

On $\text{AdS}_5 \times S^5$ String S-matrix

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ABSTRACT: Recently two interesting conjectures about the string S-matrix on $\text{AdS}_5 \times S^5$ have been made. First, assuming the existence of a Hopf algebra symmetry Janik derived a functional equation for the dressing factor of the quantum string Bethe ansatz. Second, Hernández and López proposed an explicit form of $1/\sqrt{\lambda}$ correction to the dressing factor. In this note we show that in the strong coupling expansion Janik's equation is solved by the dressing factor up to the order of its validity. This observation provides a strong evidence in favor of a conjectured Hopf algebra symmetry for strings in $\text{AdS}_5 \times S^5$ as well as the perturbative string S-matrix.

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The S-matrix of the quantum string Bethe ansatz [1] coincides with the S-matrix of the asymptotic $\mathcal{N} = 4$ SYM Bethe ansatz [2, 3] up to a scalar function called the dressing factor. It appears to be universal for all sectors [4, 5]. The leading form of the dressing factor at large λ was determined by discretizing the integral equations [6] which describe the spectrum of spinning strings in the scaling limit of [7]. The analysis of one-loop corrections to energies of spinning strings [8] revealed that the dressing factor acquires $1/\sqrt{\lambda}$ corrections [9]-[11]. The explicit form of these corrections was then conjectured in [12].

Recently Janik put forward a proposal [13] that a gauged-fixed string sigma model on a plane exhibits a Hopf algebra symmetry which allows one to derive a set of functional equations on the dressing factor. The action of the Hopf algebra antipode is an analog of the particle-to-antiparticle transformation in relativistic field theory [14]. Implementing the antipode action in a given representation of the Lie algebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ leads to nontrivial relations for the corresponding S-matrix. These relations are analogous to the crossing symmetry relations which arise in relativistic integrable models [15].

The construction of [13] implicitly assumes that the string sigma model is quantized in a gauge preserving the invariance under the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ subalgebra of $\mathfrak{psu}(2, 2|4)$, the latter being the symmetry algebra of the string sigma model on $\text{AdS}_5 \times S^5$. The two copies of $\mathfrak{su}(2|2)$ subalgebra share the same central element which is the string Hamiltonian in the gauge chosen. For instance, one can consider the string sigma model in the temporal gauge $t = \tau$, $p_\phi = J$, where p_ϕ is the canonical momentum conjugate to an angle variable ϕ of S^5 [16]. Another example is given by the uniform light-cone gauge $x^+ = \tau$ and $p^+ = \text{const}$ [17]. In these type of gauges the string Lagrangian depends on two parameters, e.g. in the temporal gauge, it depends on the string tension $\sqrt{\lambda}$ and J . For finite λ and J the gauged-fixed theory is a two-dimensional model on a cylinder and by this reason the notion of the S-matrix is not defined. On the other hand, at infinite J with λ finite the gauge-fixed string sigma model is described by a two-dimensional field theory on a plane because the J -dependence of the string Lagrangian can be absorbed into rescaling of the world-sheet σ -coordinate [18]. The rescaled range of σ is $-\pi J \leq \sigma \leq \pi J$, and in the limit $J \rightarrow \infty$ one gets a model on a plane. The S-matrix for the model can be determined by using the symmetry algebra (and choosing properly its representation) up to a scalar factor [5]. The functional equations of [13] might be further used to fix the scalar factor. Different solutions of the functional equations would correspond to different gauge choices respecting the residual $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ symmetry.

The resulting S-matrix is the main building block to derive a set of Bethe equations along the lines of Ref.[5]. However, the following is to be mentioned. First, the Bethe equations arising in this way are asymptotic and they hardly capture the exact spectrum of strings at finite J . In fact, additional exponential corrections of the form $e^{-J/\sqrt{\lambda}}$, which are due to finite-size effects, are expected [19]. Second, it

is possible that at finite J and λ the Bethe equations should be abandoned for direct diagonalization of a short-range Hubbard type Hamiltonian [20]. Third, it is presently unclear if and how the string Bethe equations turn into the *gauge theory* asymptotic Bethe equations [3] in the weak coupling limit $\lambda \rightarrow 0$, J fixed. One of the possibilities here is that starting at 4-loop order of weak coupling perturbation theory the dressing factor could lead to violation of the BMN [21] scaling. In this respect we note that the breakdown of the BMN scaling was indeed observed in the plane-wave matrix models [22].

In this letter we analyze the dressing factor taking into account the $1/\sqrt{\lambda}$ correction suggested in [12], and show that it satisfies the functional equation in the large λ limit up to the second order of perturbation theory. This result can be considered as a nontrivial test of the both proposals of [12] and [13].

To formulate the string and gauge theory Bethe equations it is convenient to use the variables x^\pm introduced in [23], which satisfy the following equation

$$x^+ + \frac{\lambda}{16\pi^2 x^+} - x^- - \frac{\lambda}{16\pi^2 x^-} = i.$$

The momentum p of a physical excitation is expressed via x^\pm as $e^{ip} = \frac{x^+}{x^-}$.

To study the strong coupling expansion it is useful to rescale x^\pm as follows

$$x^\pm \rightarrow \frac{\sqrt{\lambda}}{4\pi} x^\pm.$$

Then the rescaled x^\pm satisfy the relation

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i \frac{4\pi}{\sqrt{\lambda}} = 2i\zeta,$$

where we introduced the notation $\zeta = \frac{2\pi}{\sqrt{\lambda}}$. In fact $1/\zeta$ is equal to the effective string tension. We choose the following parametrization of x^\pm in terms of a unconstrained variable¹ x

$$x^\pm(x) = x \sqrt{1 - \frac{\zeta^2}{(x - \frac{1}{x})^2}} \pm i\zeta \frac{x}{x - \frac{1}{x}}, \quad (1)$$

The momentum p is related to x through

$$\sin \frac{p}{2} = \frac{\zeta}{x - \frac{1}{x}},$$

and the energy of a physical excitation is

$$e(x) = \frac{x + \frac{1}{x}}{x - \frac{1}{x}}.$$

¹This variable should not be confused with the variable x used in [23].

An interesting feature of the above formula is that in this parametrization the energy does not explicitly depend on the coupling constant ζ . A dependence on the coupling will arise upon solving the Bethe equations to be discussed below. Also, as we will see later on, the obvious singularity of these formulae at $x = 1$ is related to the branch cut singularity of the perturbative string S-matrix. It is not difficult to verify that the particle-to-antiparticle transformation, $x^\pm \rightarrow 1/x^\pm$, is just the inversion $x \rightarrow 1/x$

$$x^\pm(1/x) = 1/x^\pm(x)$$

and it transforms $e(x)$ to $-e(x)$.

To fix the conventions we write down the Bethe ansatz equations for rank-one sectors

$$e^{ip_j L} = \prod_{k \neq j}^M S(x_j, x_k). \quad (2)$$

Here the string S-matrix is given by

$$S(x_j, x_k) = \left(\frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \right)^\mathfrak{s} \frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_j^- x_k^+}} \sigma(x_j, x_k) \quad (3)$$

and M is a number of excitations (Bethe roots), $L = J + \frac{\mathfrak{s}+1}{2}M$, where J is a $\mathfrak{u}(1)$ -charge, and $\mathfrak{s} = 1, 0, -1$ for $\mathfrak{su}(2)$, $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$ sectors respectively. We point out that the S-matrix describes the scattering of string states in the temporal gauge $t = \tau$ and $p_\phi = J$ in the limit $J \rightarrow \infty$ with λ kept fixed.

Finally, the function $\sigma(x_j, x_k)$ appearing in the string S-matrix is called the dressing factor. Being universal to all sectors, it cannot be fixed by $\mathfrak{psu}(2, 2|4)$ symmetry and therefore is supposed to encode dynamical information about the model. The dressing factor depends on the coupling constant λ and, according to the AdS/CFT correspondence, should be equal to one at $\lambda = 0$ to recover the perturbative gauge theory results. On the other hand, at strong coupling the dressing factor can be determined by studying the spectrum of string theory states in the near plane-wave limit or, alternatively, the spectrum of semi-classical spinning strings. This analysis leads to the following structure of the dressing factor $\sigma(j, k)$

$$\sigma(x_j, x_k) = e^{i\theta(x_j, x_k)}, \quad (4)$$

where the dressing phase is a bilinear form² of local excitation charges q_r .

$$\theta(x_j, x_k) = \frac{1}{\zeta} \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} c_{r, r+1+2n}(\zeta) (q_r(x_j) q_{r+1+2n}(x_k) - q_r(x_k) q_{r+1+2n}(x_j)). \quad (5)$$

²This functional form of the dressing factor was found by analyzing the most general long-range integrable deformations of XXX spin chains [24].

The local charges are defined as follows

$$q_r(x_k) = \frac{i}{r-1} \left(\left(\frac{1}{x_k^+} \right)^{r-1} - \left(\frac{1}{x_k^-} \right)^{r-1} \right), \quad (6)$$

and the functions $c_{r,s}$ can be expanded in power series in ζ , where the first two terms of this expansion are

$$c_{r,s}(\zeta) = \delta_{r+1,s} - \zeta \frac{4}{\pi} \frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \dots \quad (7)$$

Here the leading term was found [1] by discretizing the integral equations describing the finite-gap solutions of the classical string sigma-model [6]. The subleading term was recently proposed in [12] by studying the one-loop sigma model corrections to circular spinning strings. It is worth noting that the expansion for the dressing phase is not strictly speaking an expansion in ζ because the charges q_r have non-trivial dependence on ζ . Therefore, the strong coupling expansion requires also expanding the charges q_r .

The functional equations of [13] were written for the function S_0 which is related to the dressing factor (4) as follows³

$$S_0(x_j, x_k) = \frac{x_j^- - x_k^+}{x_j^+ - x_k^-} \frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_j^- x_k^+}} \sigma(x_j, x_k). \quad (8)$$

The functional equation to be satisfied by S_0 is [13]

$$S_0(x_j, x_k) S_0(1/x_j, x_k) = f(x_j, x_k)^{-2}, \quad (9)$$

where the function $f(x_j, x_k)$ is

$$f(x_j, x_k) = \frac{1 - \frac{1}{x_j^+ x_k^-} x_j^+ - x_k^+}{1 - \frac{1}{x_j^- x_k^+} x_j^- - x_k^+}. \quad (10)$$

It follows from eq.(9) that S_0 has to satisfy the consistency condition

$$S_0(x_j, x_k) = S_0(1/x_j, 1/x_k).$$

By using this condition one can show that the Bethe equations are invariant under the particle-to-antiparticle transformation accompanied by changing the sign of the charge J .

Equation (9) rewritten for the dressing factor (4) takes the following form

$$\sigma(x_j, x_k) \sigma(1/x_j, x_k) = h(x_j, x_k)^2, \quad (11)$$

³It is worth mentioning that S_0 is equal to the S-matrix for the $\mathfrak{sl}(2)$ sector ($s = -1$).

where the function h is

$$h(x_j, x_k) = \frac{x_k^- (1 - \frac{1}{x_j^- x_k^-})(x_j^- - x_k^+)}{x_k^+ (1 - \frac{1}{x_j^+ x_k^+})(x_j^+ - x_k^+)} . \quad (12)$$

Here h is related to f as follows

$$h(x_j, x_k) = \frac{x_k^-}{x_k^+} f(x_j, x_k)^{-1} .$$

Equation (11) admits different solutions which should correspond to string S-matrices in different gauges preserving the $SU(2|2) \times SU(2|2)$ symmetry. This can be seen, for instance, by comparing the string S-matrices in the temporal [16] and the light-cone gauges [17, 25]. The light-cone Bethe equations [25] have the same form as eq.(2) with $L = P_+ + \frac{s+1}{2}M$, where the light-cone momentum P_+ is defined as $P_+ = (E + J)/2$. One can see that the temporal and light-cone gauge string S-matrices differ by dressing factors only; the ratio of the dressing factors satisfies eq.(11) with $h = 1$.

In spite of an attractive picture of the Hopf algebra symmetry leading to the tight constraints on the string S-matrix at present we do not have any firm evidence that this is indeed the case. Thus, we would like to confront equation (11) against the known leading terms in the asymptotic (strong coupling) expansion of the dressing factor.

The strong coupling expansion in the parametrization chosen is simply an expansion in powers of $\zeta = \frac{2\pi}{\sqrt{\lambda}}$ with the variable x kept fixed. It is more convenient to come to the logarithmic version of eq.(11) which reads as

$$i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 2 \log h(x_j, x_k) . \quad (13)$$

Then expanding the function $\log h$, we get

$$\begin{aligned} 2 \log h(x_j, x_k) = & -\zeta \frac{4ix_k(x_k + x_j(-2 + x_j x_k))}{(x_j - x_k)(x_j x_k - 1)(x_k^2 - 1)} \\ & + \zeta^2 \frac{4x_j^2 x_k^2 (1 - 4x_j x_k + x_j^2 + x_k^2 + x_j^2 x_k^2)}{(x_j^2 - 1)(x_k^2 - 1)(x_j - x_k)^2 (x_j x_k - 1)^2} + \dots \end{aligned} \quad (14)$$

In order to make a comparison of this expansion with the one of the l.h.s. of eq.(13) we have first to perform the sums in eq.(5) defining the dressing phase. Substituting in eq.(5) the explicit form (6) of the charges we see that the dressing phase acquires the following form

$$\begin{aligned} \theta(x_j, x_k) = & \frac{1}{\zeta} \left[\chi(x_j^-, x_k^-) - \chi(x_j^-, x_k^+) - \chi(x_j^+, x_k^-) + \chi(x_j^+, x_k^+) \right. \\ & \left. - \chi(x_k^-, x_j^-) + \chi(x_k^+, x_j^-) + \chi(x_k^-, x_j^+) - \chi(x_k^+, x_j^+) \right] , \end{aligned} \quad (15)$$

where we have introduced

$$\chi(x, y) = - \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} \frac{c_{r,r+1+2n}(\zeta)}{(r-1)(r+2n)} \frac{1}{x^{r-1}y^{r+2n}} = \chi_0 + \zeta\chi_1 + \dots \quad (16)$$

Using the explicit form of the coefficients $c_{r,s}$ we get for the leading term

$$\chi_0(x, y) = -\frac{1}{y} - \frac{xy-1}{y} \log\left(\frac{xy-1}{xy}\right). \quad (17)$$

Using this formula we develop the expansion of the l.h.s. of (13) up to the second order in ζ :

$$i\theta_0(x_j, x_k) + i\theta_0(1/x_j, x_k) = -\zeta \frac{4ix_k(x_k + x_j(-2 + x_jx_k))}{(x_j - x_k)(x_jx_k - 1)(x_k^2 - 1)} + \mathcal{O}(\zeta^3). \quad (18)$$

The expression above literally coincides with the leading term on the r.h.s. of eq. (13). Note that the subleading term of order ζ^2 is absent in this expansion!

Further, performing the sums in the first subleading correction we get

$$\begin{aligned} \chi_1(x, y) = & \frac{1}{\pi} \left[\log \frac{y-1}{y+1} \log \frac{x-\frac{1}{y}}{x-y} \right. \\ & \left. + \text{Li}_2 \frac{\sqrt{y} - \sqrt{\frac{1}{y}}}{\sqrt{y} - \sqrt{x}} - \text{Li}_2 \frac{\sqrt{\frac{1}{y}} + \sqrt{y}}{\sqrt{y} - \sqrt{x}} + \text{Li}_2 \frac{\sqrt{y} - \sqrt{\frac{1}{y}}}{\sqrt{y} + \sqrt{x}} - \text{Li}_2 \frac{\sqrt{y} + \sqrt{\frac{1}{y}}}{\sqrt{y} + \sqrt{x}} \right]. \end{aligned} \quad (19)$$

This formula was obtained under the assumption that $|xy| > 1$ and $\text{Re}(\sqrt{x}\sqrt{y}) > 1$. It is then analytically continued to the complex planes of x and y variables. As the function of two complex variables it has a rather complicated structure of singularities, in particular a branch cut singularity at $y = 1$.

Again substituting this function into the dressing phase and expanding it up to the second order in ζ we find

$$\begin{aligned} & i\theta_1(x_j, x_k) + i\theta_1(1/x_j, x_k) \\ & = 4i \frac{\partial^2}{\partial x_j \partial x_k} (\chi_1(x_j, x_k) - \chi_1(x_k, x_j) + \chi_1(1/x_j, x_k) - \chi_1(1/x_k, x_j)) \delta x_j \delta x_k, \end{aligned} \quad (20)$$

where

$$\delta x = \zeta \frac{ix^2}{x^2 - 1}.$$

Performing the differentiation and combining the logarithmic terms we obtain the following result

$$i\theta_1(x_j, x_k) + i\theta_1(1/x_j, x_k) = \frac{i}{\pi} W(x_j, x_k) \delta x_j \delta x_k,$$

where

$$\begin{aligned}
W(x_j, x_k) = & 4 \frac{(1 - 4x_j x_k + x_k^2 + x_j^2 + x_j^2 x_k^2)}{(x_j - x_k)^2 (1 - x_j x_k)^2} \left(\log \frac{x_j - 1}{x_j + 1} - \log \frac{1 - x_j}{1 + x_j} \right) \\
& - \frac{2(1 + x_j x_k)}{\sqrt{x_j x_k} (1 - x_j x_k)^2} \log \frac{-1 + \sqrt{\frac{x_k}{x_j}}}{1 + \sqrt{\frac{x_k}{x_j}}} + \frac{1 - \sqrt{x_j x_k}}{\sqrt{x_j x_k} (1 - \sqrt{x_j x_k})^3} \log \frac{-x_j + \sqrt{x_j x_k}}{x_j + \sqrt{x_j x_k}} \\
& - \frac{1 + \sqrt{x_j x_k}}{\sqrt{x_j x_k} (1 + \sqrt{x_j x_k})^3} \log \frac{x_j + \sqrt{x_j x_k}}{-x_j + \sqrt{x_j x_k}} \\
& - \left(\frac{2(x_j + x_k)}{(x_j - x_k)^2 \sqrt{x_j x_k}} + \frac{x_j(-x_k + \sqrt{x_j x_k})}{x_k(-x_j + \sqrt{x_j x_k})^3} \right) \log \frac{1 + \sqrt{x_j x_k}}{-1 + \sqrt{x_j x_k}} \\
& - \frac{x_j(x_k + \sqrt{x_j x_k})}{x_k(x_j + \sqrt{x_j x_k})^3} \log \frac{-1 + \sqrt{x_j x_k}}{1 + \sqrt{x_j x_k}}. \tag{21}
\end{aligned}$$

Note a non-trivial cancellation of all the terms which do not involve logarithms. The expression for $W(x_j, x_k)$ can be further simplified to produce the following result:

$$\begin{aligned}
W(x_j, x_k) = & 4 \frac{(1 - 4x_j x_k + x_k^2 + x_j^2 + x_j^2 x_k^2)}{(x_j - x_k)^2 (1 - x_j x_k)^2} \left(\log \frac{x_j - 1}{x_j + 1} - \log \frac{1 - x_j}{1 + x_j} \right) \\
= & 4\pi i \frac{(1 - 4x_j x_k + x_k^2 + x_j^2 + x_j^2 x_k^2)}{(x_j - x_k)^2 (1 - x_j x_k)^2}, \tag{22}
\end{aligned}$$

where we used the principle branch of \log . Thus, we finally arrive at

$$i\theta_1(x_j, x_k) + i\theta_1(1/x_j, x_k) = -4 \frac{(1 - 4x_j x_k + x_k^2 + x_j^2 + x_j^2 x_k^2)}{(x_j - x_k)^2 (1 - x_j x_k)^2} \delta x_i \delta x_j. \tag{23}$$

One can now recognize that this expression perfectly matches the ζ^2 term in the r.h.s. of eq.(13). It is interesting to note that if we would drop all Li_2 -functions in eq.(19) keeping only the logarithms we would still satisfy eq.(13) at order ζ^2 . However, dilogarithmic functions are necessary for the dressing phase to be expandable in Taylor series in local excitation charges. This is clearly related to yet to be understood analytic properties of the dressing phase.

To summarize, we have found that the perturbative string S-matrix satisfies the equation (11) on the dressing factor arising upon requiring the existence of the Hopf algebra structure up to two leading orders in the strong coupling expansion.

There are many open interesting questions. First of all it is unclear what additional (analyticity) conditions one should impose to restrict the space of solutions of the functional equation [13]. Second, one would like to understand how the representation used in [5] to derive the S-matrix from the symmetry algebra might appear by quantizing string theory in a particular gauge. In particular, one should be able to recover the central charges introduced in [5] in the symmetry algebra of gauge-fixed string theory. The derivation of [13] was based on the existence of a Hopf algebra

structure of gauge-fixed string theory. It is important to find an origin of the structure in string theory. Finally, it would be interesting to establish a connection of the approach used in [5, 13] to that of [26].

Acknowledgments

We are grateful to R. Janik, J. Plefka, M. Staudacher, A. Tseytlin and M. Zamaklar for useful discussions and comments. The work of G. A. was supported in part by the European Commission RTN programme HPRN-CT-2000-00131, by RFBI grant N05-01-00758, by NWO grant 047017015 and by the INTAS contract 03-51-6346. The work of S. F. was supported in part by the EU-RTN network *Constituents, Fundamental Forces and Symmetries of the Universe* (MRTN-CT-2004-005104).

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