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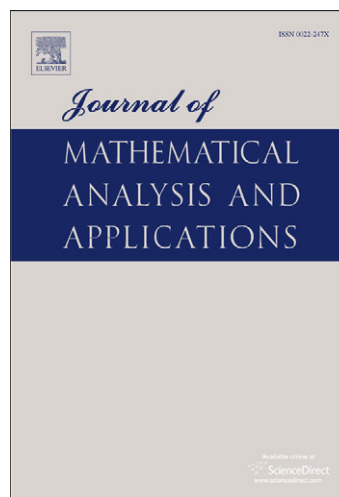
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OPERATOR RANGES AND SPACEABILITY

DEREK KITSON AND RICHARD M. TIMONEY

ABSTRACT. Recent contributions on spaceability have overlooked the applicability of results on operator range subspaces of Banach spaces or Fréchet spaces. Here we consider general results on spaceability of the complement of an operator range, some of which we extend to the complement of a union of countable chains of operator ranges. Applications we give include spaceability of the non-absolutely convergent power series in the disk algebra and of the non absolutely p -summing operators between certain pairs of Banach spaces. Another application is to ascent and descent of countably generated sets of continuous linear operators, where we establish some closed range properties of sets with finite ascent and descent.

Dedicated in memory of Nigel Kalton.

1. INTRODUCTION

The concept of *spaceability* of a subset of a topological vector space was first used in [13]. (If X is a topological vector space and $S \subset X$, then S is called *spaceable* if there is a closed infinite dimensional linear subspace $W \subset X$ with $W \subset S \cup \{0\}$.) It was highlighted further in [1], where it was pointed out that highly non-linear and apparently pathological sets can often have the property. The term ‘spaceable’ was introduced in [2] (see also [15]). There have been several further works on this notion (for example [3]) and on the weaker notion of *lineability* (which omits the closure condition on the subspace).

One of our results will be an improvement (Theorem 4.1) on recent results of Botelho *et. al.* [5, 6], which considered a problem raised in [22] about lineability of the complement $\mathcal{B}(X, Y) \setminus \Pi_p(X, Y)$ of the p -summing operators between Banach spaces X and Y ($1 \leq p < \infty$). We can establish spaceability of the intersection of these complements, under more general conditions on X and Y .

The realisation that the notion of operator range is very useful seems to be due to Dixmier [9, 10] for ranges of operators on Hilbert space. See also Fillmore & Williams [12].

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Definition 1.1. A linear subspace Y of a Banach space X is called a (*Banach*) *operator range* if there is a Banach space Z and a bounded linear operator $T: Z \rightarrow X$ with $T(Z) = Y$.

Operator ranges generalise closed linear subspaces, but yet they still have surprisingly many of the properties of closed subspaces.

One may extend the notion of operator range by allowing X and Z to be Fréchet spaces (completely metrizable locally convex topological vector spaces) or even further to be F -spaces (completely metrizable topological vector spaces). Alternatively, one may restrict the notion by imposing restrictions on Z (or on T). Dixmier considered Hilbert space operators, for example. If we assume the operator T is injective (as we may do by replacing Z by $Z/\ker T$ and T by the induced map on the quotient), then the (complete) space Z is uniquely determined up to isomorphism by Y . (If $T_i: Z_i \rightarrow X$, $i = 1, 2$, are injective continuous linear operators with $T_1(Z_1) = T_2(Z_2)$ then $T_2^{-1}T_1$ has closed graph. Hence $T_2^{-1}T_1: Z_1 \rightarrow Z_2$ is an isomorphism.) We could therefore consider ‘Hilbert operator ranges’, or ‘Fréchet operator ranges’ in addition to Banach operator ranges (by taking Z in the appropriate class and generalising to allow X to be Fréchet).

Our proof of Theorem 4.1 relies on a technique of Davis & Johnson [8] (who showed that $S = \mathcal{K}(X, Y) \setminus \bigcup_{1 \leq p < \infty} \Pi_p(X, Y)$ is nonempty when X is a super-reflexive Banach space and X, Y are infinite dimensional) and on results of Drewnowski [11] (including Proposition 2.4 below). Our notation is that $\mathcal{B}(X, Y)$ denotes the bounded linear operators from X to Y , $\mathcal{K}(X, Y)$ denotes the (closed ideal of) compact operators in $\mathcal{B}(X, Y)$ and $\Pi_p(X, Y)$ the (absolutely) p -summing operators (in $\mathcal{B}(X, Y)$).

Our second main application (Theorem 5.6) concerns sets of operators on a Fréchet space X with finite ascent and descent. These notions were considered in [20] for arbitrary sets of linear operators on X (not necessarily commuting) and [20] established an algebraic direct sum decomposition of X into a generalised kernel and range space (given finiteness of the ascent and descent). It seems much more desirable to have a topological direct sum decomposition of X , which we show in Theorem 5.6 for countably generated sets of operators.

While there are counterexamples showing that many of our results become false if the countability assumption is removed, there are nevertheless positive results in the literature (such as [3, Theorem 3] on everywhere divergent Fourier series) which can be stated as spaceability of the complement of the union (not linear span) of uncountably many operator ranges. We think it would be of interest to be able to capture such results in a general theorem.

2. BACKGROUND

We will consider spaceability of the complement of a (Fréchet) operator range, and later extend some of the results to the complement of a union of operator ranges. Perhaps the most basic case is that of a closed subspace.

Our proof for Theorem 2.2 requires the following and, since we do not have a reference, we include a proof (using a simpler argument than our original, for which we thank N. Kalton).

Lemma 2.1. *If X is an infinite dimensional Fréchet space (over the field \mathbb{K}) such that the weak topology coincides with the Fréchet topology, then X is isomorphic to $\mathbb{K}^{\mathbb{N}}$ (with the product topology).*

Proof. As the topology of X is Fréchet, hence there is a countable basis of zero neighbourhoods in X , it must be that there are countably many linear functionals determining the weak topology. That means that the dual space X' of X must have countable algebraic dimension. Let $\{\phi_1, \phi_2, \dots\}$ be an algebraic basis for X' and consider the map $\pi: X \rightarrow \mathbb{K}^{\mathbb{N}}$ given by $\pi(y) = (\phi_j(y))_{j \in \mathbb{N}}$, which is linear, continuous, injective, a homeomorphism onto its range since X has the weak topology, hence $\pi(X)$ is complete and closed in $\mathbb{K}^{\mathbb{N}}$. By the Hahn-Banach theorem and linear independence of the ϕ_j , $\pi(X) = \mathbb{K}^{\mathbb{N}}$. \square

Theorem 2.2 (Wilansky, Kalton). *If X is a Fréchet space and $Y \subset X$ is a closed linear subspace, then the complement $X \setminus Y$ is spaceable if and only if Y has infinite codimension.*

Proof. It is clear that spaceability of $X \setminus Y$ implies infinite codimension of Y . Also, as finite dimensional $Y \subset X$ are complemented, we need only consider the situation where both Y and the quotient X/Y are infinite dimensional Fréchet spaces.

In the Banach space case Wilansky [25, p. 12] has a short argument with basic sequences to prove the result. There is also a remark on [25, p. 12] (ascribed to Kalton) that the same proof works for Fréchet spaces X , as long as the case where Y is a minimal space is excluded. In the absence of a complete reference, we provide some details.

Bessaga & Pelczynski [4] is one source for the existence of basic sequences in Fréchet spaces, but something more specific is needed for the argument.

We first consider the situation where the (subspace) topology on Y coincides with the weak topology of Y . In this case, by Lemma 2.1, $Y \cong \mathbb{K}^{\mathbb{N}}$, the Hahn-Banach theorem implies that Y is complemented in X , and then the kernel of the projection is a closed subspace contained in $(X \setminus Y) \cup \{0\}$. So we have spaceability of $X \setminus Y$ in this case.

Assume then that the topology of Y is not the same as the weak topology. As the topology of Y has a zero neighbourhood basis of

closed convex sets (the closed unit balls in continuous seminorms), it has a zero neighbourhood basis of weakly closed sets.

There is a weakly null net $(y_\alpha)_\alpha$ in Y which is not strongly null. Hence a strong neighbourhood U of the origin so that $\{\alpha : y_\alpha \notin U\}$ is cofinal, and we may pass to a subnet and assume $y_\alpha \notin U$ for all α . By a result of Kalton [17, Theorem 3.2], there is a basic sequence $(y_n)_{n=1}^\infty$ in Y that is contained in the complement of U . This basic sequence is then bounded away from 0, which means that it is known as a ‘regular’ basic sequence.

Applying the considerations above to the quotient X/Y , we can conclude that X/Y contains a basic sequence $(x_n + Y)_{n=1}^\infty$ (not necessarily a regular one). We can alternatively reach this conclusion via [4].

From Kalton [17, Lemma 4.3], it follows that there are strictly positive scalars $(t_n)_{n=1}^\infty$ such that $(y_n + t_n x_n)_{n=1}^\infty$ is a basic sequence in X . Now the argument of [25] can be used. Let Z denote the closed linear span of $(y_n + t_n x_n)_{n=1}^\infty$. If $x \in Z \cap Y$, then there are scalars λ_n with $x = \sum_{n=1}^\infty \lambda_n (y_n + t_n x_n)$. Considering the quotient, we have $0 = x + Y = \sum_{n=1}^\infty \lambda_n t_n (x_n + Y)$ and so $\lambda_n = 0$ ($\forall n$). Thus $x = 0$ and $Z \cap Y = \{0\}$. We have shown that $X \setminus Y$ is spaceable. \square

Remark 2.3. A counterexample of Kalton [18] shows that there is an (infinite dimensional) F -space X with a one-dimensional subspace Y that is contained in all closed infinite dimensional subspaces of X . Thus $X \setminus Y$ is certainly not spaceable, and we cannot generalise Theorem 2.2 to the case of F -spaces X .

Proposition 2.4. *Let X and Z be Fréchet spaces and $T: Z \rightarrow X$ a continuous linear operator with range $Y = T(Z)$ not closed. Then the complement $X \setminus Y$ is spaceable.*

This is shown by Drewnowski [11] (see Theorem 5.6 (c) and the reformulation of it). We use a variation on the proof from [11] in the proof of Theorem 3.3 below.

As an application of the above proposition, we mention the following answer to a question posed to us informally by R. M. Aron. This question led to the current work and we would like to acknowledge the motivation provided by his question.

Example 2.5. The complement in the disk algebra $A(\mathbb{D})$ of the absolutely convergent power series is spaceable.

Proof. Consider the map $T: \ell_1 \rightarrow A(\mathbb{D})$ given by $T((a_n)_{n=1}^\infty)(z) = \sum_{n=0}^\infty a_{n-1} z^n$ (for $z \in \mathbb{D}$). The range $T(\ell_1)$ is the space of absolutely convergent power series, which is well-known to be a proper subspace (see [16, p. 122] or [14, 26, 21]). As $T(\ell_1)$ is dense in $A(\mathbb{D})$ (since it contains the polynomials), it is not closed. Hence we can apply Proposition 2.4. \square

Remark 2.6. Clearly there are many similar examples of dense subspaces of Banach or Fréchet spaces that are complete in their own stronger metric where we can also invoke Proposition 2.4.

3. COUNTABLE UNIONS

We now pass from considering the range of a single continuous linear operator to the (algebraic) linear span of countably many such operators.

Proposition 3.1. *Let Z_n ($n \in \mathbb{N}$) and X be Fréchet spaces and $T_n: Z_n \rightarrow X$ continuous linear operators. Let Y be the linear span of $\bigcup_{n \in \mathbb{N}} T_n(Z_n)$.*

If Y is closed, then there exists $n \geq 1$ so that $Y = \text{span} \left(\bigcup_{j=1}^n T_j(Z_j) \right)$.

Proof. Let $W_n = \bigoplus_{j=1}^n Z_j$ with the product topology (so that W_n is a Fréchet space) and define $S_n: W_n \rightarrow X$ by $S_n((z_j)_{j=1}^n) = \sum_{j=1}^n T_j(z_j)$. Then $Y = \bigcup_{n=1}^{\infty} S_n(W_n)$. By the Baire category theorem, there is $n \geq 1$ so that $S_n(W_n)$ is of second category in Y . By [19, Theorem 11.4], $S_n(W_n) = Y$. \square

Remark 3.2. Proposition 3.1 cannot be extended to uncountable sets of operators $T_\alpha: Z_\alpha \rightarrow X$.

For example, consider an arbitrary linear subspace Y of a Banach space X such that there exists a closed subspace $N \subset X$ with $Y \cap N = \{0\}$, $Y + N = X$ (for instance Y could be a non-closed hyperplane in X , and N one dimensional). Then Y is the union of its 1-dimensional subspaces. For each $y \in Y$ we could take T_y to be the linear map from the scalars to X given by $T_y(\lambda) = \lambda y$. If we add the inclusion map $i_N: N \hookrightarrow X$, we get an uncountable set $\{T_y : y \in Y\} \cup \{i_N\}$ such that the linear span of their ranges is X but no finite subset of the ranges spans X .

Theorem 3.3. *Let Z_n ($n \in \mathbb{N}$) be Banach spaces and X a Fréchet space. Let $T_n: Z_n \rightarrow X$ be continuous linear operators and Y the linear span of $\bigcup_{n \in \mathbb{N}} T_n(Z_n)$.*

If Y is not closed in X , then the complement $X \setminus Y$ is spaceable.

Proof. We retain the notation $S_n: W_n \rightarrow X$ from the proof of Proposition 3.1. We take the ℓ_1 norm on $W_n = \bigoplus_{j=1}^n Z_j$.

First Y must have infinite codimension. Otherwise Y has a finite dimensional complement F in X . Let $q: X \rightarrow X/F$ be the quotient map. Then the union of the ranges of $q \circ S_n$ (for $n \in \mathbb{N}$) is all of X/F (hence closed). Using Proposition 3.1, there is n with $Y = S_n(W_n)$. As a (Fréchet) operator range with closed complementary subspace F , Y must be closed.

Next, it is sufficient to deal with the case where X is separable. Since Y is not closed there is a sequence $(y_k)_{k=1}^{\infty}$ with $\lim_{k \rightarrow \infty} y_k = x \notin Y$.

Let \tilde{X} be the closed linear span in X of $\{x\} \cup \{y_k : k \in \mathbb{N}\}$, $\tilde{W}_n = S_n^{-1}(\tilde{X})$, \tilde{S}_n the restriction of S_n to \tilde{W}_n , and $\tilde{Y} = Y \cap \tilde{X}$. We work now with the Banach spaces \tilde{W}_n and the sequence $\tilde{S}_n : \tilde{W}_n \rightarrow \tilde{X}$ of continuous operators with $\tilde{Y} = \bigcup_{n=1}^{\infty} \tilde{S}_n(\tilde{W}_n)$ an increasing union of operator ranges. Of course \tilde{X} is separable and Fréchet.

We know that \tilde{Y} is not closed because of the sequence $(y_k)_{k=1}^{\infty}$ and so \tilde{Y} must have infinite codimension in \tilde{X} (by the argument given above for Y).

If there are infinitely many n such that $\tilde{S}_n(\tilde{W}_n)$ is closed in \tilde{X} , say $n_1 < n_2 < \dots$, then we can write each $\tilde{S}_{n_j}(\tilde{W}_{n_j}) \setminus \{0\}$ as a countable union $\bigcup_{k=1}^{\infty} C_{j,k}$ of closed convex subsets of \tilde{X} . (To see this use translates of open convex neighbourhoods of the origin in the subspace to cover the set, taking care that 0 is not in the closures of any of these translates. Then invoke the Lindelof property to get a countable subcover.) Thus

$$\tilde{Y} \setminus \{0\} = \bigcup_{k=1}^{\infty} \tilde{S}_{n_j}(\tilde{W}_{n_j}) \setminus \{0\} = \bigcup_{j,k} C_{j,k}$$

is a countable union of closed convex subsets of \tilde{X} . Since \tilde{Y} is also of infinite codimension in \tilde{X} , [11, Corollary 5.5] implies that $\tilde{X} \setminus \tilde{Y}$ is spaceable. Hence, using $\tilde{X} \setminus \tilde{Y} = \tilde{X} \cap (X \setminus Y)$, in this case $X \setminus Y$ is spaceable.

Thus we are left with the case where there are at most finitely many n such that $\tilde{S}_n(\tilde{W}_n)$ is closed. By discarding that finite number of n and renumbering, we may assume that all fail to be closed.

Consider the unit ball U_n in \tilde{W}_n . Then

$$Y_n = \text{span} \left(\overline{\tilde{S}_n(U_n)} \right) = \bigcup_{r>0} r \overline{\tilde{S}_n(U_n)}$$

is not barrelled (in the induced topology from \tilde{X}). This follows by an argument of Drewnowski [11, p. 388]. Since $\overline{\tilde{S}_n(U_n)}$ is a closed absorbing balanced subset of Y_n , if Y_n were barrelled, $\overline{\tilde{S}_n(U_n)}$ would contain a zero neighbourhood in Y_n . By completeness of \tilde{W}_n , then $\tilde{S}_n(\tilde{W}_n) = Y_n$ and \tilde{S}_n is open (see [24, Lemma III.2.1]). So \tilde{S}_n induces a linear isomorphism $\tilde{W}_n / \ker \tilde{S}_n$ onto Y_n . So $Y_n = \tilde{S}_n(\tilde{W}_n)$ is completely metrizable, hence closed in \tilde{X} — a contradiction.

We observe next that $Y_n \setminus \{0\}$ has a countable cover by closed convex subsets of \tilde{X} . Via the Lindelof argument mentioned above (and given in [11, p. 398]), we can write $\tilde{X} \setminus \{0\} = \bigcup_{k=1}^{\infty} A_k$ where A_k are convex open sets in \tilde{X} with $0 \notin \bar{A}_k$ ($\forall k$). Then $Y_n \setminus \{0\} = \bigcup_{j,k \in \mathbb{N}} \left(\bar{A}_k \cap \left(j \overline{\tilde{S}_n(U_n)} \right) \right)$.

By the constructions involved, the reader may verify that we have $\tilde{S}_n(U_n) \subset \tilde{S}_{n+1}(U_{n+1})$ and so $Y_n \subset Y_{n+1}$.

Notice that the union $Y_\infty = \bigcup_n Y_n$ is a linear subspace of \tilde{X} . As each Y_n is not barrelled, each one has infinite codimension in \tilde{X} ([23]). We claim that Y_∞ has infinite codimension also. If not consider a finite dimensional subspace $F \subset \tilde{X}$ with $F \cap Y_\infty = \{0\}$, $F + Y_\infty = \tilde{X}$, and take the quotient map $q: \tilde{X} \rightarrow \tilde{X}/F$. Then, by the Baire category theorem, there must be n so that $q(Y_n)$ is a subspace of second category in \tilde{X}/F . As

$$q(Y_n) = \bigcup_{k \in \mathbb{N}} k q(\tilde{S}_n(U_n)) \subset \bigcup_{k \in \mathbb{N}} k \overline{q(\tilde{S}_n(U_n))},$$

$\overline{q(\tilde{S}_n(U_n))}$ must have nonempty interior in \tilde{X}/F and so $q \circ \tilde{S}_n$ is surjective by [24, Lemma III.2.1]. Thus $Y_n = Y_\infty$, but this contradicts infinite codimension of Y_n .

As Y_∞ has infinite codimension, there is an infinite dimensional subspace of \tilde{X} intersecting Y_∞ only in 0. As $Y_\infty \setminus \{0\} = \bigcup_n (Y_n \setminus \{0\})$ can be expressed as a countable union of closed convex subsets of \tilde{X} , we can invoke [11, Corollary 5.5] to conclude that $\tilde{X} \setminus Y_\infty$ is spaceable. Since

$$\tilde{X} \setminus Y_\infty \subset \tilde{X} \setminus \tilde{Y} = \tilde{X} \cap (X \setminus Y),$$

we can see that $X \setminus Y$ must be spaceable. \square

Remark 3.4. It would be interesting to know if the above result can be extended to allow Z_n to be Fréchet spaces.

4. APPLICATION TO NON p -SUMMING OPERATORS

Motivated by the study of lineability, conditions were given in [22] on X and Y sufficient to ensure lineability of $\mathcal{B}(X, Y) \setminus \Pi_1(X, Y)$. The question posed in [22, Problem 2.4] was whether super-reflexivity of X (and infinite dimension of Y) is sufficient to ensure lineability of $\mathcal{B}(X, Y) \setminus \Pi_p(X, Y)$ for each p . Botelho *et. al.* [5, 6] obtained positive answers using conditions relating to existence of subspaces of X or of Y with unconditional basis (and required that the subspace be complemented in the case of X). In fact they obtained a subspace in $(\mathcal{K}(X, Y) \setminus \Pi_p(X, Y)) \cup \{0\}$ vector space isomorphic to ℓ_1 (hence of uncountable dimension) in [6].

We improve on these results (and answer [22, Problem 2.4]) by establishing spaceability, and indeed a single infinite dimensional closed subspace valid for all p .

Theorem 4.1. *Let X and Y be infinite dimensional Banach spaces and assume that X is super-reflexive. Then*

$$S = \mathcal{K}(X, Y) \setminus \bigcup_{1 \leq p < \infty} \Pi_p(X, Y)$$

is spaceable (as a subset of $\mathcal{K}(X, Y)$).

Proof. Let $\pi_p(\cdot)$ denote the p -summing norm (in which $\Pi_p(X, Y)$ is known to be a Banach space). It is also known that $\|s\| \leq \pi_q(s) \leq \pi_p(s)$ and $\Pi_p(X, Y) \subseteq \Pi_q(X, Y)$ for $s \in \Pi_p(X, Y)$, $p < q < \infty$. Since $\mathcal{K}(X, Y)$ is closed in the operator norm, it follows that $\Pi_p(X, Y) \cap \mathcal{K}(X, Y)$ is closed in $\Pi_p(X, Y)$, hence a Banach space in the norm $\pi_p(\cdot)$. Note that $\Pi_p(X, Y) \cap \mathcal{K}(X, Y)$ is an operator range (via the inclusion map) in $\mathcal{K}(X, Y)$.

In the proof of [8, Theorem], it is shown that the norm induced by $\Pi_p(X, Y)$ on the finite rank operators is not equivalent to the operator norm (under the given hypothesis on X) for $1 \leq p < \infty$. This implies that $\Pi_p(X, Y) \cap \mathcal{K}(X, Y)$ is not closed in $\mathcal{K}(X, Y)$.

As

$$\bigcup_{1 \leq p < \infty} \Pi_p(X, Y) \cap \mathcal{K}(X, Y) = \bigcup_{p \in \mathbb{N}} \Pi_p(X, Y) \cap \mathcal{K}(X, Y)$$

is a countable union of increasing operator ranges, Proposition 3.1 implies that the union is not closed and the result then follows from Theorem 3.3. \square

Note that the same proof applies if we assume instead that X does not contain ℓ_1^n uniformly for large n (in view of the proof of [8, Theorem B]).

One might ask for conditions on X and Y so that

$$\Pi_q(X, Y) \setminus \Pi_p(X, Y) \subset \Pi_q(X, Y)$$

is spaceable (for given $p < q$), but observe that, by Proposition 2.4, this is always true if $\Pi_p(X, Y)$ fails to be closed in $\Pi_q(X, Y)$.

The referee has kindly pointed out that recent considerations of Botelho, Pellegrino and Rueda [7] are relevant to seeking further results. We note that [5, §1] provides an example of a proper operator ideal in $\mathcal{B}(X)$ which has finite codimension in $\mathcal{B}(X)$ (with X hereditarily indecomposable).

The key idea we need from [7] is that the techniques in [8] can be rephrased in terms of the ideal $\mathcal{A}(X, Y)$ in $\mathcal{B}(X, Y)$ of *approximable operators*, defined as the closure of the finite ranks in the operator norm. Assume $\mathcal{I}(X, Y)$ is a nonzero operator ideal in $\mathcal{B}(X, Y)$ complete in its own intrinsic norm $\|\cdot\|_{\mathcal{I}}$ and satisfying $\|x\| \leq \|x\|_{\mathcal{I}}$ for $x \in \mathcal{I}(X, Y)$. Then $\mathcal{I}(X, Y)$ must contain the finite ranks. In [7] it is observed that the norm $\|\cdot\|_{\mathcal{I}}$ is equivalent to $\|\cdot\|$ on finite ranks if and only if $\mathcal{A}(X, Y) \subseteq \mathcal{I}(X, Y)$. It follows from Proposition 2.4 then that if $\mathcal{A}(X, Y) \not\subseteq \mathcal{I}(X, Y)$, then $\mathcal{A}(X, Y) \setminus \mathcal{I}(X, Y)$ is spaceable in $\mathcal{A}(X, Y)$ (or in $\mathcal{K}(X, Y)$).

In [7, Corollary 2.6] it is shown that if X is infinite dimensional and Y has cotype $\text{cot}(Y) > 2$, then $\mathcal{A}(X, Y) \not\subseteq \Pi_r(X, Y)$ for $1 \leq r < \text{cot}(Y)$. As in the proof of Theorem 4.1, it follows that for X and Y satisfying

these conditions

$$\mathcal{A}(X, Y) \setminus \bigcup_{1 \leq p < \text{cot}(Y)} \Pi_p(X, Y)$$

is spaceable in $\mathcal{A}(X, Y)$.

5. ASCENT AND DESCENT OF COUNTABLE SETS

We now recall some notation and terminology from [20]. If A is a nonempty subset of the (continuous) linear operators on a (Fréchet) space X , $N(A) = \bigcap_{T \in A} \ker T$ and $R(A) = \text{span} \bigcup_{T \in A} T(X)$. For $r \in \mathbb{N}$, A^r denotes the set of all products $a_1 a_2 \cdots a_r$ of elements $a_1, a_2, \dots, a_r \in A$, while A^0 means the singleton set containing the identity operator. The *ascent* $\alpha(A)$ of A is the smallest $r \geq 0$ such that $N(A) \cap R(A^r) = \{0\}$, taken to be ∞ if no such r exists. The *descent* $\delta(A)$ is the smallest $r \geq 0$ with $N(A^r) + R(A) = X$. It is shown in [20] that finiteness of both $\alpha(A)$ and $\delta(A)$ implies equality $\alpha(A) = \delta(A)$ and also that there is an algebraic direct sum decomposition $X = N(A^r) \oplus R(A^r)$ (where $R(A^r)$ need not be closed, as the following example shows).

Example 5.1. We can modify the construction in Remark 3.2 to exhibit uncountable sets A of operators so that $\alpha(A) = \delta(A) = 1$ but $R(A)$ is not closed.

As in Remark 3.2, we take an arbitrary linear subspace Y of a Banach space X such that there exists a closed $N \subset X$ with $Y \cap N = \{0\}$, $Y + N = X$. Consider bounded linear operators $T_{\phi, y}: X \rightarrow X$ with $\phi \in X^*$, $y \in Y$ given by $T_{\phi, y}(x) = \phi(x)y$.

Now $A = \{T_{\phi, y} : y \in Y, \phi \in N^\perp\}$ is a collection of bounded operators on X which has $N(A) = N$ and $R(A) = Y$.

In particular, if we take $Y \subset X$ to be a non-closed hyperplane, and N the span of a single nonzero $x \in X \setminus Y$, then A has $\alpha(A) = \delta(A) = 1$ (which means $N(A) \cap R(A) = \{0\}$, $N(A) + R(A) = X$, $\{0\} \neq N(A)$ and $R(A) \neq X$). However $R(A)$ is not closed.

Proposition 5.2. *Let W_n, Z_n ($n \in \mathbb{N}$) and X be Fréchet spaces. Let $S_n : W_n \rightarrow X$ and $T_n : Z_n \rightarrow X$ be continuous linear operators. Let Y_1 be the linear span of $\bigcup_{n \in \mathbb{N}} S_n(W_n)$ and let Y_2 be the linear span of $\bigcup_{n \in \mathbb{N}} T_n(Z_n)$. If $Y_1 \cap Y_2 = \{0\}$ and $Y_1 + Y_2 = X$ then Y_1 and Y_2 are both closed.*

Proof. By Proposition 3.1 (applied to operators S_j and T_j) there must be finite sets $\{S_1, \dots, S_n\}$ and $\{T_1, \dots, T_m\}$ so that $X = \tilde{Y}_1 + \tilde{Y}_2$ where \tilde{Y}_1 is the linear span of $\bigcup_{j=1}^n S_j(W_j)$ and \tilde{Y}_2 is the linear span of $\bigcup_{j=1}^m T_j(Z_j)$. Let $W = \bigoplus_{j=1}^n W_j$ and $Z = \bigoplus_{j=1}^m Z_j$. We now consider $S : W \rightarrow X$ given by $S(x_1, x_2, \dots, x_n) = \sum_{j=1}^n S_j(x_j)$ and $T : Z \rightarrow X$ given by $T(x_1, x_2, \dots, x_m) = \sum_{j=1}^m T_j(x_j)$. Note that the range of S is \tilde{Y}_1 and the range of T is \tilde{Y}_2 . We may assume S and T are injective (by

passing to quotient spaces). Now the map $(w, z) \mapsto S(w) + T(z)$ is a bijective linear operator from $W \oplus Z$ onto X and hence has a continuous inverse. It follows that $S(W) = Y_1$ is closed and $T(Z) = Y_2$ is closed. \square

Proposition 5.3. *Consider a (nonempty) set A of continuous linear operators on a Fréchet space X . If A is countable with $\alpha(A) < \infty$ and $\delta(A) < \infty$ then $R(A^k)$ is closed for $k \geq \min(\alpha(A), \delta(A))$.*

Proof. From [20], $\alpha(A) = \delta(A)$ and we let $r = \alpha(A)$. Moreover, $R(A^k) = R(A^r)$ for all $k > r$. We have an algebraic direct sum $N(A^r) \oplus R(A^r) = X$ which satisfies the hypothesis of Proposition 5.2. Hence $R(A^r)$ must be closed. \square

Lemma 5.4. *Let X be a (Hausdorff) locally convex topological vector space, A a nonempty set of continuous linear operators on X and \bar{A}^{wot} the closure of A in the weak operator topology (that is, the topology generated by the seminorms $T \mapsto \phi(Tx)$ with $x \in X$ and $\phi \in X^*$). If $A \subseteq B \subseteq \bar{A}^{\text{wot}}$, then $\alpha(A) \leq \alpha(B)$ and $\delta(A) \geq \delta(B)$.*

Proof. First, for all $k \in \mathbb{N}$, $N(A^k) \supseteq N(B^k)$ and $R(A^k) \subseteq R(B^k)$. In fact $N(A^k) = N(B^k)$, since if there is $x \in N(A^k) \setminus N(B^k)$, then there are $b_1, b_2, \dots, b_k \in B$ with $b_1 b_2 \cdots b_k x \neq 0$. Choosing $\phi \in X^*$ with $\phi(b_1 b_2 \cdots b_k x) \neq 0$, we can show by induction that

$$\phi(b_1 \cdots b_j a_{j+1} \cdots a_k x) = 0 \quad (0 \leq j \leq k),$$

a contradiction.

If $r = \alpha(B) < \infty$ then $N(B) \cap R(B^r) = \{0\}$. Hence $N(A) \cap R(A^r) = \{0\}$ and so $\alpha(A) \leq r$. If $s = \delta(A) < \infty$ then $N(A^s) + R(A) = X$. Hence $N(B^s) + R(B) = X$ and so $\delta(B) \leq s$. \square

The following example shows that we cannot always expect equality in Lemma 5.4.

Example 5.5. Let $X = \ell_2$ and denote by $(e_j)_{j \in \mathbb{N}}$ the usual basis in ℓ_2 . Let v be the bounded operator given by $v(e_1) = 0$ and $v(e_n) = e_{2n-3}$ for $n \geq 2$. Define bounded operators b_k by $b_k(e_n) = 0$ if $n \neq 2$ and $b_k(e_2) = (1/2^k) \sum_{j=1}^k e_{2j}$. Then let $a_k = b_k + v$ and $A = \{a_k : k = 1, 2, \dots\}$. We claim that A has finite ascent but the norm closure \bar{A} of A has infinite ascent. Note that $\|b_k\| = \sqrt{k}/2^k$ and so $b_k \rightarrow 0$. Hence $a_k \rightarrow v$ and $v \in \bar{A}$. Now if $x = \sum_{j=1}^{\infty} \lambda_j e_j \in \ell_2$ then

$$a_k x = b_k(x) + v(x) = \lambda_2 \frac{1}{2^k} \sum_{j=1}^k e_{2j} + \sum_{j=2}^{\infty} \lambda_j e_{2j-3}.$$

We have $\ker a_k = \mathbb{C}e_1$ for all k and $N(A) = \mathbb{C}e_1$. Suppose $e_1 \in R(A)$. Then $e_1 = a_{k_1} x_1 + \cdots + a_{k_n} x_n$ for some $k_1, \dots, k_n \in \mathbb{N}$ and some $x_1, \dots, x_n \in X$. We assume $k_1 < k_2 < \dots < k_n$. Writing $x_i =$

$\sum_{j=1}^{\infty} \lambda_{i,j} e_j$ and considering coefficients of e_1 we obtain $\lambda_{1,2} + \dots + \lambda_{n,2} = 1$. However, looking at the even terms we obtain the equation

$$\lambda_{1,2} \frac{1}{2^{k_1}} \sum_{j=1}^{k_1} e_{2j} + \dots + \lambda_{n,2} \frac{1}{2^{k_n}} \sum_{j=1}^{k_n} e_{2j} = 0.$$

The resulting system of equations can be expressed in the following form,

$$\begin{pmatrix} 1/2^{k_1} & 1/2^{k_2} & \dots & 1/2^{k_n} \\ 0 & 1/2^{k_2} & \dots & 1/2^{k_n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1/2^{k_n} \end{pmatrix} \begin{pmatrix} \lambda_{1,2} \\ \lambda_{2,2} \\ \vdots \\ \lambda_{n,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Solving this system of equations we find that $\lambda_{1,2} = \dots = \lambda_{n,2} = 0$ which is a contradiction. Hence $e_1 \notin R(A)$ and so $N(A) \cap R(A) = \{0\}$. As $N(A) \neq \{0\}$, this implies that $\alpha(A) = 1$. The norm closure \bar{A} of A includes v and $v(e_2) = e_1$. Thus $N(\bar{A}) = N(A)$ has nonzero intersection with $R(\bar{A})$ and so $\alpha(\bar{A}) > 1$. In fact, $e_1 \in v^k(\ell_2) \subseteq R((\bar{A})^k)$ for all $k \in \mathbb{N}$ and so $N(\bar{A}) \cap R((\bar{A})^k) \neq \{0\}$. Hence $\alpha(\bar{A}) = \infty$.

In our discussion below, an ‘algebra’ of operators is not required to have an identity.

Theorem 5.6. *Let A be a countable collection of continuous linear operators on a Fréchet space X .*

Let \mathcal{A} denote the weak operator topology closure of the algebra generated by A and suppose $A \subseteq B \subseteq \mathcal{A}$.

If A has finite ascent and finite descent and r denotes their common value then

- i) B has finite ascent and finite descent with $r = \alpha(B) = \delta(B)$;*
- ii) $N(A^r) = N(B^r)$ and $R(A^r) = R(B^r)$;*
- iii) $X = N(B^r) \oplus R(B^r)$ is a topological direct sum decomposition of B -invariant subspaces.*

Proof. We know $X = N(A^r) \oplus R(A^r)$ (algebraically). Since A is countable, $R(A^r)$ is closed by Proposition 5.3 and hence we have a topological direct sum (as $N(A^r)$ is also closed).

Let $\langle A \rangle$ denote the algebra generated by A (which is the linear span of $\bigcup_{j=1}^{\infty} A^j$). Note that $R(A^k) = R(\langle A \rangle^k)$ and $N(A^k) = N(\langle A \rangle^k)$ for all k . As in the proof of Lemma 5.4, $N(\langle A \rangle^k) = N(B^k)$ (all k).

Clearly $R(A^r) \subseteq R(B^r)$. By closure of $R(A^r)$, if there is $x \in R(B^r) \setminus R(A^r)$, then there is $\phi \in X^*$ with $\phi(x) = 1$ but $\phi(y) = 0$ for all $y \in R(A^r) = R(\langle A \rangle^k)$. Thus for $c_1, c_2, \dots, c_r \in \langle A \rangle$ and $z \in X$, $\phi(c_1 c_2 \dots c_r z) = 0$. By induction we conclude $\phi(b_1 \dots b_j c_{j+1} \dots c_r z) = 0$ for $b_1, \dots, b_j \in B$, $c_{j+1}, \dots, c_r \in \langle A \rangle$, $0 \leq j \leq k$. As x is a finite linear combination of terms $b_1 b_2 \dots b_r z$, this gives a contradiction. Hence $R(B^r) = R(A^r)$ is closed.

Thus $X = N(B^r) \oplus R(B^r)$ and so B has finite ascent and finite descent (at most r). From Lemma 5.4 $\alpha(B) = \delta(B) = \alpha(A) = r$. Finally, it is evident from their definitions that $N(B^r)$ and $R(B^r)$ are B -invariant. \square

Note that the theorem applies in particular to the norm closure B of a countable set A of bounded operators on a Banach space X , if A has finite ascent and descent. Closures other than the norm closure may also be used. From Example 5.1, starting with B satisfying $\alpha(B) = \delta(B) < \infty$, we cannot always find a countable dense subset A of finite ascent and descent.

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