

Higgs Bundles, Gauge Theories and Quantum Groups

Anton A. Gerasimov^{1,2,3} and **Samson L. Shatashvili**^{2,3,4}

¹ *Institute for Theoretical and Experimental Physics, Moscow, 117259, Russia*

² *School of Mathematics, Trinity College, Dublin 2, Ireland*

³ *The Hamilton Mathematics Institute TCD, Dublin 2, Ireland*

⁴ *IHES, 35 route de Chartres, Bures-sur-Yvette, FRANCE*

The appearance of the Bethe Ansatz equation for the Nonlinear Schrödinger equation in the equivariant integration over the moduli space of Higgs bundles is revisited. We argue that the wave functions of the corresponding two-dimensional topological $U(N)$ gauge theory reproduce quantum wave functions of the Nonlinear Schrödinger equation in the N -particle sector. This implies the full equivalence between the above gauge theory and the N -particle sub-sector of the quantum theory of Nonlinear Schrödinger equation. This also implies the explicit correspondence between the gauge theory and the representation theory of degenerate double affine Hecke algebra. We propose similar construction based on the G/G gauged WZW model leading to the representation theory of the double affine Hecke algebra. The relation with the Nahm transform and the geometric Langlands correspondence is briefly discussed.

1. Introduction

In [1] a relation between a certain type of two-dimensional Yang-Mills theory and the Bethe Ansatz equations for the quantum theory of the Nonlinear Schrödinger equation was uncovered. The topological Yang-Mills-Higgs theory considered in [1] captures the hyperkähler geometry of the moduli space of Higgs bundles introduced in [2] by Hitchin. It was shown that the path integral in this theory can be localized on the disconnected set whose components are naturally enumerated by the solutions of a system of the Bethe Ansatz equations. The conceptual explanation of the appearance of the Bethe Ansatz equations as a result of the localization in the topological Yang-Mills-Higgs theory, as well as potential consequences were missing.

To elucidate the structure of the theory we consider the space of wave functions of the two-dimensional gauge theory introduced in [1]. We argue that this space can be identified with the space of wave-functions in the N -particle sector of the quantum theory of the Nonlinear Schrödinger equation constructed in the framework of the coordinate Bethe Ansatz (see e.g. [3], [4]). This implies the equivalence between two seemingly different quantum field theories. Taking into account the interpretation of the coordinate Bethe Ansatz in the Nonlinear Schrödinger theory via the Quantum Inverse Scattering Method (algebraic Bethe Ansatz) [5] this also provides a direct correspondence between gauge theory based on the moduli problem for Higgs bundles and the representation theory of quantum groups.

The partition function of the theory considered in [1] depends on an additional parameter playing the role of the first Chern class of the tautological line bundle on the classifying space $BU(1)$ of the $U(1)$ group in the description of the equivariant cohomology of the Hitchin moduli space. In the the Nonlinear Schrödinger theory the same parameter plays the role of the coupling constant.

In addition we show that the considerations of [1] can be generalized from the Yang-Mills-Higgs theory to G/G gauged WZW model. The corresponding partition function is expressed in terms of solutions of Bethe Ansatz equations similar to the ones for XXZ spin chains. Thus presumably the wave functions for $G = U(N)$ can be identified with the wave functions of the spin chains. One can suspect that the relations discussed in this paper are more general and other examples considered in [1] have similar interpretation in terms of the representation theory of quantum groups¹. The case of the instanton moduli

¹ See in this respect [6],[7] where the role of the moduli spaces of monopoles in the representation theory of infinite-dimensional quantum groups was studied.

space in the four-dimensional Yang-Mills theory studied in [1] looks especially interesting in this regard and will be considered elsewhere.

Let us finally note that the revealed correspondence between topological quantum field theories and integrable structures captured by the Bethe Ansatz equations obviously imply some relation with the old standing wish to unify three-dimensional hyperbolic geometry, two dimensional conformal/integrable theories and (some fragments of) algebraic K-theory [8], [9], [10], [11].

The plan of the paper is as follows. In Section 2 we recall the standard facts from the two-dimensional Yang-Mills theory and the G/G gauged WZW model. This provides a template for further considerations in the Yang-Mills-Higgs theory. This part can be skipped by the reader familiar with the subject. In Section 3, following [1], the topological Yang-Mills-Higgs theory is introduced and the application of the cohomological localization technique is discussed. We also provide the explicit description of the two important limiting cases: $c \rightarrow \infty$ and $c \rightarrow 0$ - these limiting cases are instructive in establishing the correspondence between answers computed in this model and representation theory. In Section 4 we recall the description of exact N -particle wave functions in the quantum theory of the Nonlinear Schrödinger equation emphasizing the role of the degenerate double affine Hecke algebra. We also consider the same limiting cases: $c \rightarrow \infty$ and $c \rightarrow 0$. In addition, we stress the fact that the quantum wave functions of Nonlinear Schrödinger equation are $p \rightarrow 1$ limits of the (generalized) spherical functions for $GL(N, \mathbb{Q}_p)$ with p - prime given by Hall-Littlewood polynomials. In Section 5 we propose the explicit expressions for the exact wave functions in the Yang-Mills-Higgs theory and identify the bases of wave functions with the bases of eigenfunctions of the Hamiltonian in the finite-particle sector for Nonlinear Schrödinger equation. This provides the conceptual explanation of the appearance of the Bethe Ansatz equations in [1]. In Section 6 we give the equivariant cohomology description of the Hilbert space in the topological Yang-Mills-Higgs theory. In Sections 7 we discuss the connection between the gauge theory and the quantum the Nonlinear Schrödinger integrable system using the Nahm duality and comment on the relation with the geometric Langlands correspondence. In Section 8 the relevant generalizations of G/G gauge Wess-Zumino-Witten is proposed and partition function is derived. We conclude with the discussion of the possible general framework.

We include two relevant topics in the Appendices. In Appendix A we provide the twistor type description for the Yang-Mills-Higgs theory and Appendix B is devoted to the topic of quantization of a singular manifold relevant to the construction of the wave function in the main part of the paper.

2. Two-dimensional gauge theories with compact group

There is an interesting class of two-dimensional gauge theories that are exactly solvable on an arbitrary Riemann surface. The simplest examples are given by the Yang-Mills theory on a Riemann surface Σ with a gauge group $\mathcal{G}_\Sigma = \text{Map}(\Sigma, G)$ with G - a compact Lie group [12], [13],[14], [15], [16], [17], and more generally by the G/G gauged WZW theory [18], [19]. The Lagrangian of the G/G gauged WZW model depends on an integer number k and the two-dimensional Yang-Mills theory can be recovered in the limit $k \rightarrow \infty$. In all these cases the space of the classical solutions of the theory is naturally described in terms of the moduli spaces associated with the underlying Riemann surface and the partition function can be represented as a sum over the set of irreducible representations of some algebraic object. For the Yang-Mills theory it is the set of finite-dimensional representations of the group G and for the G/G gauged WZW model it is the set of finite-dimensional irreducible representations of the quantum group $U_q \mathfrak{g}$ with $\mathfrak{g} = \text{Lie}(G)$ and $q = \exp(2\pi i/(k + c_v))$, c_v - the dual Coxeter number. In this section we recall the standard facts about these two-dimensional gauge theories.

2.1. Two-dimensional Yang-Mills theory

We start with a two-dimensional Yang-Mills theory for a compact group G on a Riemann surface Σ (we mostly follow [16]). The partition function is given by:

$$Z_{YM}(\Sigma) = \frac{1}{\text{Vol}(\mathcal{G}_\Sigma)} \int D\varphi DA D\psi e^{\frac{1}{2\pi} \int_\Sigma d^2z (i\text{Tr} \varphi F(A) + \frac{1}{2} \text{Tr} \psi \wedge \psi - g_{YM}^2 \text{Tr} \varphi^2 \text{vol}_\Sigma)} , \quad (2.1)$$

where $\text{Vol}(\mathcal{G}_\Sigma)$ is a volume of the gauge group $\mathcal{G}_\Sigma = \text{Map}(\Sigma, G)$ and vol_Σ is a volume form on Σ . Here A is a connection on a principal G -bundle over Σ , $\varphi \in \mathcal{A}^0(\Sigma, \text{ad}_\mathfrak{g})$ is a section of the vector bundle $\text{ad}_\mathfrak{g}$ and $\psi \in \mathcal{A}^1(\Sigma, \text{ad}_\mathfrak{g})$ is an odd one-form taking values in $\text{ad}_\mathfrak{g}$, $\mathfrak{g} = \text{Lie}(G)$. The measure $DA D\psi$ is a canonical flat measure and the measure $D\varphi$ is defined using the standard normalization of the Killing form on \mathfrak{g} . We also imply the sum over all topological classes of the principal G -bundle over Σ_h .

The Feynman path integral (2.1) is invariant under the action of the following odd and even vector fields:

$$Q A = i\psi, \quad Q \psi = -(d\varphi + [A, \varphi]), \quad Q \varphi = 0, \quad (2.2)$$

$$\mathcal{L}_\varphi A = (d\varphi + [A, \varphi]), \quad \mathcal{L}_\varphi \psi = -[\varphi, \psi], \quad \mathcal{L}_\varphi \varphi = 0, \quad (2.3)$$

such that $Q^2 = -i\mathcal{L}_\varphi$. The invariance of the path integral under transformations (2.2), (2.3) allows to solve the theory exactly. We will be mostly interested in the (generalizations of) the two-dimensional Yang-Mills theory with $g_{YM}^2 = 0$. The theory for $g_{YM}^2 = 0$ will be called topological Yang-Mills theory because its correlation functions depend only on the topology of the underlying surface.

The observables in the theory are constructed using the standard descent procedure. Given any Ad_G -invariant function $f \in \text{Fun}(\mathfrak{g})$ the corresponding local observable is given by:

$$\mathcal{O}_f^{(0)} = \text{Tr } f(\varphi), \quad (2.4)$$

where trace is taken in the adjoint representation. The corresponding nonlocal observables are given by the solutions of the descent equations:

$$d\mathcal{O}^{(i)} = -iQ(\mathcal{O}^{(i+1)}). \quad (2.5)$$

Thus we have:

$$\mathcal{O}_f^{(1)} = \int_\Gamma dz \sum_{a=1}^{\text{rank}(\mathfrak{g})} \frac{\partial f(\varphi)}{\partial \varphi^a} \psi^a, \quad (2.6)$$

$$\mathcal{O}_f^{(2)} = \frac{1}{2} \int_\Sigma d^2z \sum_{a,b=1}^{\text{rank}(\mathfrak{g})} \frac{\partial^2 f(\varphi)}{\partial \varphi^a \partial \varphi^b} \psi^a \wedge \psi^b + i \int_\Sigma d^2z \sum_{a=1}^{\text{rank}(\mathfrak{g})} \frac{\partial f(\varphi)}{\partial \varphi^a} F(A)^a, \quad (2.7)$$

where $\varphi = \sum_{a=1}^{\text{rank}(\mathfrak{g})} \varphi^a t_a$, $\{t^a\}$ is a bases in \mathfrak{g} and $\Gamma \in \Sigma$ is a closed curve on a two-dimensional surface Σ .

To describe the Hilbert space of the the Yang-Mills theory consider the theory on a two dimensional torus $\Sigma = T^2$ supplied with the flat coordinates $(t, \sigma) \sim (t + 2\pi m, \sigma + 2\pi n)$, $n, m \in \mathbb{Z}$. The action has the form:

$$S(\varphi, A) = \frac{1}{2\pi} \int_{T^2} dt d\sigma \text{Tr} (\varphi \partial_t A_\sigma + A_t (\partial_\sigma \varphi + [A_\sigma, \varphi])). \quad (2.8)$$

Here we have integrate out fermionic degrees of freedom using the appropriate choice of the measure in the path integral. Due to the gauge invariance generated by the first-class constraint:

$$\partial_\sigma \varphi + [A_\sigma, \varphi] = 0, \quad (2.9)$$

the phase space of the theory can be reduced to the finite-dimensional space - the phase space is an orbifold:

$$\mathcal{M}_G = (T^*H)/W. \quad (2.10)$$

Its non-singular open part is given by:

$$\mathcal{M}_G^{(0)} = (T^*H_0)/W, \quad (2.11)$$

where $H_0 = H \cap G^{reg}$ is an intersection of the Cartan torus $H \subset G$ with a subset G^{reg} of regular elements of G (the set of the elements of G such that its centralizer has the dimension equal to the rank of \mathfrak{g}) and W - is Weyl group. Note that Cartan torus is given by $H = \mathfrak{h}/Q^\vee$ where Q^\vee is a coroot lattice of \mathfrak{g} . In the case of $G = U(N)$ the set $H/H_0 = \cup_{j < k} \{e^{2\pi i x_j} = e^{2\pi i x_k}\}$ is the main diagonal. Here $x = \sum_{j=1}^{\text{rank}(\mathfrak{g})} x_j e^j$ and $\{e^j\}$ is an orthonormal bases of \mathfrak{h} .

The Hilbert space of the theory can be realized as a space of Ad_G -invariant functions on G . The paring is defined by the integration with a bi-invariant normalized Haar measure:

$$\langle \Psi_1, \Psi_2 \rangle = \int_G dg \bar{\Psi}_1(g) \Psi_2(g). \quad (2.12)$$

Being restricted to the subspace of Ad_G -invariant functions it descends to the integral over Cartan torus H :

$$\langle \Psi_1, \Psi_2 \rangle = \frac{1}{|W|} \int_H dx \Delta_G^2(e^{2\pi i x}) \bar{\Psi}_1(x) \Psi_2(x), \quad (2.13)$$

where the Jacobian is given by:

$$\Delta_G^2(e^{2\pi i x}) = \prod_{\alpha \in R^+} (e^{i\pi\alpha(x)} - e^{-i\pi\alpha(x)})^2, \quad (2.14)$$

and R^+ is a set of positive roots of \mathfrak{g} .

The set of invariant operators descending onto \mathcal{M} includes a commuting family of the operators given by Ad_G -invariant polynomials of φ . In the case of $G = U(N)$ one can take:

$$\mathcal{O}_k^{(0)}(\varphi) = \frac{1}{(2\pi)^k} \text{Tr} \varphi^k. \quad (2.15)$$

where trace is taken in the N -dimensional representation. Define the bases of wavefunctions by the condition:

$$\mathcal{O}_k^{(0)}(\varphi) \Psi_\lambda(x_1, \dots, x_N) = p_k(\lambda) \Psi_\lambda(x_1, \dots, x_N), \quad (2.16)$$

$$\Psi_\lambda(x_{w(1)}, \dots, x_{w(N)}) = \Psi_\lambda(x_1, \dots, x_N), \quad w \in W.$$

where $\lambda = (\lambda_1, \dots, \lambda_N)$ are elements of the weight lattice P of \mathfrak{g} and $p_k \in \mathbb{C}[\mathfrak{h}^*]^W$ is the bases of invariant polynomials on the dual \mathfrak{h}^* to Cartan subalgebra \mathfrak{h} .

It is useful to redefine wave functions by multiplying them on:

$$\Delta_G(e^{2\pi i x}) = \prod_{\alpha \in R^+} (e^{i\pi\alpha(x)} - e^{-i\pi\alpha(x)}), \quad (2.17)$$

so that the integration measure becomes a flat measure on H :

$$\langle \Psi_1, \Psi_2 \rangle = \frac{1}{|W|} \int_H dx \bar{\Phi}_1(x) \Phi_2(x), \quad (2.18)$$

where $\Phi_i(x) = \Delta_G(e^{2\pi i x})\Psi_i(x)$. Then the action of the operators c_k on the wave-functions has a simple form:

$$\mathcal{O}_k^{(0)} = \frac{1}{(2\pi i)^k} \sum_{i=1}^N \frac{\partial^k}{\partial x_i^k}. \quad (2.19)$$

Note that after the redefinition the wave functions are skew-symmetric with respect to the action of W . To get the symmetric wave functions one can use slightly different redefinition $\Phi_i(x) = |\Delta_G(e^{2\pi i x})|\Psi_i(x)$.

The space of states of the theory allows simple description in terms of the representation theory of G . The bases of W -skew-invariant eigenfunctions can be expressed through the characters $\text{ch}_\mu(g) = \text{Tr}_{V_\mu} g$ of the finite-dimensional irreducible representations of G as:

$$\Phi_{\mu+\rho}(g) = \Delta_G(e^{2\pi i x}) \text{ch}_\mu(g), \quad \mu \in P_{++}, \quad (2.20)$$

where P_{++} is a subset of the dominant weights of G , $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ is a half-sum of positive roots and $\lambda = \mu + \rho$ in (2.16). Explicitly we have:

$$\Phi_{\mu+\rho}(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i w(\mu+\rho)(x)}, \quad (2.21)$$

where $l(w)$ is a length of a reduced decomposition of $w \in W$.

The partition function of the topological Yang-Mills theory on a Riemann surface Σ_h of a genus h can be expressed as a sum over unitary irreducible representations of G :

$$Z_{YM}(\Sigma_h) = \left(\frac{\text{Vol}(G)}{(2\pi)^{\dim(G)}} \right)^{2h-2} \sum_{\mu \in P_{++}} (\dim V_\mu)^{2-2h}, \quad (2.22)$$

where:

$$\dim V_\mu = \prod_{\alpha \in R_+} \frac{(\mu + \rho, \alpha)}{(\rho, \alpha)}, \quad (2.23)$$

is a dimension of the irreducible representation V_μ given by the Weyl formula. Here (α, β) is an invariant symmetric pairing on \mathfrak{h}^* . Thus for $\mathfrak{g} = \mathfrak{u}_N$ we have:

$$\dim V_\mu = \prod_{1 \leq i < j \leq N} \frac{(\mu_i - \mu_j + j - i)}{(j - i)}, \quad (2.24)$$

where $\mu_i \in \mathbb{Z}_+$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$.

Note that for $h = 0, 1$ the partition function (2.22) is divergent. It can be made finite by taking $g_{YM} \neq 0$ in (2.1). We use more general regularization compatible with symmetries of the theory - we consider the following deformation of the action:

$$\Delta S_{YM} = - \sum_{k=1}^{\infty} t_k \int_{\Sigma_h} d^2 z \mathcal{O}_k^{(0)}(\varphi) \text{vol}_{\Sigma_h} \quad (2.25)$$

where $\mathcal{O}_k^{(0)}(\varphi)$ is a bases of $\text{Ad}_{\mathfrak{g}}$ -invariant polynomials on φ and the number of $t_k \neq 0$ is finite. We also impose the additional condition on t_k such that the functional integral is well defined. For $\mathfrak{g} = \mathfrak{u}_N$ one can take $\mathcal{O}_k^{(0)}(\varphi) = \frac{1}{(2\pi)^k} \text{Tr} \varphi^k$. Then the partition function is given by:

$$Z_{YM}(\Sigma_h) = \left(\frac{\text{Vol}(G)}{(2\pi)^{\dim(G)}} \right)^{2h-2} \sum_{\mu \in P_{++}} (\dim V_\mu)^{2-2h} e^{-\sum_{k=1}^{\infty} t_k p_k(\mu+\rho)}, \quad (2.26)$$

where $p_k \in \mathbb{R}[\mathfrak{h}^*]^W$. Note that $p_k(\mu + \rho)$ are equal to the eigenvalues of particular combinations of the (higher) Casimir operator c_k acting on V_μ . For example in the case of the quadratic Casimir operator we have:

$$c_2|_{V_\mu} = \frac{1}{2}(\mu + 2\rho, \mu) = \frac{1}{2}(\mu + \rho, \mu + \rho) - \frac{1}{2}(\rho, \rho) = \frac{1}{2}p_2(\mu + \rho) - \frac{1}{2}(\rho, \rho)$$

In fact, not only the dimensions of the irreducible representations show up in the 2d Yang-Mills theory, but also the characters of the irreducible representations enter the explicit expressions for correlation functions. To illustrate this let us consider the correlation function of the operator $\mathcal{O}_k^{(0)} = \text{tr} \varphi^k$ inserted at the center of the disk D . Boundary conditions are defined by fixing the holonomy of the connection over the boundary. The explicit representation is given by:

$$\langle \mathcal{O}_n^{(0)} \rangle_D(x) = \left(\frac{\text{Vol}(G)}{(2\pi)^{\dim(G)}} \right)^{-1} \sum_{\mu \in P_{++}} p_n(\mu + \rho) e^{-\sum_{k=1}^{\infty} t_k p_k(\mu+\rho)} \dim V_\mu \Phi_{\mu+\rho}(x),$$

where the holonomy of the boundary connection is $g = \exp(2\pi i x) \in H$ and $\Phi_{\mu+\rho}(x) = \Delta_G(e^{2\pi i x}) \text{ch}_\mu(e^{2\pi i x})$. In particular for $n = 0$ and $t_k = 0$ we have:

$$Z_{YM}(D) \equiv \langle 1 \rangle_D(g) = \left(\frac{\text{Vol}(G)}{(2\pi)^{\dim(G)}} \right)^{-1} \sum_{\mu \in P_{++}} \dim V_\mu \Phi_{\mu+\rho}(g) = \delta_e^{(G)}(g), \quad (2.27)$$

where we lift the expression to Ad_G -invariant functions on G . The delta-function $\delta_e^{(G)}(g)$ on the group G has a support at the unit element $e \in G$ and is a vacuum wave function corresponding to a disk. Let us remark that $\delta_e^{(G)}(g)$ can be considered as a character of the regular representation of G and (2.27) is a decomposition of the regular representation over the right action of the group G . Note that the factor $\dim V_\mu$ enters in the power of the Euler characteristic of the disk $\chi(D) = 1$, naturally generalizing the representation (2.26) for the compact surface.

Finally note that (2.27) is obviously compatible with (2.22) for $h = 0$ if we represent a sphere as glued from two disks:

$$\begin{aligned} Z_{YM}(\Sigma_0) &= \int_G dg \langle 1 \rangle_D(g) \langle 1 \rangle_D(g^{-1}) = \\ &= \left(\frac{\text{Vol}(G)}{(2\pi)^{\dim(G)}} \right)^{-2} \sum_{\mu \in P_{++}} (\dim V_\mu)^2 e^{-\sum_{k=1}^{\infty} t_k p_k(\mu+\rho)}. \end{aligned} \quad (2.28)$$

2.2. One-dimensional reduction

Consider the dimensional reduction of the two-dimensional Yang-Mills theory to one dimension (quantum mechanics). We have for the partition function:

$$Z_{QM}(\Gamma) = \frac{1}{\text{Vol}(\mathcal{G}_\Gamma)} \int D\varphi Da Db D\eta D\zeta e^{\frac{1}{2} \int_\Gamma dt (i \text{Tr} \varphi (\partial_t a + [b, a]) + \eta \zeta - g_{YM}^2 \text{Tr} \varphi^2 \text{vol}_\Gamma)}, \quad (2.29)$$

where Γ is a trivalent graph supplied with the volume form vol_Γ on its edges, $\text{Vol}(\mathcal{G}_\Gamma)$ is a volume of the gauge group $\mathcal{G}_\Sigma = \text{Map}(\Gamma, G)$. Here (a, b) and (η, ζ) are 1d reductions of two-dimensional fields (A_σ, A_t) and (ψ_σ, ψ_t) . The sewing conditions for the fields on different edges of the graph are chosen in such way that the gauge invariance of (2.29) holds. As in the case of the Yang-Mills theory we will be mostly interested in (the generalizations of) the theory with $g_{YM}^2 = 0$.

The path integral (2.29) is invariant under the action of the odd and even vector fields:

$$Q a = i\eta, \quad Q b = i\zeta, \quad Q \eta = -[a, \varphi], \quad Q \zeta = -(\partial_t \varphi + [b, \varphi]), \quad Q \varphi = 0, \quad (2.30)$$

$$\mathcal{L}_\varphi a = -[\varphi, a], \quad \mathcal{L}_\varphi b = -(\partial_t \varphi + [\varphi, b]), \quad \mathcal{L}_\varphi \eta = -[\varphi, \eta], \quad \mathcal{L}_\varphi \zeta = -[\varphi, \zeta], \quad \mathcal{L}_\varphi \varphi = 0,$$

such that $Q^2 = i\mathcal{L}_\varphi$.

Consider the simplest case of the Yang-Mills theory for $G = U(N)$ on the circle $\Gamma = S^1$. The bosonic part of the action has the form:

$$S(\varphi, a, b) = \frac{1}{2} \int_{S^1} dt \operatorname{Tr} (\varphi \partial_t a + b [a, \varphi]). \quad (2.31)$$

Using the invariance with respect to the gauge transformations generated by the first-class constraint:

$$[a, \varphi] = 0, \quad (2.32)$$

the phase space of the theory can be reduced to the finite-dimensional space. The (open part of the) phase space is given by:

$$\mathcal{M}^{(0)} = (T^*\mathfrak{h}_0)/W, \quad (2.33)$$

where $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}^{reg}$ is an intersection of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the subset \mathfrak{g}^{reg} of regular elements of \mathfrak{g} and W - is Weyl group. For $G = U(N)$ the set $\mathfrak{h}/\mathfrak{h}_0 = \cup_{j < k} \{x_j = x_k\}$ is the main diagonal.

The Hilbert space of the theory can be realized as a space of Ad_G -invariant functions on \mathfrak{g} . The pairing is defined by the integration with a bi-invariant normalized Haar measure:

$$\langle \Psi_1, \Psi_2 \rangle = \int_{\mathfrak{g}} dy \bar{\Psi}_1(y) \Psi_2(y). \quad (2.34)$$

On Ad_G -invariant functions it descends to the integral over Cartan subalgebra \mathfrak{h} :

$$\langle \Psi_1, \Psi_2 \rangle = \frac{1}{|W|} \int_{\mathfrak{h}} dx \Delta_{\mathfrak{g}}^2(x) \bar{\Psi}_1(x) \Psi_2(x), \quad (2.35)$$

where the corresponding Jacobian is given by:

$$\Delta_{\mathfrak{g}}^2(x) = \prod_{\alpha \in R^+} \alpha(x)^2. \quad (2.36)$$

The same set of invariant operators (2.15) descends onto $\mathcal{M}^{(0)}$ and we fix the bases of wave functions by the conditions (2.16) where $(\lambda_1, \dots, \lambda_N)$ now take values in a $\operatorname{rank}(\mathfrak{g})$ -dimensional vector space $P(\mathbb{R}) = P \otimes \mathbb{R}$. Similar to the case of wave-function of the Yang-Mills theory it is useful to redefine wave functions by multiplying them on:

$$\Delta_{\mathfrak{g}}(x) = \prod_{\alpha \in R^+} \alpha(x), \quad (2.37)$$

so that the integration measure becomes flat:

$$\langle \Psi_1, \Psi_2 \rangle = \frac{1}{|W|} \int_{\mathfrak{h}} dx \bar{\Phi}_1(x) \Phi_2(x), \quad (2.38)$$

where $\Phi_i(x) = \Delta_{\mathfrak{g}}(x) \Psi_i(x)$.

In the case of the dimensionally reduced theory the space of states does not have a direct connection with the representation theory of G . However as a bases of W -skew-invariant eigenfunctions one can use the renormalized characters of the finite-dimensional irreducible representations continued to the arbitrary weights $(\lambda_1, \dots, \lambda_N)$ in the positive Weyl chamber $P_+(\mathbb{R})$. Explicitly we have:

$$\Phi_{\lambda}(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i w(\lambda)(x)}. \quad (2.39)$$

The partition function of the dimensionally reduced topological Yang-Mills theory on a graph Γ_h can be expressed as a formal integral over the cone $P_{\mathbb{R}}^+$:

$$Z_{YM}(\Sigma_h) = \int_{\lambda \in P_+(\mathbb{R})} d^N \lambda \ d_{\lambda}^{2-2h}, \quad (2.40)$$

where:

$$d_{\lambda} = \prod_{\alpha \in R_+} (\lambda, \alpha), \quad (2.41)$$

is a continuation of the renormalized Weyl expression for the dimension of the irreducible representation V_{λ} from P_{++} to $P_+(\mathbb{R})$.

2.3. G/G gauged Wess-Zumino-Witten model

The Yang-Mills theory in two dimensions allows the following nontrivial generalization to the G/G gauged WZW model. The partition function for the G/G gauged WZW model on Σ_h is given by the following path integral:

$$Z_{GWZW}(\Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G}_{\Sigma_h})} \int Dg DA D\psi \ e^{kS(g,A,\psi)}, \quad (2.42)$$

$$\begin{aligned} S(g, A, \psi) = & S_{WZW}(g) - \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr} (A_z g^{-1} \bar{\partial}_{\bar{z}} g + g \partial_z g^{-1} A_{\bar{z}} + g A_z g^{-1} A_{\bar{z}} - A_z A_{\bar{z}}) + \\ & + \frac{1}{4\pi} \int_{\Sigma_h} d^2z \text{Tr}(\psi \wedge \psi), \end{aligned}$$

where $S_{WZW}(g)$ is an action functional for Wess-Zumino-Witten model:

$$S_{WZW} = -\frac{1}{8\pi} \int_{\Sigma_h} d^2z \operatorname{Tr}(g^{-1} \partial_z g \cdot g^{-1} \partial_{\bar{z}} g) - i\Gamma(g), \quad (2.43)$$

$$\Gamma(g) = \frac{1}{12\pi} \int_B d^3y \epsilon^{ijk} \operatorname{Tr} g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g.$$

Here k is a positive integer and $\partial B = \Sigma_h$.

The gauged WZW model allows even and odd symmetries extending those for the Yang-Mills theory [20]:

$$Q A = i\psi, \quad Q \psi^{(1,0)} = i(A^g)^{(1,0)} - iA^{(1,0)}, \quad Q \psi^{(0,1)} = -i(A^{g^{-1}})^{(0,1)} + iA^{(0,1)}, \quad (2.44)$$

$$Q g = 0,$$

$$\mathcal{L}_g A^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)} \quad \mathcal{L}_g A_A^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}, \quad (2.45)$$

$$\mathcal{L}_g \psi^{(1,0)} = -g \psi^{(1,0)} g^{-1} + \psi^{(1,0)}, \quad \mathcal{L}_g \psi^{(0,1)} = g^{-1} \psi^{(0,1)} g - \psi^{(0,1)}, \quad \mathcal{L}_g g = 0.$$

Here $A^g = g^{-1} dg + g^{-1} A g$ is a gauge transformation. We have the following relation $Q^2 = \mathcal{L}_g$. It is useful to compare these generators with those in the pure Yang-Mills theory. Generators (2.3) and (2.2) realize infinitesimal symmetries (from the Lie algebra) of the action functional. In contrast the transformations (2.45) and (2.44) realize the finite (from the gauge group) symmetries of the action. Note that in the limit $g \rightarrow 1 + i\epsilon\varphi_0$, $\epsilon \rightarrow 0$ (2.45) and (2.44) are reduced to (2.3) and (2.2).

Consider a deformation of the theory by:

$$\Delta S = \sum_{\mu \in P_{++}} t_\mu \int_{\Sigma_h} d^2z \operatorname{Tr}_{V_\mu} g \operatorname{vol}_{\Sigma_h}. \quad (2.46)$$

We take $t_\mu = 0$ for all but finite subset of P_{++} to make the path integral well defined. Similarly to (2.22) the partition function can be represented in the following form:

$$\begin{aligned} Z_{GWZW}(\Sigma_h) &= |Z(G)|^{2h-2} \left(\frac{k + c_v}{4\pi^2} \right)^{\frac{1}{2} \dim \mathcal{M}_G(\Sigma_h)} \operatorname{Vol}_q(G)^{2h-2} \times \\ &\times \sum_{\mu \in P_{++}^k} (\dim_q V_\mu)^{2-2h} e^{-\sum_{\mu \in P_{++}^k} t_\mu \operatorname{ch}_{V_\mu}(e^{2\pi i \hat{\lambda}})}. \end{aligned} \quad (2.47)$$

where $\dim \mathcal{M}_G(\Sigma_h) = \dim G(2h-2)$ is the dimension of the moduli space of flat G -bundles on Σ_h , $|Z(G)|$ is a dimension of the center of G and:

$$\dim_q V_\mu = \text{Tr}_{V_\mu} q^{-\hat{\rho}} = \prod_{\alpha \in R_+} \frac{(q^{\frac{1}{2}(\mu+\rho, \alpha)} - q^{-\frac{1}{2}(\mu+\rho, \alpha)})}{(q^{\frac{1}{2}(\rho, \alpha)} - q^{-\frac{1}{2}(\rho, \alpha)})}, \quad (2.48)$$

$$\text{Vol}_q(G) = (2\pi)^{\dim G} (k + c_v)^{-\frac{1}{2}(\dim G - \dim H)} \prod_{\alpha \in R_+} \left(q^{\frac{1}{2}(\rho, \alpha)} - q^{-\frac{1}{2}(\rho, \alpha)} \right)^{-1}, \quad (2.49)$$

and the sum is over the set P_{++}^k of integrable representations of the affine group \widehat{LG}_k at the level k . The same set also enumerates irreducible representations of $U_q(\mathfrak{g})$ for $q = \exp(2\pi i/(k + c_v))$. We define $\exp(2\pi i \hat{\lambda}) = \exp(2\pi i \sum_j \lambda_j e_j) \in H$ and $\text{ch}_{V_\mu}(e^{2\pi i \hat{\lambda}})$ is a character of the element $\exp(2\pi i \hat{\lambda})$ taken in the representation V_μ . The expressions $\dim_q V_\mu$ are known as quantum dimensions of the representations of the quantum group. For example in the case of $\mathfrak{g} = \mathfrak{gl}_N$ we have:

$$\dim_q V_\mu = \prod_{i < j}^N \frac{(q^{\frac{1}{2}(\mu_i - \mu_j + j - i)} - q^{\frac{1}{2}(\mu_j - \mu_i + i - j)})}{(q^{\frac{1}{2}(j - i)} - q^{\frac{1}{2}(i - j)})}. \quad (2.50)$$

and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$, $\mu_i \in \mathbb{Z}_+$. The q -analog of the character is given by:

$$\Psi_\mu(x) = \text{ch}_\mu q^{-\rho+x}. \quad (2.51)$$

and corresponding q -generalizations of (2.27) holds. The representation (2.47) for the partition function of the gauged WZW model can be derived using the cohomological localization technique [20] (see also [21] for slightly different approach). The choice of the regularization scheme leading to a particular normalization of the partition function (2.47) is compatible with the interpretation of (2.47) as number of the conformal blocks in WZW theory on Σ_h . This interpretation follows from the fact that the partition function in G/G gauged WZW model coincides with partition function in 3d Chern-Simons theory for three-dimensional manifold $\Sigma_h \times S^1$.

3. Topological Yang-Mills-Higgs theory

In [1] a two dimensional gauge theory was proposed such that the space of classical solutions of the theory on a Riemann surface Σ_h is closely related with the cotangent space to the space of solutions of the dimensionally reduced (to 2d) four-dimensional self-dual

Yang-Mills equations studied by Hitchin [2]. Below we present a way to calculate the partition function of the theory following [1]. Let us given a principal G -bundle P_G over a Riemann surface Σ supplied with a complex structure. Then we have an associated vector bundle $\text{ad}_{\mathfrak{g}} = (P_G \times \mathfrak{g})/G$ with the fiber $\mathfrak{g} = \text{Lie}(G)$ supplied with a coadjoint action of G . Consider the pairs (A, Φ) where A is a connection on P_G and Φ is a one-form taking values in $\text{ad}_{\mathfrak{g}}$. Then Hitchin equations are given by:

$$F(A) - \Phi \wedge \Phi = 0, \quad \nabla_A^{(1,0)} \Phi^{(0,1)} = 0, \quad \nabla_A^{(0,1)} \Phi^{(1,0)} = 0. \quad (3.1)$$

The space of the solutions has a natural hyperkähler structure and admits compatible $U(1)$ action. The correlation functions in the theory introduced in [1] can be described by the integrals of the products of $U(1) \times G$ -equivariant cohomology classes over the moduli space of solutions of (3.1). Note that the $U(1)$ -equivariance makes the path integral well-defined.

The field content of the theory introduced in [1] can be described as follows. In addition to the triplet (A, ψ_A, φ_0) of the topological Yang-Mills theory one has:

$$(\Phi, \psi_{\Phi}) : \quad \Phi \in \mathcal{A}^1(\Sigma, \text{ad}_{\mathfrak{g}}), \quad \psi_{\Phi} \in \mathcal{A}^1(\Sigma, \text{ad}_{\mathfrak{g}}) \quad (3.2)$$

$$(\varphi_{\pm}, \chi_{\pm}) : \quad \varphi_{\pm} \in \mathcal{A}^0(\Sigma, \text{ad}_{\mathfrak{g}}), \quad \chi_{\pm} \in \mathcal{A}^0(\Sigma, \text{ad}_{\mathfrak{g}}) \quad (3.3)$$

where Φ, φ_{\pm} are even and ψ_{Φ}, χ_{\pm} are odd fields. We will use also another notations $\varphi_{\pm} = \varphi_1 \pm i\varphi_2$.

The theory is described by the following path integral:

$$Z_{YMH}(\Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G}_{\Sigma_h})} \int D\varphi_0 D\varphi_{\pm} DA D\Phi D\psi_A D\psi_{\Phi} D\chi_{\pm} e^{S(\varphi_0, \varphi_{\pm}, A, \Phi, \psi_A, \psi_{\Phi}, \chi_{\pm})}, \quad (3.4)$$

where $S = S_0 + S_1$ with:

$$S_0(\varphi_0, \varphi_{\pm}, A, \Phi, \psi_A, \psi_{\Phi}, \chi_{\pm}) = \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(i\varphi_0 (F(A) - \Phi \wedge \Phi) - c\Phi \wedge *\Phi) + \\ + \varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)} \quad (3.5)$$

and

$$S_1(\varphi_0, \varphi_{\pm}, A, \Phi, \psi_A, \psi_{\Phi}, \chi_{\pm}) = \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(\frac{1}{2}\psi_A \wedge \psi_A + \frac{1}{2}\psi_{\Phi} \wedge \psi_{\Phi} + \\ + \chi_+ [\psi_A^{(1,0)}, \Phi^{(0,1)}] + \chi_- [\psi_A^{(0,1)}, \Phi^{(1,0)}] + \chi_+ \nabla_A^{(1,0)} \psi_{\Phi}^{(0,1)} + \chi_- \nabla_A^{(0,1)} \psi_{\Phi}^{(1,0)}) \quad (3.6)$$

where the decompositions $\Phi = \Phi^{(1,0)} + \Phi^{(0,1)}$ and $\psi_\Phi = \psi_\Phi^{(1,0)} + \psi_\Phi^{(0,1)}$ correspond to the decomposition of the space of one-forms $\mathcal{A}^1(\Sigma_h) = \mathcal{A}^{(1,0)}(\Sigma_h) \oplus \mathcal{A}^{(0,1)}(\Sigma_h)$ defined in terms of a fixed complex structure on Σ_h .

The theory is invariant under the action of the following even vector field:

$$\mathcal{L}_v \Phi^{(1,0)} = +\Phi^{(1,0)}, \quad \mathcal{L}_v \Phi^{(0,1)} = -\Phi^{(0,1)}, \quad \mathcal{L}_v \psi_\Phi^{(1,0)} = +\psi_\Phi^{(1,0)}, \quad (3.7)$$

$$\mathcal{L}_v \psi_\Phi^{(0,1)} = -\psi_\Phi^{(0,1)}, \quad \mathcal{L}_v \varphi_\pm = \mp \varphi_\pm, \quad \mathcal{L}_v \chi_\pm = \pm \chi_\pm,$$

$$\mathcal{L}_{\varphi_0} A = -\nabla_A \varphi_0, \quad \mathcal{L}_{\varphi_0} \psi_A = -[\varphi_0, \psi_A], \quad \mathcal{L}_{\varphi_0} \Phi = -[\varphi_0, \Phi], \quad \mathcal{L}_{\varphi_0} \psi_\Phi = -[\varphi_0, \psi_\Phi], \quad (3.8)$$

$$\mathcal{L}_{\varphi_0} \varphi_0 = 0, \quad \mathcal{L}_{\varphi_0} \varphi_\pm = -[\varphi_0, \varphi_\pm], \quad \mathcal{L}_{\varphi_0} \chi_\pm = -[\varphi_0, \chi_\pm],$$

and an odd vector field generated by the BRST operator:

$$QA = i\psi_A, \quad Q\psi_A = -\nabla_A \varphi_0, \quad Q\varphi_0 = 0,$$

$$Q\Phi = i\psi_\Phi, \quad Q\psi_\Phi^{(1,0)} = -[\varphi_0, \Phi^{(1,0)}] + c\Phi^{(1,0)}, \quad Q\psi_\Phi^{(0,1)} = -[\varphi_0, \Phi^{(0,1)}] - c\Phi^{(0,1)}, \quad (3.9)$$

$$Q\chi_\pm = i\varphi_\pm, \quad Q\varphi_\pm = -[\varphi_0, \chi_\pm] \pm c\chi_\pm.$$

We have $Q^2 = i\mathcal{L}_{\varphi_0} + c\mathcal{L}_v$ and Q can be considered as a BRST operator on the space of \mathcal{L}_{φ_0} and \mathcal{L}_v -invariant functionals. The action functional of the topological Yang-Mills-Higgs theory can be represented as a sum of the action functional of the topological pure Yang-Mills theory (written in terms of fields ϕ_0, A, ψ_A) and an additional part which can be represented as a Q -anti-commutator:

$$S_{YMH} = S_{YM} + [Q, \int_{\Sigma_h} d^2z \operatorname{Tr} \left(\frac{1}{2} \Phi \wedge \psi_\Phi + \varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)} \right)]_+. \quad (3.10)$$

The theory given by (3.4) is a quantum field theory whose correlation functions are given by the intersections pairings of the equivariant cohomology classes on the moduli spaces of Higgs bundles.

To simplify the calculations it is useful to consider more general action given by:

$$S_{YMH} = S_{YM} + [Q, \int_{\Sigma_h} d^2z \operatorname{Tr} \left(\frac{1}{2} \Phi \wedge \psi_\Phi + \right. \quad (3.11)$$

$$\left. + \tau_1 (\varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)}) + \tau_2 (\chi_+ \varphi_- + \chi_- \varphi_+) \operatorname{vol}_{\Sigma_h} \right)].$$

Cohomological localization of the functional integral takes the simplest form for $\tau_1 = 0$, $\tau_2 \neq 0$. Note that it is not obvious that the theory for $\tau_1 = 0$, $\tau_2 \neq 0$ is equivalent to that

for $\tau_1 \neq 0$, $\tau_2 = 0$. However taking into account that the action functionals in these two cases differ on the equivariantly exact form and for $c \neq 0$ the space of fields is essentially compact one can expect that the theories are equivalent.

For $\tau_1 = 0$ the path integrals over Φ , φ_\pm and χ_\pm is quadratic. Thus we have for the partition function the following formal representation:

$$Z_{YMH}(\Sigma_h) = \tag{3.12}$$

$$= \frac{1}{\text{Vol}(\mathcal{G}_{\Sigma_h})} \int DA D\varphi_0 D\psi_A e^{\frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr} (i\varphi_0 F(A) + \frac{1}{2} \psi_A \wedge \psi_A)} \text{Sdet}_V(ad_{\varphi_0} + ic),$$

where the super-determinant is taken over the super-space:

$$V = V_{\text{even}} \oplus V_{\text{odd}} = \mathcal{A}^0(\Sigma_h, \text{adg}) \oplus \mathcal{A}^{(1,0)}(\Sigma_h, \text{adg}). \tag{3.13}$$

and should be properly understood using a regularization compatible with Q -symmetry of the path integral (e.g. $\tau_1 \neq 0$). Thus the Yang-Mills-Higgs theory can be considered as a pure Yang-Mills theory deformed by a non-local gauge invariant observable.

Let us stress that there are two interesting limiting cases $c \rightarrow \infty$ and $c \rightarrow 0$ for the theory (3.4), obvious from the definition of the corresponding Lagrangian. Note that the dependence on c is through the mass term for the field Φ in (3.5) and (3.6). Thus in the limit $c \rightarrow \infty$ the field Φ (and corresponding fermions) drops out and we get the 2d Yang-Mills theory with a compact group G . Therefore, in this limit we have to recover the known answers from 2d Yang-Mills theory. On the other hand for $c = 0$ the topological Yang-Mills-Higgs theory for group G is equivalent to 2d topological Yang-Mills theory for complex group G^c . This might be considered as a manifestation of the general relation between the hyperkähler quotient over a compact group and a Kähler quotient over its complexification. In the case of (3.4) the gauge symmetry group is $\mathcal{G} = \text{Map}(\Sigma_h, G)$ while in the complexified Yang-Mills theory one would have $\mathcal{G}^c = \text{Map}(\Sigma_h, G^c)$ as a gauge group. Thus in the limit $c \rightarrow 0$ one might expect the relation with the representation theory of the complexified group G^c .

Let us demonstrate the relation of $c = 0$ Yang-Mills-Higgs theory with the Yang-Mills theory for complexified gauge group explicitly. The Feynman path integral representation for Yang-Mills-Higgs theory at $c = 0$ in (3.4) can be considered as a result of a partial gauge fixing of the symmetry in the complex Yang-Mills theory. Consider a complex gauge field:

$$\nabla_{\mathcal{A}} = d + A^c = d + A + i\Phi, \tag{3.14}$$

where A and Φ are skew-hermitian one forms. The corresponding curvature is naturally decomposed into the skew-hermitian and hermitian parts:

$$F(A^c) = (dA + A \wedge A - \Phi \wedge \Phi) + i(d\Phi + A \wedge \Phi). \quad (3.15)$$

Define two-dimensional topological Yang-Mills theory for the complex group G^c as:

$$Z_{YM^c}(\Sigma_h) = \frac{1}{\text{Vol}(\mathcal{G}_{\Sigma_h}^c)} \int D\varphi_0 D\varphi_- DA D\Phi e^{S_{YM^c}(\varphi_{\pm}, A, \Phi)}, \quad (3.16)$$

where:

$$\begin{aligned} S_{YM^c}(\varphi_0, \varphi_1, A, \Phi) &= \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(\varphi_c F(A^c) + \bar{\varphi}_c F(\bar{A}^c)) = \\ &= \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(\varphi_0(dA + A \wedge A - \Phi \wedge \Phi) + \varphi_1(d\Phi + A \wedge \Phi)), \end{aligned} \quad (3.17)$$

and $\varphi_c = \varphi_0 + i\varphi_1$. This theory has an infinitesimal gauge symmetry:

$$A \rightarrow d\epsilon + [\epsilon, A] - [\eta, \Phi], \quad \Phi \rightarrow d\eta + [\eta, A] + [\epsilon, \Phi], \quad (3.18)$$

with the gauge parameter $(\epsilon + i\eta) \in \mathcal{A}^0(\Sigma_h, \mathfrak{g}^c)$. To make contact with (3.4) one should partially fix the gauge freedom generated by the η -dependent part of (3.18) by adding:

$$\begin{aligned} \Delta S &= \int_{\Sigma_h} d^2z \text{Tr}(\varphi_2 \nabla_A(*\Phi)) + \\ &+ \int_{\Sigma_h} d^2z \text{Tr} \frac{1}{2} (\nabla_A^{1,0} \chi_+ + i[\Phi^{(1,0)}, \chi_+]) (\nabla_A^{0,1} \chi_- + i[\Phi^{(0,1)}, \chi_-]), \end{aligned} \quad (3.19)$$

where last term is the ghost-antighost contribution. Note that this term is invariant with respect to ϵ -symmetry if the field φ_- takes values in the coadjoint representation of the group. Taking $c = 0$ and integrating over ψ_A and ψ_Φ in (3.4) one can see that the theory (3.4) is equivalent to this, partially gauge fixed, complex Yang-Mills theory.

The partition function (3.4) of the Yang-Mills-Higgs theory on a compact Riemann surface can be calculated using the standard methods of the cohomological localization [17]. As in the case of Yang-Mills theory we consider the deformation of the action of the theory:

$$\Delta S_{YMH} = - \sum_{k=1}^{\infty} t_k \int_{\Sigma_h} d^2z \text{Tr} \varphi_0^k \text{vol}_{\Sigma_h}. \quad (3.20)$$

where we impose the condition that t_k for all but finite set of indexes.

Path integral with the action (3.11) at $\tau_1 = 0$ and $\tau_2 = 1$ is easily reduced to the integral over abelian gauge fields. The contribution of the additional nonlocal observable in (3.12) can be calculated as follows. The purely bosonic part of the nonlocal observable after reduction to abelian fields can be easily evaluated using any suitable regularization (i.e. zeta function regularization) and result is the change of the bosonic part of the abelian action $\int d^2z (\varphi_0)_i F^i(A)$ by:

$$\begin{aligned} \Delta S = & \int_{\Sigma_h} d^2z \sum_{i,j=1}^N \log \left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic} \right) F(A)^i \\ & + \frac{1}{2} \int_{\Sigma_h} d^2z \sum_{i,j=1}^N \log((\varphi_0)_i - (\varphi_0)_j + ic) R^{(2)} \sqrt{g}; \end{aligned} \quad (3.21)$$

where $F(A)^i$ is i -th component of the curvature of the abelian connection A and $R^{(2)}(g)$ is curvature on Σ_h for 2d metric g used to regularize non-local observable. We will use the notation $(\varphi_0)_i = \lambda_i$ in remaining part of the paper. This leads to the unique Q -closed completion. The completion of the term in (3.21) containing the curvature of the gauge field is given by the two-observable $\mathcal{O}_f^{(2)}$ corresponding to the descendent of the following function on the Cartan subalgebra isomorphic to \mathbb{R}^N :

$$f(\text{diag}(\lambda_1, \dots, \lambda_N)) = \sum_{k,j=1}^N \int_0^{\lambda_j - \lambda_k} \text{arctg } \lambda/c d\lambda. \quad (3.22)$$

Thus, the abelianized action is defined by two-observable descending from:

$$I(\lambda) = \sum_{j=1}^N \left(\frac{1}{2} \lambda_j^2 - 2\pi n_j \lambda_j \right) + \sum_{k,j=1}^N \int_0^{\lambda_j - \lambda_k} \text{arctg } \lambda/c d\lambda, \quad (3.23)$$

according to the formula (2.7). On the other hand the term containing the metric curvature R in (3.21) is Q -closed and thus does not need any completion. It can be considered as an integral of the zero observable:

$$\mathcal{O}^{(0)} = \sum_{i,j=1}^N \log((\varphi_0)_i - (\varphi_0)_j + ic) \quad (3.24)$$

over Σ_h weighted by the half of the metric curvature. Note that the function $I(\lambda)$ plays an important role in Nonlinear Schrödinger theory which we explain in the next section.

After integrating out the fermionic partners of abelian connection A the standard localization procedure leads to following final finite-dimensional integral representation for the the partition function [1]:

$$Z_{YMH}(\Sigma_h) = \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbb{R}^N} d^N \lambda \mu(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \times \quad (3.25)$$

$$\times \prod_{k \neq j} (\lambda_k - \lambda_j)^{n_k - n_j + 1 - h} \prod_{k, j} (\lambda_k - \lambda_j - ic)^{n_k - n_j + 1 - h} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)},$$

where

$$\mu(\lambda) = \det \left\| \frac{\partial^2 I(\lambda)}{\partial \lambda_i \partial \lambda_j} \right\|, \quad (3.26)$$

and $p_k(\lambda)$ are S_N -invariant polynomial functions of degree k on \mathbb{R}^N and $a(c)$ is a h -independent constant defined by the appropriate choice of the regularization of the functional integral. One can write the n_i -dependent parts of the products in (3.25) as the exponent of the sum:

$$Z_{YMH}(\Sigma_h) = \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbb{R}^N} d^N \lambda \mu(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_j n_j \alpha_j(\lambda)} \quad (3.27)$$

$$\times \prod_{k \neq j} (\lambda_k - \lambda_j)^{1-h} \prod_{k, j} (\lambda_k - \lambda_j - ic)^{1-h} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)},$$

with notation:

$$e^{2\pi i \alpha_j(\lambda)} = \mathcal{F}_j(\lambda) \equiv e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic} \quad (3.28)$$

After taking the sum over $(n_1, \dots, n_N) \in \mathbb{Z}^N$ using:

$$\mu(\lambda) \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_j n_j \alpha_j(\lambda)} = \mu(\lambda) \sum_{(m_1, \dots, m_N) \in \mathbb{Z}^N} \prod_j \delta(\alpha_j(\lambda) - m_j) \quad (3.29)$$

$$= \sum_{(\lambda_1^*, \dots, \lambda_N^*) \in \mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda_j^*)$$

(see definition of \mathcal{R}_N below) and integral over $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ we see that only $\alpha_j(\lambda) \in \mathbb{Z}$, or the same - $\mathcal{F}_j(\lambda) = 1$, contribute to the partition function which now can be written in the form similar to (2.22), (2.47) for $\mathfrak{g} = \mathfrak{u}_N$:

$$Z_{YMH}(\Sigma_h) = e^{(1-h)a(c)} \sum_{\lambda \in \mathcal{R}_N} D_\lambda^{2-2h} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)}, \quad (3.30)$$

where:

$$D_\lambda = \mu(\lambda)^{-1/2} \prod_{i < j} (\lambda_i - \lambda_j)(c^2 + (\lambda_i - \lambda_j)^2)^{1/2}, \quad (3.31)$$

and the \mathcal{R}_N in (3.29) and (3.30) denotes a set of the solutions of the Bethe Ansatz equations $\mathcal{F}_j(\lambda) = 1$:

$$e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j + ic}{\lambda_k - \lambda_j - ic} = 1, \quad k = 1, \dots, N, \quad (3.32)$$

for the N -particle sector of the quantum theory of Nonlinear Schrödinger equation (see e.g. [3], [4]). Note that the sum in (3.30) is taken over the classes of the solutions up to action of the symmetric group on λ_i . This set can be enumerated by the multiplets of the integer numbers $(p_1, \dots, p_N) \in \mathbb{Z}^N$ such that $p_1 \geq p_2 \geq \dots \geq p_N$, $p_i \in \mathbb{Z}$. Thus, the sum in (3.30) is the sum over the same set of partitions as in 2d Yang-Mills theory. The structure of the representation (3.30) for the partition function of Yang-Mills-Higgs theory is very similar to the analogous representation (2.22), (2.47) of the partition functions for Yang-Mills and gauged Wess-Zumino-Witten theories.

3.1. Reduction of Yang-Mills-Higgs theory to one dimension

Let us use the following notations for the one-dimensional reduction of the non-scalar fields entering the description of Yang-Mills-Higgs theory:

$$A \rightarrow (a, b), \quad \Phi \rightarrow (\phi, \rho), \quad \psi_A \rightarrow (\eta_a, \zeta_b), \quad \psi_\Phi \rightarrow (\eta_\Phi, \zeta_\Phi) \quad (3.33)$$

Reduction of topological Yang-Mills-Higgs theory to one dimension (Yang-Mills-Higgs Quantum Mechanics) is described by the following path integral:

$$Z_{YMHQM}(\Gamma_h) = \int D(\varphi_0, \varphi_\pm, a, b, \phi, \rho, \eta_a, \zeta_b, \eta_\Phi, \zeta_\Phi, \chi_\pm) e^{S(\varphi_0, \varphi_\pm, a, b, \phi, \rho, \eta_a, \zeta_b, \eta_\Phi, \zeta_\Phi, \chi_\pm)},$$

where the action is given by $S = S_0 + S_1$:

$$S_0 = \frac{1}{4\pi} \int dt \text{Tr} (i\varphi_0 (\partial_t a + [b, a] + [\phi, \rho]) - c(\phi^2 + \rho^2) + \quad (3.34)$$

$$+ \varphi_1 (\partial_t \phi + [b, \phi] - [a, \rho]) + \varphi_2 (\partial_t \rho + [b, \rho] + [a, \phi]),$$

$$S_1 = \frac{1}{2\pi} \int dt \text{Tr} \left(\frac{1}{2} \eta_a \zeta_b + \frac{1}{2} \eta_\Phi \zeta_\Phi + \chi_1 ([\eta_a, \phi] + [\zeta_b, \rho]) + \chi_2 ([\zeta_b, \phi] - [\eta_a, \rho]) + \quad (3.35)$$

$$+ \chi_1 (\partial_t \eta_\Phi + [b, \eta_\Phi] - [a, \zeta_\Phi]) + \chi_2 (\partial_t \zeta_\Phi + [b, \zeta_\Phi] + [a, \eta_\Phi]) \right).$$

The theory is invariant under the action of the vector fields:

$$\begin{aligned}
\mathcal{L}_v \phi &= +\rho, & \mathcal{L}_v \rho &= -\phi, & \mathcal{L}_v \eta_\Phi &= -\zeta_\Phi, \\
\mathcal{L}_v \zeta_\Phi &= +\eta_\Phi & \mathcal{L}_v \varphi_\pm &= \mp \varphi_\pm, & \mathcal{L}_v \chi_\pm &= \pm \chi_\pm, \\
\mathcal{L}_{\varphi_0} a &= -[a, \varphi_0,], & \mathcal{L}_{\varphi_0} b &= -(\partial_t \varphi_0 + [b, \varphi_0,]), & \mathcal{L}_{\varphi_0} \eta_a &= -[\varphi_0, \eta_a], & \mathcal{L}_{\varphi_0} \zeta_b &= -[\varphi_0, \zeta_b], \\
\mathcal{L}_{\varphi_0} \phi &= -[\varphi_0, \phi], & \mathcal{L}_{\varphi_0} \rho &= -[\varphi_0, \rho], & \mathcal{L}_{\varphi_0} \eta_\Phi &= -[\varphi_0, \eta_\Phi], & \mathcal{L}_{\varphi_0} \zeta_\Phi &= -[\varphi_0, \zeta_\Phi], \\
\mathcal{L}_{\varphi_0} \varphi_0 &= 0, & \mathcal{L}_{\varphi_0} \chi_\pm &= -[\varphi_0, \chi_\pm],
\end{aligned} \tag{3.36}$$

and a fermionic symmetry generated by the BRST operator:

$$Qa = i\eta_a, \quad Qb = i\zeta_b, \quad Q\eta_a = -[a, \varphi_0], \quad Q\zeta_b = -(\partial_t b + [b, \varphi_0]), \quad Q\varphi_0 = 0, \tag{3.37}$$

$$Q\phi = i\eta_\Phi, \quad Q\rho = i\zeta_\Phi, \quad Q\eta_\Phi = -[\varphi_0, \phi] + c\rho, \quad Q\zeta_\Phi = -[\varphi_0, \rho] - c\phi, \tag{3.38}$$

$$Q\chi_\pm = i\varphi_\pm, \quad Q\varphi_\pm = -[\varphi_0, \chi_\pm] \pm c\chi_\pm. \tag{3.39}$$

We have $Q^2 = i\mathcal{L}_{\varphi_0} + c\mathcal{L}_v$ and Q can be considered as a BRST operator on the space of \mathcal{L}_{φ_0} and \mathcal{L}_v -invariant functionals.

The partition function on the graph Γ_h for $\mathfrak{g} = \mathfrak{u}_N$ after localization is given by:

$$\begin{aligned}
Z_{YMQM}(\Gamma_h) &= \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbb{R}^N/S_N} d^N \lambda \prod_{k \neq j} (\lambda_k - \lambda_j)^{1-h} \prod_{k \neq j} (\lambda_k - \lambda_j - ic)^{1-h} = \\
&= \frac{e^{(1-h)a(c)}}{|W|} \int_{\mathbb{R}^N/S_N} d^N \lambda D_\lambda^{2-2h},
\end{aligned} \tag{3.40}$$

where:

$$D_\lambda = \prod_{i < j} (\lambda_i - \lambda_j)(c^2 + (\lambda_i - \lambda_j)^2)^{1/2}. \tag{3.41}$$

In contrast with the two-dimensional case we have integral over \mathbb{R}^N/S_N instead of the sum over the solutions of Bethe Ansatz equations. Note also that the factor $\mu(\lambda)$ is a constant for the dimensionally reduced theory and is included into the proper normalization of the partition function. If we compare (3.27) and (3.40) we see that the only difference is that in (3.27) we have additional insertion under integral over λ 's of the sum over integers (n_1, \dots, n_N) with the exponential factor that reduces the integral to the sum over the zeros of the exponent, $\alpha_i(\lambda)$, which is the same as a reduction to those λ 's that solve

Bethe Ansatz equation. This simple fact will be important later in computation of wave functions for Yang-Mills-Higgs theory.

Obvious similarities between (3.30) and (2.22), (2.47) suggests that there should be the full analog of the results discussed in the Section 2, including the interpretation of D_λ as a (formal) dimension of the representation of some algebraic structure together with the identification of the eigenfunctions of the operators $\mathcal{O}_k^{(0)} = \frac{1}{(2\pi)^k} \text{Tr} \varphi_0^k$ in the appropriate polarization with the corresponding characters. The obvious candidate for the replacement of (2.21) in the Yang-Mills-Higgs theory is a set of wave-functions in the N -particle sector of Nonlinear Schrödinger theory. The basis in this space is defined in terms of the eigenfunctions of the quantum Hamiltonian operator of Nonlinear Schrödinger equation and has an interpretation in terms of the representation theory of degenerate (double) affine Hecke algebra. Before we consider this proposal in details let us discuss two limiting cases of the representation (3.30), (3.31) that have connections with representation theory of the classical Lie groups.

3.2. $c \rightarrow \infty$

In the limit $c \rightarrow \infty$ the c -dependent term in the action (3.34) transforms into the delta-function of the fields Φ and the path integral reduces effectively to the path integral of the two-dimensional Yang-Mills theory discussed in Section 2. One should expect that the representation for the partition function (3.30) to reproduce (2.22) in this limit. Indeed in the limit $c \rightarrow \infty$ we have $\mu(\lambda) \rightarrow 1$ and :

$$\lim_{c \rightarrow \infty} D_\lambda = \Delta_{\mathfrak{g}}(\lambda) = \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k), \quad (3.42)$$

and λ_i are the solutions of the limiting Bethe Ansatz equations:

$$(-1)^{N-1} e^{2\pi i \lambda_k} = 1. \quad (3.43)$$

Thus $\lambda_i = m_i + \frac{(N-1)}{2}$, $m_i \in \mathbb{Z}$ and one can identify $m_i = \mu_i - i$ in the expressions (2.24) and (3.42).

3.3. $c \rightarrow 0$

In the opposite limit $c \rightarrow 0$ we again have $\mu(\lambda) \rightarrow 1$ and :

$$\lim_{c \rightarrow 0} c^{\frac{N(N-1)}{2}} D_\lambda = \Delta_{\mathfrak{g}}(\lambda)^2 = \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k)^2, \quad (3.44)$$

and λ_i are solutions of the limiting Bethe Ansatz equation:

$$e^{2\pi i \lambda_k} = 1. \quad (3.45)$$

The interpretation of the limit is not so obvious because the localization technique does not straightforwardly applicable for $c = 0$. However let us remind that in the case of $c = 0$ the theory (3.4) is equivalent to the two-dimensional Yang-Mills theory with complex gauge group G^c . Thus one might expect that in the limit $c \rightarrow 0$ one gets an answer with at least some interpretation in terms of the representation theory of the complexified group G^c . In the Yang-Mills theory for G^c one expects to have a sum (more exactly integral and the sum) over the set of unitary representations arising in the decomposition of the regular representation of G in $L^2(G)$, i.e. over the principal series of unitary representations. In order to compare this with the limit $c \rightarrow 0$ let us first recall standard facts in the representation theory of complex groups (see e.g. [22], [23]). For simplicity we discuss only the case $G = GL(N, \mathbb{C})$. Principal series of unitary representations of $GL(N, \mathbb{C})$ can be studied by inducing them from a Borel subgroup $B \subset GL(N, \mathbb{C})$ using the character of B :

$$\chi(b) = \prod_{j=1}^N |b_{jj}|^{i\rho_j - m_j} b_{jj}^{m_j} \quad b \in B, \quad \rho_j \in \mathbb{R}, \quad m_j \in \mathbb{Z}. \quad (3.46)$$

For complex groups all unitary representations are infinite-dimensional and thus the definitions of the character of the representation and the dimension of the representation deserves some care. The character of the representation $\pi : G \rightarrow \text{End}(V)$ is defined as follows. Let $f(g)$ be a smooth function with compact support. Then define the trace of f as:

$$\text{Tr}_V f \equiv \text{Tr} \left(\int dg f(g) \pi(g) \right). \quad (3.47)$$

Under some conditions (3.47) is well defined (i.e. the operator (3.47) is of trace class) and one calls the generalized function ch_V on G^c a character if

$$\text{Tr}_V f = \langle \text{ch}_V, f \rangle. \quad (3.48)$$

It was shown by Harich-Chandra that thus defined generalized function is an ordinary function and therefore ch_V can be considered as a generalization of the characters of finite dimensional representations.

The simplest example is a representation of $GL(N, \mathbb{C})$ obtained by quantization of the regular coadjoint orbit generalizing two-sheet hyperboloid for $GL(2, \mathbb{C})$. The corresponding character is given by:

$$\text{ch}_\lambda(e^x) = \frac{1}{|\Delta_G(e^x)|} \sum_{w \in S_N} e^{2\pi i \sum_{j=1}^N \lambda_{w(j)} x_j} \quad (3.49)$$

where $\lambda_j = m_j + i\rho_j$ and S_N is a Weyl group of $GL(N, \mathbb{C})$.

In the case of the finite-dimensional representations the dimension of the representation is given by the value of the corresponding character at the unit element of the group. However in the case of the infinite-dimensional representations this relation can not be used to define the notion of dimension even formally. In particular the value of the corresponding character at the unit element can be infinite. For example (3.49) tends to infinity when $x \rightarrow 0$ which is a manifestation of the fact that the corresponding representation is infinite-dimensional.

The correct definition of the dimension \mathcal{D}_λ of the principal series unitary representations is provided by the decomposition:

$$\delta_e^{(G)}(g) = \sum_{\lambda \in \widehat{G}} \mathcal{D}_\lambda \text{ch}_\lambda(g), \quad (3.50)$$

where $\delta_e^{(G)}(g)$ is a delta-function with the support at the unit element $e \in G$ of the group, $\text{ch}_\lambda(g)$ is a character and \widehat{G} is a unitary dual to G (i.e. the set of isomorphism classes of the unitary representations entering the decomposition of the regular representation). The dimension \mathcal{D}_λ defined in such way (known as a formal degree of the representation) coincides with the ratio of the Plancherel measure arising in the decomposition of the regular representation and the flat measure on the space of characters (see e.g. [22], [23]). Explicitly we have for \mathcal{D}_λ :

$$\mathcal{D}_\lambda = \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^2. \quad (3.51)$$

Comparing (3.51) with (3.44) one infers that in the limit $c \rightarrow 0$ one obtains the subset of the principal series of representations corresponding to $\lambda_k = m_k \in \mathbb{Z}$ (i.e. $\rho_k = 0$). It is reasonable to guess that in the limit $c \rightarrow 0$ the only information that remains is a class of

functions and a particular class of representations that arise in the spectral decomposition corresponds to this class of functions². This is natural because the localization demands a compactification of the configuration space and $c \neq 0$ term just provides effectively this compactification. In the limit $c \rightarrow 0$ not all elements of $L^2(G)$ arise in the description of Hilbert space of the theory and we get a subset of the representations.

Thus the wave-functions of Yang-Mills-Higgs theory for $c \neq 0$ should interpolate between characters of finite-dimensional representations of G and characters of a class of infinite-dimensional representations of G^c . As we will demonstrate in the next Section the wave-functions in the N -particle sector of Nonlinear Schrödinger theory provides exactly this interpolation.

4. N -particle wave functions in Nonlinear Schrödinger theory

The appearance of a particular form of Bethe Ansatz equations (3.32) strongly suggests the relevance of quantum integrable theories in the description of wave-functions in topological Yang-Mills-Higgs theory. Precisely this form of Bethe Ansatz equations (3.32) arises in the description of the N -particle wave functions for the quantum Nonlinear Schrödinger theory with the coupling constant $c \neq 0$ [24], [25], [26], [27]. In this section we recall the standard facts about the construction of these wave-functions using the coordinate Bethe Ansatz. We also discuss the relation with the representation theory of the degenerate (double) affine Hecke algebras and the representation theory of the Lie groups over complex and p -adic numbers. For the application of the quantum inverse scattering method to Nonlinear Schrödinger theory see [28], [29]. One can also recommend [30] as a quite readable introduction into the Bethe Ansatz machinery.

The Hamiltonian of Nonlinear Schrödinger theory with a coupling constant c is given by:

$$\mathcal{H}_2 = \int dx \left(\frac{1}{2} \frac{\partial \phi^*(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} + c(\phi^*(x)\phi(x))^2 \right), \quad (4.1)$$

with the following Poisson structure for bosonic fields:

$$\{\phi^*(x), \phi(x')\} = \delta(x - x'). \quad (4.2)$$

² This may be compared with the Bogomolony limit of the monopole equations where the only information on the potential that survives in the limit is encoded in the asymptotic behaviour of the solutions.

The operator of the number of particles:

$$\mathcal{H}_0 = \int dx \phi^*(x)\phi(x), \quad (4.3)$$

commutes with the Hamiltonian \mathcal{H}_2 and thus one can solve the eigenfunction problem in the sub-sector for a given number of particles $\mathcal{H}_0 = N$. We will consider the both the theory on infinite interval ($x \in \mathbb{R}$) and its periodic version $x \in S^1$. The equation for eigenfunctions in the N -particle sector has the following form:

$$\left(-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)\right) \Phi_\lambda(x) = 2\pi^2 \left(\sum_{i=1}^N \lambda_i^2\right) \Phi_\lambda(x) \quad i = 1, \dots, N. \quad (4.4)$$

This equation is obviously symmetric with respect to the action of symmetric group S_N on the coordinates x_i . Thus the solutions are classified according to the representations of S_N . Quantum integrability of the Nonlinear Schrödinger theory implies the existence of the complete set of the commuting Hamiltonian operators. The corresponding eigenvalues are given by the symmetric polynomials $p_k(\lambda)$.

Finite-particle sub-sectors of the Nonlinear Schrödinger theory can be described in terms of the representation theory of a particular kind of Hecke algebra [31], [32], [33], [34]. Let $R = \{\alpha_1, \dots, \alpha_l\}$ be a root system, W - corresponding Weyl group and P - a weight lattice. Degenerate affine Hecke algebra $\mathcal{H}_{R,c}$ associated to R is defined as an algebra with the basis S_w , $w \in W$ and $\{D_\lambda, \lambda \in P\}$ such that S_w $w \in W$ generate subalgebra isomorphic to group algebra $\mathbb{C}[W]$ and the elements D_λ , $\lambda \in P$ generate the group algebra $\mathbb{C}[P]$ of the weight lattice P . In addition one has the relations:

$$S_{s_i} D_\lambda - D_{s_i(\lambda)} S_{s_i} = c \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, \quad i = 1, \dots, n. \quad (4.5)$$

Here s_i are the generators of the Weyl algebra corresponding to the reflection with respect to the simple roots α_i . The center of $\mathcal{H}_{R,c}$ is isomorphic to the algebra of W -invariant polynomial functions on $R \otimes \mathbb{C}$. The degenerate affine Hecke algebras were introduced by Drinfeld [35] and independently by Lusztig [36].

Below we consider only the case of \mathfrak{gl}_N root system and thus we have $W = S_N$. Let us introduce the following differential operators (Dunkle operators [37]):

$$\mathcal{D}_i = -i \frac{\partial}{\partial x_i} + i \frac{c}{2} \sum_{j=i+1}^N (\epsilon(x_i - x_j) + 1) s_{ij}. \quad (4.6)$$

Here $\epsilon(x)$ is a sign-function and $s_{ij} \in S_N$ is a transposition (ij) . These operators together with the action of the symmetric group (4.6) provide a representation of the degenerate affine algebra $\mathcal{H}_{N,c}$ for $\mathfrak{g} = \mathfrak{gl}(N)$:

$$S_{s_i} \rightarrow s_i, \quad D_i \rightarrow \mathcal{D}_i, \quad i = 1, \dots, N. \quad (4.7)$$

The image of the quadratic element of the center is given by:

$$\frac{1}{2} \sum_{i=1}^N \mathcal{D}_i^2 = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \quad (4.8)$$

and thus coincides with the restriction of the quantum Hamiltonian on the N -particle sector of Nonlinear Schrödinger theory on the infinite interval.

We are interested in S_N -invariant solutions of (4.4). They play the role of spherical vectors (with respect to the spherical subalgebra $\mathbb{C}[W] \in \mathcal{H}_{N,c}$) in the representation theory of degenerate affine Hecke algebra.

The eigenvalue problem (4.4) allows the equivalent reformulation as an eigenvalue problem in the domain $x_1 \leq x_2 \leq \dots \leq x_N$ for the differential operator:

$$\left(-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}\right) \Phi_\lambda(x) = 2\pi^2 \left(\sum_{i=1}^N \lambda_i^2\right) \Phi_\lambda(x) \quad i = 1, \dots, N, \quad (4.9)$$

with the boundary conditions:

$$(\partial_{x_{i+1}} \Phi_\lambda(x) - \partial_{x_i} \Phi_\lambda(x))_{x_{i+1}-x_i=+0} = 4\pi c \Phi_\lambda(x)_{x_{i+1}-x_i=0}. \quad (4.10)$$

The solution is given by:

$$\Phi_\lambda^{(0)}(x) = \sum_{w \in W} \prod_{1 \leq i < j \leq N} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)}} \right) \exp(2\pi i \sum_k \lambda_{w(k)} x_k), \quad (4.11)$$

or equivalently:

$$\Phi_\lambda^{(0)}(x) = \frac{1}{\Delta_{\mathfrak{g}}(\lambda)} \sum_{w \in W} (-1)^{l(w)} \prod_{1 \leq i < j \leq N} (\lambda_{w(i)} - \lambda_{w(j)} + ic) \exp(2\pi i \sum_k \lambda_{w(k)} x_k). \quad (4.12)$$

where $\Delta_{Fg}(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)$. Note that the wave-function is explicitly symmetric under the action of symmetric group S_N on $\lambda = (\lambda_1, \dots, \lambda_N)$. This solution can be also

constructed using the representation theory of degenerate affine Hecke algebra $\mathcal{H}_{N,c}$ (see [31], [32], [33], [34]).

Given a solution of the equations (4.9) with boundary conditions (4.10) S_N -symmetric solutions of (4.4) on \mathbb{R}^N can be represented in the following form:

$$\Phi_\lambda^{(0)}(x) = \sum_{w \in W} \left(\prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic\epsilon(x_i - x_j)}{\lambda_{w(i)} - \lambda_{w(j)}} \right) \exp(2\pi i \sum_k \lambda_{w(k)} x_k) \right), \quad (4.13)$$

where $\epsilon(x)$ is a sign-function. This gives the full set of solutions of (4.4) for $(\lambda_1 \leq \dots \leq \lambda_N) \in \mathbb{R}^N$ satisfying the orthogonality condition with respect to the natural pairing:

$$\langle \Phi_\lambda, \Phi_\mu \rangle = \frac{1}{N!} \int dx_1 \cdots dx_N \bar{\Phi}_\lambda(x) \Phi_\mu(x) = G(\lambda) \prod_{i=1}^N \delta(\lambda_i - \mu_i), \quad (4.14)$$

where:

$$G(\lambda) = \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2 + c^2}{(\lambda_i - \lambda_j)^2}, \quad (4.15)$$

Therefore the normalized wave functions are given by:

$$\Phi_\lambda(x) = \sum_{w \in W} (-1)^{l(w)} \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic\epsilon(x_i - x_j)}{\lambda_{w(i)} - \lambda_{w(j)} - ic\epsilon(x_i - x_j)} \right)^{\frac{1}{2}} \exp(2\pi i \sum_k \lambda_{w(k)} x_k). \quad (4.16)$$

The eigenvalue problem for periodic N -particle Hamiltonian of Nonlinear Schrödinger theory can be reformulated in the following way. Consider the eigenfunction problem for the differential operator:

$$\left(-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{n \in \mathbb{Z}} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j + n) \right) \Phi_\lambda(x) = 2\pi^2 \left(\sum_{i=1}^N \lambda_i^2 \right) \Phi_\lambda(x) \quad i = 1, \dots, N. \quad (4.17)$$

The wave function of the periodic Nonlinear Schrödinger equation are the eigenfunction of (4.17) satisfying the following invariance conditions:

$$\Phi_\lambda(x_1, \dots, x_j + 1, \dots, x_N) = \Phi_\lambda(x_1, \dots, x_N), \quad j = 1, \dots, N, \quad (4.18)$$

$$\Phi_\lambda(x_{w(1)}, \dots, x_{w(N)}) = \Phi_\lambda(x_1, \dots, x_N), \quad w \in S_N.$$

These are the conditions of invariance under the action of the affine Weyl group on the space of wave functions.

The solutions can be obtained imposing the additional periodicity conditions on the wave functions (4.16). This leads to the following set of the Bethe Ansatz equations for $(\lambda_1, \dots, \lambda_N)$:

$$\mathcal{F}_j(\lambda) \equiv e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic} = 1, \quad j = 1, \dots, N. \quad (4.19)$$

The set of solutions of these equations can be enumerated by sets of integer numbers $(p_1 \geq \dots \geq p_N)$ - for each ordered set of these integers there is exactly one solution to Bethe Ansatz equations [27]. Let us remark that there is the following equivalent representation for the periodic wave-functions:

$$\tilde{\Phi}_\lambda(x) = \sum_{w \in W} (-1)^{l(w)} \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)} - ic} \right)^{\frac{1}{2} + [x_i - x_j]} \exp(2\pi i \sum_k \lambda_{w(k)} x_k), \quad (4.20)$$

where $[x]$ is an integer part of x defined by the conditions: $[x] = 0$ for $0 \leq x < 1$ and $[x + n] = [x] + n$. It is easy to see that these wave functions are periodic and descend to the wave functions (4.16) if $\lambda = (\lambda_1, \dots, \lambda_N)$ satisfy (4.19). The normalized wave functions in the periodic case are given by:

$$\Phi_\lambda^{norm}(x) = \left(\det \left\| \frac{\partial \log \mathcal{F}_j(\lambda)}{\partial \lambda_k} \right\| \right)^{-1/2} \Phi_\lambda(x) = \mu(\lambda)^{-1/2} \Phi_\lambda(x). \quad (4.21)$$

Note that the normalization factor is closely related to the factor (3.26) arising in the representation of the partition function of Yang-Mills-Higgs theory. Indeed the function $I(\lambda)$ introduced in (3.26) is known in the theory of Nonlinear Schrödinger equations as Yang function [27]; critical points of Yang function are in one to one correspondence with the solutions of Bethe Ansatz equations:

$$\alpha_j(\lambda) = \log \mathcal{F}_j(\lambda) = \frac{\partial I(\lambda)}{\partial \lambda_j} = n_j. \quad (4.22)$$

Below we will see that this is not accidental. Finally note that the periodic Nonlinear Schrödinger theory has an interpretation in terms of the representation theory of the degenerate double affine Hecke algebras introduced by Cherednik [38]. For the details in this regard see [34].

4.1. $c \rightarrow \infty$: Representation theory of compact Lie groups

The limit $c \rightarrow \infty$ corresponds to the case of impenetrable bosons and the correlation functions are naturally represented in terms of free fermions. Indeed in the limit $c \rightarrow \infty$ Bethe Ansatz equations are reduced to the condition:

$$\lambda_j = \frac{N-1}{2} + m_i, \quad m_i \in \mathbb{Z}, \quad (4.23)$$

and the wave-function is given by the wave-function of free fermions (up to a simple sign factor):

$$\Phi_\lambda^{c=\infty}(x) = \frac{|\Delta(e^x)|}{\Delta(e^x)} \det \|e^{i\lambda_k x_j}\| = \frac{|\Delta(e^x)|}{\Delta(e^x)} \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \sum_{k=1}^N \lambda_{w(k)} x_k}. \quad (4.24)$$

Note that we have a simple relation with the characters of the finite-dimensional representations of $U(N)$:

$$\text{ch}_\lambda(x) = \frac{1}{|\Delta(e^x)|} \Phi_\lambda^{c=\infty}(x). \quad (4.25)$$

Thus in the limit $c \rightarrow \infty$ the wave function $\Phi_\lambda(x)$ in Nonlinear Schrödinger theory can be considered as a wave-function in two-dimensional Yang-Mills theory renormalized according to $\Phi_i(x) = |\Delta_G(e^x)| \Psi_i(x)$ (see Section 2).

4.2. $c \rightarrow 0$: Representation theory of complex Lie groups

In the limit $c \rightarrow 0$ Bethe Ansatz equations are reduced to the condition:

$$\lambda_i = m_i, \quad m_i \in \mathbb{Z}, \quad (4.26)$$

and the wave-functions are given by:

$$\Phi_\lambda^{c=0}(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \sum_{k=1}^N \lambda_{w(k)} x_k}. \quad (4.27)$$

Note that wave functions (4.27) are normalized with respect to the standard scalar product. Now we have a simple relation with the characters (3.49) of the infinite-dimensional representations of $GL(N, \mathbb{C})$:

$$\text{ch}_\lambda(x) = \frac{1}{|\Delta(e^x)|} \Phi_\lambda^{c=0}(x). \quad (4.28)$$

4.3. $c \neq 0, \infty$: Representation theory of p -adic Lie groups

Wave functions of Nonlinear Schrödinger theory for $c \neq 0, \infty$ has also a connection with the representation theory of Lie groups due to the relation between the representation theory of the degenerate affine Hecke algebras $\mathcal{H}_{N,c}$ and the representation theory of $GL(N, \mathbb{Q}_p)$ where \mathbb{Q}_p is a field of p -adic numbers. More specifically the wave functions of Nonlinear Schrödinger theory can be obtained as a limit of Hall-Littlewood polynomials that can be considered as generalized zonal spherical functions for $GL(N, \mathbb{Q}_p)$. The limit is a kind of $p \rightarrow 1$ limit³. For the detailed discussion of p -adic zonal spherical functions see Macdonald [40] and for the relation with Nonlinear Schrödinger theory through representation theory of degenerate Hecke algebras see [33].

Let us start with the definition of Hall-Littlewood polynomials (see e.g. [41]). Let $\{\Lambda_i\}$, $i = 1, \dots, N$ be a set of formal variables and $\mu = (\mu_1, \dots, \mu_N)$ be a partition of length N . Then Hall-Littlewood polynomial depending on additional formal variable t is defined as:

$$\begin{aligned} P_\mu(\Lambda_1, \dots, \Lambda_N | t) &= \frac{1}{v_\mu(t)} \sum_{w \in S_N} w \left(\Lambda_1^{\mu_1} \cdots \Lambda_N^{\mu_N} \prod_{i < j} \frac{\Lambda_i - \Lambda_j t}{\Lambda_i - \Lambda_j} \right) = \\ &= \frac{1}{v_\mu(t) \Delta(\Lambda)} \sum_{w \in S_N} (-1)^{l(w)} w \left(\Lambda_1^{X_1} \cdots \Lambda_N^{X_N} \prod_{i < j} (\Lambda_i - \Lambda_j t) \right), \end{aligned} \quad (4.29)$$

where for the partition $\mu = (1^{m_1}, 2^{m_2}, \dots, r^{m_r}, \dots)$:

$$v_\mu = \prod_{j=1}^N \prod_{i=1}^{m_j} \frac{1-t^i}{1-t}, \quad \Delta(\Lambda) = \prod_{i < j} (\Lambda_i - \Lambda_j). \quad (4.30)$$

Hall-Littlewood polynomials enter the explicit formulas for the zonal spherical functions for p -adic Lie groups. The zonal spherical functions are defined as follows. Given a p -adic Lie group G and its maximal compact subgroup $K \subset G$ zonal spherical function $\omega(g)$ on G is a continuous complex-valued functions satisfying the following conditions: (1) the function is invariant with respect to the left and right action of the compact subgroup $\omega(kgk') = \omega(g)$, $k, k' \in K$, (2) normalization condition $\omega(1) = 1$, (3) the function is eigenfunction for the convolution with any function with compact support on G satisfying (1). The spherical functions for $G = GL(N, \mathbb{Q}_p)$, $K = GL(N, \mathbb{Z}_p)$ (here \mathbb{Z}_p is a ring of p -adic integers) has

³ For another example where the same limit $p \rightarrow 1$ is relevant in string theory see [39].

the following representation in terms Hall-Littlewood polynomials. Note that the set of the representatives of the double-coset $K \backslash G / K$ can be identified with the elements of the form $(p^{\mu_1}, \dots, p^{\mu_N}) \in K \backslash G / K$ where $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_N)$ is a partition. Then for the zonal spherical functions we have:

$$\omega_s(p^{\mu_1}, \dots, p^{\mu_N}) = p^{-\sum_{i=1}^N (n-i)\mu_i} \frac{v_\mu(p^{-1})}{v_N(p^{-1})} P_\mu(p^{-s_1}, \dots, p^{-s_N} | p^{-1}), \quad (4.31)$$

where $s = (s_1, \dots, s_N) \in \mathbb{Z}^N$ and

$$v_N(t) = \prod_{i=1}^N \frac{1-t^i}{1-t}.$$

Note that the ‘‘spectral’’ indexes in the spherical function and Hall-Littlewood polynomial are interchanged. There is a generalization of the notion of the spherical function which is similar to the multivariable hypergeometric functions for general root systems introduced by Heckman and Opdam (see e.g. [42]). For the case of the p -adic Lie groups the generalized spherical function depends on the additional integer parameter $k \in \mathbb{Z}$ and is given by:

$$\omega_s^{(k)}(p^{\mu_1}, \dots, p^{\mu_N}) = p^{-\sum_{i=1}^N (n-i)\mu_i} \frac{v_\mu(p^{-k})}{v_N(p^{-k})} P_\mu(p^{-s_1}, \dots, p^{-s_N} | p^{-k}). \quad (4.32)$$

Now one can see how in a particular limit the wave functions of N -particle sector of Nonlinear Schrödinger theory on \mathbb{R} obtained using the coordinate Bethe Ansatz arise. Taking $\mu_i = \epsilon^{-1} x_i$, $\Lambda_i = \exp(2\pi\epsilon\lambda_i)$ and $t = e^{2\pi i c \epsilon}$ while $\epsilon \rightarrow 0$ (we use analytical continuation over parameters here) we have for (4.29):

$$\frac{v_\mu(t)}{v_N(t)} P_\mu(\Lambda_1, \dots, \Lambda_N; t) \rightarrow \frac{1}{(c\epsilon)^N} \frac{1}{N!} \sum_{w \in S_N} w \left(e^{2\pi i \sum_{k=1}^N x_k \lambda_k} \prod_{k < j} \frac{\lambda_k - \lambda_j + ic}{\lambda_k - \lambda_j} \right) \quad (4.33)$$

This expression coincides with the restriction of the Bethe wave function on the subspace $x_1 < x_2 < \dots < x_N$. Thus taking into account (4.32) one can conclude that the generalized zonal spherical functions in the formal limit $\epsilon \rightarrow 0$ while $p = e^{2\pi i \epsilon}$, $\mu_i = \frac{1}{\epsilon} x_i$ are given by the wave functions for the N -particle sector of Nonlinear Schrödinger equation with $c = k$.

It known that Hall-Littlewood polynomials are a special case of the Macdonald polynomials and one can expect that Hall-Littlewood polynomials before degeneration and more general Macdonald polynomials should be related with the quantum integrable/topological field theories these along the line discussed in this section. Below in Section 8 we propose the generalization of the topological Yang-Mills-Higgs theory that provides a realization of these more general polynomials.

5. Wave-function in topological Yang-Mills-Higgs theory

In this section we provide the evidences for the identification of a bases of the wave functions of topological Yang-Mills-Higgs theory for $G = U(N)$ (given by a path integral on a disk with the insertion of observables in the center) with the eigenfunctions of the N -particle Hamiltonian operator of Nonlinear Schrödinger theory. First by counting the observables of the theory we show that the phase space of the Yang-Mills-Higgs theory can be considered as a deformation of the phase space of Yang-Mills theory. This implies that the bases of wave functions in Yang-Mills-Higgs theory can be obtained by a deformation of the bases of wave functions in Yang-Mills theory. Next, using the explicit representation of the partition function on the two-dimensional torus we derive the transformation properties of the wave functions under large gauge transformations. They are in agreement with the known explicit transformation properties of wave function in Nonlinear Schrödinger theory. Finally, we compute the cylinder path integral (Green function) and torus partition function in Nonlinear Schrödinger theory (with all higher Hamiltonians) and show that latter coincides with torus partition function in Yang-Mills-Higgs theory (with arbitrary observables turned on). Taking into account that the constructed wave functions in the gauge theory are the eigenfunctions of the full set of the Hamiltonian operators these considerations presumably uniquely fix the set of wave functions.

5.1. Local Q -cohomology

We start with a description of the Hilbert space of the Yang-Mills-Higgs theory using the operator-state correspondence. In the simplest form the operator-state correspondence is as follows. Each operator, by acting on the vacuum state, defines a state in the Hilbert space. In turn for each state there is an operator, creating the state from the vacuum state. Moreover, for the maximal commutative subalgebra of the operators this correspondence should be one to one. For example, the space of local gauge-invariant Q -cohomology classes in topological $U(N)$ Yang-Mills theory is spanned, linearly, by the operators:

$$\mathcal{O}_k^{(0)} = \frac{1}{(2\pi i)^k} \text{Tr } \varphi^k, \quad (5.1)$$

and thus this space coincides with the space of Ad_G -invariant regular functions on the Lie algebra \mathfrak{u}_N . This is in accordance with the description of the Hilbert space of the theory given in Section 2.

We would like to apply the same reasoning to the topological Yang-Mills-Higgs theory. To get economical description of the Hilbert space of the theory one should find a maximal (Poisson) commutative subalgebra of local Q -cohomology classes (where Q given by (3.37) acts on the space of functions invariant under the symmetries generated by (3.7), (3.8)). Obviously operators (5.1) provide non-trivial cohomology classes. One can show that these operators provide a maximal commutative subalgebra for $c \neq 0$ and therefore the reduced phase space in Yang-Mills-Higgs system can be identified with a phase space of pure Yang-Mills theory. Thus the Hilbert space of Yang-Mills-Higgs theory ($c \neq 0$) can be naturally identified with the Hilbert space of Yang-Mills theory (identified with $c \rightarrow \infty$). The fact that the Hilbert space of Yang-Mills-Higgs theory is the same for all $c \neq 0$ implies that the bases of wave-functions for $c \neq 0$ should be a deformation of the bases for $c = \infty$.

One should stress that this reasoning is not applicable to the case $c = 0$ ⁴. The local cohomology for $c = 0$ contains additional operators. For example, the following operators provide non-trivial cohomology classes for arbitrary $t \in \mathbb{C}$ and $c = 0$:

$$\mathcal{O}_k^{(0)}(t) = \frac{1}{(2\pi i)^k} \text{Tr}(\varphi_0 + t\varphi_+ - \frac{t}{2}\chi_+^2)^k. \quad (5.2)$$

This is a manifestation of the fact that $c = 0$ theory is a Yang-Mills theory for the complexified group G^c and thus its phase space is given by $\mathcal{M}_{\mathbb{C}} = T^*H^c/W$.

The identification of the Hilbert spaces of Yang-Mills-Higgs theory and of pure Yang-Mills theory supports the idea to use wave functions in Nonlinear Schrödinger theory discussed in the previous section as a bases in the Hilbert space of Yang-Mills-Higgs theory. Below we provide further evidences for this identification.

5.2. Gauge transformations of wave function

The discreteness of the spectrum of N -particle Hamiltonian operator in periodic Nonlinear Schrödinger theory arises due to periodicity condition on wave functions. Thus the eigenfunctions in the periodic case are given by a subset of eigenfunctions on \mathbb{R}^N descending to S_N -invariant functions on $(S^1)^N$. The eigenfunction (4.16) of the Hamiltonian operator

⁴ We distinguish the case corresponding to Yang-Mills theory for a complex group ($c = 0$) and $c \rightarrow 0$ in the Yang-Mills-Higgs theory. In the latter case, as we explained before, the Hilbert space is the same as for Yang-Mills-Higgs theory with $c \neq 0$ and the corresponding bases of wave functions is given by the formal characters (3.49) for a subset of the unitary representations of the complex group.

on \mathbb{R}^N represented as sum over elements of symmetric group of simple wave functions. For generic eigenvalues each term of the sum is multiplied by some function under the shift $x_i \rightarrow x_i + n_i$, $n_i \in \mathbb{Z}$ of the coordinates. Below we will show how these multiplicative factors arising in Nonlinear Schrödinger theory can be derived in Yang-Mills-Higgs gauge theory.

We start with a simple case of Yang-Mills theory. The partition function of $U(N)$ Yang-Mills theory on the torus Σ_1 is given by:

$$\begin{aligned} Z_{YM}(\Sigma_1) &= \int_{\mathbb{R}^N/S_N} d^N \lambda \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)} = \quad (5.3) \\ &= \sum_{(m_1, \dots, m_N) \in P_{++}} e^{-\sum_{k=1}^{\infty} t_k p_k(m+\rho)} \end{aligned}$$

where P_{++} is a set of the dominant weights of $U(N)$. The sum over $(n_1, \dots, n_N) \in \mathbb{Z}^N$ has a meaning of the sum over topological classes of $U(1)^N$ -principle bundles on the torus Σ_1 . It results in the replacement of the integration over λ by a sum over a discrete subset. This should be compared with the partition function of dimensionally reduced $U(N)$ Yang-Mills theory on S^1 :

$$Z_{QM}(S^1) = \int_{\mathbb{R}^N/S_N} d^N \lambda e^{-\sum_{k=1}^{\infty} t_k p_k(\lambda)}. \quad (5.4)$$

Contrary to the two-dimensional Yang-Mills theory in the last case we do not have any additional restriction on the spectrum $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N/S_N$. The appearance of the additional sum in (3.40) can be traced back to the difference between the Hilbert spaces of dimensionally reduced and non-reduced theories.

The mechanism of the spectrum restriction via the sum over the topological sectors can be explained in terms of the structure of the Hilbert space of the theory as follows. In the Hamiltonian formalism the partition function on a torus is given by the trace of the evolution operator over the Hilbert space of the theory. Let us consider first the dimensionally reduced $U(N)$ Yang-Mills theory. The phase space of the theory is $T^*\mathbb{R}^N/S_N$ where we divide over Weyl group $W = S_N$. To construct the Hilbert space we quantize the phase space using the following polarization. Consider Lagrangian projection $\pi : T^*\mathbb{R}^N \rightarrow \mathbb{R}^N$ supplied with a section. We chose the coordinates on the base as position variables and the coordinates on the fibers as the corresponding momenta. Thus the Hilbert space in this polarization is realized as a space of S_N (skew)-invariant functions on the base \mathbb{R}^N of the projection.

Now consider two-dimensional Yang-Mills theory. For the phase space we have T^*H/S_N where H is Cartan subgroup. We use similar polarization associated with the projection $\pi : T^*H \rightarrow H$. Thus the wave functions are S_N invariant functions on a torus H or equivalently the functions on \mathbb{R}^N invariant under action of the semidirect product of the lattice $P_0 = \pi_1(H)$ and Weyl $W = S_N$ group (i.e. under the action of the affine Weyl group W^{aff}). The lattice P_0 can be interpreted as a lattice of the \mathbb{R}^N -valued constant connections on S^1 which are gauge equivalent to the zero connection. The corresponding gauge transformations act on the wave functions by the shifts $x_j \rightarrow x_j + n_j$, $n_j \in \mathbb{Z}$ of the argument of the wave functions in the chosen polarization and the wave functions in two-dimensional Yang-Mills theory can be obtained by the averaging over this gauge transformations and global gauge transformations by the nontrivial elements of the normalizer of Cartan torus $W = N(H)/H$.

It is possible to relate the averaging over the topologically non-trivial transformations with the sum over topological classes of H bundle on the torus. Note that the maps of S^1 to the gauge group H are topologically classified by $\pi_1(H) = \mathbb{Z}^N$. Consider a connection $A = (A_1, \dots, A_N)$ on a H bundle over a cylinder L , $\partial L = S^1_+ \cup S^1_-$ such that the holonomies along the boundaries S^1_+ and S^1_- are in the different topological classes $[(m_1, \dots, m_N)] \in \pi_1(H)$ and $[(m_1 + n_1, \dots, m_N + n_N)] \in \pi_1(H)$. Gluing boundaries of the cylinder L we obtain a torus supplied with a connection ∇_A such that the first Chern classes of the bundles corresponding to each $U(1)$ -factor are given by $c_1(\nabla_{A_i}) = \frac{1}{2\pi i} \int_L F(A_j) = n_j$, $j = 1, \dots, N$. Thus we see that the sum over the topologically non-trivial gauge transformations on S^1 can be translated into the sum over topological classes of the H -bundles on the torus.

Let us re-derive the partition function of Yang-Mills theory on the torus (5.3) using the averaging procedure. We start with the dimensionally reduced theory. Let us choose a bases in the Hilbert space of the dimensionally reduced Yang-Mills theory given by the S_N skew-invariant eigenfunctions of the quadratic operator $H_2^{(0)} = \text{tr}\varphi^2$. In the polarization discussed above we have:

$$H_2^{(0)} \psi_\lambda(x) = -\frac{1}{2} \left(\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \right) \psi_\lambda(x) = 2\pi^2 \sum_{j=1}^N \lambda_j^2 \psi_\lambda(x), \quad (5.5)$$

where $(x_1, \dots, x_N) \in \mathbb{R}^N$. The set of normalized skew-invariant eigenfunctions is given by:

$$\psi_\lambda(x) = \sum_{w \in S_N} (-1)^{l(w)} \exp(2\pi i \sum_{j=1}^N \lambda_{w(j)} x_j), \quad (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N/W. \quad (5.6)$$

$$\frac{1}{N!} \int_{\mathbb{R}^N} d^N x \overline{\psi_\lambda(x)} \psi_{\lambda'}(x) = (2\pi)^N \sum_{w \in S_N} (-1)^{l(w)} \prod_{j=1}^N \delta(\lambda_{w(j)} - \lambda'_j) = \delta^{(S_N)}(\lambda - \lambda'). \quad (5.7)$$

The integral kernel of the identity operator acting on the skew-symmetric functions can be represented (due to translation invariance it is the function of difference $x - x'$) as:

$$K_0(x, x') = K_0(x - x') = \delta^{(S_N)}(x - x') = \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi_\lambda(x)} \psi_\lambda(x'). \quad (5.8)$$

The partition function of the dimensionally reduced Yang-Mills theory on S^1 is given by the trace of a evolution operator and can be written explicitly as:

$$\begin{aligned} Z_{QM}(S^1) &= \text{Tr} e^{-t_2 H_2(\hat{p}, \hat{q})} = \int_{(\mathbb{R}^N \times \mathbb{R}^N)/S_N} d^N x d^N \lambda \overline{\psi_\lambda(x)} e^{-t_2 H_2^{(0)}(i\partial_x, x)} \psi_\lambda(x) = \\ &= \int_{(\mathbb{R}^N \times \mathbb{R}^N)/S_N} d^N x d^N \lambda e^{-t_2 p_2(\lambda)}. \end{aligned} \quad (5.9)$$

The Green function of the theory is:

$$\begin{aligned} G_0(x, x') &= \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi_\lambda(x')} e^{-t_2 H_2^{(0)}(i\partial_x, x)} \psi_\lambda(x) = \\ &= \int_{\mathbb{R}^N/S_N} d^N \lambda \psi_\lambda(x') e^{-t_2 p_2(\lambda)} \psi_\lambda(x). \end{aligned} \quad (5.10)$$

Up to the infinite factor given by the integral over $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ the integral in (5.9) coincides with the expression (5.4) for the partition function for $t_{i \neq 2} = 0$.

Now consider two-dimensional Yang-Mills theory. In this case we have the periodic eigenvalue problem for (5.5). Then for the normalized eigenfunctions of H_2 we have:

$$\psi_n(x) = \sum_{w \in S_N} (-1)^{l(w)} \exp(2\pi i \sum_{j=1}^N (n_{w(j)} + \rho_{w(j)}) x_j), \quad (n_1, \dots, n_N) \in P_{++}, \quad (5.11)$$

$$\frac{1}{N!} \int_{(S^1)^N} d^N x \overline{\psi_n(x)} \psi_{n'}(x) = \sum_{w \in S_N} (-1)^{l(w)} \prod_{j=1}^N \delta_{n_{w(j)}, n'_j} = \delta_{n, n'}^{S_N}. \quad (5.12)$$

Here $\rho = (\rho_1, \dots, \rho_N)$ is a half-sum of the positive roots of \mathfrak{u}_N . The integral kernel of the identity operator can be represented as:

$$K(x, x') = K(x - x') = \delta_{(S_N)}(x - x') = \sum_{n \in P_{++}} \overline{\psi_n(x)} \psi_n(x'). \quad (5.13)$$

The partition function of the Yang-Mills theory on a torus Σ_1 is given by the trace of a evolution operator and:

$$Z_{YM}(\Sigma_1) = \text{Tr} e^{-t_2 H_2(\hat{p}, \hat{q})} = \sum_{n \in P_{++}} \int_{(S^1)^N} d^N x \overline{\psi}_n(x) e^{-t_2 H_2^{(0)}(i\partial_x, x)} \psi_n(x). \quad (5.14)$$

The kernel for the periodic case can be obviously represented as a matrix element of the projection operator as follows:

$$K(x, x') = \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(x) P(\lambda) \psi_\lambda(x'), \quad (5.15)$$

where the wave-functions $\psi_\lambda(x)$ are given by (5.6) and:

$$P(\lambda) = \sum_{m \in \mathbb{Z}^N} \prod_{j=1}^N \delta(\lambda_j - m_j) = \sum_{k \in \mathbb{Z}^N} e^{2\pi i \sum_{j=1}^N \lambda_j k_j}. \quad (5.16)$$

Equivalently we have:

$$\begin{aligned} K(x, x') &= \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(x) e^{2\pi i \sum_{j=1}^N \lambda_j k_j} \psi_\lambda(x') = \\ &= \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(x) \psi_\lambda(x' + k). \end{aligned} \quad (5.17)$$

We conclude that the Green function $G(x, x')$ (the path integral on the cylinder with insertion of $\exp(-t_2 H_2^{(0)})$) is represented as:

$$G_{YM}(x, x') = \sum_{n \in \mathbb{Z}^N} \int_{\lambda \in \mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(x) e^{2\pi i \sum_{j=1}^N \lambda_j n_j} e^{-t_2 p_2(\lambda)} \psi_\lambda(x') \quad (5.18)$$

or equivalently as:

$$G_{YM}(x, x') = \sum_{k \in \mathbb{Z}^N} \int_{\lambda \in \mathbb{R}^N/S_N} d^N \lambda \overline{\psi}_\lambda(x) e^{-t_2 p_2(\lambda)} \psi_\lambda(x' + k). \quad (5.19)$$

Let us note that the identities in (5.17), (5.19) are based on the following transformation property of the complete set of skew-symmetric normalized wave-functions on \mathbb{R}^N :

$$\psi_\lambda(x + k) = \sum_{w \in S_N} (-1)^{l(w)} e^{2\pi i \sum_{j=1}^N \lambda_{w(j)} k_j} e^{2\pi i \sum_{j=1}^N \lambda_{w(j)} x_j}. \quad (5.20)$$

Thus each elementary term in the averaging over S_N is multiplied on the simple exponent factor entering the description of the projector (5.16). Let us also note that the shift transformations in (5.20) can be interpreted as large gauge transformations in Yang-Mills theory discussed above.

The representation (5.19) can be written in the following form:

$$G_{YM}(x, x') = \sum_{k \in \mathbb{Z}^N} G_0(x, x' + k). \quad (5.21)$$

If we set the coupling t_2 to zero, $t_2 = 0$, we recover the formula (5.17) for $K(x, x')$:

$$K(x, x') = \sum_{k \in \mathbb{Z}^N} K_0(x, x' + k). \quad (5.22)$$

For the partition function of Yang-Mills theory on a torus we get (after setting $x = x'$ above and integrating over x):

$$\begin{aligned} Z_{YM}(\Sigma_1) &= \sum_{n \in \mathbb{Z}^N} \int_{\lambda \in \mathbb{R}^N / S_N} d^N \lambda \int_{(S^1)^N} d^N x \overline{\psi_\lambda}(x) e^{2\pi i \sum_{j=1}^N \lambda_j n_j} e^{-t_2 p_2(\lambda)} \psi_\lambda(x) = \\ &= \sum_{m \in P_{++}} e^{-t_2 p_2(m+\rho)}. \end{aligned} \quad (5.23)$$

and this coincides with the representation (5.3). Note that obvious relations between (5.17), (5.21), (5.22) and the averaging over the topologically non-trivial gauge transformations discussed above.

Let us remark that the averaging procedure represented by (5.21), (5.22) is a standard tool in construction of Green functions on non-simply connected spaces. At the first step one computes the Green function on the universal covering space and then averages with respect to the action of the action of π_1 of the underlying non-simply connected space. For example this procedure has been used in similar problem of the quantization of the coadjoint orbits of compact Lie groups in [43]. We will apply this procedure to the Yang-Mills-Higgs theory in a fashion described above for 2d Yang-Mills theory.

Now we are finally ready to consider the case of Yang-Mills-Higgs theory. As it was conjectured above one can chose as a bases of the wave-functions the bases of eigenfunctions of the set of the Hamiltonian operators in N -particle subsector of Nonlinear Schrödinger theory. Below we construct the Green function and partition function in Nonlinear Schrödinger theory and demonstrate that identifying the Hamiltonian operator

with quadratic observable $\mathcal{O}_2^{(0)} = \frac{1}{(2\pi)^2} \text{Tr} \varphi_0^2$ in Yang-Mills-Higgs theory we reproduce the partition function of Yang-Mills-Higgs theory on a torus.

Let us start with the construction of the kernel of the unit operator in the bases of the N -particle eigenfunctions of the Nonlinear Schrödinger theory. The representation for the kernel (5.17) can be straightforwardly generalized to this case:

$$\tilde{K}(x, x') = \sum_{(\lambda_1, \dots, \lambda_N) \in \mathcal{R}_N} \overline{\Phi_\lambda^{norm}}(x) \Phi_\lambda^{norm}(x') = \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\Phi_\lambda}(x) P(\lambda) \Phi_\lambda(x'), \quad (5.24)$$

where $\Phi_\lambda(x)$ are normalized skew-invariant eigenfunctions on \mathbb{R}^N given by (4.16), $\Phi_\lambda^{norm}(x)$ are normalized periodic eigenfunctions given by (4.21) and the sum goes over the set \mathcal{R}_N of the solutions of Baxter Ansatz equations. The projector here is given by:

$$P(\lambda) = \mu(\lambda) \sum_{m \in \mathbb{Z}^N} \prod_{j=1}^N \delta(\alpha_j(\lambda) - m_j) = \sum_{(\lambda_1^*, \dots, \lambda_N^*) \in \mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda_j^*) \quad (5.25)$$

where $\alpha_j(\lambda)$ are defined as follows (compare with (3.28)):

$$\alpha_j(\lambda) = \lambda_j + \frac{1}{2\pi i} \sum_{k \neq j} \log \left(\frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic} \right). \quad (5.26)$$

Then we have:

$$\begin{aligned} \tilde{K}(x, x') &= \sum_{n \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \mu(\lambda) \overline{\Phi_\lambda}(x) e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{l \neq j} \left(\frac{\lambda_l - \lambda_j - ic}{\lambda_l - \lambda_j + ic} \right)^{n_j} \Phi_\lambda(x') = \\ &= \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\Phi_\lambda}(x) \Phi_\lambda(x' + n). \end{aligned} \quad (5.27)$$

The last equality follows from the following property of the eigenfunctions (4.20) of the N -particle Hamiltonian in Nonlinear Schrödinger theory:

$$\begin{aligned} \Phi_\lambda(x + n) &= \sum_{w \in W} (-1)^{l(w)} \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)} - ic} \right)^{n_i} \exp(2\pi i \sum_m \lambda_{w(m)} n_m) \times \\ &\times \prod_{i < j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic}{\lambda_{w(i)} - \lambda_{w(j)} - ic} \right)^{\frac{1}{2} + [x_i - x_j]} \exp(2\pi i \sum_k \lambda_{w(k)} x_k), \end{aligned} \quad (5.28)$$

These wave functions are periodic and descend to the wave functions (4.16) if $\lambda = (\lambda_1, \dots, \lambda_N)$ satisfy (4.19).

The representation for the kernel (5.27) leads to the following representation for the Green function (cylinder path integral) and torus partition function for $U(N)$ Yang-Mills-Higgs theory for $t_{i \neq 2} = 0$:

$$G_{YMH}(x, x') = \tag{5.29}$$

$$= \sum_{n \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \mu(\lambda) \overline{\Phi_\lambda(x)} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{l \neq j} \left(\frac{\lambda_l - \lambda_j - ic}{\lambda_l - \lambda_j + ic} \right)^{n_j} e^{-t_2 p_2(\lambda)} \Phi_\lambda(x'),$$

or same:

$$G_{YMH}(x, x') = \sum_{k \in \mathbb{Z}^N} G_{YMH}^0(x, x' + k) = \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{R}^N/S_N} d^N \lambda \overline{\Phi_\lambda(x)} e^{-t_2 p_2(\lambda)} \Phi_\lambda(x' + k).$$

Similarly for kernel: $\tilde{K}(x, x') = \sum_{k \in \mathbb{Z}^N} \tilde{K}_0(x, x' + k)$ since the kernel is a Green function at $t_2 = 0$. Integrating over x after setting $x = x'$ we obtain the representation for the partition function on the torus:

$$Z_{YMH}(\Sigma_1) = \tag{5.30}$$

$$= \int_{\mathbb{R}^N/S_N} d^N \lambda \mu(\lambda) \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m} \prod_{l \neq j} \left(\frac{\lambda_l - \lambda_j - ic}{\lambda_l - \lambda_j + ic} \right)^{n_j} e^{-t_2 p_2(\lambda)}.$$

This is in a complete agreement with a representation for the partition function of $U(N)$ Yang-Mills-Higgs theory on a torus discussed in Section 3, formula (3.30). All above expressions have the property to recover corresponding well-known answers of 2d YM theory in the limit $c \rightarrow \infty$, as they should from the general arguments presented before. Note that one can repeat the same arguments for all observables and higher differential operators of Nonlinear Schrödinger theory, traces of higher powers of Dunkle operator from Section 4, by simply turning on all other couplings t_k .

The identification of the representation of the partition function of Nonlinear Schrödinger operator and Yang-Mills-Higgs theory on the torus strongly suggests that the full equivalence of the theories. It would be very desirable to obtain same answer for Green function and wave-function (cylinder path integral) directly from path integral for cylinder topology using the cohomology localization technique.

Finally let us comment on the explicit form of the wave function (5.28) from the gauge theory point of view. The appearance of an integer part $[x]$ in (5.28) is not quite unexpected phenomena from the point of view of the proposed identification of the wave functions in Yang-Mills-Higgs theory with the wave functions in Nonlinear Schrödinger theory. Let us remark that the interpretation of the topologically non-trivial bundles as

an interpolation between topologically nontrivial gauge transformations naturally arises in the discussion of the spectral flow of the eigenfunctions of the gauge invariant operators on the boundary (see [44], [45] for the details and examples). To make a closer contact with this interpretation let us recall the simplest instance of Atiyah-Patodi-Singer index theorem on an two-dimensional manifold with non-empty boundary [46].

Let us given an even-dimensional Riemann spin manifold M with a boundary ∂M , and a vector bundle \mathcal{E} supplied with a connection ∇_A . Consider Dirac operator on a vector bundle $\mathcal{E} \otimes S$ where S is a spinor bundle supplied with a connection ∇_S . The definition of the index of the Dirac operator on a non-compact manifold M relies on the correct treatment of the boundary conditions. In [46] specific non-local boundary conditions were defined corresponding to a vacuum state in the Hilbert space of the Dirac fermions. We would like to apply the index theorem to the two-dimensional cylinder L , $\partial L = S_+^1 \cup S_-^1$ with a flat metric and $U(1)$ bundles supplied with a connection ∇_A . The spinor bundle on L can be identified with $\Omega^{0,0}(L) \oplus \Omega^{0,1}(L)$ and the Dirac operator is given by $\mathcal{D} = \bar{\partial}_A + \bar{\partial}_A^+$. We have the following expression for the index of \mathcal{D} which coincides with the index of $\bar{\partial}_A$:

$$\text{Index}_{\mathcal{D}} = \int_L c_1(\nabla) + \frac{1}{2} \eta(S_+^1 \cup S_-^1) \quad (5.31)$$

where the η -invariant $\eta(S_+^1 \cup S_-^1)$ is defined in terms of the spectrum of the restriction of \mathcal{D} to the boundary :

$$\eta(\partial L) = \lim_{s \rightarrow 0} \sum_{\lambda_i \in \text{Spec}(\mathcal{D}|_{\partial L})} \text{sign}(\lambda_i) |\lambda_i|^s. \quad (5.32)$$

Taking into account the relation $\int_L F(A) = \int_{S_+^1} A - \int_{S_-^1} A$ we have:

$$\left(\eta(S_+^1) - \frac{i}{2\pi} \int_{S_+^1} A \right) - \left(\eta(S_-^1) - \frac{i}{2\pi} \int_{S_-^1} A \right) = \text{Index}_{\mathcal{D}} \in \mathbb{Z}. \quad (5.33)$$

An easy calculation shows that the η -invariant for a a constant connection $\nabla_A = \partial_t + x$, on S^1 is given by:

$$-\frac{1}{2} \eta(S^1) = x - [x] - \frac{1}{2}. \quad (5.34)$$

where $[x]$ is an integer part of x . Thus:

$$\frac{i}{2\pi} \int_{S^1} A - \frac{1}{2} \eta(S^1) = -x + (x - [x] - \frac{1}{2}) = [x] + \frac{1}{2}. \quad (5.35)$$

This seems makes the appearance of integer values in (5.28) less mysterious. One should stress however that the proper derivation of the wave function (5.28) using this reasoning

should use a more refined form of the η -invariant also introduced in [46]. Let B be an operator commuting with \mathcal{D} . Then character-valued η -invariant is given by:

$$\eta_B(\partial M) = \lim_{s \rightarrow 0} \sum_{\lambda_i \in \text{Spec}(\mathcal{D}|_{\partial M})} \text{sign}(\lambda_i) |\lambda_i|^s \text{tr}_{V_{\lambda_i}} B, \quad (5.36)$$

where V_{λ_i} is an eigenspace of \mathcal{D} corresponding to the eigenvalue λ_i . Using an appropriate operator B one can reproduce the phase factor in (5.28).

Finally note that η -invariant is defined using the vacuum boundary condition on the quantum fields [46]. Thus the appearance of η -invariant in the explicit expression for a wave function can be traced back to the fact that the additional fields $(\Phi, \psi_\Phi, \varphi_\pm, \chi_\pm)$ entering the description of the Yang-Mills-Higgs theory are in the vacuum state. Thus they only contribute to the total wave function is a phase factor.

6. On equivariant cohomology description of the Hilbert space

The realization of the representations of the degenerate affine Hecke algebras $\mathcal{H}_{\mathfrak{g},c}$ in the space of the $S^1 \times G$ -equivariant cohomology of the flag spaces for the Lie group G , such that $\text{Lie}(G) = \mathfrak{g}$ considered in [47], [48] (see also [49], [50]), bears an obvious resemblance with the constructions discussed in this paper. Below we make some preliminary remarks regarding this relation. The detailed consideration will be postponed for another occasion.

We start with the simplest case of the dimensionally reduced Yang-Mills theory. The one-dimensional Yang-Mills theory in the Hamiltonian formulation is a system with a first class constraint and thus the Hilbert space is naturally described as a cohomology of the corresponding BRST operator. The standard approach is to realize a Hilbert space as the cohomology of the BRST operator acting in some extended space including ghost variables. One considers a pair of canonically conjugate ghost-antighost fields (b, c) of ghost numbers $(-1, 1)$ such that BRST operator acts on the space of functions of φ and c , and is given by:

$$Q_{BRST} = \text{Tr} \left(c \left[\varphi, \frac{\partial}{\partial \varphi} \right] \right) - \frac{1}{2} \text{Tr} \left([c, c], \frac{\partial}{\partial c} \right). \quad (6.1)$$

The Hilbert space $\mathcal{H} = H_{Q_{BRST}}^*$ of the theory is naturally graded by the ghost number. The relevant cohomology can be interpreted as a Lie algebra cohomology of $\mathfrak{g} = \text{Lie}(G)$ with coefficients in the space of functions on \mathfrak{g} considered as a \mathfrak{g} -module with respect to the adjoint action:

$$\mathcal{H} = H_{Q_{BRST}}^* = H^*(\mathfrak{g}, \text{Fun}(\mathfrak{g})). \quad (6.2)$$

Note that the higher cohomologies are non-trivial and result differs from the naive expectation of finding the space of the gauge invariant function on \mathfrak{g} as a realization of the Hilbert space. Let us remark that (6.2) is close to $H_G(G)^*$.

One can try to use a more economical way to quantize the theory. Let us consider the same set of fields but use a modified BRST operator:

$$Q_{BRST} = Tr(c[\varphi, \frac{\partial}{\partial\varphi}]) - \frac{1}{2}Tr([c, c], \frac{\partial}{\partial c}) + \varphi \frac{\partial}{\partial c}. \quad (6.3)$$

The cohomology of this BRST operator provides a BRST model for G -equivariant cohomology of the point (see [51] for discussion of various models for equivariant cohomology):

$$\mathcal{H} = H_G^*(pt) \equiv H^*(BG) = \mathbb{C}[G]^G \quad (6.4)$$

We see that in this formulation we get the correct Hilbert space.

The interpretation of the Hilbert space of the dimensionally reduced Yang-Mills theory in terms of the equivariant cohomology is very natural. Let us recall that according to [52] the Hilbert space of the G/G gauged Wess-Zumino-Witten theory at the level k can be described in terms of twisted equivariant cohomology $K_G(G)^{*+k+c_v+\dim(G)}$. In the limit $k \rightarrow \infty$ this provides a description of the states in Yang-Mills theory in terms of equivariant cohomologies $H_G^*(G)$. After the reduction to one dimensions one gets the equivariant cohomology $H_G^*(G) \otimes \mathbb{C}$ with coefficients in \mathbb{C} which provide a model for Hilbert space of Yang-Mills theory.

7. On Nahm transform and Langlands duality

Taking into account previous considerations it is natural to look for more direct correspondence between two-dimensional $U(N)$ Yang-Mills-Higgs theory and quantum non-relativistic Nonlinear Schrödinger theory associated with $U(2)$ (Nonlinear Schrödinger theory exists for any group see e. g. [29], here we only utilized the $U(2)$ version of it). In this section we briefly comment on this issue leaving the detailed considerations for the future work.

Let us consider a covariant description of the phase space of the theory as a space of solutions of the equations of motions (see [53] for discussion of the general formalism). Applying the general construction to the two-dimensional theory one can identify the phase space with a space of classical solutions on $S^1 \times \mathbb{R}$. Consider first the Yang-Mills

theory. Equations of motion for φ lead to the flatness condition $F(A) = 0$ and thus the moduli space of the classical solutions has a natural projection onto the space of unitary flat G -connections on $S^1 \times \mathbb{R}$. Under appropriate boundary conditions this space can be identified with $G/\text{Ad}_G = H/W$. The fiber of the projection is given by the space of covariantly constant sections $\nabla_A \varphi = 0$ and thus the total phase space can be identified with the cotangent bundle to a moduli space of flat connections. Thus, indeed the covariant phase space coincides with the phase space $\mathcal{M} = T^*H/W$ described in Section 2.

Consider now Yang-Mills-Higgs theory for $c = 0$. Equations of motion obtained by the variation over $(\varphi_0, \varphi_+, \varphi_-)$ are given by:

$$F(A) - \Phi \wedge \Phi = 0, \quad \nabla_A^{(1,0)} \Phi^{(0,1)} = 0, \quad \nabla_A^{(0,1)} \Phi^{(1,0)} = 0, \quad (7.1)$$

and those given by the variation over (A, Φ) are:

$$\begin{aligned} \nabla_A^{(1,0)} \varphi_+ &= c\Phi^{(1,0)} + [\Phi^{(1,0)}, \varphi_0], \\ \nabla_A^{(0,1)} \varphi_- &= -c\Phi^{(0,1)} + [\Phi^{(0,1)}, \varphi_0]. \\ \nabla_A \varphi_0 &= [\Phi^{(1,0)}, \varphi_+] - [\Phi^{(0,1)}, \varphi_-]. \end{aligned} \quad (7.2)$$

Let us start with the space \mathcal{M}_H of the solutions of the first set of equations. The equations:

$$F(A) - \Phi \wedge \Phi = 0, \quad \nabla_A \Phi = 0, \quad (7.3)$$

are equivalent to a flatness condition of the modified connection:

$$(d + A + i\Phi)^2 = 0, \quad (7.4)$$

and have a simple solution on $\mathbb{C}^* = S^1 \times \mathbb{R}$:

$$A^c = A + i\Phi = g_c^{-1} dg_c + g_c^{-1} A_D^c g_c, \quad (7.5)$$

where A_D^c is constant one form taking values in the diagonal matrices. It is useful to represent complex matrix g_c as $g_c = bg$ where $b \in GL(N, \mathbb{C})/U(N)$ is a Hermitian matrix and $g \in U(N)$ (Cartan decomposition). Then g can be gauged away and we have:

$$A = A_D^b - (A_D^b)^+, \quad \Phi = -i(A_D^b + (A_D^b)^+), \quad (7.6)$$

where $A_D^b = b^{-1}db + b^{-1}A_D^c b$. The third equation from the first set provides a constraint on b which fixes it up to a holomorphic map $\mathbb{C}^* = S^1 \times \mathbb{R} \rightarrow G$ (harmonicity condition). Variant of Narasimhan-Seshadri and Ramanathan arguments [54], [55] allow to describe, in the holomorphic terms, the moduli space of flat G -bundles for a compact complex curve. Consider the equation $\nabla_A^{(0,1)}\Phi^{1,0} = 0$. It describes a holomorphic section of the holomorphic bundle. The rest of the equations define the unitary structure and have unique solutions on the compact surface. In the non-compact case the same arguments work for appropriate boundary conditions. Thus, we can think of the phase space of Yang-Mills-Higgs theory as a moduli space of Hitchin equations on $\mathbb{C}^* = S^1 \times \mathbb{R}$.

Now we can try to apply Nahm duality to characterize this moduli space in other terms. Indeed, the solutions of Hitchin equations on $S^1 \times \mathbb{R}$ can be considered as solutions of four-dimensional (anti)self-dual YM equations on $\mathbb{R} \times S_{R_1}^1 \times S_{R_2}^1 \times S_{R_2}^1$ when $R_2 \rightarrow 0$. By Nahm duality [56], [57] the moduli space of (anti)self-dual gauge fields on $S_{R_0}^1 \times S_{R_1}^1 \times S_{R_2}^1 \times S_{R_2}^1$ for a gauge group $U(N)$ is equivalent to the moduli space of (anti)self-dual gauge fields on $S_{1/R_0}^1 \times S_{1/R_1}^1 \times S_{1/R_2}^1 \times S_{1/R_2}^1$ for another gauge group $U(M)$ where M is second Chern class of the gauge field (instanton number). Therefore taking $R_0 \rightarrow \infty$, $R_2 \rightarrow 0$ we get an equivalence of the moduli space of solutions of Hitchin equations for $G = U(N)$ with the moduli space of periodic monopoles on $S_{R_1}^1 \times \mathbb{R} \times \mathbb{R}$ with the gauge group $U(M)$ where M is an appropriately defined topological characteristic of the solutions with fixed boundary conditions. Taking into account that the cylinder has two boundaries the simplest nontrivial boundary conditions leads to $M = 2$. Thus, one can expect that the moduli space for $U(N)$ -gauge theory will be equivalent to N -monopole solution in a $U(2)$ gauge theory. These considerations correspond to the case $c = 0$. One can hope that considering S^1 -equivariant version of the Nahm correspondence one obtains the analogous relations for $c \neq 0$. This would provide a hint for the direct connection between Yang-Mills-Higgs theory for $G = U(N)$ and Nonlinear Schrödinger theory associated with $U(2)$.

Let us finally note that under Nahm duality G -bundles on a four dimensional torus $S_{R_0}^1 \times S_{R_1}^1 \times S_{R_2}^1 \times S_{R_3}^1$ map to G^L -bundles on the dual torus $S_{1/R_0}^1 \times S_{1/R_1}^1 \times S_{1/R_2}^1 \times S_{1/R_3}^1$ where G^L is a Langlands dual group (at least for classical groups). Thus one would expect that the proposed relation between Yang-Mills-Higgs theory and Nonlinear Schrödinger theory being generalized to the case of an arbitrary semisimple Lie group should be related to the Langlands duality.

8. Generalization of G/G gauged WZW model

It is natural to expect that the story presented in previous sections extends from the Yang-Mills-Higgs theory to the certain generalization of G/G gauged WZW theory. In this section we describe such construction and show that the partition function can be represented as a sum over solutions of a certain generalization of Bethe Ansatz equation.

We start with the definition of the set of fields and the action of the odd and even symmetries in the spirit of (3.7), (3.8), (3.37). Let us note that the gauged Wess-Zumino-Witten model can be obtained from the topological Yang-Mills theory by using the group-valued field g instead of algebra-valued field φ . Correspondingly, we replace the generators of the Lie algebra actions with the parameter φ (2.2), (3.8) by the generators of the Lie group action (2.45), (2.44) with the parameter g . Thus it is natural to introduce the set of fields $(A, \psi_A, \Phi, \psi_\Phi, \chi_\pm, \varphi_\pm, g)$ and $t \in \mathbb{R}^*$ with the following action of the odd and even symmetries:

$$\mathcal{L}_{(g,t)} A^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)} \quad \mathcal{L}_{(g,t)} A_A^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}, \quad (8.1)$$

$$\mathcal{L}_{(g,t)} \psi_A^{(1,0)} = -g\psi_A^{(1,0)}g^{-1} + \psi_A^{(1,0)}, \quad \mathcal{L}_{(g,t)} \psi_A^{(0,1)} = g^{-1}\psi_A^{(0,1)}g - \psi_A^{(0,1)}, \quad \mathcal{L}_{(g,t)} g = 0,$$

$$\mathcal{L}_{(g,t)} \Phi^{(1,0)} = tg\Phi^{(1,0)}g^{-1} - \Phi^{(1,0)}, \quad \mathcal{L}_{(g,t)} \Phi^{(0,1)} = -t^{-1}g^{-1}\Phi^{(0,1)}g + \Phi^{(0,1)},$$

$$\mathcal{L}_{(g,t)} \psi_\Phi^{(1,0)} = tg\psi_\Phi^{(1,0)}g^{-1} - \psi_\Phi^{(1,0)}, \quad \mathcal{L}_{(g,t)} \psi_\Phi^{(0,1)} = -t^{-1}g^{-1}\psi_\Phi^{(0,1)}g + \psi_\Phi^{(0,1)},$$

$$\mathcal{L}_{(g,t)} \chi_+ = tg\chi_+g^{-1} - \chi_+, \quad \mathcal{L}_{(g,t)} \chi_- = -t^{-1}g^{-1}\chi_-g + \chi_-$$

$$\mathcal{L}_{(g,t)} \varphi_+ = t^{-1}g\varphi_+g^{-1} - \varphi_+, \quad \mathcal{L}_{(g,t)} \varphi_- = -tg^{-1}\varphi_-g + \varphi_-$$

$$Q A = i\psi_A, \quad Q \psi_A^{(1,0)} = i(A^g)^{(1,0)} - iA^{(1,0)}, \quad Q \psi_A^{(0,1)} = -i(A^{g^{-1}})^{(0,1)} + iA^{(0,1)}, \quad (8.2)$$

$$Q g = 0,$$

$$Q \Phi = i\psi_\Phi, \quad Q \psi_\Phi^{(1,0)} = tg\Phi^{(1,0)}g^{-1} - \Phi^{(1,0)}, \quad Q \psi_\Phi^{(0,1)} = -t^{-1}g^{-1}\Phi^{(0,1)}g - \Phi^{(0,1)}, \quad (8.3)$$

$$Q \chi_\pm = i\varphi_\pm, \quad Q \varphi_+ = tg\chi_+g^{-1} - \chi_+, \quad Q \varphi_- = -t^{-1}g^{-1}\chi_-g + \chi_-.$$

We have $Q^2 = \mathcal{L}_{(g,t)}$ and Q can be considered as a BRST operator on the space of $\mathcal{L}_{(g,t)}$ -invariant functionals.

We define the action of the theory in analogy with the construction of the action for Yang-Mills-Higgs theory as follows:

$$S = S_{GWZW} + [Q, \int_{\Sigma_h} d^2z \operatorname{Tr} \left(\frac{1}{2} \Phi \wedge \psi_\Phi + \right. \quad (8.4)$$

$$+\tau_1 (\varphi_+ \nabla_A^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_A^{(0,1)} \Phi^{(1,0)}) + \tau_2 (\chi_+ \varphi_- + \chi_- \varphi_+) \text{vol}_{\Sigma_h} \Big]_+.$$

Taking $\tau_1 = 0$, $\tau_2 = 1$ and applying the standard localization technique to this theory we obtain for the partition function:

$$Z_{GWZWH}(\Sigma_h) = \frac{e^{(1-h)a(t)}}{|W|} \int_H d^N \lambda \mu_q(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \lambda_m n_m (k+c_v)} \times \quad (8.5)$$

$$\times \prod_{j \neq k} (e^{2\pi i(\lambda_j - \lambda_k)} - 1)^{n_j - n_k + 1 - h} \prod_{j, k} (te^{2\pi i(\lambda_j - \lambda_k)} - 1)^{n_j - n_k + 1 - h},$$

where $a(t)$ is a h -independent constant, the integral goes over the Cartan torus $H = (S^1)^N$ and

$$\mu_q(\lambda) = \det \left\| \frac{\partial \beta_j(\lambda)}{\partial \lambda_k} \right\|, \quad (8.6)$$

with:

$$e^{2\pi i \beta_j(\lambda)} = e^{2\pi i \lambda_j (k+c_v)} \prod_{k \neq j} \frac{te^{2\pi i(\lambda_j - \lambda_k)} - 1}{te^{2\pi i(\lambda_k - \lambda_j)} - 1}. \quad (8.7)$$

We can rewrite this formula in the form similar to (3.27):

$$Z_{GWZWH}(\Sigma_h) = \frac{e^{(1-h)a(t)}}{|W|} \int_H d^N \lambda \mu_q(\lambda)^h \sum_{(n_1, \dots, n_N) \in \mathbb{Z}^N} e^{2\pi i \sum_{m=1}^N \beta_m(\lambda) n_m} \times \quad (8.8)$$

$$\times \prod_{j < k} (e^{i\pi(\lambda_j - \lambda_k)} - e^{i\pi(\lambda_k - \lambda_j)})^{2-2h} \prod_{j < k} |te^{i\pi(\lambda_j - \lambda_k)} - e^{i\pi(\lambda_k - \lambda_j)}|^{2-2h}.$$

Summation over integers in (8.5) leads to the following restriction on the integration parameters:

$$e^{2\pi i \lambda_j (k+c_v)} \prod_{k \neq j} \frac{te^{2\pi i(\lambda_j - \lambda_k)} - 1}{te^{2\pi i(\lambda_k - \lambda_j)} - 1} = 1 \quad , \quad i = 1, \dots, N, \quad (8.9)$$

It is useful to rewrite the equations (8.9) in the standard form of the Bethe Ansatz equations:

$$e^{2\pi i \lambda_j (k+c_v)} \prod_{k \neq j} \frac{\sin(i\pi(\lambda_j - \lambda_k + ic))}{\sin(i\pi(\lambda_j - \lambda_k - ic))} = 1 \quad , \quad i = 1, \dots, N, \quad (8.10)$$

This clearly shows that we are dealing with a kind of XXZ quantum integrable chain. The particular form (8.10) can be obtained by the taking the limit $s \rightarrow -i\infty$ in the following Bethe equations:

$$\left(\frac{\sin(i\pi(\lambda_j - isc))}{\sin(i\pi(\lambda_j + isc))} \right)^{(k+c_v)} \prod_{k \neq j} \frac{\sin(i\pi(\lambda_j - \lambda_k + ic))}{\sin(i\pi(\lambda_j - \lambda_k - ic))} = 1 \quad , \quad i = 1, \dots, N, \quad (8.11)$$

corresponding to formal limit of the infinite spin s of XXZ chain.

The partition function is the generalization of Yang-Mills-Higgs theory, discussed above, and can be written in the following form:

$$Z_{GWZWH}(\Sigma_h) = \sum_{\lambda_i \in \mathcal{R}_q} (D_\lambda^q)^{2-2h}, \quad (8.12)$$

where \mathcal{R}_q is a set of the solutions of (8.9) and:

$$D_\lambda^q = \mu_q(\lambda)^{-1/2} \prod_{i < j} (q^{\frac{1}{2}(\lambda_i - \lambda_j)} - q^{\frac{1}{2}(\lambda_j - \lambda_i)}) \prod_{i < j} |t q^{\frac{1}{2}(\lambda_i - \lambda_j)} - q^{\frac{1}{2}(\lambda_j - \lambda_i)}|, \quad (8.13)$$

where we use the standard parametrization $q = \exp(2\pi i/(k + c_v))$. Note that in the limit $t \rightarrow \infty$ equation (8.9) and the expression for the partition function (8.13) up to an overall scaling factor become the corresponding expressions for a gauged Wess-Zumino-Witten model. Finally note that the form of (8.9) and the explicit expressions for the q -Casimir operators, playing the role of the Hamiltonians, strongly imply the description of the wave functions of the theory in terms of the wave functions in a particular XXZ finite spin chain. This proposition will be discussed in details elsewhere.

9. Conclusion

Let us put the results of this paper in a more general perspective. Any two-dimensional topological theory satisfying the appropriate cutting/gluing relations [58] can be described by a commutative Frobenius algebra. Generally this Frobenius algebra comes from the chiral ring \mathcal{R} of some Conformal Field Theory (CFT). For example the gauged WZW model leads to finite-dimensional Frobenius algebra associated with the representation theory of the finite-dimensional quantum groups. The corresponding CFT is a WZW model. Similarly, the two-dimensional topological Yang-Mills theory is related to the infinite-dimensional Frobenius algebra constructed in terms of the representation theory of finite-dimensional Lie groups. The associated CFT is a particular degeneration of the WZW model. Thus, one should expect that the topological Yang-Mills-Higgs theory introduced in [1] and its generalizations defined in this paper correspond to some interesting classes of two-dimensional CFT. One can speculate that such CFT should be constructed by an appropriate deformation of the WZW model for complex gauge groups. Taking into account the relation between WZW models for complex groups and the (generalizations) two-dimensional quantum gravity this might be elaborated in precise form.

In this paper we mostly restrict ourselves to the case $G = U(N)$. This restriction is not essential. One can study these theories for an arbitrary semisimple Lie group (for the construction of finite-particle wave functions for arbitrary G see e. g. [3]).

Let us stress that the simple expression for the partition function as a sum over the solutions of Bethe Ansatz equations can be considered as a kind of nonlinear Fourier transform of the similar, but more conventional, representation as a sum over a set of \mathbb{C}^* -fixed point components of the Higgs bundle moduli spaces (see [1] for details). The equality of the two dual representations of the partition function can be considered as an example of the Arthur-Selberg trace formula if one takes into account the interpretation of \mathbb{C}^* -fixed points as variations of the Hodge structures on the underlying curve. Note also that the existence of the two dual representations leads to sum rules for the solutions of the Bethe Ansatz equations and the corresponding nonlinear Fourier transform looks very similar to the Quantum Inverse Scattering Method for quantum integrable systems [5]. It would be interesting to make this analogy more precise. Non-linear Fourier transforms become standard tools in the study of topological sigma models and mirror symmetry and here we see another important appearance. Let us note that it would be also interesting to establish the relation between results of the present paper and the other type of connection between 2d Yang-Mills theory and many-body systems described in [59].

Given a $U(N)$ gauge theory it is natural to consider its t'Hooft limit $N \rightarrow \infty$. In this limit one expects to find a dual description of the theory in terms of stringy expansion. The known results for the dual string description of two-dimensional Yang-Mills theory [60], [61], [62] implies that the similar description of $N \rightarrow \infty$ limit of Yang-Mills-Higgs theory and its generalizations can be very instructive. In this respect let us note that recently the same program has been completed for q -deformed two-dimensional Yang-Mills theory [63]. The q -deformed theories are closely related to the gauged WZW deformations of two-dimensional Yang-Mills theory. Thus the deformation of the Yang-Mills-Higgs theory proposed in Section 10 can provide a dual description of interesting string backgrounds. In this respect it is interesting to note that the proper quasi-classical expansion of the Bethe Ansatz solutions, given via the double scaling limit $c \rightarrow 0, N \rightarrow \infty$ with Nc -fixed, is known to lead to Riemann surfaces and genus expansion [64].

The gauge theories considered in this paper seem to provide a proper framework for the quantum field theory version of a Kazhdan-Lusztig type construction of the representations of double affine Hecke algebras (DAHA). Indeed the essential role played by $S^1 \times G$ -equivariant cohomology groups and K -groups both in the Kazhdan-Lusztig constructions

and in the quantum gauge theory construction implies a deep relation between these two subjects. In the topological gauge theories considered here the natural objects are gauge invariant and thus reproduce only the properties of the center of DAHA. One can expect that the consideration of the corresponding CFT discussed above should provide a more deeper connection with the representation theory of DAHA. This seems compatible with the ideas regarding the relation between the representation theory of affine Hecke algebra and hyperkähler geometry of Higgs bundles advanced in [65].

Note that the Hecke algebras are important ingredients of the explicit construction of the Langlands correspondence (see [66],[67] for modern introductions into the subject). Thus it is tempting to suggest that the relation between gauge theories and double affine Hecke algebras considered in this paper can lead to a better understanding of this correspondence. The latter might be related to recent studies in [68] where an interpretation of the geometric Langlands correspondence in terms of quantum field theory was described.

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Appendix A. Twistor type description

In this Appendix we give a Chern-Simons type representation of the bosonic part of the Yang-Mills-Higgs action using a twistor approach. Consider the one-dimensional family of flat connections:

$$\mathcal{D} = d + \mathcal{A}(\lambda) = d + \frac{1}{\lambda} \Phi^{(1,0)} + A + \lambda \Phi^{(0,1)}. \quad (\text{A.1})$$

The flatness condition for \mathcal{D} :

$$\mathcal{F}(\mathcal{A}) \equiv \mathcal{D}^2 = 0. \quad (\text{A.2})$$

is equivalent to the Hitchin equations for the pair (A, Φ) . Then we have the following representation for the bosonic part of the action S_0 (3.34):

$$S(A, \Phi) = \text{Res}_{\lambda=0} \int d^2z \text{Tr}(\varphi(z, \bar{z}, \lambda) \mathcal{F}(\mathcal{A}(z, \bar{z}, \lambda)) + c \mathcal{A}(z, \bar{z}, \lambda) \frac{\partial}{\partial \lambda} \mathcal{A}(z, \bar{z}, \lambda)). \quad (\text{A.3})$$

Here $\varphi(z, \bar{z}, \lambda) = \varphi_+(z, \bar{z}) + \lambda^{-1} \varphi_0(z, \bar{z}) + \lambda^{-2} \varphi_-(z, \bar{z})$. Let us note that the elements of the gauge group are considered to be independent of λ . In this representation it is natural to denote $\varphi(z, \bar{z}, \lambda) \equiv \mathcal{A}_\lambda(z, \bar{z}, \lambda)$ to get, formally, a three-dimensional action. Note however that there are severe constraints on the dependence of the fields on λ . Thus, it might be considered as a reduction of three dimensional Chern-Simons theory.

Appendix B. On Lagrangian geometry of the singular manifold

In this Appendix we discuss the classical gauge theory counterpart of the c -dependent boundary conditions (4.10) used for the construction of the eigenfunctions in Nonlinear Schrödinger theory. We will consider only the case of $G = U(2)$ in the gauge theory reduced to one dimension. The phase space in this case is $\mathcal{M} = T^*\mathbb{R}^2/\mathbb{Z}_2$. Choose the coordinates (x_1, x_2, π_1, π_2) on $T^*\mathbb{R}^2$ such that (x_1, x_2) are coordinates on \mathbb{R}^2 and (π_1, π_2) are coordinates on the fibers of the projection $T^*\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Let $X = x_2 - x_1$, $Y = \pi_2 - \pi_1$ and the action of \mathbb{Z}_2 be given by:

$$w_0 : (X, Y) \rightarrow (-X, -Y). \quad (\text{B.1})$$

The space of functions on the phase space \mathcal{M} is generated by the invariant functions $a_1 = \frac{1}{2}X^2$, $a_2 = \frac{1}{2}Y^2$ and $a_3 = XY$ satisfying:

$$F(a) = 4a_1a_2 - a_3^2 = 0. \quad (\text{B.2})$$

One has $a_1 \geq 0$, $a_2 \geq 0$ and the fiber of the projection on the subspace $a_1 > 0$, $a_2 > 0$ consists of two points while the fiber over $(a_1 = 0) \cup (a_2 = 0)$ consists of one point. The point $a_1 = a_2 = 0$ is singular (i.e. $F = 0$, $\partial_i F = 0$ at this point). The quantization of this space in the standard representation $\hat{X} = x$, $\hat{Y} = \partial_x$ is given by the representation of the algebra $\mathfrak{sl}(2)$:

$$[\hat{a}_2, \hat{a}_1] = \hat{a}_3, \quad [\hat{a}_3, \hat{a}_1] = 2\hat{a}_1, \quad [\hat{a}_3, \hat{a}_2] = -2\hat{a}_2. \quad (\text{B.3})$$

where:

$$\hat{a}_1 = \frac{1}{2}x^2, \quad \hat{a}_2 = \frac{1}{2}\partial_x^2, \quad \hat{a}_3 = \frac{1}{2}(x\partial_x + \partial_x x). \quad (\text{B.4})$$

The equation $4a_1a_2 - a_3^2 = 0$ translates into the restriction on the value of the second Casimir operator in this representation. Thus we are dealing here with the representations associated with the nilpotent orbits of $\mathfrak{sl}(2)$. The representations can be described in terms of the space of regular functions on Lagrangian submanifold \mathcal{L}_0 defined by the equation $a_2 = 0$. The functions on \mathcal{L}_0 are the functions of a_1 ($a_3 = 0$ on \mathcal{L}_0) and the action of \hat{a}_i is given by:

$$\hat{a}_1 f(a_1) = a_1 f(a_1), \quad (\text{B.5})$$

$$\hat{a}_2 f(a_1) = \frac{\partial f(a_1)}{\partial a_1} + 4a_1 \frac{\partial^2 f}{\partial a_1^2}, \quad (\text{B.6})$$

$$\hat{a}_3 f(a_1) = 4a_1 \frac{\partial f(a_1)}{\partial a_1} + \frac{1}{2} f(a_1). \quad (\text{B.7})$$

Regular functions depending on x^2 can be characterized by the following conditions:

$$\delta(x) \partial_x (x^{-1} \partial_x)^n f(x) = 0, \quad n \geq 0. \quad (\text{B.8})$$

It is easy to see that these conditions are invariant with respect to the action of \mathbb{Z}_2 and compatible with the action of $\mathfrak{sl}(2)$. The first condition can be written in a more usual form as:

$$\partial_X f(X)|_{X=0} = 0$$

which is the boundary condition discussed above.

Now consider the case $c \neq 0$. The change of the boundary condition can be understood in geometric terms as follows. As it was discussed previously the phase space is defined by the equations:

$$a_1 a_2 - a_3^2 = 0, \quad a_1 \geq 0, \quad a_2 \geq 0. \quad (\text{B.9})$$

In the case of the Yang-Mills theory we quantize the cone (B.9) using the Lagrangian submanifold \mathcal{L}_0 defined by the equation $a_2 = 0$. This Lagrangian submanifold is rather special. Note that the associated representation of $\mathfrak{sl}(2)$ is irreducible. Its naive deformation \mathcal{L}_c defined by the equation $a_2 = c^2$ provides *reducible* representation. To understand it better consider the lift $\tilde{\mathcal{L}}_c$ of \mathcal{L}_c to the (x, y) -plane. The lift of \mathcal{L}_0 is given by the connected submanifold $x = 0$. On the other hand the lift $\tilde{\mathcal{L}}_{c \neq 0}$ of $\mathcal{L}_{c \neq 0}$ consists of two components $y = \pm c$. This means that the representation associated with $\mathcal{L}_{c \neq 0}$ can be reducible. Indeed one can choose as a \mathbb{Z}_2 -invariant Lagrangian submanifold the submanifold $\hat{\mathcal{L}}_{c \neq 0}^+$ defined by the equations:

$$\tilde{\mathcal{L}}_{c \neq 0}^+ = (y = +c, x > 0) \cup (y = -c, x < 0). \quad (\text{B.10})$$

Taking the factor over \mathbb{Z}_2 we obtain:

$$\mathcal{L}_{c \neq 0}^+ = (a_1 > 0, a_3 > 0, a_2 = c^2). \quad (\text{B.11})$$

Let us describe the corresponding spaces of functions and the action of the generators of $\mathfrak{sl}(2)$. We start with the space of functions on $\tilde{\mathcal{L}}_c$ where $c \neq 0$. As functions of x it is given by a pair of functions with the action of \hat{y} by the following differential operator:

$$\hat{y} \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix} = \begin{pmatrix} \partial_x + c & 0 \\ 0 & \partial_x - c \end{pmatrix} \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix}, \quad (\text{B.12})$$

and the action of the generator w_0 of \mathbb{Z}_2 is given by:

$$w_0 \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix} = \begin{pmatrix} f_-(-x) \\ f_+(-x) \end{pmatrix}. \quad (\text{B.13})$$

Equivalently we have:

$$\hat{y} \begin{pmatrix} f_+(x)e^{cx} \\ f_-(x)e^{-cx} \end{pmatrix} = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} f_+(x)e^{cx} \\ f_-(x)e^{-cx} \end{pmatrix}, \quad (\text{B.14})$$

and

$$w_0 \begin{pmatrix} f_+(x)e^{cx} \\ f_-(x)e^{-cx} \end{pmatrix} = \begin{pmatrix} f_-(-x)e^{cx} \\ f_+(-x)e^{-cx} \end{pmatrix}. \quad (\text{B.15})$$

Thus the symmetry condition is reduced to $f_+(x) = f_-(-x)$. Note that we automatically have $f_+(0) = f_-(0)$. Now consider the functions defined on the lift of the Lagrangian submanifold \mathcal{L}_c^+ . They can be described as follows:

$$f(x) = f_+(x)e^{cx}, \quad x > 0, \quad (\text{B.16})$$

$$f(x) = f_+(-x)e^{-cx}, \quad x < 0. \quad (\text{B.17})$$

The symmetry condition implies that $f_+(x)$ is a symmetric function. Note that: $f(x)|_{x \rightarrow +0} = f(x)|_{x \rightarrow -0}$ and $\partial_x f(x)|_{x \rightarrow +0} = cf(0)$, $\partial_x f(x)|_{x \rightarrow -0} = -cf(0)$. Thus we have the *deformed* boundary conditions. As an example consider the function that is a solution of the equation $\partial_x^2 f(x) = \lambda^2 f(x)$ for $x \neq 0$ and satisfy the boundary conditions at $x = 0$:

$$f_\lambda(x) = (\lambda - ic) e^{i\lambda x} + (\lambda + ic) e^{-i\lambda x}, \quad x > 0, \quad (\text{B.18})$$

$$f_\lambda(x) = (\lambda + ic) e^{i\lambda x} + (\lambda - ic) e^{-i\lambda x}, \quad x < 0. \quad (\text{B.19})$$

This function can be easily represented in the form discussed above:

$$f_\lambda(x) = \left((\lambda - ic) e^{i(\lambda+ic)x} + (\lambda + ic) e^{-i(\lambda-ic)x} \right) e^{cx}, \quad x > 0, \quad (\text{B.20})$$

$$f_\lambda(x) = \left((\lambda + ic) e^{i(\lambda-ic)x} + (\lambda - ic) e^{-i(\lambda+ic)x} \right) e^{-cx}, \quad x < 0. \quad (\text{B.21})$$

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