

Two-dimensional Gauge Theories and Quantum Integrable Systems

Anton A. Gerasimov^{1,2,3} and **Samson L. Shatashvili**^{2,3,4}

¹ *Institute for Theoretical and Experimental Physics, Moscow, 117259, Russia*

² *School of Mathematics, Trinity College, Dublin 2, Ireland*

³ *The Hamilton Mathematics Institute TCD, Dublin 2, Ireland*

⁴ *IHES, 35 route de Chartres, Bures-sur-Yvette, FRANCE*

In this paper the relation between 2d topological gauge theories and Bethe Ansatz equations is reviewed.¹ In addition we present some new results and clarifications. We hope the relations discussed here are particular examples of more general relations between quantum topological fields theories in dimensions $d \leq 4$ and quantum integrable systems.

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1. Introduction

In [1] a relation between a certain type of two-dimensional Yang-Mills theory and the Bethe Ansatz equations for the quantum theory of the Nonlinear Schrödinger equation was uncovered. The topological Yang-Mills-Higgs theory considered in [1] captures the hyperkähler geometry of the moduli space of Higgs bundles introduced in [2] by Hitchin. The further step in the understanding of the role of Bethe Ansatz equations in two-dimensional gauge theories was made in [3] where the wave functions in gauge theory were identified with eigenfunctions of the quantum Hamiltonian in the N -particle sector for the quantum theory of the Nonlinear Schrödinger equation, constructed in the framework of the coordinate Bethe Ansatz. This implies the equivalence between two seemingly different quantum field theories. Moreover, in [3] it was argued that this phenomena is not isolated and was shown how to generalize the considerations of [1] from the Yang-Mills-Higgs theory to G/G gauged WZW model with Higgs fields. The corresponding partition function is expressed in terms of solutions of Bethe Ansatz equations describing a particular limit of XXZ spin chains. One expects that generic two-dimensional topological quiver gauge theories with additional matter fields in various representations should reproduce the Bethe Ansatz equations of other known quantum systems with finite number of degrees of freedom.

Let us remark that topological quantum field theories on the manifolds fibered over circle S^1 can be effectively reduced to Quantum mechanical systems with finite number of degrees of freedom. Being perturbed by topological observables the higher-dimensional topological theories lead to multi-parametric evolution in the effective, one-dimensional, quantum theories. One can expect that their integrability also is a property of three and four dimensional gauge theories among which the topological theories calculating Donaldson invariants look especially interesting.

2. On symplectic and hyperkähler reductions

To motivate the introduction of the topological Yang-Mills-Higgs theory in the next Section we start recalling the representation of integrals over symplectic and hyperkähler quotients of a G -manifold X in terms of integrals over X . This is basically covered by [1]; here we give integral over X , expectedly invariant under $SO(3)$ -rotations of symplectic structures.

Consider a symplectic manifold X of dimension $d = 2n$ with a Hamiltonian action of a compact Lie group G . Define a volume form on X as a maximal non-zero exterior product $\text{vol}_X = \omega_X \wedge \cdots \wedge \omega_X$ of the symplectic two-form ω_X . Given a Hamiltonian action of Lie group G one has a corresponding moment map:

$$\mu : X \rightarrow \mathfrak{g}^*, \quad \mathfrak{g} = \text{Lie}(G), \quad (2.1)$$

and the symplectic quotient over the coadjoint orbit \mathcal{O}_u of an element $u \in \mathfrak{g}^*$ is given by:

$$Y = \mu^{-1}(\mathcal{O}_u)/G = \mu^{-1}(u)/G_u. \quad (2.2)$$

Here $G_u \subset G$ is a stabilizer of u . Let $\pi : \mu^{-1}(u) \rightarrow Y = \mu^{-1}(u)/G_u$ be the projection on Y and $\text{vol}_{\mu^{-1}(u)} = \text{vol}_X / \wedge^{\dim \mathfrak{g}} d\mu$ be a volume form on $\mu^{-1}(u)$ induced by the volume form vol_X on X . Then the volume form vol_Y on the quotient Y is given by:

$$\text{vol}_Y(y) = \frac{1}{\text{Vol}(G_u)} \pi^*(\text{vol}_{\mu^{-1}(u)}) = \frac{1}{\text{Vol}(G_u)} \int_{\pi^{-1}(y)} \text{vol}_{\mu^{-1}(u)}. \quad (2.3)$$

More generally the integral of any function f over Y can be expressed as an integral over X as follows:

$$\langle f \rangle = \int_Y f \text{vol}_Y = \frac{1}{\text{Vol}(G_u)} \int_{X \times \mathfrak{g}^*} F e^{i\langle \varphi, (\mu(x)-u) \rangle} \text{vol}_X \wedge d\varphi, \quad (2.4)$$

where \langle, \rangle is a natural pairing between \mathfrak{g}^* and \mathfrak{g} , F is an arbitrary smooth extension of $\pi^*(f)$ to a function on X and $d\varphi$ as a flat measure on \mathfrak{g}^* . The integrand is explicitly invariant with respect to the action of G_u and thus instead of integration over G_u one can use ‘‘Faddeev-Popov’’ gauge fixing procedure:

$$\begin{aligned} \langle f \rangle = & \int_{X \times \mathfrak{g}^* \times \Pi(\mathfrak{g}_C)} F e^{i\langle \varphi, (\mu(x)-u) \rangle + i\langle \chi, \nu(x) \rangle + \sum_{a,b=1}^{\dim(G_u)} \bar{\mathcal{C}}^a i_{V^b} d\nu_a(y) \mathcal{C}^b} \times \\ & \times \text{vol}_X \wedge d\varphi \wedge d\chi \wedge d\bar{\mathcal{C}}^a \wedge d\mathcal{C}, \end{aligned} \quad (2.5)$$

where $V^b \in \text{Vect}_X$, $a = 1, \dots, \dim(G_u)$ is a bases of vector fields generating the action of $\mathfrak{g}_u = \text{Lie}(G_u)$, $\nu_a(x)$, $a = 1, \dots, \dim(G_u)$ is a generic set of the gauge fixing functions, $(\bar{\mathcal{C}}, \mathcal{C})$ is a pair of odd \mathfrak{g}_u -valued ghosts and χ is \mathfrak{g}_u -valued even.

Now consider a generalization of representation (2.5) to the case of hyperkähler reduction. Given a manifold X with three compatible symplectic structures ω_α , $\alpha = 1, 2, 3$ and an action of a compact Lie group G which is Hamiltonian with respect to any of the

symplectic structures. We have a family of symplectic structures with an obvious action of $SO(3)$ group through a three-dimensional representation. It will be useful to consider a pair of real and complex symplectic form $\omega_{\mathbb{R}} = \omega_3$, $\omega_{\mathbb{C}} = \omega_1 + i\omega_2$. The hyperkähler reduction of X with respect to G is define as follows:

$$Y = (\mu_{\mathbb{C}}^{-1}(u_c) \cap \overline{\mu_{\mathbb{C}}^{-1}(u_c)} \cap \mu_{\mathbb{R}}^{-1}(u_r))/G_{(u_c, u_r)}, \quad (2.6)$$

where $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ are moment maps for $\omega_{\mathbb{C}}$ and $\omega_{\mathbb{R}}$ and $G_{(u_c, u_r)} \subset G$ is a stabilizer of the triple $u_r \in \mathfrak{g}^*$, $u_c \in \mathfrak{g}^* \otimes \mathbb{C}$. The same space (up to some subtleties with a compactification of the space of stable orbits) can be obtained as a holomorphic symplectic manifold by a holomorphic version of symplectic reduction:

$$Y = \mu_{\mathbb{C}}^{-1}(u_c)/\tilde{G}_u^c, \quad (2.7)$$

where $\tilde{G}_u^c \subset G^c$ is a stabilizer of $u_c \in \mathfrak{g}^* \otimes \mathbb{C}$ with respect to the action of the complexification G^c of G . Note that \tilde{G}_u^c is not necessary a complexification of G_{u_c, u_r} . However in the following we will consider only the case when \tilde{G}_u^c is indeed a complexification of G_{u_c, u_r} .

It would be natural to define the integral of a function f over the quotient Y by an analog of (2.4):

$$\int_Y f \sim \frac{1}{\text{Vol}(\tilde{G}_u^c)} \int_{\pi^{-1}(y)} \pi^*(f) \text{vol}_{\mu_{\mathbb{C}}^{-1}(u_c)}, \quad (2.8)$$

where $\pi : \mu_{\mathbb{C}}^{-1}(u_c) \rightarrow Y$, $\text{vol}_X = (\omega_{\mathbb{C}} \wedge \overline{\omega_{\mathbb{C}}}) \wedge \dots \wedge (\omega_{\mathbb{C}} \wedge \overline{\omega_{\mathbb{C}}})$ is a volume form on X . However this expression is not well defined because the volume of \tilde{G}_u^c is not finite and the fibres of the projection π are not compact.

To represent integral over the hyperkähler quotient one should consider an analog of (2.5). It is useful to impose gauge-fixing conditions only on a troublesome non-compact part of G_u^c so that the compact subgroup $G_{u_c, u_r} \subset G_u^c$ remains unfixed. The natural choice of the gauge-fixing conditions is a set of real moments $\mu_{\mathbb{R}}$. Thus we arrive at the following representation:

$$\begin{aligned} \langle f \rangle &= \int_Y f = \frac{1}{\text{Vol}(G_{u_c, u_r})} \times \\ &\times \int_{X \times (\mathfrak{g}^* \otimes \mathbb{R}^3)} F e^{i \sum_{\alpha=1}^3 \langle \varphi_{\alpha}, (\mu^{\alpha}(x) - u^{\alpha}) \rangle + \sum_{a,b=1}^{\dim(G_{u_c, u_r})} \bar{c}^a i_{v^b} d\mu_{\mathbb{R}}^a(y) c^b} \text{vol}_X \wedge d\varphi_{\alpha} \wedge d\bar{c}^a \wedge dc^b \end{aligned} \quad (2.9)$$

This integral allows an explicitly $SU(2)$ -invariant reformulation:

$$\langle f \rangle = \frac{1}{\text{Vol}(G_{u_c, u_r})} \int_{X \times (\mathfrak{g}^* \otimes \mathbb{R}^3) \times \Pi(\mathfrak{g}_{\mathbb{C}})} F e^{i\mathcal{F}(x, \phi, \mathcal{C}, \bar{\mathcal{C}})} \text{vol}_X \wedge d\varphi_{\alpha} \wedge d\bar{c}^a \wedge dc^b, \quad (2.10)$$

$$\mathcal{F}(x, \phi, \mathcal{C}, \bar{\mathcal{C}}) = \sum_{\alpha=1}^3 \langle \varphi_{\alpha}, (\mu^{\alpha}(x) - u^{\alpha}) \rangle + \sum_{\alpha, \beta, \gamma=1}^3 \sum_{a, b=1}^{\dim(G_{u_c, u_r})} \bar{\mathcal{C}}^a \{ \mu_{\alpha}^a, \mu_{\beta}^b \}_{\gamma} \mathcal{C}^b \epsilon^{\alpha\beta\gamma}$$

where μ_{α} , $\alpha = 1, 2, 3$ is a triple of moment maps corresponding to symplectic forms ω_{α} , $\{, \}_{\alpha}$ are Poisson brackets given by inverse of symplectic forms ω_{α} and F is extension to X of a lift of f to $\mu_{\mathbb{C}}^{-1}(u_c) \cap \overline{\mu_{\mathbb{C}}^{-1}(u_c)} \cap \mu_{\mathbb{R}}^{-1}(u_r)$.

In next chapter we consider the example of infinite-dimensional hyperkähler quotient give by Hitchin equation.

3. Topological Yang-Mills-Higgs theory

In [1] a two dimensional gauge theory was proposed such that the space of classical solutions of the theory on a Riemann surface Σ_h is closely related with the cotangent space to the space of solutions of the dimensionally reduced (to 2d) four-dimensional self-dual Yang-Mills equations studied by Hitchin [2]. Let us given a principal G -bundle P_G over a Riemann surface Σ supplied with a complex structure. Then we have an associated vector bundle $\text{ad}_{\mathfrak{g}} = (P_G \times \mathfrak{g})/G$ with the fiber $\mathfrak{g} = \text{Lie}(G)$ supplied with a coadjoint action of G . Consider the pairs (A, Φ) where A is a connection on P_G and Φ is a one-form taking values in $\text{ad}_{\mathfrak{g}}$. Then Hitchin equations are given by:

$$F(A) - \Phi \wedge \Phi = 0, \quad \nabla_A^{(1,0)} \Phi^{(0,1)} = 0, \quad \nabla_A^{(0,1)} \Phi^{(1,0)} = 0. \quad (3.1)$$

The space of the solutions has a natural hyperkähler structure and admits compatible $U(1)$ action. The correlation functions in the theory introduced in [1] can be described by the integrals of the products of $U(1) \times G$ -equivariant cohomology classes over the moduli space of solutions of (3.1). Note that the $U(1)$ -equivariance makes the path integral well-defined.

The field content of the theory introduced in [1] can be described as follows. In addition to the triplet (A, ψ_A, φ_0) of the topological Yang-Mills theory one has:

$$(\Phi, \psi_{\Phi}) : \quad \Phi \in \mathcal{A}^1(\Sigma, \text{ad}_{\mathfrak{g}}), \quad \psi_{\Phi} \in \mathcal{A}^1(\Sigma, \text{ad}_{\mathfrak{g}}) \quad (3.2)$$

$$(\varphi_{\pm}, \chi_{\pm}) : \quad \varphi_{\pm} \in \mathcal{A}^0(\Sigma, \text{ad}_{\mathfrak{g}}), \quad \chi_{\pm} \in \mathcal{A}^0(\Sigma, \text{ad}_{\mathfrak{g}}) \quad (3.3)$$

where Φ , φ_{\pm} are even and ψ_{Φ} , χ_{\pm} are odd fields. We will use also another notations $\varphi_{\pm} = \varphi_1 \pm i\varphi_2$.

The theory is described by the following path integral:

$$Z_{YMH}(\Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G}_{\Sigma_h})} \int D\varphi_0 D\varphi_{\pm} DA D\Phi D\psi_A D\psi_{\Phi} D\chi_{\pm} e^{S(\varphi_0, \varphi_{\pm}, A, \Phi, \psi_A, \psi_{\Phi}, \chi_{\pm})}, \quad (3.4)$$

where $S = S_0 + S_1$ with:

$$S_0(\varphi_0, \varphi_{\pm}, A, \Phi, \psi_A, \psi_{\Phi}, \chi_{\pm}) = \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(i\varphi_0 (F(A) - \Phi \wedge \Phi) - c\Phi \wedge * \Phi) + \varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)} \quad (3.5)$$

and

$$S_1(\varphi_0, \varphi_{\pm}, A, \Phi, \psi_A, \psi_{\Phi}, \chi_{\pm}) = \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(\frac{1}{2}\psi_A \wedge \psi_A + \frac{1}{2}\psi_{\Phi} \wedge \psi_{\Phi} + \chi_+ [\psi_A^{(1,0)}, \Phi^{(0,1)}] + \chi_- [\psi_A^{(0,1)}, \Phi^{(1,0)}] + \chi_+ \nabla_A^{(1,0)} \psi_{\Phi}^{(0,1)} + \chi_- \nabla_A^{(0,1)} \psi_{\Phi}^{(1,0)}) \quad (3.6)$$

where the decompositions $\Phi = \Phi^{(1,0)} + \Phi^{(0,1)}$ and $\psi_{\Phi} = \psi_{\Phi}^{(1,0)} + \psi_{\Phi}^{(0,1)}$ correspond to the decomposition of the space of one-forms $\mathcal{A}^1(\Sigma_h) = \mathcal{A}^{(1,0)}(\Sigma_h) \oplus \mathcal{A}^{(0,1)}(\Sigma_h)$ defined in terms of a fixed complex structure on Σ_h .

The theory is invariant under the action of the following even vector field:

$$\mathcal{L}_v \Phi^{(1,0)} = +\Phi^{(1,0)}, \quad \mathcal{L}_v \Phi^{(0,1)} = -\Phi^{(0,1)}, \quad \mathcal{L}_v \psi_{\Phi}^{(1,0)} = +\psi_{\Phi}^{(1,0)}, \quad (3.7)$$

$$\mathcal{L}_v \psi_{\Phi}^{(0,1)} = -\psi_{\Phi}^{(0,1)}, \quad \mathcal{L}_v \varphi_{\pm} = \mp \varphi_{\pm}, \quad \mathcal{L}_v \chi_{\pm} = \pm \chi_{\pm},$$

$$\mathcal{L}_{\varphi_0} A = -\nabla_A \varphi_0, \quad \mathcal{L}_{\varphi_0} \psi_A = -[\varphi_0, \psi_A], \quad \mathcal{L}_{\varphi_0} \Phi = -[\varphi_0, \Phi], \quad \mathcal{L}_{\varphi_0} \psi_{\Phi} = -[\varphi_0, \psi_{\Phi}], \quad (3.8)$$

$$\mathcal{L}_{\varphi_0} \varphi_0 = 0, \quad \mathcal{L}_{\varphi_0} \varphi_{\pm} = -[\varphi_0, \varphi_{\pm}], \quad \mathcal{L}_{\varphi_0} \chi_{\pm} = -[\varphi_0, \chi_{\pm}],$$

and an odd vector field generated by the BRST operator:

$$QA = i\psi_A, \quad Q\psi_A = -\nabla_A \varphi_0, \quad Q\varphi_0 = 0,$$

$$Q\Phi = i\psi_{\Phi}, \quad Q\psi_{\Phi}^{(1,0)} = -[\varphi_0, \Phi^{(1,0)}] + c\Phi^{(1,0)}, \quad Q\psi_{\Phi}^{(0,1)} = -[\varphi_0, \Phi^{(0,1)}] - c\Phi^{(0,1)}, \quad (3.9)$$

$$Q\chi_{\pm} = i\varphi_{\pm}, \quad Q\varphi_{\pm} = -[\varphi_0, \chi_{\pm}] \pm c\chi_{\pm}.$$

We have $Q^2 = i\mathcal{L}_{\varphi_0} + c\mathcal{L}_v$ and Q can be considered as a BRST operator on the space of \mathcal{L}_{φ_0} and \mathcal{L}_v -invariant functionals. The action functional of the topological Yang-Mills-Higgs theory can be represented as a sum of the action functional of the topological pure

Yang-Mills theory (written in terms of fields ϕ_0, A, ψ_A) and an additional part which can be represented as a Q -anti-commutator:

$$S_{YMH} = S_{YM} + [Q, \int_{\Sigma_h} d^2z \text{Tr} (\frac{1}{2} \Phi \wedge \psi_\Phi + \varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)})]_+. \quad (3.10)$$

The theory given by (3.4) is a quantum field theory whose correlation functions are given by the intersections pairings of the equivariant cohomology classes on the moduli spaces of Higgs bundles.

To simplify the calculations it is useful to consider more general action given by:

$$S_{YMH} = S_{YM} + [Q, \int_{\Sigma_h} d^2z \text{Tr} (\frac{1}{2} \Phi \wedge \psi_\Phi + \tau_1 (\varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)}) + \tau_2 (\chi_+ \varphi_- + \chi_- \varphi_+) \text{vol}_{\Sigma_h})]. \quad (3.11)$$

Cohomological localization of the functional integral takes the simplest form for $\tau_1 = 0$, $\tau_2 \neq 0$. Note that it is not obvious that the theory for $\tau_1 = 0$, $\tau_2 \neq 0$ is equivalent to that for $\tau_1 \neq 0$, $\tau_2 = 0$. However taking into account that the action functionals in these two cases differ on the equivariantly exact form and for $c \neq 0$ the space of fields is essentially compact one can expect that the theories are equivalent.

For $\tau_1 = 0$ the path integrals over Φ , φ_\pm and χ_\pm is quadratic. Thus we have for the partition function the following formal representation:

$$Z_{YMH}(\Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G}_{\Sigma_h})} \int DA D\varphi_0 D\psi_A e^{\frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr} (i\varphi_0 F(A) + \frac{1}{2} \psi_A \wedge \psi_A)} \text{Sdet}_V (ad_{\varphi_0} + ic), \quad (3.12)$$

where the super-determinant is taken over the super-space:

$$V = V_{even} \oplus V_{odd} = \mathcal{A}^0(\Sigma_h, \text{adg}) \oplus \mathcal{A}^{(1,0)}(\Sigma_h, \text{adg}). \quad (3.13)$$

and should be properly understood using a regularization compatible with Q -symmetry of the path integral (e.g. $\tau_1 \neq 0$). Thus the Yang-Mills-Higgs theory can be considered as a pure Yang-Mills theory deformed by a non-local gauge invariant observable. Note that for $c \rightarrow \infty$ the action functional (3.5) reduces to that of 2d topological Yang-Mills theory, with corresponding consequences. At the same time for $c = 0$ it is again 2d topological Yang-Mills action but now for complexified gauge group, thus the theory interpolates for

non-zero c between topological Yang-Mills theory for compact and complex gauge groups, which becomes also apparent from final answers.

The partition function (3.4) of the Yang-Mills-Higgs theory on a compact Riemann surface can be calculated using the standard methods of the cohomological localization. As in the case of Yang-Mills theory we consider the deformation of the action of the theory:

$$\Delta S_{YMH} = - \sum_{k=1}^{\infty} t_k \int_{\Sigma_h} d^2 z \operatorname{Tr} \varphi_0^k \operatorname{vol}_{\Sigma_h}. \quad (3.14)$$

where we impose the condition that $t_k \neq 0$ for a finite subset of indexes $k \in \mathbb{Z}$.

Path integral with the action (3.11) at $\tau_1 = 0$ and $\tau_2 = 1$ is easily reduced to the integral over abelian gauge fields. The contribution of the additional nonlocal observable in (3.12) can be calculated as follows. The purely bosonic part of the nonlocal observable after reduction to abelian fields can be easily evaluated using any suitable regularization (i.e. zeta function regularization) and result is the change of the bosonic part of the abelian action $\int d^2 z (\varphi_0)_i F^i(A)$ by:

$$\begin{aligned} \Delta S = & \int_{\Sigma_h} d^2 z \sum_{i,j=1}^N \log \left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic} \right) F(A)^i \\ & + \frac{1}{2} \int_{\Sigma_h} d^2 z \sum_{i,j=1}^N \log((\varphi_0)_i - (\varphi_0)_j + ic) R^{(2)} \sqrt{g}; \end{aligned} \quad (3.15)$$

where $F(A)^i$ is i -th component of the curvature of the abelian connection A and $R^{(2)}(g)$ is curvature on Σ_h for 2d metric g used to regularize non-local observable. We will use the notation $(\varphi_0)_i = \lambda_i$ in remaining part of the paper. This leads to the unique Q -closed completion. The completion of the term in (3.15) containing the curvature of the gauge field is given by the two-observable $\mathcal{O}_f^{(2)}$ corresponding to the descendent of the following function on the Cartan subalgebra isomorphic to \mathbb{R}^N :

$$f(\operatorname{diag}(\lambda_1, \dots, \lambda_N)) = \sum_{k,j=1}^N \int_0^{\lambda_j - \lambda_k} \operatorname{arctg} \lambda/c d\lambda. \quad (3.16)$$

Thus, the abelianized action is defined by two-observable descending from:

$$I(\lambda) = \sum_{j=1}^N \left(\frac{1}{2} \lambda_j^2 - 2\pi n_j \lambda_j \right) + \sum_{k,j=1}^N \int_0^{\lambda_j - \lambda_k} \operatorname{arctg} \lambda/c d\lambda. \quad (3.17)$$

On the other hand the term containing the metric curvature R in (3.15) is Q -closed and thus does not need any completion. It can be considered as an integral of the zero observable:

$$\mathcal{O}^{(0)} = \sum_{i,j=1}^N \log((\varphi_0)_i - (\varphi_0)_j + ic) \quad (3.18)$$

over Σ_h weighted by the half of the metric curvature. Note that the function $I(\lambda)$ plays important role in Nonlinear Schrödinger theory which we explain in next section.

After integrating out the fermionic partners of abelian connection A the standard localization procedure leads to following final finite-dimensional integral representation for the the partition function [1]:

$$\begin{aligned} Z_{YMH}(\Sigma_h) &= \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbb{R}^N} d^N \lambda \mu(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \times \\ &\times \prod_{k \neq j} (\lambda_k - \lambda_j)^{n_k - n_j + 1 - h} \prod_{k,j} (\lambda_k - \lambda_j - ic)^{n_k - n_j + 1 - h} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)}, \end{aligned} \quad (3.19)$$

where

$$\mu(\lambda) = \det \left\| \frac{\partial^2 I(\lambda)}{\partial \lambda_i \partial \lambda_j} \right\|, \quad (3.20)$$

and $p_k(\lambda)$ are S_N -invariant polynomial functions of degree k on \mathbb{R}^N and $a(c)$ is a h -independent constant defined by the appropriate choice of the regularization of the functional integral. One can write the n_i -dependent parts of the products in (3.19) as the exponent of the sum:

$$\begin{aligned} Z_{YMH}(\Sigma_h) &= \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbb{R}^N} d^N \lambda \mu(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_j n_j \alpha_j(\lambda)} \\ &\times \prod_{k \neq j} (\lambda_k - \lambda_j)^{1-h} \prod_{k,j} (\lambda_k - \lambda_j - ic)^{1-h} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)}, \end{aligned} \quad (3.21)$$

with notation:

$$e^{2\pi i \alpha_j(\lambda)} = \mathcal{F}_j(\lambda) \equiv e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic} \quad (3.22)$$

After taking the sum over $(n_1, \dots, n_N) \in \mathbb{Z}^N$ using:

$$\begin{aligned} \mu(\lambda) \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_j n_j \alpha_j(\lambda)} &= \mu(\lambda) \sum_{(m_1, \dots, m_N) \in \mathbb{Z}^N} \prod_j \delta(\alpha_j(\lambda) - m_j) \\ &= \sum_{(\lambda_1^*, \dots, \lambda_N^*) \in \mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda_j^*) \end{aligned} \quad (3.23)$$

(see definition of \mathcal{R}_N below) and integral over $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ we see that only $\alpha_j(\lambda) \in \mathbb{Z}$, or the same - $\mathcal{F}_j(\lambda) = 1$, contribute to the partition function which now can be written as:

$$Z_{YMH}(\Sigma_h) = e^{(1-h)a(c)} \sum_{\lambda \in \mathcal{R}_N} D_\lambda^{2-2h} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)}, \quad (3.24)$$

where:

$$D_\lambda = \mu(\lambda)^{-1/2} \prod_{i < j} (\lambda_i - \lambda_j)(c^2 + (\lambda_i - \lambda_j)^2)^{1/2}, \quad (3.25)$$

and the \mathcal{R}_N in (3.23) and (3.24) denotes a set of the solutions of the Bethe Ansatz equations $\mathcal{F}_j(\lambda) = 1$:

$$e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j + ic}{\lambda_k - \lambda_j - ic} = 1, \quad k = 1, \dots, N, \quad (3.26)$$

for the N -particle sector of the quantum theory of Nonlinear Schrödinger equation. Note that the sum in (3.24) is taken over the classes of the solutions up to action of the symmetric group on λ_i . This set can be enumerated by the multiplets of the integer numbers $(p_1, \dots, p_N) \in \mathbb{Z}^N$ such that $p_1 \geq p_2 \geq \dots \geq p_N$, $p_i \in \mathbb{Z}$. Thus, the sum in (3.24) is the sum over the same set of partitions as in 2d Yang-Mills theory.

4. N -particle wave functions in Nonlinear Schrödinger theory

The appearance of a particular form of Bethe Ansatz equations (3.26) strongly suggests the relevance of quantum integrable theories in the description of wave-functions in topological Yang-Mills-Higgs theory. Precisely this form of Bethe Ansatz equations (3.26) arises in the description of the N -particle wave functions for the quantum Nonlinear Schrödinger theory with the coupling constant $c \neq 0$ [4], [5], [6], [7]. In this section we recall the standard facts about the construction of the these wave-functions using the coordinate Bethe Ansatz. We also discuss the relation with the representation theory of the degenerate (double) affine Hecke algebras and the representation theory of the Lie groups over complex and p -adic numbers. For the application of the quantum inverse scattering method to Nonlinear Schrödinger theory see [8], [9]. One can also recommend [10] as a quite readable introduction into the Bethe Ansatz machinery.

The Hamiltonian of Nonlinear Schrödinger theory with a coupling constant c is given by:

$$\mathcal{H}_2 = \int dx \left(\frac{1}{2} \frac{\partial \phi^*(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} + c(\phi^*(x)\phi(x))^2 \right), \quad (4.1)$$

with the following Poisson structure for bosonic fields:

$$\{\phi^*(x), \phi(x')\} = \delta(x - x'). \quad (4.2)$$

The operator of the number of particles:

$$\mathcal{H}_0 = \int dx \phi^*(x)\phi(x), \quad (4.3)$$

commutes with the Hamiltonian \mathcal{H}_2 and thus one can solve the eigenfunction problem in the sub-sector for a given number of particles $\mathcal{H}_0 = N$. We will consider the both the theory on infinite interval ($x \in \mathbb{R}$) and its periodic version $x \in S^1$. The equation for eigenfunctions in the N -particle sector has the following form:

$$\left(-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)\right) \Phi_\lambda(x) = 2\pi^2 \left(\sum_{i=1}^N \lambda_i^2\right) \Phi_\lambda(x) \quad i = 1, \dots, N. \quad (4.4)$$

This equation is obviously symmetric with respect to the action of symmetric group S_N on the coordinates x_i . Thus the solutions are classified according to the representations of S_N . Quantum integrability of the Nonlinear Schrödinger theory implies the existence of the complete set of the commuting Hamiltonian operators. The corresponding eigenvalues are given by the symmetric polynomials $p_k(\lambda)$.

Finite-particle sub-sectors of the Nonlinear Schrödinger theory can be described in terms of the representation theory of a particular kind of Hecke algebra [11], [12], [13], [14]. Let $R = \{\alpha_1, \dots, \alpha_l\}$ be a root system, W - corresponding Weyl group and P - a weight lattice. Degenerate affine Hecke algebra $\mathcal{H}_{R,c}$ associated to R is defined as an algebra with the basis S_w , $w \in W$ and $\{D_\lambda, \lambda \in P\}$ such that S_w $w \in W$ generate subalgebra isomorphic to group algebra $\mathbb{C}[W]$ and the elements D_λ , $\lambda \in P$ generate the group algebra $\mathbb{C}[P]$ of the weight lattice P . In addition one has the relations:

$$S_{s_i} D_\lambda - D_{s_i(\lambda)} S_{s_i} = c \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, \quad i = 1, \dots, n. \quad (4.5)$$

Here s_i are the generators of the Weyl algebra corresponding to the reflection with respect to the simple roots α_i . The center of $\mathcal{H}_{R,c}$ is isomorphic to the algebra of W -invariant polynomial functions on $R \otimes \mathbb{C}$. The degenerate affine Hecke algebras were introduced by Drinfeld [15] and independently by Lusztig [16].

Below we consider only the case of \mathfrak{gl}_N root system and thus we have $W = S_N$. Let us introduce the following differential operators (Dunkle operators [17]):

$$\mathcal{D}_i = -i \frac{\partial}{\partial x_i} + i \frac{c}{2} \sum_{j=i+1}^N (\epsilon(x_i - x_j) + 1) s_{ij}. \quad (4.6)$$

Here $\epsilon(x)$ is a sign-function and $s_{ij} \in S_N$ is a transposition (ij) . These operators together with the action of the symmetric group (4.6) provide a representation of the degenerate affine algebra $\mathcal{H}_{N,c}$ for $\mathfrak{g} = \mathfrak{gl}(N)$:

$$S_{s_i} \rightarrow s_i, \quad D_i \rightarrow \mathcal{D}_i, \quad i = 1, \dots, N. \quad (4.7)$$

The image of the quadratic element of the center is given by:

$$\frac{1}{2} \sum_{i=1}^N \mathcal{D}_i^2 = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \quad (4.8)$$

and thus coincides with the restriction of the quantum Hamiltonian on the N -particle sector of Nonlinear Schrödinger theory on the infinite interval.

We are interested in S_N -invariant solutions of (4.4). They play the role of spherical vectors (with respect to the subalgebra $\mathbb{C}[W] \in \mathcal{H}_{N,c}$) in the representation theory of degenerate affine Hecke algebra.

The eigenvalue problem (4.4) allows the equivalent reformulation as an eigenvalue problem in the domain $x_1 \leq x_2 \leq \dots \leq x_N$ for the differential operator:

$$\left(-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}\right) \Phi_\lambda(x) = 2\pi^2 \left(\sum_{i=1}^N \lambda_i^2\right) \Phi_\lambda(x) \quad i = 1, \dots, N, \quad (4.9)$$

with the boundary conditions:

$$(\partial_{x_{i+1}} \Phi_\lambda(x) - \partial_{x_i} \Phi_\lambda(x))_{x_{i+1}-x_i=+0} = 4\pi c \Phi_\lambda(x)_{x_{i+1}-x_i=0}. \quad (4.10)$$

The solution is given by:

$$\Phi_\lambda^{(0)}(x) = \sum_{w \in W} \prod_{1 \leq i < j \leq N} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)}} \right) \exp(2\pi i \sum_k \lambda_{w(k)} x_k), \quad (4.11)$$

or equivalently:

$$\Phi_\lambda^{(0)}(x) = \frac{1}{\Delta_{\mathfrak{g}}(\lambda)} \sum_{w \in W} (-1)^{l(w)} \prod_{1 \leq i < j \leq N} (\lambda_{w(i)} - \lambda_{w(j)} + ic) \exp(2\pi i \sum_k \lambda_{w(k)} x_k). \quad (4.12)$$

where $\Delta_{Fg}(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)$. Note that the wave-function is explicitly symmetric under the action of symmetric group S_N on $\lambda = (\lambda_1, \dots, \lambda_N)$. This solution can be also constructed using the representation theory of degenerate affine Hecke algebra $\mathcal{H}_{N,c}$ (see [11], [12], [13], [14]).

Given a solution of the equations (4.9) with boundary conditions (4.10) S_N -symmetric solutions of (4.4) on \mathbb{R}^N can be represented in the following form:

$$\Phi_\lambda^{(0)}(x) = \sum_{w \in W} \left(\prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic\epsilon(x_i - x_j)}{\lambda_{w(i)} - \lambda_{w(j)}} \right) \exp(2\pi i \sum_k \lambda_{w(k)} x_k) \right), \quad (4.13)$$

where $\epsilon(x)$ is a sign-function. This gives the full set of solutions of (4.4) for $(\lambda_1 \leq \dots \leq \lambda_N) \in \mathbb{R}^N$ satisfying the orthogonality condition with respect to the natural pairing:

$$\langle \Phi_\lambda, \Phi_\mu \rangle = \frac{1}{N!} \int dx_1 \cdots dx_N \bar{\Phi}_\lambda(x) \Phi_\mu(x) = G(\lambda) \prod_{i=1}^N \delta(\lambda_i - \mu_i), \quad (4.14)$$

where:

$$G(\lambda) = \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2 + c^2}{(\lambda_i - \lambda_j)^2}, \quad (4.15)$$

Therefore the normalized wave functions are given by:

$$\Phi_\lambda(x) = \sum_{w \in W} (-1)^{l(w)} \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic\epsilon(x_i - x_j)}{\lambda_{w(i)} - \lambda_{w(j)} - ic\epsilon(x_i - x_j)} \right)^{\frac{1}{2}} \exp(2\pi i \sum_k \lambda_{w(k)} x_k). \quad (4.16)$$

The eigenvalue problem for periodic N -particle Hamiltonian of Nonlinear Schrödinger theory can be reformulated in the following way. Consider the eigenfunction problem for the differential operator:

$$\left(-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{n \in \mathbb{Z}} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j + n) \right) \Phi_\lambda(x) = 2\pi^2 \left(\sum_{i=1}^N \lambda_i^2 \right) \Phi_\lambda(x) \quad i = 1, \dots, N. \quad (4.17)$$

The wave function of the periodic Nonlinear Schrödinger equation are the eigenfunction of (4.17) satisfying the following invariance conditions:

$$\Phi_\lambda(x_1, \dots, x_j + 1, \dots, x_N) = \Phi_\lambda(x_1, \dots, x_N), \quad j = 1, \dots, N, \quad (4.18)$$

$$\Phi_\lambda(x_{w(1)}, \dots, x_{w(N)}) = \Phi_\lambda(x_1, \dots, x_N), \quad w \in S_N.$$

These are the conditions of invariance under the action of the affine Weyl group on the space of wave functions.

The solutions can be obtained imposing the additional periodicity conditions on the wave functions (4.16). This leads to the following set of the Bethe Ansatz equations for $(\lambda_1, \dots, \lambda_N)$:

$$\mathcal{F}_j(\lambda) \equiv e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic} = 1, \quad j = 1, \dots, N. \quad (4.19)$$

The set of solutions of these equations can be enumerated by sets of integer numbers $(p_1 \geq \dots \geq p_N)$ - for each ordered set of these integers there is exactly one solution to Bethe Ansatz equations [7]. Let us remark that there is the following equivalent representation for the periodic wave-functions:

$$\tilde{\Phi}_\lambda(x) = \sum_{w \in W} (-1)^{l(w)} \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)} - ic} \right)^{\frac{1}{2} + [x_i - x_j]} \exp(2\pi i \sum_k \lambda_{w(k)} x_k), \quad (4.20)$$

where $[x]$ is an integer part of x defined by the conditions: $[x] = 0$ for $0 \leq x < 1$ and $[x + n] = [x] + n$. It is easy to see that these wave functions are periodic and descend to the wave functions (4.16) if $\lambda = (\lambda_1, \dots, \lambda_N)$ satisfy (4.19). The normalized wave functions in the periodic case are given by:

$$\Phi_\lambda^{norm}(x) = \left(\det \left\| \frac{\partial \log \mathcal{F}_j(\lambda)}{\partial \lambda_k} \right\| \right)^{-1/2} \tilde{\Phi}_\lambda(x) = \mu(\lambda)^{-1/2} \tilde{\Phi}_\lambda(x). \quad (4.21)$$

Note that the normalization factor is closely related to the factor (3.20) arising in the representation of the partition function of Yang-Mills-Higgs theory. Indeed the function $I(\lambda)$ introduced in (3.20) is known in the theory of Nonlinear Schrödinger equations as Yang function [7]; critical points of Yang function are in one to one correspondence with the solutions of Bethe Ansatz equations:

$$\alpha_j(\lambda) = \log \mathcal{F}_j(\lambda) = \frac{\partial I(\lambda)}{\partial \lambda_j} = n_j. \quad (4.22)$$

Below we will see that this is not accidental. Finally note that the periodic Nonlinear Schrödinger theory has an interpretation in terms of the representation theory of the degenerate double affine Hecke algebras introduced by Cherednik [18]. For the details in this regard see [14].

5. Wave-function in topological Yang-Mills-Higgs theory

In this section we provide the evidences for the identification of a natural bases of wave functions of topological Yang-Mills-Higgs theory for $G = U(N)$ (given by a path integral on a disk with the insertion of observables in the center) with the eigenfunctions of the N -particle Hamiltonian operator of Nonlinear Schrödinger theory. First by counting the observables of the theory we show that the phase space of the Yang-Mills-Higgs theory can be considered as a deformation of the phase space of Yang-Mills theory. We compute the cylinder path integral (Green function) and torus partition function in Nonlinear Schrödinger theory (with all higher Hamiltonians) and show that latter coincides with torus partition function in Yang-Mills-Higgs theory (with arbitrary observables turned on). Then, using the explicit representation of the partition function on the two-dimensional torus we derive the transformation properties of the wave functions under large gauge transformations. They are in agreement with the known explicit transformation properties of wave function in Nonlinear Schrödinger theory. Finally, we demonstrate that eigenfunctions of the N -particle Hamiltonian operator of Nonlinear Schrödinger theory indeed coincide with the bases of Yang-Mills-Higgs wave-functions in appropriate polarization.

5.1. Local Q -cohomology

We start with a description of the Hilbert space of the Yang-Mills-Higgs theory using the operator-state correspondence. In the simplest form the operator-state correspondence is as follows. Each operator, by acting on the vacuum state, defines a state in the Hilbert space. In turn for each state there is an operator, creating the state from the vacuum state. Moreover, for the maximal commutative subalgebra of the operators this correspondence should be one to one. For example, the space of local gauge-invariant Q -cohomology classes in topological $U(N)$ Yang-Mills theory is spanned, linearly, by the operators:

$$\mathcal{O}_k^{(0)} = \frac{1}{(2\pi i)^k} \text{Tr} \varphi^k, \quad (5.1)$$

and thus this space coincides with the space of Ad_G -invariant regular functions on the Lie algebra \mathfrak{u}_N . This is in accordance with the description of the Hilbert space of the theory given in Section 2.

We would like to apply the same reasoning to the topological Yang-Mills-Higgs theory. To get economical description of the Hilbert space of the theory one should find a maximal (Poisson) commutative subalgebra of local Q -cohomology classes (where Q given by (3.9)

acts on the space of functions invariant under the symmetries generated by (3.7), (3.8)). Obviously operators (5.1) provide non-trivial cohomology classes. One can show that these operators provide a maximal commutative subalgebra for $c \neq 0$ and therefore the reduced phase space in Yang-Mills-Higgs system can be identified with a phase space of pure Yang-Mills theory. Thus the Hilbert space of Yang-Mills-Higgs theory ($c \neq 0$) can be naturally identified with the Hilbert space of Yang-Mills theory (identified with $c \rightarrow \infty$). The fact that the Hilbert space of Yang-Mills-Higgs theory is the same for all $c \neq 0$ implies that the bases of wave-functions for $c \neq 0$ should be a deformation of the bases for $c = \infty$.

One should stress that this reasoning is not applicable to the case $c = 0$. The local cohomology for $c = 0$ contains additional operators. For example, the following operators provide non-trivial cohomology classes for arbitrary $t \in \mathbb{C}$ and $c = 0$:

$$\mathcal{O}_k^{(0)}(t) = \frac{1}{(2\pi i)^k} \text{Tr}(\varphi_0 + t\varphi_+ - \frac{t}{2}\chi_+^2)^k. \quad (5.2)$$

This is a manifestation of the fact that $c = 0$ theory is a Yang-Mills theory for the complexified group G^c and thus its phase space is given by $\mathcal{M}_C = T^*H^c/W$.

5.2. Gauge transformations of wave function

The discreteness of the spectrum of N -particle Hamiltonian operator in periodic Non-linear Schrödinger theory arises due to periodicity condition on wave functions. Thus the eigenfunctions in the periodic case are given by a subset of eigenfunctions on \mathbb{R}^N descending to S_N -invariant functions on $(S^1)^N$. The eigenfunction (4.16) of the Hamiltonian operator on \mathbb{R}^N represented as sum over elements of symmetric group of simple wave functions. For generic eigenvalues each term of the sum is multiplied by some function under the shift $x_i \rightarrow x_i + n_i$, $n_i \in \mathbb{Z}$ of the coordinates. Below we will show how these multiplicative factors arising in Nonlinear Schrödinger theory can be derived in Yang-Mills-Higgs gauge theory.

We start with a simple case of Yang-Mills theory. The partition function of $U(N)$ Yang-Mills theory on the torus Σ_1 is given by:

$$\begin{aligned} Z_{YM}(\Sigma_1) &= \int_{\mathbb{R}^N/S_N} d^N \lambda \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)} = \\ &= \sum_{(m_1, \dots, m_N) \in P_{++}} e^{-\sum_{k=1}^{\infty} t_k p_k(m+\rho)} \end{aligned} \quad (5.3)$$

where P_{++} is a set of the dominant weights of $U(N)$. The sum over $(n_1, \dots, n_N) \in \mathbb{Z}^N$ has a meaning of the sum over topological classes of $U(1)^N$ -principle bundles on the torus Σ_1 . It results in the replacement of the integration over λ by a sum over a discrete subset. This should be compared with the partition function of dimensionally reduced $U(N)$ Yang-Mills theory on S^1 :

$$Z_{QM}(S^1) = \int_{\mathbb{R}^N/S_N} d^N \lambda e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)}. \quad (5.4)$$

Contrary to the two-dimensional Yang-Mills theory in the last case we do not have any additional restriction on the spectrum $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N/S_N$. The appearance of the additional sum can be traced back to the difference between the Hilbert spaces of dimensionally reduced and non-reduced theories.

The mechanism of the spectrum restriction via the sum over the topological sectors can be explained in terms of the structure of the Hilbert space of the theory as follows. In the Hamiltonian formalism the partition function on a torus is given by the trace of the evolution operator over the Hilbert space of the theory. Let us consider first the dimensionally reduced $U(N)$ Yang-Mills theory. The phase space of the theory is $T^*\mathbb{R}^N/S_N$ where we divide over Weyl group $W = S_N$. To construct the Hilbert space we quantize the phase space using the following polarization. Consider Lagrangian projection $\pi : T^*\mathbb{R}^N \rightarrow \mathbb{R}^N$ supplied with a section. We chose the coordinates on the base as position variables and the coordinates on the fibers as the corresponding momenta. Thus the Hilbert space in this polarization is realized as a space of S_N (skew)-invariant functions on the base \mathbb{R}^N of the projection.

Now consider two-dimensional Yang-Mills theory. For the phase space we have T^*H/S_N where H is Cartan subgroup. We use similar polarization associated with the projection $\pi : T^*H \rightarrow H$. Thus the wave functions are S_N invariant functions on a torus H or equivalently the functions on \mathbb{R}^N invariant under action of the semidirect product of the lattice $P_0 = \pi_1(H)$ and Weyl $W = S_N$ group (i.e. under the action of the affine Weyl group W^{aff}). The lattice P_0 can be interpreted as a lattice of the \mathbb{R}^N -valued constant connections on S^1 which are gauge equivalent to the zero connection. The corresponding gauge transformations act on the wave functions by the shifts $x_j \rightarrow x_j + n_j$, $n_j \in \mathbb{Z}$ of the argument of the wave functions in the chosen polarization and the wave functions in two-dimensional Yang-Mills theory can be obtained by the averaging over this gauge transformations and global gauge transformations by the nontrivial elements of the normalizer of Cartan torus $W = N(H)/H$.

It is possible to relate the averaging over the topologically non-trivial transformations with the sum over topological classes of H bundle on the torus. Note that the maps of S^1 to the gauge group H are topologically classified by $\pi_1(H) = \mathbb{Z}^N$. Consider a connection $A = (A_1, \dots, A_N)$ on a H bundle over a cylinder L , $\partial L = S_+^1 \cup S_-^1$ such that the holonomies along the boundaries S_+^1 and S_-^1 are in the different topological classes $[(m_1, \dots, m_N)] \in \pi_1(H)$ and $[(m_1 + n_1, \dots, m_N + n_N)] \in \pi_1(H)$. Gluing boundaries of the cylinder L we obtain a torus supplied with a connection ∇_A such that the first Chern classes of the bundles corresponding to each $U(1)$ -factor are given by $c_1(\nabla_{A_i}) = \frac{1}{2\pi i} \int_L F(A_j) = n_j$, $j = 1, \dots, N$. Thus we see that the sum over the topologically non-trivial gauge transformations on S^1 can be translated into the sum over topological classes of the H -bundles on the torus.

Let us rederive the partition function of Yang-Mills theory on the torus (5.3) using the averaging procedure. We start with the dimensionally reduced theory. Let us choose a bases in the Hilbert space of the dimensionally reduced Yang-Mills theory given by the S_N skew-invariant eigenfunctions of the quadratic operator $H_2^{(0)} = \text{tr}\varphi^2$. In the polarization discussed above we have:

$$H_2^{(0)} \psi_\lambda(\theta) = -\frac{1}{2} \left(\sum_{j=1}^N \frac{\partial^2}{\partial \theta_j^2} \right) \psi_\lambda(\theta) = 2\pi^2 \sum_{j=1}^N \lambda_j^2 \psi_\lambda(\theta), \quad (5.5)$$

where $(\theta_1, \dots, \theta_N) \in \mathbb{R}^N$. The set of normalized skew-invariant eigenfunctions is given by:

$$\psi_\lambda(\theta) = \sum_{w \in S_N} (-1)^{l(w)} \exp(2\pi i \sum_{j=1}^N \lambda_{w(j)} \theta_j), \quad (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N / W. \quad (5.6)$$

$$\frac{1}{N!} \int_{\mathbb{R}^N} d^N \theta \overline{\psi_\lambda(\theta)} \psi_{\lambda'}(\theta) = (2\pi)^N \sum_{w \in S_N} (-1)^{l(w)} \prod_{j=1}^N \delta(\lambda_{w(j)} - \lambda'_j) = \delta^{(S_N)}(\lambda - \lambda'). \quad (5.7)$$

The integral kernel of the identity operator acting on the skew-symmetric functions can be represented (due to translation invariance it is the function of difference $\theta - \theta'$) as:

$$K_0(\theta, \theta') = K_0(\theta - \theta') = \delta^{(S_N)}(\theta - \theta') = \int_{\mathbb{R}^N / S_N} d^N \lambda \overline{\psi_\lambda(\theta)} \psi_\lambda(\theta'). \quad (5.8)$$

The partition function of the dimensionally reduced Yang-Mills theory on S^1 is given by the trace of a evolution operator and can be written explicitly as:

$$Z_{QM}(S^1) = \text{Tr} e^{-t_2 H_2(\hat{p}, \hat{q})} = \int_{(\mathbb{R}^N \times \mathbb{R}^N) / S_N} d^N \theta d^N \lambda \overline{\psi_\lambda(\theta)} e^{-t_2 H_2^{(0)}(i\partial_\theta, \theta)} \psi_\lambda(\theta) = \quad (5.9)$$

$$= \int_{(\mathbb{R}^N \times \mathbb{R}^N)/S_N} d^N \theta d^N \lambda e^{-t_2 p_2(\lambda)}.$$

The Green function of the theory is:

$$\begin{aligned} G_0(\theta, \theta') &= \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(\theta') e^{-t_2 H_2^{(0)}(i\partial_\theta, \theta)} \psi_\lambda(\theta) = \\ &= \int_{\mathbb{R}^N/S_N} d^N \lambda \psi_\lambda(\theta') e^{-t_2 p_2(\lambda)} \psi_\lambda(\theta). \end{aligned} \quad (5.10)$$

Up to the infinite factor given by the integral over $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$ the integral in (5.9) coincides with the expression (5.4) for the partition function for $t_{i \neq 2} = 0$.

Now consider two-dimensional Yang-Mills theory. In this case we have the periodic eigenvalue problem for (5.5). Then for the normalized eigenfunctions of H_2 we have:

$$\psi_n(\theta) = \sum_{w \in S_N} (-1)^{l(w)} \exp(2\pi i \sum_{j=1}^N (n_{w(j)} + \rho_{w(j)}) \theta_j), \quad (n_1, \dots, n_N) \in P_{++}, \quad (5.11)$$

$$\frac{1}{N!} \int_{(S^1)^N} d^N \theta \overline{\psi}_n(\theta) \psi_{n'}(\theta) = \sum_{w \in S_N} (-1)^{l(w)} \prod_{j=1}^N \delta_{n_{w(j)}, n'_j} = \delta_{n, n'}. \quad (5.12)$$

Here $\rho = (\rho_1, \dots, \rho_N)$ is a half-sum of the positive roots of \mathfrak{u}_N . The integral kernel of the identity operator can be represented as:

$$K(\theta, \theta') = K(\theta - \theta') = \delta_{(S^1)^N}(\theta - \theta') = \sum_{n \in P_{++}} \overline{\psi}_n(\theta) \psi_n(\theta'). \quad (5.13)$$

The partition function of the Yang-Mills theory on a torus Σ_1 is given by the trace of a evolution operator and:

$$Z_{YM}(\Sigma_1) = \text{Tr} e^{-t_2 H_2(\hat{p}, \hat{q})} = \sum_{n \in P_{++}} \int_{(S^1)^N} d^N x \overline{\psi}_n(\theta) e^{-t_2 H_2^{(0)}(i\partial_\theta, \theta)} \psi_n(\theta). \quad (5.14)$$

The kernel for the periodic case can be obviously represented as a matrix element of the projection operator as follows:

$$K(\theta, \theta') = \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(\theta) P(\lambda) \psi_\lambda(\theta'), \quad (5.15)$$

where the wave-functions $\psi_\lambda(\theta)$ are given by (5.6) and:

$$P(\lambda) = \sum_{m \in \mathbb{Z}^N} \prod_{j=1}^N \delta(\lambda_j - m_j) = \sum_{k \in \mathbb{Z}^N} e^{2\pi i \sum_{j=1}^N \lambda_j k_j}. \quad (5.16)$$

Equivalently we have:

$$\begin{aligned}
K(\theta, \theta') &= \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi_\lambda}(\theta) e^{2\pi i \sum_{j=1}^N \lambda_j k_j} \psi_\lambda(\theta') = \\
&= \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi_\lambda}(\theta) \psi_\lambda(\theta' + k).
\end{aligned} \tag{5.17}$$

We conclude that the Green function $G(\theta, \theta')$ (the path integral on the cylinder with insertion of $\exp(-t_2 H_2^{(0)})$) is represented as:

$$G_{YM}(\theta, \theta') = \sum_{n \in \mathbb{Z}^N} \int_{\lambda \in \mathbb{R}^N/S_N} d^N \lambda \overline{\psi_\lambda}(\theta) e^{2\pi i \sum_{j=1}^N \lambda_j n_j} e^{-t_2 p_2(\lambda)} \psi_\lambda(\theta') \tag{5.18}$$

or equivalently as:

$$G_{YM}(\theta, \theta') = \sum_{k \in \mathbb{Z}^N} \int_{\lambda \in \mathbb{R}^N/S_N} d^N \lambda \overline{\psi_\lambda}(\theta) e^{-t_2 p_2(\lambda)} \psi_\lambda(\theta' + k). \tag{5.19}$$

Let us note that the identities in (5.17), (5.19) are based on the following transformation property of the complete set of skew-symmetric normalized wave-functions on \mathbb{R}^N :

$$\psi_\lambda(\theta + k) = \sum_{w \in S_N} (-1)^{l(w)} e^{2\pi i \sum_{j=1}^N \lambda_{w(j)} k_j} e^{2\pi i \sum_{j=1}^N \lambda_{w(j)} \theta_j}. \tag{5.20}$$

Thus each elementary term in the averaging over S_N is multiplied on the simple exponent factor entering the description of the projector (5.16). Let us also note that the shift transformations in (5.20) can be interpreted as large gauge transformations in Yang-Mills theory discussed above.

The representation (5.19) can be written in the following form:

$$G_{YM}(\theta, \theta') = \sum_{k \in \mathbb{Z}^N} G_0(\theta, \theta' + k). \tag{5.21}$$

If we set the coupling t_2 to zero, $t_2 = 0$, we recover the formula (5.17) for $K(\theta, \theta')$:

$$K(\theta, \theta') = \sum_{k \in \mathbb{Z}^N} K_0(\theta, \theta' + k). \tag{5.22}$$

For the partition function of Yang-Mills theory on a torus we get (after setting $\theta = \theta'$ above and integrating over x):

$$Z_{YM}(\Sigma_1) = \sum_{n \in \mathbb{Z}^N} \int_{\lambda \in \mathbb{R}^N/S_N} d^N \lambda \int_{(S^1)^N} d^N \theta \overline{\psi_\lambda}(\theta) e^{2\pi i \sum_{j=1}^N \lambda_j n_j} e^{-t_2 p_2(\lambda)} \psi_\lambda(\theta) =$$

$$= \sum_{m \in P_{++}} e^{-t_2 p_2(m+\rho)}.$$

and this coincides with the representation (5.3). Note that obvious relations between (5.17), (5.21), (5.22) and the averaging over the topologically non-trivial gauge transformations discussed above.

Now we are finally ready to consider the case of Yang-Mills-Higgs theory. As it was conjectured above one can chose as a bases of the wave-functions the bases of eigenfunctions of the set of the Hamiltonian operators in N -particle subsector of Nonlinear Schrödinger theory. Below we construct the Green function and partition function in Nonlinear Schrödinger theory and demonstrate that identifying the Hamiltonian operator with quadratic observable $\mathcal{O}_2^{(0)} = \frac{1}{(2\pi)^2} \text{Tr} \varphi_0^2$ in Yang-Mills-Higgs theory we reproduce the partition function of Yang-Mills-Higgs theory on a torus.

Let us start with the construction of the kernel of the unit operator in the bases of the N -particle eigenfunctions of the Nonlinear Schrödinger theory. The representation for the kernel (5.17) can be straightforwardly generalized to this case:

$$\tilde{K}(x, x') = \sum_{(\lambda_1, \dots, \lambda_N) \in \mathcal{R}_N} \overline{\Phi_\lambda^{norm}}(x) \Phi_\lambda^{norm}(x') = \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\Phi_\lambda}(x) P(\lambda) \Phi_\lambda(x'), \quad (5.23)$$

where $\Phi_\lambda(x)$ are normalized skew-invariant eigenfunctions on \mathbb{R}^N given by (4.16), $\Phi_\lambda^{norm}(x)$ are normalized periodic eigenfunctions given by (4.21) and the sum goes over the set \mathcal{R}_N of the solutions of Baxter Ansatz equations. The projector here is given by:

$$P(\lambda) = \mu(\lambda) \sum_{m \in \mathbb{Z}^N} \prod_{j=1}^N \delta(\alpha_j(\lambda) - m_j) = \sum_{(\lambda_1^*, \dots, \lambda_N^*) \in \mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda_j^*) \quad (5.24)$$

where $\alpha_j(\lambda)$ are defined as follows (compare with (3.22)):

$$\alpha_j(\lambda) = \lambda_j + \frac{1}{2\pi i} \sum_{k \neq j} \log \left(\frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic} \right). \quad (5.25)$$

Then we have:

$$\begin{aligned} \tilde{K}(x, x') &= \sum_{n \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \mu(\lambda) \overline{\Phi_\lambda}(x) e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{l \neq j} \left(\frac{\lambda_l - \lambda_j - ic}{\lambda_l - \lambda_j + ic} \right)^{n_j} \Phi_\lambda(x') = \\ &= \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\Phi_\lambda}(x) \Phi_\lambda(x' + n). \end{aligned} \quad (5.26)$$

The last equality is a consequence of the following property of the eigenfunctions (4.20) of the N -particle Hamiltonian in Nonlinear Schrödinger theory:

$$\begin{aligned} \Phi_\lambda(x+n) &= \sum_{w \in W} (-1)^{l(w)} \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)} - ic} \right)^{n_i} \exp(2\pi i \sum_m \lambda_{w(m)} n_m) \times \\ &\times \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)} - ic} \right)^{\frac{1}{2} + [x_i - x_j]} \exp(2\pi i \sum_k \lambda_{w(k)} x_k), \end{aligned} \quad (5.27)$$

These wave functions are periodic and descend to the wave functions (4.16) if $\lambda = (\lambda_1, \dots, \lambda_N)$ satisfy (4.19).

The representation for the kernel (5.26) leads to the following representation for the Green function (cylinder path integral) and torus partition function for $t_{i \neq 2} = 0$:

$$\begin{aligned} G_{NLS}(x, x') &= \\ &= \sum_{n \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \mu(\lambda) \overline{\Phi_\lambda(x)} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{l \neq j} \left(\frac{\lambda_l - \lambda_j - ic}{\lambda_l - \lambda_j + ic} \right)^{n_j} e^{-t_2 p_2(\lambda)} \Phi_\lambda(x'), \end{aligned} \quad (5.28)$$

or same:

$$G_{NLS}(x, x') = \sum_{k \in \mathbb{Z}^N} G_{NLS}^0(x, x' + k) = \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\Phi_\lambda(x)} e^{-t_2 p_2(\lambda)} \Phi_\lambda(x' + k).$$

Similarly for kernel: $\tilde{K}(x, x') = \sum_{k \in \mathbb{Z}^N} \tilde{K}_0(x, x' + k)$ since the kernel is a Green function at $t_2 = 0$. Integrating over x after setting $x = x'$ we obtain the representation for the partition function on the torus:

$$\begin{aligned} Z_{NLS}(\Sigma_1) &= \\ &= \int_{\mathbb{R}^N/S_N} d^N \lambda \mu(\lambda) \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{l \neq j} \left(\frac{\lambda_l - \lambda_j - ic}{\lambda_l - \lambda_j + ic} \right)^{n_j} e^{-t_2 p_2(\lambda)}. \end{aligned} \quad (5.29)$$

This is in a complete agreement with a representation for the partition function of $U(N)$ Yang-Mills-Higgs theory on a torus discussed in Section 3, formula (3.24). Note that one can repeat the same arguments for all observables and higher differential operators of Nonlinear Schrödinger theory, traces of higher powers of Dunkle operator from Section 4, by simply turning on all other couplings t_k . The identification of the representation of the partition function of Nonlinear Schrödinger operator and Yang-Mills-Higgs theory on the torus strongly suggests that the full equivalence of the theories.

5.3. More precise relation between wave-functions

Now we establish a precise relation between wave-functions in Yang-Mills-Higgs $U(N)$ gauge theory and N -particle sector in Nonlinear Schrödinger equation theory. Let us stress that the localization procedure in topological theories is most straightforward in the case of compact manifolds. Thus for example in the case of the torus in Yang-Mills-Higgs theory we recover unambiguously the spectral invariants of the operators $\mathcal{O}_k^{(0)}$ acting on the Hilbert space of the theory. On the classical level this means that the results do not depend on the total derivative terms in the Lagrangian density. It was shown above that the spectrum of operators $\mathcal{O}_k^{(0)} = \frac{1}{(2\pi i)^k} \text{Tr} \varphi_0^k$ coincides with the spectrum of quantum Hamiltonians of Nonlinear Schrödinger theory restricted to N -particle sub-sector. The identification of the bases of eigenfunctions implies a fixing additional data - the choice of polarization for symplectic form in the Hamiltonian quantization. Below we provide a choice of polarization in Yang-Mills-Higgs theory leading to identification of wave-functions with eigenfunctions in Nonlinear Schrödinger theory. To simplify the presentation below we omit symmetrization of wave-functions.

Let consider Yang-Mills-Higgs theory on Σ_h . The result of abelianization can be formulated in the form of an effective gauge theory with the gauge group $U(1)^N$ and the bosonic part of the action (3.15):

$$\begin{aligned} \Delta S = & \int_{\Sigma_h} d^2 z \sum_{i,j=1}^N \log \left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic} \right) F(A)^i + \\ & + \frac{1}{2} \int_{\Sigma_h} d^2 z \sum_{i,j=1}^N \log((\varphi_0)_i - (\varphi_0)_j + ic) R^{(2)} \sqrt{g}; \end{aligned} \quad (5.30)$$

We consider a family of deformations (3.14) of the theory:

$$\Delta S_{YMH} = - \sum_{k=1}^{\infty} t_k \int_{\Sigma_h} d^2 z \text{Tr} \varphi_0^k \text{vol}_{\Sigma_h} = - \sum_{k=1}^{\infty} \sum_{j=1}^N t_k \int (\varphi_0)_j^k \text{vol}_{\Sigma_h} \quad (5.31)$$

where we impose the condition that $t_k \neq 0$ for a finite subset of the indexes $k \in \mathbb{Z}$.

In the case of $\Sigma_h = T^2$ the Feynman path integral in the two-dimensional quantum field theory with the action (5.30) is equivalent to the path integral in the one-dimensional theory on S^1 with the phase space T^*H supplied with the following symplectic form:

$$\omega = d \left(\sum_{j=1}^N (\lambda_j + \sum_{k \neq j} \log \left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \right)) d\theta_j \right), \quad \theta_j \sim \theta_j + 2\pi i n_j, \quad n_j \in \mathbb{Z}, \quad (5.32)$$

which is a c -deformation of the standard symplectic form $\omega = \sum_{j=1}^N d\lambda_j \wedge d\theta_j$ on T^*H where $\{\theta_j\}$ are coordinates on H and $\{\lambda_j\}$ are coordinates in the fiber of the projection $T^*H \rightarrow H$. Note that λ_j can be identified with (z, \bar{z}) -independent $(\varphi_0)_i$.

This symplectic form (5.32) can be easily transformed into a canonical one:

$$\omega = d\left(\sum_{j=1}^N \alpha_j(\lambda) d\theta_j\right) = \sum_{j=1}^N d\alpha_j(\lambda) \wedge d\theta_j, \quad (5.33)$$

where

$$\alpha_j(\lambda) = \lambda_j + \sum_{k=1, k \neq j}^N \log\left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}\right). \quad (5.34)$$

Quantization in θ -polarization is given by the following realization:

$$\hat{\alpha}_j = -i \frac{\partial}{\partial \theta_j}, \quad \hat{\theta}_j = \theta_j. \quad (5.35)$$

A bases of wave functions then can be constructed using a complete set of common eigenfunctions of the operators $\hat{\alpha}_j$:

$$\Psi_\lambda(\theta) = e^{i \sum_{j=1}^N \alpha_j(\lambda) \theta_j} = \prod_{j,k=1}^N \left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}\right)^{\theta_j} e^{i \sum_{j=1}^N \lambda_j \theta_j}, \quad (5.36)$$

$$\hat{\alpha}_j \Psi_\lambda(\theta) = \alpha_j(\lambda) \Psi_\lambda(\theta). \quad (5.37)$$

The deformation (5.31) leads to a consideration of the following set of Hamiltonians in the effective one-dimensional theory:

$$\mathcal{H}_k = \sum_{j=1}^N \hat{\lambda}_j^k, \quad (5.38)$$

where $\hat{\lambda}$ is related to the derivative in θ through:

$$\hat{\alpha}_j = \hat{\lambda}_j + \sum_{k \neq j} \log\left(\frac{\hat{\lambda}_j - \hat{\lambda}_k + ic}{\hat{\lambda}_j - \hat{\lambda}_k - ic}\right) = -i \frac{\partial}{\partial \theta_j}. \quad (5.39)$$

Computations from Section 3 show that the eigenvalues of the Hamiltonians are given by $E_k = \sum_{j=1}^N \lambda_j^k$ with λ 's subject to (for periodicity conditions $\theta_j \sim \theta_j + 2\pi m_j$, $m_j \in \mathbb{Z}$):

$$\alpha_j(\lambda) = \lambda_j + \sum_{k \neq j} \log\left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}\right) = n_j, \quad n_j \in \mathbb{Z}. \quad (5.40)$$

Let us remark that in θ -representation Hamiltonians (5.38) act on wave-functions by complicated integral operators due to the non-trivial relations (5.34) between $\widehat{\alpha}_j$ and $\widehat{\lambda}_k$.

Compare now the bases (5.36) of common eigenfunctions of \mathcal{H}_k , $k \in \mathbb{Z}_+$ with the bases of common eigenfunctions for the Yang N -particle system:

$$\Psi_\lambda^Y(x) = \prod_{j,k=1}^N \left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \right)^{[x_j]} e^{i \sum_{j=1}^N \lambda_j x_j}, \quad (5.41)$$

where $[x]$ is an integer part of x . Here we also impose periodicity conditions $x_j \sim x_j + 2\pi m_j$, $m_j \in \mathbb{Z}$ and thus:

$$\lambda_j + \sum_{k \neq j} \log \left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \right) = n_j, \quad n_j \in \mathbb{Z}. \quad (5.42)$$

The Yang system can be considered as a quantization of T^*H with the symplectic structure:

$$\omega = \sum_{j=1}^N dp_j \wedge dx_j, \quad x_j \sim x_j + 2\pi n_j, \quad n_j \in \mathbb{Z}. \quad (5.43)$$

In x -polarization we have:

$$\widehat{p}_j = -i \frac{\partial}{\partial x_j}, \quad \widehat{x}_j = x_j. \quad (5.44)$$

The Hamiltonians \mathcal{H}_k of Yang system can be expressed as symmetric functions of Dunkle operators \mathcal{D}_j (see Section 4), and for $k = 2$ we have:

$$\mathcal{H}_2 = \frac{1}{2} \sum_{i=1}^N \mathcal{D}_i^2 = \frac{1}{2} \sum_{i=1}^N \widehat{p}_i^2 + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j), \quad (5.45)$$

The eigenvalues E_k of the Hamiltonians H_k are given by:

$$E_k = \sum_{j=1}^N \lambda_j^k, \quad \lambda_j + \sum_{k \neq j} \log \left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \right) = n_j, \quad (5.46)$$

and coincides with the spectrum in Yang-Mills-Higgs theory.

The two bases of wave functions (5.36) and (5.41) are related by a unitary operator with the integral kernel given by:

$$K(\theta, x) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \Psi_{\lambda_{\mathbf{n}}}(\theta) \overline{\Psi_{\lambda_{\mathbf{n}}}^Y(x)}, \quad (5.47)$$

where the sum is over solutions $\lambda_{\mathbf{n}} = (\lambda_1(\mathbf{n}), \dots, \lambda_N(\mathbf{n}))$ of the equations (5.42) for all $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$. The only invariant that does not depend on the choice of the bases of wave functions is the common spectrum of the operators \mathcal{H}_k which is captured by partition function. The coincidence of the partition functions for Yang-Mills-Higgs theory and Yang system was demonstrated in the previous subsection.

Let us stress that in θ -representation there is no simple way to write eigenfunction problem for wave-functions in terms of the differential operators while in x -representation the eigenfunction problem for Hamiltonians \mathcal{H}_k acting on $\Psi_\lambda^Y(x)$ is given by the Yang equations and its higher-derivative analogs. This explains the relevance of the x -polarization in Yang-Mills-Higgs theory deformed by local operators $\mathcal{O}_k^{(0)}$.

6. Generalization of G/G gauged WZW model

In this section we describe following [3] another instance of the relation between two-dimensional gauge theories and quantum integrability in many-body integrable systems. We consider a generalization of GWZW theory and demonstrate that partition function can be represented as a sum over solutions of a certain generalization of Bethe Ansatz equation.

We start with the definition of the set of fields and the action of the odd and even symmetries in the spirit of (3.7), (3.8), (3.9). Let us note that the gauged Wess-Zumino-Witten model can be obtained from the topological Yang-Mills theory by using the group-valued field g instead of algebra-valued field φ . Introduce the set of fields $(A, \psi_A, \Phi, \psi_\Phi, \chi_\pm, \varphi_\pm, g)$ and $t \in \mathbb{R}^*$ with the following action of the odd and even symmetries:

$$\begin{aligned}
\mathcal{L}_{(g,t)} A^{(1,0)} &= (A^g)^{(1,0)} - A^{(1,0)} & \mathcal{L}_{(g,t)} A_A^{(0,1)} &= -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}, & (6.1) \\
\mathcal{L}_{(g,t)} \psi_A^{(1,0)} &= -g\psi_A^{(1,0)}g^{-1} + \psi_A^{(1,0)}, & \mathcal{L}_{(g,t)} \psi_A^{(0,1)} &= g^{-1}\psi_A^{(0,1)}g - \psi_A^{(0,1)}, & \mathcal{L}_{(g,t)} g &= 0, \\
\mathcal{L}_{(g,t)} \Phi^{(1,0)} &= tg\Phi^{(1,0)}g^{-1} - \Phi^{(1,0)}, & \mathcal{L}_{(g,t)} \Phi^{(0,1)} &= -t^{-1}g^{-1}\Phi^{(0,1)}g + \Phi^{(0,1)}, \\
\mathcal{L}_{(g,t)} \psi_\Phi^{(1,0)} &= tg\psi_\Phi^{(1,0)}g^{-1} - \psi_\Phi^{(1,0)}, & \mathcal{L}_{(g,t)} \psi_\Phi^{(0,1)} &= -t^{-1}g^{-1}\psi_\Phi^{(0,1)}g + \psi_\Phi^{(0,1)}, \\
\mathcal{L}_{(g,t)} \chi_+ &= tg\chi_+g^{-1} - \chi_+, & \mathcal{L}_{(g,t)} \chi_- &= -t^{-1}g^{-1}\chi_-g + \chi_- \\
\mathcal{L}_{(g,t)} \varphi_+ &= t^{-1}g\varphi_+g^{-1} - \varphi_+, & \mathcal{L}_{(g,t)} \varphi_- &= -tg^{-1}\varphi_-g + \varphi_- \\
Q A &= i\psi_A, & Q \psi_A^{(1,0)} &= i(A^g)^{(1,0)} - iA^{(1,0)}, & Q \psi_A^{(0,1)} &= -i(A^{g^{-1}})^{(0,1)} + iA^{(0,1)}, & (6.2)
\end{aligned}$$

$$Qg = 0,$$

$$Q\Phi = i\psi_\Phi, \quad Q\psi_\Phi^{(1,0)} = tg\Phi^{(1,0)}g^{-1} - \Phi^{(1,0)}, \quad Q\psi_\Phi^{(0,1)} = -t^{-1}g^{-1}\Phi^{(0,1)}g - \Phi^{(0,1)}, \quad (6.3)$$

$$Q\chi_\pm = i\varphi_\pm, \quad Q\varphi_+ = tg\chi_+g^{-1} - \chi_+, \quad Q\varphi_- = -t^{-1}g^{-1}\chi_-g + \chi_-.$$

We have $Q^2 = \mathcal{L}_{(g,t)}$ and Q can be considered as a BRST operator on the space of $\mathcal{L}_{(g,t)}$ -invariant functionals.

We define the action of the theory in analogy with the construction of the action for Yang-Mills-Higgs theory as follows:

$$S = S_{GWZW} + [Q, \int_{\Sigma_h} d^2z \operatorname{Tr} \left(\frac{1}{2} \Phi \wedge \psi_\Phi + \right. \quad (6.4)$$

$$\left. + \tau_1 (\varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)}) + \tau_2 (\chi_+ \varphi_- + \chi_- \varphi_+) \operatorname{vol}_{\Sigma_h} \right]_+.$$

Taking $\tau_1 = 0$, $\tau_2 = 1$ and applying the standard localization technique to this theory we obtain for the partition function:

$$Z_{GWZWH}(\Sigma_h) = \frac{e^{(1-h)a(t)}}{|W|} \int_H d^N \lambda \mu_q(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m (k+c_v)} \times \quad (6.5)$$

$$\times \prod_{j \neq k} (e^{2\pi i (\lambda_j - \lambda_k)} - 1)^{n_j - n_k + 1 - h} \prod_{j, k} (te^{2\pi i (\lambda_j - \lambda_k)} - 1)^{n_j - n_k + 1 - h},$$

where $a(t)$ is a h -independent constant, the integral goes over the Cartan torus $H = (S^1)^N$ and

$$\mu_q(\lambda) = \det \left\| \frac{\partial \beta_j(\lambda)}{\partial \lambda_k} \right\|, \quad (6.6)$$

with:

$$e^{2\pi i \beta_j(\lambda)} = e^{2\pi i \lambda_j (k+c_v)} \prod_{k \neq j} \frac{te^{2\pi i (\lambda_j - \lambda_k)} - 1}{te^{2\pi i (\lambda_k - \lambda_j)} - 1}. \quad (6.7)$$

We can rewrite this formula in the form similar to (3.21):

$$Z_{GWZWH}(\Sigma_h) = \frac{e^{(1-h)a(t)}}{|W|} \int_H d^N \lambda \mu_q(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \beta_m(\lambda) n_m} \times \quad (6.8)$$

$$\times \prod_{j < k} (e^{i\pi (\lambda_j - \lambda_k)} - e^{i\pi (\lambda_k - \lambda_j)})^{2-2h} \prod_{j < k} |te^{i\pi (\lambda_j - \lambda_k)} - e^{i\pi (\lambda_k - \lambda_j)}|^{2-2h}.$$

Summation over integers in (6.5) leads to the following restriction on the integration parameters:

$$e^{2\pi i \lambda_j (k+c_v)} \prod_{k \neq j} \frac{te^{2\pi i(\lambda_j - \lambda_k)} - 1}{te^{2\pi i(\lambda_k - \lambda_j)} - 1} = 1 \quad , \quad i = 1, \dots, N, \quad (6.9)$$

It is useful to rewrite the equations (6.9) in the standard form of the Bethe Ansatz equations:

$$e^{2\pi i \lambda_j (k+c_v)} \prod_{k \neq j} \frac{\sin(i\pi(\lambda_j - \lambda_k + ic))}{\sin(i\pi(\lambda_j - \lambda_k - ic))} = 1 \quad , \quad i = 1, \dots, N, \quad (6.10)$$

This clearly shows that we are dealing with a kind of XXZ quantum integrable chain. The particular form (6.10) can be obtained by the taking the limit $s \rightarrow -i\infty$ in the following Bethe equations:

$$\left(\frac{\sin(i\pi(\lambda_j - isc))}{\sin(i\pi(\lambda_j + isc))} \right)^{(k+c_v)} \prod_{k \neq j} \frac{\sin(i\pi(\lambda_j - \lambda_k + ic))}{\sin(i\pi(\lambda_j - \lambda_k - ic))} = 1 \quad , \quad i = 1, \dots, N, \quad (6.11)$$

corresponding to formal limit of the infinite spin s of XXZ chain.

The partition function is the generalization of Yang-Mills-Higgs theory, discussed above, and can be written in the following form:

$$Z_{GWZWH}(\Sigma_h) = \sum_{\lambda_i \in \mathcal{R}_q} (D_\lambda^q)^{2-2h}, \quad (6.12)$$

where \mathcal{R}_q is a set of the solutions of (6.9) and:

$$D_\lambda^q = \mu_q(\lambda)^{-1/2} \prod_{i < j} (q^{\frac{1}{2}(\lambda_i - \lambda_j)} - q^{\frac{1}{2}(\lambda_j - \lambda_i)}) \prod_{i < j} |tq^{\frac{1}{2}(\lambda_i - \lambda_j)} - q^{\frac{1}{2}(\lambda_j - \lambda_i)}|, \quad (6.13)$$

where we use the standard parametrization $q = \exp(2\pi i/(k + c_v))$. Note that in the limit $t \rightarrow \infty$ equation (6.9) and the expression for the partition function (6.13) up to an overall scaling factor become the corresponding expressions for a gauged Wess-Zumino-Witten model. Finally note that the form of (6.9) and the explicit expressions for the q -Casimir operators, playing the role of the Hamiltonians, strongly imply the description of the wave functions of the theory in terms of the wave functions in a particular XXZ finite spin chain. This proposition will be discussed in details elsewhere.

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References

- [1] G. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs Branches*, Commun. Math. Phys. **209** (2000) 97, [arXiv:hep-th/9712241].
- [2] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. 55 (1987), 59-126.
- [3] A. Gerasimov and S. Shatashvili, *Higgs bundles, gauge theories and quantum groups*, [arXiv:hep-th/0609024].
- [4] E. H. Lieb, W. Liniger, *Exact analysis of an interacting Bose gas I. The general solution and the ground state*, Phys. Rev. (2), **130** (1963), 1605.
- [5] F. A. Berezin, G. P. Pohil, V. M. Finkelberg, Vestnik MGU **1**, 1 (1964) 21.
- [6] C. N. Yang, *Some exact results for the many-body problems in one dimension with repulsive delta-function interaction*, Phys.Rev.Lett. **19** (1967), 1312.
- [7] C. N. Yang, C. P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. **10** (1969), 1115.
- [8] P. P. Kulish, S. V. Manakov, L. D. Faddeev, *Comparison of the exact quantum and quasiclassical results for the nonlinear Schrödinger equation*, Theor. Math. Phys. **28** (1976) 615.
- [9] L. D. Faddeev, L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, (1980).
- [10] L. D. Faddeev, *How Algebraic Bethe Ansatz works for integrable model*, “Relativistic gravitation and gravitational radiation”, Proceedings of Les Houches School of Physics (1995) p. 149, [arXiv:hep-th/9605187].
- [11] S. Murakami, M. Wadati *Connection between Yangian symmetry and the quantum inverse scattering method*, J. Phys. A: Math. Gen. **29** (1993) 7903.
- [12] K. Hikami, *Notes on the structure of the δ -function interacting gas. Intertwining operator in the degenerate affine Hecke algebra*, J. Phys. A: Math. Gen **31** (1998) No. 4.
- [13] G. J. Heckman, E. M. Opdam, *Yang’s system of particles and Hecke algebras*, Annals of Math. **145** (1997), 139.
- [14] E. Emsiz, E. M. Opdam, J. V. Stokman, *Periodic integrable systems with delta-potentials*, [arXiv:math.RT/0503034].
- [15] V. G. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funct. Anal. Appl. **20** (1986) 58.
- [16] G. Lusztig, *Affine Hecke algebras and their graded version*, J. AMS **2**, (1989) 3, 599.
- [17] T. C. Dunkl, *Differential-Difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311**, no.1 (1989), 167.
- [18] I. Cherednik, *A unification of Knizhnik-Zamolodchikov and Dunkle operators via affine Hecke algebras*, Invent. Math. J. **106** (1991), 411.