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Removing fermion doublers in chiral gauge theories on the lattice

S.A. FROLOV¹

and

A.A. SLAVNOV²

*Max-Planck-Institut für Physik,
Werner-Heisenberg-Institut für Physik
Föhringer Ring 6, 8000 München 40, Germany*

Abstract

A method for removing fermion doublers in anomaly free chiral gauge theories on the lattice is proposed. It is proven that the resulting continuum theory is gauge invariant and does not require noninvariant counterterms of fine tuning of parameters.

¹LPTHE. Tour 16, 1-er etage, Universite Pierre et Marie Curie 4; Place Jussieu 75252 Paris CEDEX 5, France, on leave from Steclov Mathematical Institute, Moscow. e-mail: frolov@lpthe.jussieu.fr.

²On leave from Steclov Mathematical Institute, Vavilov 42, GSP-1, 117966, Moscow, Russia, e-mail"slavnov@qft.mian.su

1 Introduction

In this paper we propose a method for removing fermion doublers in anomaly free chiral gauge theories which preserves the gauge invariance in the continuum limit. This problem is closely related to the problem of constructing an explicitly invariant regularization for anomaly free chiral gauge models in the continuum case. In spite of the numerous efforts [1]-[5], this problem for a long time had no satisfactory solution although some proposals still need further investigations [6]-[9]. (See also the reviews [10], [11] for more complete references.) Moreover a “no go” theorem has been proven [12] stating that under some plausible conditions such a regularization cannot exist. However, recently it was shown that this “no go” theorem may be avoided. In our paper [13] the explicitly invariant regularization for the continuum $SO(10)$ model as well as for the standard model was constructed with the help of the infinite series of the Pauli-Villars (PV) fields. At the same time D. Kaplan [14] proposed the lattice formulation of chiral gauge models based on the introduction of the extra dimension. In this model all the doublers may be given large invariant masses. It was argued that for anomaly free models this construction leads to an acceptable gauge invariant continuum theory although the proof still has to be given.

It was indicated in the paper [15] that the Kaplan’s procedure is in fact also equivalent to introducing infinitely many regulator fields and in this sense both proposals [13, 14] use a similar mechanism.

In this paper we shall show that the method proposed in our paper [13] can be generalized to lattice gauge models. The regularization which will be described below breaks a manifest gauge invariance for a finite lattice spacing. However, it will be proven that in the continuum limit the gauge invariance is restored and no noninvariant counterterms or fine tuning are needed.

The idea to use a regularization breaking the gauge invariance for a finite lattice spacing was discussed before [6, 7], but in general such a procedure requires gauge noninvariant counterterms and fine tuning of the parameters. In our approach this problem is absent.

The paper is organized as follows. In section 2 we discuss the $SO(10)$ model with an even number of generations. This case is technically simpler and allows to

present the main idea in a more transparent way. We firstly remind the construction of the invariant regularization for the continuum model and then describe the lattice formulation which leads in the continuum limit to the gauge invariant theory without doublers. Section 3 is devoted to the discussion of the $SO(10)$ model and the standard model with the odd number of generations.

2 $SO(10)$ model with an even number of generations

We start by reminding the main idea of the regularization proposed in our paper [13].

The unified $SO(10)$ model may be described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^{ij})^2 + i \sum_k \bar{\psi}_+^k (\partial_\mu - igA_\mu^{ij} \sigma_{ij}) \psi_+^k \quad (1)$$

Euclidean space formulation is used although it is not essential. We use the notations of ref. [16]. Here ψ_+ are the 16-component chiral spinors describing quark and lepton fields. Index k numerates generations. The matrices σ_{ij} are defined as follows: $\sigma_{ij} = \frac{1}{2}[\Gamma_i, \Gamma_j]$, where Γ_i are Hermitian 32×32 matrices which satisfy the Clifford algebra:

$$[\Gamma_i, \Gamma_j]_+ = \delta_{ij} \quad (2)$$

The following equation holds

$$\begin{aligned} U^{-1}(R)\Gamma_k U(R) &= R_{kl}(\omega)\Gamma_l \\ U(R) &= \exp\{i\omega_{kl}\sigma_{kl}\}, \end{aligned} \quad (3)$$

where the matrices $R_{kl}(\omega)$ determine a rotation in a 32-dimensional space.

The mapping $R \rightarrow U(R)$ defines a 32-dimensional representation of $SO(10)$. This representation is reducible and to construct the irreducible 16-dimensional representation one uses the ‘‘chiral’’ projections

$$\psi_{\pm} = \frac{1}{2}(1 \pm \Gamma_{11})\psi, \quad (4)$$

where

$$\Gamma_{11} = -\Gamma_1\Gamma_2\cdots\Gamma_{10} \quad (5)$$

We assume also that the spinors ψ_{\pm} are Weyl spinors: $\psi_{\pm} \equiv \frac{1}{2}(1 + \gamma_5)\psi_{\pm}$.

In the following we shall consider only the regularization of spinorial loops having in mind that the Yang-Mills fields may be regularized in a gauge invariant way by using the higher covariant derivative method [17]. For definiteness the number of generations is chosen to be equal to 2.

We shall use for the regularization of the spinorial loops some modification of the Pauli-Villars method introducing the interaction with the auxiliary fermionic spinor fields ψ_r and the bosonic spinors ϕ_r . The usual obstacle for using the Pauli-Villars method in chiral theories is the impossibility to introduce the mass term for the auxiliary fields as the combination $\bar{\psi}\psi$ is equal to zero for Weyl spinors. However in our case the Majorana mass term may be introduced due to the existence of a ‘‘conjugation’’ matrix C , satisfying the relation

$$\sigma_{ij}^T C = -C\sigma_{ij}. \quad (6)$$

The expression

$$M_r(\psi_r^T C_D C \Gamma_{11} \psi_r + \text{h.c.}), \quad (7)$$

where C_D is the usual charge conjugation matrix, provides a gauge invariant mass term for the $P - V$ fields. However, in this eq. ψ_r are the 32-component spinors $\psi_r = \psi_+^r + \psi_-^r$ and not the 16-component ‘‘chiral’’ ones. One cannot write a gauge invariant mass term using only positive chirality spinors as the combination $\psi_+^T C C_D \Gamma_{11} \psi_+$ is identically zero.

At first sight we meet again the same problem: the original theory includes only positive chirality spinors and to introduce the $P - V$ fields we need both positive and negative chirality spinors. The crucial observation is that for the SO(10) model the positive and negative chirality SO(10) spinors give the same contribution to the divergent diagrams. The difference between the positive and negative chirality

spontaneous arises only in diagrams with the number of external lines > 4 , which are convergent.

Alternatively, replacing the right-handed spinors by charge conjugated left-handed ones one can see that the regularization is needed only for parity conserving part of the diagrams. The parity odd part is different from zero only for the convergent diagrams. This property holds also for the standard model and presumably for all anomaly free chiral models. Let us prove it for the model under consideration.

The interaction vertex in (1) includes the projection operator $\frac{1}{2}(1 + \Gamma_{11})$. Calculating a spinorial loop one should take a trace

$$Tr(\sigma_{ij}\sigma_{kl} \dots (1 + \Gamma_{11})) \quad (8)$$

Due to the definition of Γ_{11} , (5), the trace

$$Tr(\sigma_{i_1 j_1} \sigma_{i_2 j_2} \dots \sigma_{i_n j_n} \Gamma_{11}) \quad (9)$$

is equal to zero if $n < 5$. Therefore for the divergent diagrams with $n \leq 4$ the part proportional to Γ_{11} vanishes and the positive and negative chirality spinors give the same contribution. The total contribution of the 32-component spinor $\psi = \psi_+ + \psi_-$ to the divergent diagrams is twice as big as the contribution of the 16-component ‘‘chiral’’ spinor. Hence if the number of generations in the original Lagrangian (1) is equal to 2, one can use for its regularization the 32-component spinors ψ_r with the mass term (7).

Having this in mind one can take as a regularized Lagrangian the following expression:

$$\begin{aligned} \mathcal{L}_{reg} = & i \sum_k \bar{\psi}_+^k \gamma^\mu (\partial_\mu - ig A_\mu^{nl} \sigma_{nl}) \psi_+^k + \\ & + i \sum_r \bar{\psi}^r \gamma^\mu (\partial_\mu - ig A_\mu^{nl} \sigma_{nl}) \psi^r + \\ & + i \sum_s \bar{\phi}^s \gamma^\mu \Gamma_{11} (\partial_\mu - ig A_\mu^{nl} \sigma_{nl}) \phi^s - \\ & - \left(\sum_r \frac{M_r}{2} \bar{\psi}_r C_D C \Gamma_{11} \bar{\psi}_r^T - \sum_s \frac{M_s}{2} \bar{\phi}_s C_D C \bar{\phi}_s^T + \text{h.c.} \right). \end{aligned} \quad (10)$$

Here ψ_r are the fermionic P-V fields and ϕ_r are the bosonic ones.

The Pauli-Villars conditions

$$2 \sum_r C_r - 2 \sum_s C_s + 2 = 0 \quad (11)$$

$$\sum_r C_r M_r^2 - \sum_s C_s M_s^2 = 0 \quad (12)$$

are assumed. In these equations C_t is the number of the P-V fields with the mass M_t . The last term in the eq.(11) is 2 because in our case there are 2 generations of the original fields giving the identical contributions to the spinorial loops. The factor 2 multiplying C_r and C_s is due to the presence of the P-V fields of both chiralities giving the identical contribution to the divergent diagrams.

The propagators generated by the Lagrangian (10) look as follows

$$\begin{aligned} S_{\bar{\psi}_r^+ \psi_r^+} &= S_{\bar{\psi}_r^- \psi_r^-} = S_{\bar{\phi}_r^+ \phi_r^+} = S_{\bar{\phi}_r^- \phi_r^-} = \frac{\widehat{k}}{k^2 + M_r^2}, \\ S_{\bar{\psi}_r^- \bar{\psi}_r^+} &= S_{\psi_r^+ \psi_r^-} = S_{\bar{\phi}_r^- \bar{\phi}_r^+} = S_{\phi_r^+ \phi_r^-} = \frac{M_r C_D C \Gamma_{11}}{k^2 + M_r^2}. \end{aligned} \quad (13)$$

One sees that eqs.(10-13) define a standard Pauli-Villars regularization. If the conditions (11-12) are fulfilled all spinor loops are finite. At the same time the regularized Lagrangian (10) is manifestly gauge invariant. It is worthwhile to emphasize that the number of generations being even was crucial for the above discussion. Due to the presence of P-V fields of both chiralities their contribution to the divergent diagrams is twice as big as the contribution of one generation of the original fields ψ_+ . In the case of the odd number of generations the last term in the eq. (11) would be replaced by $(2n + 1)$ and this equation cannot be satisfied by a finite number of P-V fields. Of course, if one allows for fractional values of C_r the eq.(11) may be satisfied for the odd number of generations as well. But in this case one cannot interpret C_r as the number of the P-V fields with the mass M_r and the locality of the regularized Lagrangian may be lost.

It was shown in our paper [13] that in this case the problem may be solved by introducing an infinite number of P-V fields.

Obviously the regularization (10) works also for the standard model. The mass terms being invariant under $SO(10)$ transformations are also invariant with respect to any subgroups of $SO(10)$ in particular $SU(3) \times SU(2) \times U(1)$. So to get the invariant regularization of the standard model it is sufficient to keep in the eq. (10)

only the gauge fields A^{kl} corresponding to the gluons and electroweak bosons and put all other gauge fields equal to zero.

Let us generalize the regularization described above to the case of lattice gauge models. It is well known that in the lattice models all fermionic states are accompanied by doubler states due to the fact that the lattice fermion propagator

$$S(p) = \left(\sum_{\mu} a^{-1} \sin p_{\mu} a \right)^{-1} \quad (14)$$

has poles not only at $p = 0$, but also in the vicinity of the points $p = (\pi a^{-1}, 0, 0, 0)$, $(\pi a^{-1}, \pi a^{-1}, 0, 0)$, etc. Following Wilson [18] one can kill these unwanted states by adding to the lattice action the term

$$\frac{\kappa}{2a} \sum_{x,\mu} \left(\bar{\psi}(x)\psi(x+a_{\mu}) + \bar{\psi}(x+a_{\mu})\psi(x) - 2\bar{\psi}(x)\psi(x) \right) \quad (15)$$

This term provides all the states but one with the masses of the order a^{-1} and therefore cures the disease. In the case of vector gauge theories it may be easily done gauge invariant by replacing lattice derivatives by the covariant ones. However it cannot be done for chiral gauge theories and the Wilson term inevitably breaks the gauge invariance.

Nevertheless it will be shown that using the regularized action of the type (10) one can introduce Wilson like mass terms in such a way that all doubler states will acquire the masses of the order of the cut-off and the gauge invariance, although broken for a finite lattice spacing is restored in the continuum limit. In this process no gauge non invariant counterterms or fine tuning of the parameters is needed. The idea is to compensate gauge noninvariant contributions of the original fields ψ_+ by the corresponding contributions of the PV fields. To do that one introduces the same Wilson like mass term for all the fields ψ_+, ψ_r, ϕ_r and chooses the P-V masses $M_r \ll a^{-1}$. Then in the vicinity of $p = 0$ one can neglect the Wilson terms recovering the gauge invariant continuum result. On the other hand in the vicinity of $p = (\pi/a, 0, 0, 0)$ etc. the leading terms in the integrands of Feynman diagrams are zero due to the P-V conditions and the remaining contributions vanish when $a \rightarrow 0$. The formal proof will be given below.

The lattice action for the SO(10) model looks as follows

$$\begin{aligned}
I = & \sum_{x,\mu,k} \left[-\frac{1}{2ia} \bar{\psi}_k^+(x) \gamma_\mu U_\mu(x) \psi_k^+(x+a_\mu) - \right. \\
& - \left. \frac{\kappa}{a} \left(\bar{\psi}_k^+(x) C_D \bar{\psi}_k^{+T}(x+a_\mu) - \bar{\psi}_k^+(x) C_D \bar{\psi}_k^{+T}(x) \right) + \text{h.c.} \right] + \\
& + \sum_{x,\mu,r} \left[-\frac{1}{2ia} \bar{\psi}_r(x) \gamma^\mu U_\mu(x) \psi_r(x+a_\mu) - \right. \\
& - \left. \frac{\kappa}{a} \left(\bar{\psi}_r(x) C_D \bar{\psi}_r^T(x+a_\mu) - \bar{\psi}_r(x) C_D \bar{\psi}_r^T(x) \right) - \frac{M_r}{2} \bar{\psi}_r(x) C_D C \Gamma_{11} \bar{\psi}_r^T(x) + \text{h.c.} \right] \\
& + \sum_{x,\mu,r} \left[-\frac{1}{2ia} \bar{\phi}_r(x) \gamma^\mu \Gamma_{11} U_\mu(x) \phi_r(x+a_\mu) - \frac{1}{2ia} \bar{\phi}_r(x) \gamma_\mu \Gamma_{11} U_\mu(x) \tilde{\phi}_r(x+a_\mu) - \right. \\
& - \left. \left(\bar{\phi}_r(x) C_D \bar{\phi}_r^T(x+a_\mu) + \bar{\phi}_r(x+a_\mu) C_D \bar{\phi}_r^T(x) - 2\bar{\phi}_r(x) C_D \bar{\phi}_r^T(x) \right) + \right. \\
& + \left. \frac{M_r}{2} \left(\bar{\phi}_r(x) C_D C \bar{\phi}_r^T(x) + \bar{\phi}_r(x) C_D C \bar{\phi}_r^T \right) + \text{h.c.} \right]
\end{aligned} \tag{16}$$

In this equation the index k as before numerates generations. For the moment we shall take $k = 1, 2$.

$$U_\mu = \exp\{iga\sigma^{ij} A_\mu^{ij}\} \tag{17}$$

All other notations are the same as in eq. (10) except for the new set of bosonic P-V fields $\tilde{\phi}_r$. The fields $\tilde{\phi}_r$ are necessary to make a nonzero mass term $\bar{\phi} C_D \bar{\phi}^T$.

In our model there are several different dimensional parameters like M_r, a^{-1} , which in the continuum limit become infinite. It is convenient to introduce one fixed mass scale λ and take all other masses to be proportional to λ : $a^{-1} = \lambda N$, $M_r = \lambda N^\delta$, etc. The continuum limit corresponds to $N \rightarrow \infty$. In the following we assume that $M_r \ll a^{-1}$, i.e. $\delta < 1$. More precise condition will be specified below.

One sees that the action (16) is nothing but a discretization of the gauge invariant continuum Lagrangian (10) except for the presence of the Wilson mass terms breaking the gauge invariance. Due to these terms each generation of the original fields ψ_k has only one massless state. Correspondingly each $P - V$ field ψ_r, ϕ_r describes one state with the mass M_r and 15 doublers with the masses $\sim \kappa a^{-1}$. We shall see that when $a \rightarrow 0$ the contribution of the doubler states vanishes and hence we are left with the same set of the Feynman rules as the one defined by the manifestly gauge invariant Lagrangian (10).

The action (16) generates the propagators of the same type as given by the

eq. (13) and the additional propagators $\bar{\psi}_\pm \bar{\psi}_\pm$, $\bar{\phi}_\pm \bar{\phi}_\pm$, $\tilde{\phi}_\pm \tilde{\phi}_\pm$.

They look as follows

$$S_{\bar{\psi}_+^k \psi_+^k} = \frac{\hat{s}}{s^2 + m^2}, \quad (18)$$

$$S_{\bar{\psi}_r^+ \psi_r^+} = S_{\bar{\psi}_r^- \psi_r^-} = S_{\bar{\psi}_r^+ \psi_r^+} = S_{\tilde{\phi}_r^+ \tilde{\phi}_r^+} = -S_{\bar{\phi}_r^- \phi_r^-} = -S_{\tilde{\phi}_r^- \tilde{\phi}_r^-} = \frac{\hat{s}}{s^2 + m^2 + M_r^2} \quad (19)$$

$$S_{\bar{\psi}_k^+ \psi_k^+} = S_{\psi_k^+ \psi_k^+} = \frac{C_D m}{s^2 + m^2 + M_r^2}, \quad (20)$$

$$S_{\bar{\psi}_r^+ \bar{\psi}_r^+} = S_{\psi_r^+ \psi_r^+} = S_{\bar{\psi}_r^- \bar{\psi}_r^-} = S_{\psi_r^- \psi_r^-} = S_{\bar{\phi}_r^+ \tilde{\phi}_r^+} = S_{\bar{\phi}_r^- \tilde{\phi}_r^-} = \frac{C_D m}{s^2 + (m^2 + M_r^2)}, \quad (21)$$

$$S_{\bar{\psi}_r^- \bar{\psi}_r^-} = S_{\psi_r^+ \psi_r^-} = S_{\bar{\phi}_r^- \bar{\phi}_r^+} = S_{\tilde{\phi}_r^- \tilde{\phi}_r^+} = S_{\phi_r^+ \phi_r^-} = S_{\tilde{\phi}_r^+ \tilde{\phi}_r^-} = \frac{M_r C_D C \Gamma_{11}}{s^2 + (m^2 + M_r^2)}, \quad (22)$$

Here

$$s_\mu = a^{-1} \sin(p_\mu a), \quad (23)$$

$$m = \kappa a^{-1} \sum_\mu (1 - \cos(p_\mu a)). \quad (24)$$

Let us show that the doublers contribution vanishes in the limit $a \rightarrow 0$. Consider for example the polarization operator $\Pi_{\mu\nu}$. It includes the contributions of all the fields $\psi_k, \psi_r, \phi_r, \tilde{\phi}_r$ and consists of the different pieces corresponding to the different types of the propagators (18-22) entering the diagram.

The generic form of $\Pi_{\mu\nu}$ is:

$$\Pi_{\mu\nu}^{(ij)(kl)}(k) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} Tr[V_\mu^{ij}(p, q) S(p) V_\nu^{kl}(p, q) S(q)] d^4 p \quad , \quad p + q + k = 0 \quad (25)$$

where $S(p)$ stands for one of the propagators (18-22) and V_μ is the interaction vertex

$$V_\mu^{ij} = g \gamma_\mu \frac{(1 + \gamma_5)}{2} \sigma_{ij} \frac{(1 + \Gamma_{11})}{2} \cos \left[\frac{1}{2} (p - q)_\mu a \right]. \quad (26)$$

We separate the integration domain in eq. (25) into two parts $V_{\text{in}}, V_{\text{out}}$, defined as follows

$$V_{\text{in}}: |p| < \lambda N^\gamma \ll a^{-1}; V_{\text{out}}: |p| > \lambda N^\gamma, \quad \gamma < \frac{1}{2}. \quad (27)$$

In the domain $V_{\text{in}} |pa| \ll 1$ and one can use the expansion over (pa) . We shall show that the integral over V_{in} in the limit $a \rightarrow 0$ coincides with the corresponding

integral generated by the manifestly gauge invariant continuum Lagrangian (10). Consider firstly the diagrams including the propagators (18-19).

$$\Pi_{\mu\nu}^{(a)} \sim g^2 \int \frac{\text{Tr} \left[\gamma_\mu \sigma^{ij} \hat{p} \gamma_\nu \sigma^{kl} (\hat{p} + \hat{k}) \left(\frac{1+\gamma_5}{2} \right) \left(\frac{1 \pm \Gamma_{11}}{2} \right) \right]}{[p^2 + \kappa^2 a^2 p^4 + M_r^2][(p+k)^2 + \kappa^2 (p+k)^4 a^2 + M_r^2]} d^4 p. \quad (28)$$

Expanding the denominator in terms of (pa) one gets

$$\Pi_{\mu\nu}^{(a)} \simeq g^2 \int \text{Tr} \left[\gamma_\mu \sigma_{ij} \hat{p} \sigma^{kl} \gamma_\nu (\hat{p} + \hat{k}) \left(\frac{1 + \gamma_5}{2} \right) \left(\frac{1 \pm \Gamma_{11}}{2} \right) \right] \cdot \left[\frac{1}{[p^2 + M_r^2][(p+k)^2 + M_r^2]} - \frac{\kappa^2 p^4 a^2}{[p^2 + M_r^2]^2 [(p+k)^2 + M_r^2]} + \dots \right] d^4 p. \quad (29)$$

The first term in this expression coincides exactly with the continuum expression generated by the Lagrangian (10). The next terms are majorated by $\lambda^2 N^{4\gamma-2} \sim a^\varepsilon, \varepsilon > 0$ and vanish in the limit $a \rightarrow 0$.

The diagrams including the propagators (22) are analyzed in the same way:

$$\Pi_{\mu\nu}^{(b)} \sim g^2 \int_{V_{\text{in}}} \text{Tr} \left[\gamma_\mu \sigma_{ij} \gamma_\nu \sigma^{kl} \left(\frac{1 + \gamma_5}{2} \right) \left(\frac{1 \pm \Gamma_{11}}{2} \right) \right] \cdot \left\{ \frac{M_r^2}{[p^2 + M_r^2][(p+k)^2 + M_r^2]} - \frac{\kappa^2 p^4 a^2 M_r^2}{[p^2 + M_r^2]^2 [(p+k)^2 + M_r^2]} + \dots \right\} d^4 p. \quad (30)$$

Again the first term coincides with the corresponding continuum expression and the next terms are majorated by $N^{2\gamma-2} M_r^2$ and vanish in the limit $a \rightarrow 0$, if $M_r = \lambda N^\delta, \delta \leq \gamma$.

Finally the contribution of the propagators (20-21) is proportional to the integral

$$\Pi_{\mu\nu}^c \sim \int_{V_{\text{in}}} g^2 d^4 p \frac{\kappa^2 a^2 p^4 d^4 p}{[p^2 + \kappa^2 a^2 p^4 + M_r^2][(p+k)^2 + \kappa^2 a^2 (p+k)^4 + M_r^2]} \quad (31)$$

and vanishes in the limit $a \rightarrow 0$.

Obviously the same arguments may be repeated for the spinorial loops with three and more external lines. For all these diagrams the integrals over V_{in} coincide in the limit $a \rightarrow 0$ with the corresponding integrals generated by the manifestly gauge invariant Lagrangian (1).

Now we shall show that the integrals over V_{out} do not contribute at all to the continuum limit. Again consider as an example the polarization operator $\Pi_{\mu\nu}$. The

sum of the diagrams contributing to $\Pi_{\mu\nu}^{(a)}$ looks as follows

$$\begin{aligned}
\Pi_{\mu\nu}^{(a)} = & \int_{V_{\text{out}}} \{Tr[\gamma_\mu \sigma^{ij} \cos(\frac{1}{2}(p-q)_\mu a) \hat{s}(p) \gamma_\nu \sigma^{kl} \cos(\frac{1}{2}(p-q)_\nu a) \\
& \cdot \hat{s}(-(p+k)) (\frac{1+\gamma_5}{2}) [2(\frac{1+\Gamma_{11}}{2})(s^2(p)+m^2(p))^{-1}(s^2(p+k)+m^2(p+k))^{-1} \\
& + \sum_{r,\pm} (\frac{1\pm\Gamma_{11}}{2})(s^2(p+k)+m^2(p+k)+M_r^2)^{-1}(s^2(p)+m^2(p)+M_r^2)^{-1} \\
& - 2 \sum_{r,\pm} (\frac{1\pm\Gamma_{11}}{2})(s^2(p+k)+m^2(p+k)+M_r^2)^{-1}(s^2(p)+m^2(p)+M_r^2)^{-1}]\} d^4p
\end{aligned} \tag{32}$$

Here the first term describes the contribution of the two generations of the original fields ψ_k^+ (hence the factor 2). The second term describes the contribution of the fermionic P-V fields of both chiralities (hence the summation over \pm). The last term describes the contribution of the bosonic P-V fields (the factor 2 is due to the presence of the two sets of bosonic fields $\phi_s, \tilde{\phi}_s$).

As we have already discussed the part proportional to Γ_{11} vanishes as

$$Tr(\sigma^{ij} \sigma^{kl} \Gamma_{11}) = 0. \tag{33}$$

Therefore the summation over \pm simply doubles the individual contribution.

In the domain $|p| > \lambda N^\gamma$ the integrand in eq. (32) may be expanded in terms of M_r^2 . The zero order term is proportional to

$$(2 + 2 \sum_r C_r - 4 \sum_s C_s), \tag{34}$$

where C_r is the number of fermionic P-V fields with the mass M_r and C_s is the number of the bosonic P-V fields with the mass M_s . Using the Pauli-Villars conditions one can make this sum equal to zero.

The first order term is proportional to

$$\sum_r C_r M_r^2 - 2 \sum_s C_s M_s^2, \tag{35}$$

which again may be done equal to zero by P-V conditions. The remaining terms are majorated by $a^2 M_r^4$ and vanish in the limit $a \rightarrow 0$.

The same reasoning may be applied to the diagrams $\Pi_{\mu\nu}^{(b)}$ and $\Pi_{\mu\nu}^{(c)}$. For example the integrand in the diagram $\Pi_{\mu\nu}^{(c)}$ is proportional to

$$\begin{aligned}
& 2m^2(p) \left\{ \frac{1}{[s^2(p) + m^2(p)][s^2(p+k) + m^2(p+k)]} + \right. \\
& + \sum_r \frac{1}{[s^2(p) + m^2(p) + M_r^2][s^2(p+k) + m^2(p+k) + M_r^2]} - \\
& \left. - \sum_r \frac{1}{[s^2(p) + m^2(p) + M_r^2][s^2(p+k) + m^2(p+k) + M_r^2]} \right\} \quad (36)
\end{aligned}$$

Expanding it in terms of M_r^2 one sees that the first two terms are zero due to the P-V conditions and the remaining terms are majorated by $a^2 M_r^4$.

In the lattice case there are also additional tadpole diagrams which are absent in the continuum theory. They arise when one expands $U_\mu(x)$ in terms of A_μ and considers the higher order terms. These diagrams are analysed exactly in the same way and one sees that they do not contribute to the continuum limit.

Generalization to the spinor loops with more than two external lines is absolutely straightforward. Integrals over V_{Out} vanish in the limit $a \rightarrow 0$ for all spinor loops.

Therefore we proved that for small lattice spacing the Feynman rules for the spinor loops which follow from the lattice action (16) are identical to the manifestly gauge invariant rules generated by the continuum Lagrangian (10). To remove ultraviolet divergencies one needs only the usual gauge invariant counterterms.

3 Regularization of the chiral models with an odd number of generations.

Now we pass to the discussion of the SO(10) model with an odd number of generations. Again we start by reminding the procedure for the continuum model [13].

The difficulty with the regularization of the odd number generation model is due to the fact that the P-V fields always enter with both chiralities, and one cannot satisfy the P-V condition

$$2 \sum C_r - 2 \sum C_s + (2n + 1) = 0 \quad (37)$$

by a finite number of P-V fields.

Instead it was proposed in the paper [13] to introduce an infinite system of P-V fields ψ_r with the masses $M_r = M |r|$ and Grassmanian parity $\varepsilon(\psi_r) = (-1)^{r-1}$. The index r changes from $-\infty$ to $+\infty$, $r = 0$ corresponds to the original field ψ_+ , positive r numerating the positive chirality P-V spinors, negative r numerating the negative chirality ones.

As it was discussed above the contribution of the positive and negative chirality spinors to the divergent diagrams are equal. For definites we shall take the number of generations equal to 1.

The propagators are given by the eqs.(13). The integral corresponding to the diagrams $\Pi_{\mu\nu}^{(a)}$ looks as follows

$$\Pi_{\mu\nu}^{(a)}(k) = \int dp \frac{\sum (-1)^r Tr[\gamma_\mu \sigma_{ij} \hat{p} \gamma_\nu \sigma_{kl} (\hat{p} + \hat{k}) \left(\frac{1+\gamma_5}{2}\right)]}{[p^2 - M^2 r^2][(p+k)^2 - M^2 r^2]}. \quad (38)$$

Here we omitted the term proportional to Γ_{11} because its contribution is equal to zero as we have shown above.

The leading term in the integrand for $p \rightarrow \infty$ is

$$\sim Tr[\gamma_\mu \sigma^{ij} \hat{p} \gamma_\nu \sigma^{kl} (\hat{p} + \hat{k}) \left(\frac{1+\gamma_5}{2}\right)] \sum_{r=-\infty}^{+\infty} \frac{(-1)^r}{(p^2 + M^2 r^2)^2}. \quad (39)$$

Using the representation

$$\sum_{r=-\infty}^{+\infty} \frac{(-1)^r}{(p^2 + M^2 r^2)^2} = -\frac{\partial}{\partial p^2} \sum_{r=-\infty}^{+\infty} \frac{(-1)^r}{p^2 + M^2 r^2}, \quad (40)$$

one can do the summation over r explicitley

$$\sum_{r=-\infty}^{+\infty} \frac{(-1)^r}{p^2 + M^2 r^2} = \frac{\pi}{MR \sin h\left(\frac{\pi R}{M}\right)}, R^2 = p_0^2 + p_1^2 + p_2^2 + p_3^2. \quad (41)$$

One sees that the leading term in the integrand decreases exponentially.

Next to leading terms are analyzed analogously by expanding the integrand in the eq.(38) in a series over k_μ . The integrands of the corresponding terms are proportional to

$$\sum_{r=-\infty}^{-\infty} \frac{(-1)^r}{(p^2 + M^2 r^2)^n}, n > 2 \quad (42)$$

and can be calculated by differentiating the eq.(41) with respect to p^2 . All these terms are decreasing exponentially and therefore the integral (38) is convergent.

The diagrams $\Pi_{\mu\nu}^{(b)}$ including the propagators $\psi_r^+\psi_r^-$ are treated in the same way. For example the integrand for $\Pi_{\mu\nu}^{(b)}$ is proportional to

$$\sum_{r=-\infty}^{\infty} (-1)^r \frac{M^2 r^2}{[(p+k)^2 + M^2 r](p^2 + M^2 r^2)}. \quad (43)$$

Representing M_r^2 in the numerator as $(M^2 r^2 + p^2) - p^2$ one reduces the problem of summation over r to the case considered above. The corresponding functions decrease exponentially providing the convergence of the integrals.

Generalization to the diagrams with more than two external lines is obvious: all these diagrams correspond to the convergent integrals.

To analyze the convergence of the regularized diagrams it is not necessary in fact to expand them in series over k_μ . Instead one can use the Feynman representation

$$\frac{1}{[p^2 + M_r^2](p+k)^2 + M_r^2]} = \int_0^1 \frac{d\alpha}{[(1-\alpha)(p^2 + M_r^2) + \alpha[(p+k)^2 + M_r^2]]^2}. \quad (44)$$

Shifting the integration variables $p_\mu, p_\mu \rightarrow p_\mu - \alpha k_\mu$, one can write this expression in the standard form

$$\frac{1}{[p^2 + \alpha(1-\alpha)k^2 + M_r^2]^2}. \quad (45)$$

Now the summation over r can be done explicitly for arbitrary k leading to the same conclusion about the convergence of the integrals. The representation (45) is also useful for practical calculations.

Therefore the infinite set of P-V fields with the Majorana masses provides a gauge invariant regularization in the case of the odd number of generations as well.

Generalization to the lattice models goes along the same lines as for the case of the even number of generations. The lattice Lagrangian is given by the eq.(16) where now there is no summation over $k, k = 1$.

The propagators are given by the eqs.(18-22). To get the analog of the eq.(38) we choose the following set of P-V fields. The fields $\psi_{(\pm 1)}, \psi_{(\pm 2)}, \dots, \psi_{(\pm n)}$ are the fermions with the masses $M|r|$, and the fields $\phi_{(\pm 1)}, \phi_{(\pm 3)}, \dots, \phi_{(\pm(2n+1))}; \tilde{\phi}_{(\pm 1)}$,

$\tilde{\phi}_{(\pm 3)}, \dots, \tilde{\phi}_{(\pm(2n+1))}$ are the bosons with the masses $M|r|$. As before positive r correspond to the positive chirality fields and negative r correspond to the negative chirality ones. With this choice the contribution of the fields with the number r to the integrand of the polarization operator $\Pi_{\mu\nu}$ is proportional to

$$\frac{(-1)^r}{[s^2(p) + m^2(p) + M^2r^2][s^2(p+k) + m^2(p+k) + M^2r^2]}. \quad (46)$$

The analog of the eq.(38) for $\Pi_{\mu\nu}^a$ now looks as follows

$$\begin{aligned} \Pi_{\mu\nu}^{(a)} &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^4p \sum_{r=-\infty}^{+\infty} (-1)^r \cdot \\ &\cdot \frac{\text{Tr}[\gamma_\mu \sigma^{ij} \hat{s}(p) \cos[\frac{1}{2}(p-q)a_\mu] \gamma_\mu \sigma^{kl} \hat{s}(p+k) \cos[\frac{1}{2}(p-q)a_\nu] \left(\frac{1+\gamma_5}{2}\right)]}{[s^2(p) + m^2(p) + M^2r^2][s^2(p+k) + m^2(p+k) + M^2r^2]} \end{aligned} \quad (47)$$

As before we separate the integration domain into $V_{\text{in}}: |p| < \lambda N^\gamma \ll a^{-1}; \gamma < \frac{1}{2}$, $V_{\text{out}}: |p| > \lambda N^\gamma$. In the domain V_{in} we can expand the integrand in terms of (pa) . In this way one gets the expression which differs from the continuum expression (38) only by the presence of the Wilson mass term $m^2 \approx \kappa^2 a^2 p^4$:

$$\Pi_{\mu\nu}^{(a)}(k) = \int_{V_{\text{in}}} d^4p \sum_{r=-\infty}^{\infty} (-1)^r \frac{\text{Tr}[\gamma_\mu \sigma^{ij} \hat{p} \gamma_\nu \sigma^{kl} (\hat{p} + \hat{k}) \left(\frac{1+\gamma_5}{2}\right)]}{[p^2 + \kappa^2 p^4 a^2 + M^2 r^2][(p+k)^2 + \kappa^2 (p+k)^4 a^2 + M^2 r^2]} \quad (48)$$

In the domain $|p| < \lambda N^\gamma$ the Wilson term $\kappa^2 a^2 p^4 \ll p^2 + M^2 r^2$. The series in eq. (48) converges for any p^2 and one can neglect the Wilson term, recovering the continuum expression (38). In the domain V_{out} we can expand the integrand of the eq. (47) in a series over k^μ . The zero order term is

$$\Pi_{\mu\nu}^{(a)}(k) = \int_{V_{\text{out}}} d^4p \sum_{r=-\infty}^{+\infty} (-1)^r \frac{\text{Tr}[\gamma_\mu \sigma^{ij} \hat{s}(p) \gamma_\nu \hat{s}(p) \cos(pa_\mu) \cos(pa_\nu) \left(\frac{1+\gamma_5}{2}\right)]}{[s^2(p) + m^2(p) + M^2 r^2]^2} \quad (49)$$

The summation over r is done in the same way as in the continuum case

$$\sum_{r=-\infty}^{+\infty} \frac{(-1)^r}{[s^2(p) + m^2(p) + M^2 r^2]} = -\frac{\partial}{\partial s^2} \left(\frac{\pi}{\sqrt{M^2(s^2 + m^2)} \sin h\left(\frac{\pi\sqrt{s^2+m^2}}{M}\right)} \right). \quad (50)$$

For small a this expression decreases exponentially. Therefore the integral over V_{out} is equal to zero in the limit $a \rightarrow 0$. Next order terms in the series over k^μ are

analyzed in the same way. These arguments are easily extended to any spinor loop. The corresponding integrals can be written in the form

$$I_n = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^4p \sum_{r=-\infty}^{+\infty} \sum_{l=0}^{n-1} \frac{A_l(p, Q, M_r)}{s^2(p + Q_l) + m^2(p + Q_l) + M^2 r^2} \quad (51)$$

$$Q_l = k_1 + \dots + k_l.$$

where A_l is a polynomial in M_r^2 . The summation over r can be done explicitly using eq. (50). One gets for I_n

$$I_n = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \sum_{l=0}^{n-1} \frac{\tilde{A}_l(P, Q)}{\sqrt{M^2(s^2 + m^2)} \sin h\left(\frac{\pi\sqrt{s+m^2}}{M}\right)} d^4p \quad (52)$$

In the domain V_{in} one can expand the integrand in terms of (pa) . The Wilson term $\sim \kappa^2 a^2 p^4 \ll p^2$ and can be neglected. Hence we recover the gauge invariant continuum result.

In the domain V_{out} , $\sqrt{s^2 + m^2} \gg M$ and the integrand vanishes exponentially when $a \rightarrow 0$.

The final conclusion is that for small a the expression for the spinorial loops in our lattice model coincides with the manifestly gauge invariant expression generated by the continuum Lagrangian (10) both in the case of even and odd number of generations. It is worthwhile to note that in the lattice case it is not necessary to take infinite number of P-V fields. It is sufficient to take a finite number $N^1(a)$ of such fields which becomes infinite when $a \rightarrow 0$. Indeed the series in eq.(52) is convergent for any p and the integration domain is finite for $a \neq 0$. Therefore choosing N^1 big enough one can always make the contribution of the remaining fields $\sum_{|n|=N_1}^{\infty}$ as small as one wishes.

In the continuum limit the number of P-V fields becomes infinite and one gets the same gauge invariant result.

Up to now we discussed only the spinor loops. There are other diagrams to be worried about, in particular spinor particle self energy diagrams. In the continuum case we assumed that the higher covariant derivative regularization was introduced for the gauge fields. It happens that in the lattice model such a regularization is also needed. Although for a finite lattice spacing the self energy diagrams are finite

without any additional regularization in the continuum limit they may require noninvariant counterterms like fermion mass renormalization. Higher covariant derivatives for gauge fields cure this decease.

One modifies the lattice Yang-Mills action as follows:

$$\begin{aligned}
S_w &= -\frac{1}{g^2} \sum \text{Tr}(U^{\mu\nu} + (U^{\mu\nu})^+) \rightarrow \\
&\rightarrow -\frac{1}{g^2} \sum \left\{ \text{Tr} U^{\mu\nu} + \frac{(\Lambda^2 a^2)}{g^2} \left[\text{Tr} U^{\mu\nu}(x) U_\rho(x) (U^{\mu\nu}(x + a_\rho))^+ U_\rho^+(x) \right. \right. \\
&\quad \left. \left. - \text{Tr} U^{\mu\nu}(x) (U^{\mu\nu}(x))^+ \right] \right\} + \text{h.c.}
\end{aligned} \tag{53}$$

where

$$U_{\mu\nu} = U_\mu(x) U_\nu(x + a_\mu) U_\mu^+(x + a_\nu) U_\nu^+(x) \text{ (no trace)}. \tag{54}$$

It leads to the following modification of the gauge field propagator (assuming a diagonal gauge):

$$\begin{aligned}
G(p) &= \left[\frac{1}{a^2} \left(\sum_\mu \cos(p_\mu a) - 1 \right) \right]^{-1} \rightarrow \\
&\rightarrow \left[a^2 \left(\sum_\mu (\cos(p_\mu a) - 1) + \Lambda^2 \left(\sum_\mu (\cos(p_\mu a) - 1) \right) \left(\sum_\nu (\cos(p_\nu a) - 1) \right) \right) \right]^{-1}.
\end{aligned} \tag{55}$$

Choosing $a^{-\frac{3}{2}} \ll \Lambda \ll a^{-2}$ we can suppress the gauge field propagator so that the doublers contribution to the fermion self energy and all other diagrams vanishes in the limit $a \rightarrow 0$. To show it one again separates the integration domain into V_{in} and V_{out} and proves that the integrals over V_{in} coincide with the gauge invariant continuum limit and the ingetrals over V_{out} are zero in this limit. Let us consider as an example the self energy diagram for the original fermion field including the propagator $\bar{\psi}^+ \psi^+$. In the domain V_{in} it is given by the integral

$$\sum_{\text{in}} \sim \int_{V_{\text{in}}} \frac{\kappa a p^2 d^4 p}{[p^2 + \kappa^2 a^2 p^4][(p+k)^2 + \Lambda^2 a^4 (p+k)^4]} d^4 p. \tag{56}$$

This diagram is majorated by $N^{2\gamma-1}$ and as $\gamma < \frac{1}{2}$ vanishes in the limit $a \rightarrow 0$.

The integral over V_{out} looks as follows

$$\sum_{\text{out}} \sim \int_{V_{\text{out}}} \frac{m(p) \left[\cos \frac{1}{2} (p-q)_\mu a \right]^2}{[s^2(p) + m^2(p)] \left[a^{-2} \sum_\mu (\cos(p_\mu a) - 1) + \Lambda^2 \left(\sum_\mu (\cos(p_\mu a) - 1) \right)^2 \right]} \tag{57}$$

In the limit $a \rightarrow 0$, $\Sigma_{\text{out}} \leq \Lambda^{-2}a^{-3} \rightarrow 0$.

Note that if the higher covariant derivatives for gauge fields were absent ($\Lambda = 0$) this term would produce an infinite mass renormalization for the fields ψ .

All other diagrams including internal gauge lines may be treated in the same way demonstrating a manifestly gauge invariant continuum limit.

It completes the proof of the gauge invariance of our construction. The lattice action (16) together with the higher derivative regularized gauge field action (51) leads to the continuum theory which is free of doublers and gauge invariant.

4 Discussion

We demonstrated above that the problem of removing the fermion doublers in anomaly free chiral gauge models may be solved by introducing the lattice action defined by the eqs. (16),(51). This action breaks the gauge invariance for finite lattice spacing but the invariance is restored in the continuum limit. The only counterterms which are needed to take a continuum limit are the usual gauge invariant counterterms. From the point of view of calculations the procedure described above is quite simple in the case of even number of generations

In this case it is essentially a discretization of the usual Pauli-Villars method. So in many practical calculations one probably can neglect the third generation which contains heavy particles and use this simple procedure.

It goes without saying that the arguments presented in this paper used the weak coupling expansion. It would be of great interest to check their validity in nonperturbative calculations.

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