A Appendix Mathematical Results used in the Model

The model uses a 3-good utility function. The mathematics of 2-good utility functions are widely known from undergraduate economics. the 3-good utility function is a simple extension of the 2-good case. However, for the sake of completeness, a few of the elementary results for 3-good utility functions that are used in the paper are first presented here. In addition, results that are specific to this model and have been used in the paper are also presented.

1. *If utility functions are Cobb-Douglas, the nominal demand for any good is a fixed proportion of nominal income (equation 7):*

   In a two-good world, it is not difficult to show that the nominal demand for any good is a fixed proportion of nominal income. This is also true for a three-good world and this well-known result is presented here.

   The three-good utility function is specified as follows:

   $$U = Y_f^{\psi}\, Y_e^{\chi}\, Y_n^{\theta}$$

   with

   $$\psi + \chi + \theta = 1$$

   The Lagrangian is set up to maximize utility subject to the constraint that all income is spent:

   $$L = {Y_f}^{\psi} {Y_e}^{\chi} {Y_n}^{\theta} + \lambda\,(M - p_f Y_f - p_e Y_e - p_n Y_n)$$

   Take the derivative with respect to the quantity demanded in each sector of the economy:

   $$\begin{aligned} \frac{\partial}{\partial Y_f}\Big({Y_f}^{\psi} {Y_e}^{\chi} {Y_n}^{\theta} + \lambda\,(M - p_f Y_f - p_e Y_e - p_n Y_n)\Big) &= \frac{{Y_f}^{\psi}\psi\, {Y_e}^{\chi} {Y_n}^{\theta}}{Y_f} - \lambda\, p_f \\ \Rightarrow \lambda p_f &= \left(\frac{\psi}{Y_f}\right) U \end{aligned}$$

   Similarly,

   $$\begin{aligned} \lambda p_e &= \left(\frac{\chi}{Y_e}\right) U \\ \lambda p_n &= \left(\frac{\theta}{Y_f}\right) U \end{aligned}$$

Using these expressions for prices, take the ratio of prices in the euro sector of the economy to prices in the non-tradable sector of the economy to find an expression for nominal income in the euro sector in terms of nominal income in the non-tradable sector:

$$\begin{aligned} \frac{p_e}{p_n} &= \Big(\frac{\chi}{\theta}\Big)\Big(\frac{Y_n}{Y_e}\Big) \\ \Rightarrow p_e Y_e &= \Big(\frac{\chi}{\theta}\Big) p_n Y_n \end{aligned} \tag{1}$$

Repeat the process to with foreign prices to get an expression of nominal income in the foreign sector in terms of nominal income in the non-tradable sector:

$$\begin{aligned} \frac{p_f}{p_n} &= \Big(\frac{\psi}{\theta}\Big)\Big(\frac{Y_n}{Y_f}\Big) \\ \Rightarrow p_f Y_f &= \Big(\frac{\psi}{\theta}\Big) p_n Y_n \end{aligned} \tag{2}$$

From the lagrangian above, the expression for income is given by:

$$M = p_e Y_e + p_f Y_f + p_n Y_n$$

Substitute the expressions for nominal income in foreign and non-tradable sectors into the expression for income:

$$\begin{aligned} M &= \Big(\frac{\psi}{\theta}\Big) p_n Y_n + \Big(\frac{\chi}{\theta}\Big) p_n Y_n + p_n Y_n \\ &= p_n Y_n \Big(\frac{\psi}{\theta} + \frac{\chi}{\theta} + 1\Big) \\ &= p_n Y_n \Big(\frac{\psi + \chi + \theta}{\theta}\Big) \\ &= p_n Y_n \left(\frac{1}{\theta}\right) \\ \Rightarrow p_n Y_n &= \theta\, M \end{aligned}$$

This final equation is equivalent to equation (7) in the paper. This expression shows that nominal demand for non-tradable goods is a fixed proportion of nominal income. Furthermore, the exponent on the non-tradable variable in the utility function gives that fixed proportion. It is readily apparent that this result is equally applicable to nominal demand in the other sectors of the economy.

2. *Expression for equilibrium condition in non-tradable sector (equation 8):*

   The equilibrium condition for prices in the non-tradable sector of the economy follows very simply from the previous result. Recall equations (13) and (14) from the previous section:

$$
\begin{aligned}
p_e Y_e &= \left(\frac{\chi}{\theta}\right) p_n Y_n \\
p_f Y_f &= \left(\frac{\psi}{\theta}\right) p_n Y_n
\end{aligned}
$$

   Add these two expressions together:

$$
\begin{aligned}
p_e Y_e + p_f Y_f &= \left(\frac{\psi}{\theta}\right) p_n Y_n + \left(\frac{\chi}{\theta}\right) p_n Y_n \\
&= p_n Y_n \left(\frac{\psi + \chi}{\theta}\right) \\
&= p_n Y_n \left(\frac{1-\theta}{\theta}\right) \\
\Rightarrow p_n Y_n &= \left(\frac{\theta}{1-\theta}\right) p_e Y_e + p_f Y_f
\end{aligned}
$$

   This is equation (9) from the body of the paper.

3. *The elasticity of output with respect to the sectoral real wage is equal to one plus the elasticity of labour demand with respect to the sectoral real wage i.e.* $\epsilon(Y_i, w_i) = 1 + \epsilon(L_i, w_i)$

   Production technology is approximated with Cobb-Douglas functions:

$$
\begin{aligned}
Y &= K^{\alpha} L^{1-\alpha} \\
\Rightarrow \hat{Y} &= \alpha \hat{K} + (1-\alpha)\hat{L} \\
\Rightarrow \hat{Y} &= \alpha(\hat{K} - \hat{L}) + \hat{L}
\end{aligned}
$$

   It has been assumed that capital stocks adjust fully so that $\hat{K} \neq 0$. Now, using the first-order condition that the marginal product of labour equals the real wage:

$$
\begin{aligned}
\frac{w}{p} &= (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \\
\Rightarrow \widehat{\left(\frac{w}{p}\right)} &= \alpha\left(\hat{K} - \hat{L}\right)
\end{aligned}
$$

Substitute this into the expression for $\hat{Y}$:

$$\begin{aligned} \Rightarrow \hat{Y} &= \hat{\left(\frac{w}{p}\right)} + \hat{L} \\ \Rightarrow \frac{d\hat{Y}}{d\hat{\left(\frac{w}{p}\right)}} &= 1 + \frac{d\hat{L}}{d\hat{\left(\frac{w}{p}\right)}} \\ \Leftrightarrow \epsilon(Y_i, w_i) &= 1 + \epsilon(L_i, w_i) \end{aligned}$$

4. *Derivation of equation (10)*

Equation (9) from the body of the paper states that:

$$p_n Y_n = \frac{\theta}{1-\theta}\left(p_e Y_e + p_f Y_f\right)$$

where

$$Y_i = f\left(\frac{w}{p_i}\right) = f(w_i)$$

Taking the total derivative of equation (9):

$$p_n dY_n + Y_n dp_n = \frac{\theta}{1-\theta}\Big(p_e dY_e + Y_e dp_e + p_f dY_f + Y_f dp_f\Big)$$

Given that each Y is a function of the sectoral real wage, this is re-written:

$$p_n\Big(\frac{dY_n}{dw_n}\Big)dw_n + Y_n dp_n = \frac{\theta}{1-\theta}\left\{p_e\Big(\frac{dY_e}{dw_e}\Big)dw_e + Y_e dp_e + p_f\Big(\frac{dY_f}{dw_f}\Big)dw_f + Y_f dp_f\right\}$$

Consider the sectoral real wage differentials; $w_i = w/p_i$. Thus,

$$dw_i = \left(\frac{d}{dw}w_i\right)dw + \left(\frac{d}{dp_i}w_i\right)dp_i = \frac{dw}{p_i} - \Big(\frac{w}{p_i{}^2}\Big)dp_i$$

Substituting these results back into the original equation:

$$\begin{aligned} p_n\left[\frac{dY_n}{dw_n}\right]\left(\frac{dw}{p_n} - \left[\frac{w}{p_n{}^2}\right]dp_n\right) + Y_n dp_n = &\frac{\theta}{1-\theta}\left\{p_e\left[\frac{dY_e}{dw_e}\right]\Big(\frac{dw}{p_e} - \left[\frac{w}{p_e{}^2}\right]dp_e\Big) + Y_e dp_e\right\} \\ &+ \frac{\theta}{1-\theta}\left\{p_f\left[\frac{dY_f}{dw_f}\right]\Big(\frac{dw}{p_f} - \left[\frac{w}{p_f{}^2}\right]dp_f\Big) + Y_f dp_f\right\} \end{aligned}$$

Algebraic manipulation yields:

$$
\begin{aligned}
dp_n\left(Y_n-\frac{dy_n}{dw_n}\left[\frac{w}{p_n}\right]\right)&=dw\left(\frac{\theta}{1-\theta}\left\{\frac{dY_f}{dw_f}+\frac{dY_e}{dw_e}\right\}-\frac{dY_n}{dw_n}\right)\\
&+\frac{\theta}{1-\theta}\left\{dp_f\left[Y_f-\frac{dY_f}{dw_f}\left(\frac{w}{p_f}\right)\right]+dp_e\left[Y_e-\frac{dY_e}{dw_e}\left(\frac{w}{p_e}\right)\right]\right\}
\end{aligned}
$$

Barry (1997) proceeds similarly up to this point. However, he uses the assumption that $p_e = p_f = p_n = 1$ to derive a non-tradable price equation in terms of wages and British prices in his model. Consequently, the accuracy of the model depends on a PPP relationship holding between sectoral prices. This is not necessary. The model already has a sufficient number of assumptions to derive a non-tradable equation. Here, a non-tradable price equation is derived without a sectoral PPP assumption, thereby making the model more widely applicable.

In the previous section, it was found that $\epsilon(Y_i, w_i) = 1 + \epsilon(L_i, w_i)$. If sectoral labour demand elasticities equal -1, $\epsilon(Y_i, w_i) = 0$, and thus sectoral real wage changes have no effect on sectoral output. This is an unrealistic scenario and therefore labour demand elasticities are restricted so that they cannot equal -1. In this case, $dY_i/dw_i \neq 0$. Using this result and the fact that $dY_i/dw_i = \epsilon(Y_i, w_i)(Y_i/w_i)$, the equation above can be written:

$$
\begin{aligned}
dp_n\left(Y_n-\epsilon\left(Y_n,w_n\right)\frac{Y_n}{w_n}\left[\frac{w}{p_n}\right]\right)&=dw\left(\frac{\theta}{1-\theta}\left\{\epsilon\left(Y_f,w_f\right)\frac{Y_f}{w_f}+\epsilon\left(Y_e,w_e\right)\frac{Y_e}{w_e}\right\}\right)\\
-\,dw\,\epsilon\left(Y_n,w_n\right)\frac{Y_n}{w_n}&+\frac{\theta}{1-\theta}\left\{dp_f\left[Y_f-\epsilon\left(Y_f,w_f\right)\frac{Y_f}{w_f}\left(\frac{w}{p_f}\right)\right]\right\}\\
&+\frac{\theta}{1-\theta}\left\{dp_e\left[Y_e-\epsilon\left(Y_e,w_e\right)\frac{Y_e}{w_e}\left(\frac{w}{p_e}\right)\right]\right\}
\end{aligned}
$$

This is equivalent to:

$$
\begin{aligned}
\frac{dp_n}{p_n}\left(p_nY_n-\epsilon\left(Y_n,w_n\right)p_nY_n\right)&=\frac{dw}{w}\left(\frac{\theta}{1-\theta}\left\{\epsilon\left(Y_f,w_f\right)p_fY_f+\epsilon\left(Y_e,w_e\right)p_eY_e\right\}\right)\\
-\frac{dw}{w}\left\{\epsilon\left(Y_n,w_n\right)p_nY_n\right\}&+\left(\frac{\theta}{1-\theta}\right)\left[\frac{dp_f}{p_f}\left(p_fY_f-\epsilon\left(Y_f,w_f\right)p_fY_f\right)\right]\\
&+\left(\frac{\theta}{1-\theta}\right)\left[\frac{dp_e}{p_e}\left(p_eY_e-\epsilon\left(Y_e,w_e\right)p_eY_e\right)\right]
\end{aligned}
$$

It was shown in the first proof that the nominal demand in the non-tradable sector is proportional to aggregate income $p_n Y_n = \theta M$. The constant of proportionality is the exponent on non-tradable goods in the utility function. Similarly, for the tradable sectors $p_f Y_f = \psi M$ and $p_e Y_e = \chi M$. Every term in the previous equation contains a nominal demand expression:

$$\begin{aligned}\frac{dp_n}{p_n}\left(\theta M - \epsilon\left(Y_n, w_n\right)\theta M\right) = \frac{dw}{w}\left(\frac{\theta}{1-\theta}\left\{\epsilon\left(Y_f, w_f\right)\psi M + \epsilon\left(Y_e, w_e\right)\chi M\right\} - \epsilon\left(Y_n, w_n\right)\theta M\right) \\ +\left(\frac{\theta}{1-\theta}\right)\left[\frac{dp_f}{p_f}\left(\psi M - \epsilon\left(Y_f, w_f\right)\psi M\right) + \frac{dp_e}{p_e}\left(\chi M - \epsilon\left(Y_e, w_e\right)\chi M\right)\right]\end{aligned}$$

Dividing accross by θM:

$$\begin{aligned}\frac{dp_n}{p_n}\left[1 - \epsilon\left(Y_n, w_n\right)\right] = \frac{dw}{w}\left(\frac{1}{1-\theta}\left\{\epsilon\left(Y_f, w_f\right)\psi + \epsilon\left(Y_e, w_e\right)\chi\right\} - \epsilon\left(Y_n, w_n\right)\right) \\ +\frac{dp_f}{p_f}\left(\frac{\psi}{1-\theta}\right)\left[1 - \epsilon\left(Y_f, w_f\right)\right] + \frac{dp_e}{p_e}\left(\frac{\chi}{1-\theta}\right)\left[1 - \epsilon\left(Y_e, w_e\right)\right]\end{aligned}$$

The tradable sectors of the economy are considered to be broadly similar. Specifically, they have the same labour shares and the same labour demand elasticities. This means that $\epsilon\left(Y_e, w_e\right) = \epsilon\left(Y_f, w_f\right)$. This allows the equation to be simplified even further:

$$\frac{dp_n}{p_n}\left[1 - \epsilon\left(Y_n, w_n\right)\right] = \frac{dw}{w}\left[\epsilon\left(Y_f, w_f\right) - \epsilon\left(Y_n, w_n\right)\right] + \left[\frac{1 - \epsilon\left(Y_f, w_f\right)}{1-\theta}\right]\left\{\chi\,\frac{dp_e}{p_e} + \psi\,\frac{dp_f}{p_f}\right\} \tag{3}$$

In the last derivation, it was seen that the assumption that capital stocks fully adjust means that $\epsilon(Y_i, w_i) = 1 + \epsilon(L_i, w_i)$. Making this substitution into the equation above means that changes in non-tradable prices can be expressed:

$$\frac{dp_n}{p_n} = \left[1 - \frac{\epsilon\left(L_f, w_f\right)}{\epsilon\left(L_n, w_n\right)}\right]\frac{dw}{w} + \left[\frac{\epsilon\left(L_f, w_f\right)}{(1-\theta)\,\epsilon\left(L_n, w_n\right)}\right]\left\{\chi\,\frac{dp_e}{p_e} + \psi\,\frac{dp_f}{p_f}\right\}$$

This is equation (10) from the body of the paper.

5. *Derivation of equation (11)*

Equation (11) again finds the derivative of equation (9) but under the assumption that capital stocks are fixed. This means that $\hat{K} = 0$. Examining the Cobb-Douglas production function with capital fixed:

$$\begin{aligned} Y &= K^{\alpha} L^{1-\alpha} \\ \Rightarrow \hat{Y} &= (1-\alpha)\,\hat{L} \\ \Rightarrow \frac{dY_i}{Y_i} &= s_i \frac{dL_i}{L_i} \end{aligned}$$

where s_i, the sectoral labour share, equals $1-\alpha$. Therefore,

$$\epsilon(Y_i, w_i) = s_i \epsilon(L_i, w_i)$$

Substituting this expression for the output elasticites into equation (15):

$$\frac{dp_n}{p_n}\left[1 - s_n \epsilon\left(L_n, w_n\right)\right] = \frac{dw}{w}\left[s_f \epsilon\left(L_f, w_f\right) - s_n \epsilon\left(L_n, w_n\right)\right] + \left[\frac{1 - s_f \epsilon\left(Y_f, w_f\right)}{1-\theta}\right]\left\{\chi\,\frac{dp_e}{p_e} + \psi\,\frac{dp_f}{p_f}\right\}$$

This means that non-tradable prices in the fixed capital stock case can be expressed:

$$\frac{dp_n}{p_n} = \left[\frac{s_f \epsilon\left(L_f, w_f\right) - s_n \epsilon\left(L_n, w_n\right)}{1 - s_n \epsilon\left(L_n, w_n\right)}\right]\frac{dw}{w} + \left[\frac{1 - s_f \epsilon\left(Y_f, w_f\right)}{(1-\theta)\left[1 - s_n \epsilon\left(L_n, w_n\right)\right]}\right]\left\{\chi\,\frac{dp_e}{p_e} + \psi\,\frac{dp_f}{p_f}\right\}$$

This is equation (11) from the body of the paper.