorem 7, there exist primitive ideals $P_1, \dots, P_N$ of A such that $G \subseteq P_i$ for $1 \leqslant i \leqslant N$ and $J := P_1 \cap \cdots \cap P_N$ is not primal. Furthermore, there exist mutually orthogonal positive elements $b_1, \dots, b_N$ of A such that $\|b_i\| = \|b_i + J\| = 1$ for $1 \leqslant i \leqslant N$ and such that for each $P \in \mathrm{Prim}(A)$ there exists $i \in \{1, \dots, N\}$ for which $b_i \in P$. We now re-label these N elements as $d_1, \dots, d_N$ (to avoid confusion with the elements $b_1, \dots, b_\ell$ which we are about to import from Lemma 16). Let $v = \sum_{j=1}^{\ell} a_j \otimes b_j \in \mathbb{C}^N \otimes \mathbb{C}^N$ have the properties of Lemma 16, let

$$\beta := \max\{\|v_F\|_h : F \text{ a proper non-empty subset of } \{1, \dots, N\}\} < 1$$

and let

$$u := \sum_{j=1}^{\ell} \left(\sum_{i=1}^{N} a_{j,i} d_i \right) \otimes \left(\sum_{i=1}^{N} b_{j,i} d_i \right) \in A \otimes A.$$

On the one hand,

$$\|u\|_{Z,h} \geqslant \|u^G\|_h \geqslant \|u^J\|_h = \|v\|_h = 1,$$

where the penultimate equality follows by applying Lemma 5 to the linear map $\phi : \mathbb{C}^N \to \mathrm{span}\{d_1 + J, \dots, d_N + J\}$ given by $\phi(e_{ii}) = d_i + J$ (where e_{ii} is the ith standard basis vector). On the other hand, if $P \in \mathrm{Prim}(A)$ then there exists $i' \in \{1, \dots, N\}$ such that $d_{i'} \in P$. Let $F = \{1, \dots, N\} \setminus \{i'\}$. Then

$$\|u^P\|_h = \left\| \sum_{j=1}^{\ell} \left(\sum_{i \neq i'} a_{j,i}(d_i + P) \right) \otimes \left(\sum_{i \neq i'} b_{j,i}(d_i + P) \right) \right\|_h \leqslant \|v_F\|_h \leqslant \beta,$$

where the penultimate inequality follows by applying Lemma 5 to the linear map $\phi : \mathrm{span}\{e_{ii} : i \neq i'\} \to \mathrm{span}\{d_i + P : i \neq i'\}$ given by $\phi(e_{ii}) = d_i + P$. Hence

$$\|\theta_Z(u)\|_{cb} = \sup_{P \in \mathrm{Prim}(A)} \|u^P\|_h \leqslant \beta < 1. \qquad \square$$

Finally, we note that we can extend Theorem 17 to the non-unital case in the same way as Corollary 9 extends Theorem 8.

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