On symmetric Cauchy-Riemann manifolds

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1. Introduction

The Riemannian symmetric spaces play an important role in different branches of mathematics. By definition, a (connected) Riemannian manifold M is called symmetric if, to every $a \in M$, there exists an involutory isometric diffeomorphism $s_a\colon M \to M$ having a as isolated fixed point in M (or equivalently, if the differential $d_a s_a$ is the negative identity on the the tangent space $T_a = T_a M$ of M at a). In case such a transformation s_a exists for $a \in M$, it is uniquely determined and is the geodesic reflection of M about the point a. As a consequence, for every Riemannian symmetric space M, the group $G = G_M$ generated by all symmetries s_a, $a \in M$, is a Lie group acting transitively on M. In particular, M can be identified with the homogeneous space G/K for some compact subgroup $K \subset G$. Using the elaborate theory of Lie groups and Lie algebras E. Cartan classified all Riemannian symmetric spaces.

The complex analogues of the Riemannian symmetric spaces are the Hermitian symmetric spaces. By definition a Hermitian symmetric space is a Riemannian symmetric space M together with an almost complex structure on M such that the metric is Hermitian and such that every symmetry s_a is holomorphic (i.e. satisfies the Cauchy-Riemann partial differential equations with respect to the almost complex structure). Also all Hermitian symmetric spaces were completely classified by E. Cartan. In particular, every Hermitian symmetric space M can be written in a unique way as an orthogonal direct product $M = M_+ \times M_0 \times M_-$ of Hermitian symmetric spaces M_ε with holomorphic curvature of sign ε everywhere (possibly of dimension 0, i.e. a single point). M_+ is a compact simply connected complex manifold, M_- is a bounded domain in some $\mathbb{C}^n$ and M_0 can be realized as the flat space $\mathbb{C}^m/\Omega$ for some discrete subgroup $\Omega \subset \mathbb{C}^m$. In particular, the almost complex structure of M is integrable. Furthermore, there exists a remarkable duality between symmetric spaces which for instance gives a one-to-one correspondence between those of compact type (i.e. $M = M_+$) and those of non-compact type (i.e. $M = M_-$), see [14] for details.

A joint generalization of real smooth as well of complex manifolds are the Cauchy-Riemann manifolds (CR-manifolds for short) or, more generally, the CR-spaces, where the integrability condition is dropped and thus also arbitrary almost complex manifolds are incorporated. These objects are smooth manifolds M such that at every point $a \in M$ the Cauchy-Riemann equations only apply in the direction of a certain linear subspace $H_a \subset T_a$ of the tangent space to M, see f.i. [9] or section 2 for details. The tangent space T_a is an $\mathbb{R}$-linear space while H_a, also called the *holomorphic* tangent space at $a \in M$, is a $\mathbb{C}$-linear space. The two extremal cases $H_a = 0$ and $H_a = T_a$ for all $a \in M$ represent the two cases of smooth and of almost complex manifolds respectively.

The main objective of this paper is to generalize the notion of symmetry to the category of CR-spaces. It turns out that for symmetries in this more general context the requirement of isolated fixed points is no longer adequate. In fact, this would happen only for Levi-flat CR-spaces (see Proposition 3.8) and hence would not be interesting. Let us illustrate our concept on a simple example (compare also section 4). Consider $E := \mathbb{C}^n$, $n > 1$, with the standard inner product as a complex Hilbert space and denote by $S := \{z \in E : \|z\| = 1\}$ the euclidean unit sphere with Riemannian metric induced from E. Then M is symmetric when considered as Riemannian manifold. But S also has a canonical structure of a CR-manifold – define for every $a \in S$ the holomorphic tangent space H_a to be the maximal complex subspace of E contained in $T_a S \subset E$, i.e. the *complex* orthogonal complement of a in E. Then T_a is the orthogonal sum $H_a \oplus i\mathbb{R}a$. It can be seen that for every isometric CR-diffeomorphism φ of S, the differential $d_a\varphi$ is the identity on $i\mathbb{R}a$ as soon as it is the negative identity on H_a, i.e. there does not exist a CR-symmetry of S at a in the strict sense. On the other hand, the unitary reflection $s_a(z) := 2(a|z) - z$ defines an involutory isometric CR-diffeomorphism of S with differential at a being the negative identity on H_a. We call this the symmetry of the CR-manifold S at a and take it as a guideline for our general definition 3.5.

Among all symmetric CR-manifolds we distinguish a large subclass generalizing the above example. This class consists of the Shilov boundaries S of bounded symmetric domains $D \subset \mathbb{C}^n$ in their circular convex realizations (Theorem 8.5). A remarkable feature of these CR-submanifolds is the fact that various geometric and analytic constructions, hard to calculate in general, can be obtained here in very explicit forms. We illustrate this on the case of polynomial and rational convex hulls.

Recall that the polynomial (resp. rational) convex hull of a compact subset $K \subset \mathbb{C}^n$ is the set of all $z \in \mathbb{C}^n$ such that $|f(z)| \le \sup_K |f|$ for every polynomial f (resp. every rational function f holomorphic on K). If K is a connected real-analytic curve, $p(K)\backslash K$ is a complex analytic subset of $\mathbb{C}^n \backslash K$, due to J. Wermer [28], where $p(K)$ denotes the polynomial convex hull. Later, the analyticity of $p(K)\backslash K$ was proved by H. Alexander [1], [2] for compact sets of finite length and recently by T.C. Dinh [13] for rectifiable closed (1,1)-currents under very weak assumptions (see also E. Bishop [8], G. Stolzenberg [24] and M.G. Lawrence[18] for related results). On the other hand, if K is not a smooth submanifold, $p(K)\backslash K$ is not analytic in general (see e.g. G. Stolzenberg [23] or J. Wermer [29]).

In the present paper we calculate the polynomial and the rational convex hulls of S, where $S \subset \partial D$ is a Shilov boundary as above (see Corollary 8.17). Here it turns out that $p(S)\backslash S$ is not necessarily analytic but rather a *submanifold with "real-analytic corners"*, even though S itself is a connected real-analytic submanifold. Similar is the behaviour of the rational convex hull of S. Both hulls are canonically stratified into real-analytic CR-submanifolds such that the (unique) stratum of the highest dimension is complex for the polynomial and Levi-flat for the rational convex hull.

The paper is organized as follows. Preliminaries on CR-spaces are given in section 2. In section 3 we introduce symmetric CR-spaces and establish their main properties: *The uniqueness of symmetries and the transitivity of the spanned group.* Example 3.11 (a generalized Heisenberg group) shows that there exist symmetric CR-manifolds M of arbitrary CR-dimension and arbitrary CR-codimension having arbitrary Levi form at a given point.

In section 4 we study more intensively the unit sphere S in the complex space $\mathbb{C}^n$, some symmetric domains in S and their coverings. In particular, we obtain uncountable families of pairwise non-isomorphic symmetric CR-manifolds (see Example 4.5). In section 5 we associate to every symmetric CR-space M a canonical fibration and discuss the situation when the base is a symmetric CR-space itself. In fact, every symmetric CR-space can be obtained in this way. As mentioned above, the underlined CR-structure of a symmetric CR-space does not need to be integrable. In section 6 we give a construction principle for symmetric CR-spaces in terms of Lie groups and illustrate this with various examples. In section 7 we give Lie theoretic conditions for M to be embeddable into a complex manifold. We show by an example (see 7.4) that this in general is not possible – in contrast to the case of Hermitian symmetric spaces.

Finally, in section 8, we consider symmetric CR-manifolds arising from bounded symmetric domains $D \subset \mathbb{C}^n$. To be a little more specific, we assume without loss of generality that D is realized as bounded circular convex domain in $\mathbb{C}^n$. Then it is known that the Shilov boundary $S \subset \partial D$ of D (which coincides here with the set of all extreme points of the convex set $\overline{D}$) is an orbit of the group $\mathsf{Aut}(D)$ of all biholomorphic automorphisms of D. Furthermore, every maximal compact subgroup K of $\mathsf{Aut}(D)$ still acts transitively on S, and, with respect to a suitable K-invariant Hermitian metric, S is a symmetric CR-manifold. Our main result states: In case D does not have a factor of tube type, (i) every smooth CR-function on S extends to a continuous function on $\overline{D}$ holomorphic on D, and (ii) the group $\mathsf{Aut}_{\mathrm{CR}}(S)$ of all smooth CR-diffeomorphisms of S coincides with the group $\mathsf{Aut}(D)$.

Notation. For every vector space E over the base field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we denote by $\mathcal{L}(E)$ the space of all $\mathbb{K}$-linear endomorphisms of E and by $\mathsf{GL}(E) \subset \mathcal{L}(E)$ the subgroup of all invertible operators. More generally, for every total subset $S \subset E$ put $\mathsf{GL}(S){:=} \{g \in \mathsf{GL}(E) : g(S) = S\}$ and denote by $\mathsf{Aff}(S)$ the group of all affine transformations of E mapping S onto itself.

$\mathbb{K}^{n\times m}$ is the $\mathbb{K}$-Hilbert space of all $n \times m$-matrices over $\mathbb{K}$ (n = row-index) with the inner product $(u|v) = \mathsf{tr}(u^*v)$ and $u^* = \overline{u}' \in \mathbb{K}^{m\times n}$ the adjoint.

By a complex structure we always understand a linear operator J on a real vector space with $J^2 = -\,\mathsf{id}$. If misunderstanding is unlikely we simply write ix instead of $J(x)$.

For complex vector spaces U, V, W a sesqui-linear map $\Phi{:}\, U \times V \to W$ is always understood to be conjugate linear in the first and complex linear in the second variable.

With $\mathsf{U}(n)$, $\mathsf{O}(n)$, $\mathsf{Sp}(n)$ etc. we denote the unitary, orthogonal and symplectic groups (see [14] for related groups). In particular, we put $\mathbb{T} := \mathsf{U}(1) = \exp(i\mathbb{R})$ and $\mathbb{R}^+{:=} \exp(\mathbb{R})$. For every topological group G we denote by G^0 the connected identity component. A continuous action of G on a locally compact space M is called proper if the mapping $G \times M \to M \times M$ defined by $(g, a) \mapsto (g(a), a)$ is proper, i.e. pre-images of compact sets are compact.

For every set S and every map $\sigma{:}\, S \to S$ we denote by $\mathsf{Fix}(\sigma){:=} \{s \in S : \sigma(s) = s\}$ the set of all fixed points.

For Lie groups G, H etc., the corresponding Lie algebras are denoted by small Gothic letters $\mathfrak{g}$, $\mathfrak{h}$ etc. For linear subspaces $\mathfrak{m}, \mathfrak{n} \subset \mathfrak{g}$ we denote by $[\mathfrak{m}, \mathfrak{n}]$ always the linear span of all $[x, y]$ with $x \in \mathfrak{m}$, $y \in \mathfrak{n}$.

2. Preliminaries

Suppose that M is a connected smooth manifold of (finite real) dimension n. Denote by $T_a{:=} T_aM$, $a \in M$, the tangent space which is a real vector space of dimension n. An *almost Cauchy-Riemann structure* (almost CR-structure for short) on M assigns to every $a \in M$ a linear subspace $H_a = H_aM \subset T_a$ (called the *holomorphic tangent space* to M at a) together with a complex structure on H_a in such a way that H_a and the complex structure depend smoothly on a. Smooth dependence can be expressed in the following way: *Every point of M admits an open neighbourhood $U \subset M$ together with a linear endomorphism j_a of T_a for every $a \in U$ such that $-j_a^2$ is a projection onto H_a with $j_a v = iv$ for all $v \in H_a$, and j_a depending smoothly on $a \in U$.* Then, in particular, all H_a have the same dimension. A connected smooth manifold together with an almost CR-structure on it is called in the sequel an *almost CR-manifold*, or a *CR-space* for short. For more details on (almost) CR-manifolds see e.g. [5], [9], [12], [15], [25].

In the following M always denotes a CR-space. Denote by $\mathfrak{V} = \mathfrak{V}(M)$ the Lie algebra of all smooth vector fields on M and by $\mathfrak{H} = \mathfrak{H}(M)$ the subspace of all vector fields X with $X_a \in H_a$ for all $a \in M$. Then, for all $a \in M$, $\mathfrak{H}_a{:=} \{X_a : X \in \mathfrak{H}\} = H_a$ holds and $(JX)_a = i(X_a)$ canonically defines a complex structure J on $\mathfrak{H}$. Define inductively

$$\mathfrak{H}^k{:=} \mathfrak{H}^{k-1} + [\mathfrak{H}, \mathfrak{H}^{k-1}]\,, \quad \text{where} \quad \mathfrak{H}^1{:=} \mathfrak{H} \quad \text{and} \quad \mathfrak{H}^j{:=} 0 \;\text{ for }\; j \le 0\,. \tag{2.1}$$

Then $[\mathfrak{H}^r, \mathfrak{H}^s] \subset \mathfrak{H}^{r+s}$ holds for all integers r, s and $\mathfrak{H}^\infty{:=} \bigcup_k \mathfrak{H}^k$ is the Lie subalgebra of $\mathfrak{V}$ generated by $\mathfrak{H}$. Also, we call the quotient vector spaces

$$T_a^r{:=} T_a/H_a \quad \text{and} \quad T_a^{rr}{:=} T_a/\mathfrak{H}_a^\infty$$

the *real* and the *totally real* part of T_a. The complex dimension of H_a and the real dimension of T_a^r do not depend on $a \in M$ - they are called the *CR-dimension* and the *CR-codimension* of M. Finally, M is called *minimal* (in the sense of TUMANOV [26]) if $U = N$ holds for every domain $U \subset M$ and every closed smooth submanifold $N \subset U$ with $H_a M \subset T_a N$ for all $a \in N$. It is known that in case M is real-analytic (all CR-spaces we discuss later will have this property, see 3.7) minimality is equivalent to $T_a^{rr} = 0$ for all $a \in M$, i.e. to the finite type in the sense of BLOOM-GRAHAM [7]. M is called *totally real* if M has CR-dimension 0. The CR-spaces of CR-codimension 0, i.e. satisfying $H_a M = T_a M$, are also called almost complex manifolds.

The CR-spaces form a category in a natural way. By definition, a smooth map $\varphi: M \to N$ of CR-spaces is called a CR-*map* if, for every $a \in M$ and $b = \varphi(a) \in N$, the differential $d_a\varphi: T_a M \to T_b N$ maps $H_a M$ complex linearly into $H_b N$. For every M we denote by $\mathsf{Aut}(M)$ or $\mathsf{Aut}_{\mathrm{CR}}(M)$ the group of all CR-diffeomorphisms $\varphi: M \to M$ and endow this group with the compact open topology.

Suppose, N is a CR-space and $M \subset N$ is a submanifold. We call M a CR-*subspace* of N if the dimension of $H_a M := (T_a M \cap H_a N) \cap i(T_a M \cap H_a N)$ does not depend on $a \in M$. Then M is a CR-space with the induced CR-structure. A CR-space M is called *integrable* or a *CR-manifold* if the following integrability condition is satisfied:

$$(2.2) \qquad Z := [JX, Y] + [X, JY] \in \mathfrak{H} \quad \text{and} \quad JZ = [JX, JY] - [X, Y] \quad \text{for all} \quad X, Y \in \mathfrak{H}\,.$$

In the special case, where M is real-analytic (which includes that the holomorphic tangent space $H_a M$ depends in a real-analytic way on $a \in M$) it is known (compare f.i. [5] or [9]) that (2.2) is equivalent to the existence of local realizations of M as a CR-submanifold of some $\mathbb{C}^n$. In that case there even exist a complex manifold N and a (global) realization of M as a real-analytic CR-submanifold of N which is generic, i.e. $T_a M + iT_a M = T_a N$ for all $a \in M$ (see [3]).

An important invariant of a CR-space M is the so-called *Levi form* defined at every point $a \in M$ in the following way. Denote by $\pi_a: T_a \to T_a/H_a$ the canonical projection. Then it is easy to see that there exists a map

$$\omega_a: H_a \times H_a \;\to\; T_a/H_a \quad \text{with} \quad \omega_a(X_a, Y_a) = \pi_a([X, Y]_a) \quad \text{for all} \quad X, Y \in \mathfrak{H}$$

(in the proof of Proposition 3.3 a more general satement actually will be shown). Then ω_a is $\mathbb{R}$-bilinear and skew-symmetric. For all $\varepsilon, \mu = \pm 1$, there exist uniquely determined $\mathbb{R}$-bilinear maps

$$\omega_a^{\varepsilon\mu}: H_a \times H_a \to (T_a/H_a) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{with} \quad \omega_a^{\varepsilon\mu}(sx, ty) = s^\varepsilon t^\mu \omega_a^{\varepsilon\mu}(x, y)$$

for all $s, t \in \mathbb{T}$, $x, y \in H_a$ such that $\omega_a = \sum \omega_a^{\varepsilon\mu}$.

2.3 Definition. The sesqui-linear part $L_a := \omega_a^{-1,1}$ of ω_a is called the *Levi form* of M at a, i.e.

$$4\, L_a(x, y) = \big(\omega_a(x, y) + \omega_a(ix, iy)\big) \,+\, i\big(\omega_a(ix, y) - \omega_a(x, iy)\big)$$

and in particular $2L_a(x, x) = i\omega_a(ix, x) \in i(T_a/H_a)$ for all $x, y \in H_a$. The convex hull of $\{L_a(x, x) : x \in H_a\}$ is called the *Levi cone* at $a \in M$ and its interior always refers to the linear space $i(T_a/H_a)$. The CR-space M is called *Levi flat* if $L_a = 0$ holds for every $a \in M$.

Denote by * the conjugation of $(T_a/H_a) \otimes_{\mathbb{R}} \mathbb{C}$ given by $(\xi + i\eta)^* := (-\xi + i\eta)$ for all $\xi, \eta \in T_a/H_a$. The following statement is obvious.

2.4 Lemma. *The Levi form L_a is *-Hermitian, that is, $L_a(x, y) = L_a(y, x)^*$ for all $x, y \in H_a$. In the case, the integrability condition* (2.2) *holds, we have $\omega_a^{1,1} = \omega_a^{-1,-1} = 0$ and*

$$2\, L_a(x, y) \;=\; \omega_a(x, y) + i\omega_a(ix, y)\,. \qquad \square$$

Various authors call $-2iL_a$ the Levi form. In [9] $L = L_a$ is called the *extrinsic Levi form*. If M is a CR-subspace of a complex manifold U, $(T_a/H_a) \otimes_{\mathbb{R}} \mathbb{C}$ can be canonically identified with a complex subspace of T_aU/H_aM. Then, for every $x\in H_a$ the vector $L(x,x)\in iT_a/H_a$ is transversal to M and points into the 'pseudoconvex direction' of M. Bogges and Polking [10] proved that all CR-functions on M extend holomorphically to a wedge in U 'in the direction of the Levi form'. This was generalized by Tumanov [26] to the case M is minimal, whereas Baouendi and Rothschild [6] proved the necessity of the minimality condition.

Let us illustrate the Levi form at a simple example: Let $M\colon=\{(z,w)\in\mathbb{C}^2 : z\overline{z}+w\overline{w}=1\}$ be the euclidean sphere and put $a\colon=(1,0)$, $e\colon=(0,1)$. Then M is a CR-submanifold with $T_aM=i\mathbb{R}a\oplus\mathbb{C}e$ and $H_aM=\mathbb{C}e$. We identify $(T_aM/H_aM)\otimes_{\mathbb{R}}\mathbb{C}$ in the obvious way with the complex line $\mathbb{C}a\subset\mathbb{C}^2$. Consider the vector field $X\in\mathfrak{H}$ defined by $X_{(z,w)}=(-\overline{w},\overline{z})$ for all $(z,w)\in M$. Then $[JX,X]_a=2ia=\omega_a(ie,e)$ and hence $L_a(e,e)=-a$. This vector points from $a\in M$ into the interior of the sphere M.

For the rest of the section assume that on M there is given a *Riemannian metric*, i.e. every tangent space T_a is a real Hilbert space with respect to an inner product $\langle u|v\rangle_a$ depending smoothly on $a\in M$. Then we call M an *Hermitian CR-space* if the metric is compatible with the CR-structure in the sense that $\|iv\|=\|v\|$ holds for all $a\in M$, $v\in H_a$, where $\|v\|=\sqrt{\langle v,v\rangle}$. In particular, every H_a is a complex Hilbert space. If the CR-structure of a Hermitian CR-space M is integrable, we call M an *Hermitian CR-manifold*. In the case of vanishing CR-codimension, i.e. $H_aM=T_aM$, Hermitian CR-spaces are usually called *Hermitian manifolds*.

For every $a\in M$ and every integer $k\geq 0$, let $\mathfrak{H}^k$ be as in (2.1) and let $H_a^k\subset T_a$ be the orthogonal complement of $\mathfrak{H}_a^{k-1}$ in $\mathfrak{H}_a^k$. Then $H_a^0=0$ and $H_a^1=H_a$ is the holomorphic tangent space in a. If we denote by $H_a^{-1}\cong T_a^{rr}$ the orthogonal complement of $\mathfrak{H}_a^\infty$ in T_a we obtain the following orthogonal decomposition

$$T_a=\bigoplus_{k\geq -1}H_a^k\,. \tag{2.5}$$

Denote by $\pi_a^k\in\mathcal{L}(T_a)$ the orthogonal projection onto H_a^k. Then $a\mapsto\pi_a^k$ defines a (not necessarily continuous) tensor field π^k of type $(1,1)$ on M. For later use we define

$$T_a^+\colon=\bigoplus_{k\ \text{even}}H_a^k\quad\text{and}\quad T_a^-\colon=\bigoplus_{k\ \text{odd}}H_a^k\,. \tag{2.6}$$

The Hermitian CR-spaces form a category together with the contractive CR-mappings as morphisms (i.e. $\|d_a\varphi(v)\|\leq\|v\|$ for all $a\in M$, $v\in T_aM$). We always denote by $I_M\subset\mathsf{Aut}_{\mathrm{CR}}(M)$ the closed subgroup of all isometric CR-diffeomorphisms. Then it is known that I_M is a Lie group acting smoothly and properly on M. In particular, I_M has dimension $\leq n(n{+}2)+m(m{+}1)/2$, where M has CR-dimension n and CR-codimension m. The full CR-automorphism group $\mathsf{Aut}_{\mathrm{CR}}(M)$ can be infinite-dimensional. In case of vanishing CR-dimension we have the full sub-category of (connected) Riemannian manifolds and in case of vanishing CR-codimension we have the full sub-category of Hermitian manifolds. In both sub-categories there exists the classical notion of a symmetric space. In the following we want to extend this concept to arbitrary Hermitian CR-spaces.

3. Symmetric CR-spaces.

3.1 Definition. Let M be an Hermitian CR-space and let $\sigma\colon M\to M$ be an isometric CR-diffeomorphism. Then σ is called a *symmetry* at the point $a\in M$ (and a is called a *symmetry point* of M) if a is a (not necessarily isolated) fixed point of σ and if the differential of σ at a coincides with the negative identity on the subspace $H_a^{-1}\oplus H_a^1$ of T_a.

3.2 Proposition. *At every point of M there exists at most one symmetry. Furthermore, every symmetry is involutive.*

The statement is an easy consequence of the following.

3.3 Uniqueness Theorem. *Let φ, ψ be isometric CR-diffeomorphisms of the Hermitian CR-space M with $\varphi(a) = \psi(a)$ for some $a \in M$. Then $\varphi = \psi$ holds if the differentials $d_a\varphi$ and $d_a\psi$ coincide on the subspace $H_a^{-1} \oplus H_a^1$ of $T_a = T_aM$.*

Proof. Without loss of generality we may assume that ψ is the identity transformation of M. Then a is a fixed point of φ and by a well known fact (compare f.i. [14] p. 62) we only have to show that the differential $\lambda := d_a\varphi$ is the identity on T_a. Now $\lambda(X_a) = (\tau X)_a$ holds for every vector field $X \in \mathfrak{V}$, where τ is the Lie automorphism of $\mathfrak{V}$ induced by φ. For all $r, s > 0$ and $k := r + s$, every smooth function f on M and all $X \in \mathfrak{H}^r$, $Y \in \mathfrak{H}^s$ the formula

$$\pi_a^k\big([fX, Y]_a\big) \;=\; f(a) \cdot \pi_a^k\big([X, Y]_a\big)$$

is easily derived, where the orthogonal projection π_a^k is defined as above. On the other hand, every $X \in \mathfrak{H}^r$ with $X_a = 0$ can be written as finite sum $X = X_0 + f_1X_1 + \cdots + f_mX_m$ with $X_0 \in \mathfrak{H}^{r-1}$, $X_1, \ldots, X_m \in \mathfrak{H}^r$ and smooth functions $f_1, \ldots, f_m$ on M vanishing in a. Therefore, $\pi_a^k\big([X, Y]_a\big)$ only depends on the vectors $\pi_a^r(X_a)$ and $\pi_a^s(Y_a)$ for $X \in \mathfrak{H}^r$, $Y \in \mathfrak{H}^s$.
Since λ is an isometry of T_a and τ leaves invariant all subspaces $\mathfrak{H}^k \subset \mathfrak{V}$ also all H_a^k must be left invariant by λ. We show by induction on k that actually λ is the identity on every H_a^k. For $k \le 1$ this follows from the assumptions. For $k > 1$, fix $X \in \mathfrak{H}$, $Y \in \mathfrak{H}^{k-1}$ and consider the vector $v := \pi_a^k\big([X, Y]_a\big) \in H_a^k$. By induction hypothesis we then have $(\tau X)_a = X_a$, $(\tau Y)_a = Y_a$ and hence

$$\lambda(v) = \pi_a^k(\lambda(v)) = \pi_a^k\big((\tau[X, Y])_a\big) = \pi_a^k\big([\tau X, \tau Y]_a\big) = \pi_a^k\big([X, Y]_a\big) = v\,. \qquad \square$$

3.4 Remark. The proof of Proposition 3.3 shows that for every symmetry σ of M at a the differential $\lambda = d_a\sigma$ satisfies $\lambda(v) = (-1)^k v$ for every $v \in H_a^k$ and every $k \ge -1$, i.e. $\lambda = \Sigma_k(-1)^k\pi_a^k$, or equivalently, $\mathsf{Fix}(\pm\lambda) = T_a^{\pm}$. $\square$

3.5 Definition. A connected Hermitian CR-space M is called *symmetric* (or an SCR-space for short) if every $a \in M$ is a symmetry point. The corresponding symmetry at a is denoted by s_a. $\square$

In the sequel we adopt the following notation: For a given SCR-space M we denote as in section 2 by $I = I_M$ the Lie group of all isometric CR-diffeomorphisms of M. Let $G = G_M$ be the closed subgroup of I generated by all symmetries s_a, $a \in M$. Fix a base point $o \in M$ and denote by $K := \{g \in G : g(o) = o\}$ the isotropy subgroup at o.

3.6 Proposition. *G is a Lie group acting transitively and properly on M. The connected identity component G^0 of G has index ≤ 2 in G and coincides with the closed subgroup of I_M generated by all transformations $s_a \circ s_b$ with $a, b \in M$. The isotropy subgroup K is compact and M can be canonically identified with the homogeneous manifold G/K via $g(o) \mapsto gK$. M is compact if and only if G is a compact Lie group.*

Proof. There exists an open subset $U \ne \emptyset$ of M such that for every $k \ge -1$ the dimension of H_a^k does not depend on $a \in U$. Therefore, every tensor field π^k is smooth over U, i.e. the orthogonal decomposition (2.5) depends smoothly on a as long as a stays in U. We may assume without loss of generality that for a fixed $\varepsilon > 0$ and every $a \in U$, the exponential mapping Exp_a is defined on the open ball B_a of radius ε about the origin in T_a and that Exp_a is a diffeomorphism from B_a onto a neighbourhood $N_a \subset M$ of a. Every isometric diffeomorphism $\varphi \in I_M$ is linear in local normal coordinates, more precisely, for every $a \in U$ with $c := \varphi(a) \in U$, the diagram

$$\begin{array}{ccc} B_a & \xrightarrow{\;d_a\varphi\;} & B_c \\ {\scriptstyle \mathrm{Exp}_a}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Exp}_c} \\ N_a & \xrightarrow{\;\varphi\;} & N_c \end{array}$$

commutes. Since $d_a s_a = \sum_k(-1)^k\pi_a^k$ depends smoothly on $a \in U$ and since G consists of isometries this implies the smoothness of the mapping $U \times U \to M$ defined by $(a, b) \mapsto s_a(b)$.

Now fix $u \in U$ and denote by $A := G(u)$ the orbit of u under the group G. Then A is a closed smooth submanifold of M since G acts properly on M. Fix $a \in A$ and $v \in T_a^- M$ arbitrarily. Choose a smooth curve $\gamma: [0,1] \to U$ with $\gamma(0) = a$ and $\gamma'(0) = v$. Then $\alpha(t) = s_{\gamma(t)}(a)$ defines a smooth curve in A with $\alpha(0) = a$ and $\alpha'(0) = 2v$. This proves $T_a^- M \subset T_a A$ and hence $T_c^{-1} M \subset T_c A$ for all $c \in A$ since $a \in A$ was arbitrarily chosen. Now fix a vector $w \in T_a^+ M$. Then there exist smooth vector fields $X^1, \ldots, X^{2k}$ on M such that $X_c^j \in T_c^- M$ for all $1 \le j \le 2k$, $c \in A$ and such that $w = \sum_{j=1}^k [X^j, X^{k+j}]_a$. But we know already that all X^j's are tangent to the submanifold $A \subset M$, i.e. all their brackets are tangent to A and therefore $w \in T_a A$. This implies $T_a A = T_a M$ and $A = M$ since A is closed in M. Therefore G acts transitively on M. □

The proof of Proposition 3.6 shows that an Hermitian CR-space M is already symmetric as soon as the set of symmetry points of M has an interior point in M. The transitivity of the G-action has several consequences.

3.7 Corollary. *Every SCR-space M has a unique structure of a real-analytic CR-manifold in such a way that every isometric diffeomorphism of M is real-analytic. All tensors π^k are real-analytic on M and also the mapping $a \mapsto s_a$ from M to G is real-analytic. In particular, the dimension of $H_a^k \subset T_a$ does not depend on $a \in M$ for every k.* □

In the following we will always consider SCR-spaces as real-analytic manifolds according to 3.7. The number $\kappa = \kappa(M) := \max\{k \ge -1 : H_a^k \ne 0\}$ does not depend on $a \in M$. Example 7.5 will show that arbitrary values of $\kappa \ne 0$ occur. $\mathbb{C}^n \times \mathbb{R}^m$ is a Levi flat CR-submanifold of $\mathbb{C}^{n+m}$ and as another corollary of 3.6 we have:

3.8 Proposition. *Let M be a symmetric CR-space with CR-dimension n and CR-codimension m. Then for every $a \in M$ the following conditions are equivalent.*

(i) *a is an isolated fixed point of the the symmetry s_a.*
(ii) *M is Levi flat.*
(iii) *M is locally CR-isomorphic to an open subset of $\mathbb{C}^n \times \mathbb{R}^m$. In particular, M is a CR-manifold.*

Proof. (i) $\Longrightarrow$ (ii). The differential $d_a s_a$ is the identity on the subspace $H_a^2 \subset T_a$ due to Remark 3.4. Therefore, if a is isolated in $\mathsf{Fix}(s_a)$, we have $H_a^2 = 0$ and hence $L_a = 0$. By homogeneity then the Levi form vanishes at every point of M, i.e. M is Levi flat.
(ii) $\Longrightarrow$ (iii). Suppose that M is Levi flat. Then the holomorphic tangent spaces form an involutive distribution on M and define a foliation of M. Let N be the leaf through a, i.e. the maximal connected immersed smooth submanifold N of M with $T_c N = H_c M$ for all $c \in N$. Then N is an Hermitian almost complex manifold in the leaf topology invariant under every symmetry s_c, $c \in N$. Therefore, N is an Hermitian symmetric space and in particular a complex manifold, see [14]. But M locally is CR-isomorphic to a direct product $U \times V$, where $U \subset N$ and $V \subset \mathbb{R}^m$ are open subsets.
(iii) $\Longrightarrow$ (i). Condition (iii) implies $H_a^{-1} \oplus H_a^1 = T_a$ and hence $d_a s_a = -\mathsf{id}$, i.e. a is an isolated fixed point of s_a. □

Every SCR-space M may be considered as a reflection space in the sense of [19], i.e. if a 'multiplication' on M is defined by $x \cdot y := s_x y$ for all $x, y \in M$, the following rules hold: $x \cdot x = x$, $x \cdot (x \cdot y) = y$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ for all $x, y, z \in M$. The SCR-spaces form in various ways a category. We prefer here the following notion (ignoring the Riemannian metrics on M and N):

3.9 Definition. A CR-map $\varphi: M \to N$ is called an SCR-map, if $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ for all $x, y \in M$.

If φ in addition is contractive we also call it a *metric SCR-map*. In this sense it is clear what (metric, isometric) SCR-isomorphisms, SCR-automorphisms are. For instance, G_M consists of isometric SCR-automorphisms of M. Also, the universal covering $\pi: \widetilde{M} \to M$ of an SCR-space M has a unique structure of an SCR-space such that locally π is an isometric CR-diffeomorphism.

The following statement can be used to construct SCR-spaces.

3.10 Lemma. *Let M be a connected Hermitian CR-space with a base point o and let $H \subset I_M$ be a subgroup acting transitively on M. Suppose that $\sigma: M \to M$ is a diffeomorphism with $\sigma(o) = o$ and $\sigma^2 = \mathsf{id}$ such that the following conditions are satisfied:*
(i) $\sigma \circ H = H \circ \sigma$,
(ii) *the differential $d_o\sigma \in \mathcal{L}(T_oM)$ is a linear isometry with $(H_oM \oplus T_o^{rr}M) \subset \mathsf{Fix}(-d_o\sigma)$.*
Then σ is a symmetry of M.

Proof. Fix $a \in M$ and choose $g, h \in H$ with $g(a) = o$ and $\sigma = h \circ \sigma \circ g$. The claim follows from the identity $d_a\sigma = d_oh \circ d_o\sigma \circ d_ag$. □

3.11 Example. Let E, F be complex Hilbert spaces of finite dimension. Suppose that $w \mapsto w^*$ is a conjugation of F (i.e. a conjugate linear, involutive isometry of E) and let $\Phi: E \times E \to F$ be an Hermitian mapping with respect to the conjugation * (i.e. Φ is sesqui-linear and $\Phi(v, u) = \Phi(u, v)^*$ for all $u, v \in E$). Set

$$V := \{w \in F : w + w^* = 0\} \quad \text{and} \quad M := \{(z, w) \in E \oplus F : w + w^* = \Phi(z, z)\}.$$

Then M is a CR-submanifold of $E \oplus F$ with

$$\begin{aligned} T_aM &= \{(z, w) \in E \oplus F : w + w^* = \Phi(e, z) + \Phi(z, e)\} \\ H_aM &= \{(z, w) \in E \oplus F : w = \Phi(e, z)\} \end{aligned}$$

for every $a = (e, c) \in M$. The group

$$\Lambda := \big\{(z, w) \mapsto (z + e, w + \Phi(e, z) + \Phi(e, e)/2 + v) : e \in E, v \in V\big\}$$

acts transitively and freely on M by affine CR-diffeomorphisms. Therefore M has a group structure, a generalization of the Heisenberg group. For $o := (0, 0) \in M$, there exists a unique Λ-invariant Riemannian metric on M such that $T_oM = E \oplus V$ is the orthogonal sum of E and V. By Lemma 3.10, M is a symmetric CR-manifold – for every $a = (e, c) \in M$, the corresponding symmetry s_a is given by $(z, w) \mapsto \big(2e - z, w + \Phi(2e, e - z)\big)$ and $G_M = \Lambda \cup s_o\Lambda$, $G^0_M = \Lambda$. The full group $\mathsf{Aut}_{\mathrm{CR}}(M)$ does not act properly on M since it contains all transformations of the form $(z, w) \mapsto (tz, t\bar{t}w)$, $t \in \mathbb{C}^*$.
For every $\xi \in E$, the vector field X on $E \oplus F$ defined by $X_{(z,w)} = (\xi, \Phi(z, \xi))$ satisfies $X_a \in H_aM$ for all $a \in M$. From this it is easily derived that the Levi form L_o at $o \in M$ as defined in (2.3) coincides with the Hermitian map $\Phi: E \times E \to F$ after the identification $H_oM = E$ and $(T_oM/H_oM) \otimes_{\mathbb{R}} \mathbb{C} \cong F$. □

The next statement will be used later.

3.12 Lemma. *Let $E, V \subset F, M, \Phi, \Lambda$ be as in Example 3.11. Assume that Φ is non-degenerate in the following sense: For every $e \in E$ with $e \neq 0$ there exists $c \in E$ with $\Phi(e, c) \neq 0$. Then*

$$\mathsf{Aff}(M) = \Lambda \rtimes \mathsf{GL}(M) \ \text{ and}$$

$$\mathsf{GL}(M) = \{(\eta \times \varepsilon) \in \mathsf{GL}(E) \times \mathsf{GL}(V) : \Phi(\eta z, \eta z) = \varepsilon\Phi(z, z) \quad \text{for all} \quad z \in E\}.$$

Furthermore, the group I_M of all isometric CR-diffeomorphisms of M is given by

$$I_M = \Lambda \rtimes \Gamma \subset \mathsf{Aff}(M), \quad \text{where} \quad \Gamma := \{(\eta \times \varepsilon) \in \mathsf{GL}(M) : \eta \text{ unitary}, \varepsilon \text{ orthogonal}\}.$$

Proof. Let us start with an arbitrary real-analytic CR-diffeomorphism φ of M satisfying $\varphi(o) = o$. Then φ extends to a holomorphichic map $\varphi: U \to E \oplus F$ for a suitable open connected neighbourhood U of M in $E \oplus F$ (see e.g. [5], §1.7). The differential $g := d_o\varphi \in \mathsf{GL}(E \oplus F)$ leaves $H_oM = E$ invariant and hence can be written as operator matrix $g = \left(\begin{smallmatrix} \eta & \alpha \\ 0 & \varepsilon \end{smallmatrix}\right) \in \mathsf{GL}(E \oplus F)$ with a linear operator $\alpha: F \to E$. Since also $T_oM = E \oplus V$ is invariant under g, the operator $\varepsilon \in \mathsf{GL}(F)$ must leave invariant the subspace $V \subset F$, i.e. $\varepsilon \in \mathsf{GL}(V) \subset \mathsf{GL}(F)$ and, in particular,

$\varepsilon(w^*) = (\varepsilon w)^*$ for all $w \in F$. There exist holomorphic functions $h\colon U \to E$, $f\colon U \to F$ vanishing of order ≥ 2 at o such that

$$\varphi(z,w) \;=\; \big(\eta z + \alpha w + h(z,w), \varepsilon w + f(z,w)\big)$$

for all $(z,w) \in M$. For every $z \in E$, $v \in V$ and

$$w = w(z,v) \colon= v + \Phi(z,z)/2$$

we have $(z,w) \in M$ and hence

$$(*) \qquad \varepsilon\,\Phi(z,z) + f(z,w) + f(z,w)^* = \Phi(\eta z + \alpha w + h(z,w), \eta z + \alpha w + h(z,w))$$

for all $z \in E$ and $v \in V$. Comparing terms in $(*)$ we derive $\varepsilon\,\Phi(z,z) = \Phi(\eta z, \eta z)$ for all $z \in E$. Now suppose that φ is affine, i.e. $f = 0$ and $h = 0$. Comparing terms in $(*)$ again we get $\Phi(\alpha v, \eta z) = 0$ for all $z \in E$ and $v \in V$. But then the non-degeneracy of Φ implies $\alpha v = 0$ for all $v \in V$, i.e. $\alpha = 0$. This proves that the groups $\mathsf{Aff}(M)$ and $\mathsf{GL}(M)$ have the claimed forms.
Now suppose that $\varphi \in I_M$ is an isometry. Since there is a unique real-analytic structure on M such that the Lie group I_M acts as a real-analytic transformation group. Since the same holds for the Lie subgroup Λ, these two structures must coincide, i.e. φ is real-analytic. Furthermore, $g = d_o\varphi$ is a linear isometry of $E \oplus V$, i.e. $g = \left(\begin{smallmatrix}\eta & 0\\ 0 & \varepsilon\end{smallmatrix}\right)$ with η unitary and ε orthogonal. Together with $\varepsilon\,\Phi(z,z) = \Phi(\eta z, \eta z)$ for all $z \in E$ this implies $g \in I_M$ and hence $\varphi = g \in \mathsf{GL}(M)$ as a consequence of the Uniqueness Theorem 3.3. □

For all CR-manifolds M in Example 3.11 with $\Phi \neq 0$ the group G^0 is nilpotent of nilpotency class $2 = \kappa(M)$. For examples with nilpotent groups of higher class compare section 7. An explicit example of class 3 with lowest possible dimension is the following.

3.13 Example. Set

$$M \colon= \big\{(z,w,v) \in \mathbb{C}^3 : \mathrm{Im}(w) = z\overline{z},\ \mathrm{Im}(v) = \mathrm{Im}(w\overline{z})\big\}\,.$$

Then M is a CR-submanifold of $\mathbb{C}^3$ with CR-dimension 1 and CR-codimension 2 and every $(a,b,c) \in M$ induces an affine CR-automorphism of M by

$$(*) \qquad \big(z,w,v\big) \;\longmapsto\; \big(z + a,\ w + 2i\overline{a}z + b,\ v + (2i\overline{a}^2 - \overline{b})z + (a + 2\overline{a})w + c\big)\,.$$

Indeed, if we denote the right hand side of $(*)$ by $(\mathbf{z},\mathbf{w},\mathbf{v})$ then

$$\begin{aligned}\mathrm{Im}(\mathbf{w}\overline{\mathbf{z}}) \;&=\; \mathrm{Im}(w\overline{z} + 2iz\overline{z}\overline{a} + \overline{z}b + 2i\overline{a}^2 z + \overline{a}w + \overline{a}b)\\ &=\; \mathrm{Im}(v + (w - \overline{w})\overline{a} - \overline{b}z + 2i\overline{a}^2 z + \overline{a}w + c) \;=\; \mathrm{Im}(\mathbf{v})\,.\end{aligned}$$

An elementary calculation shows that the transformations $(*)$ form a nilpotent Lie group G^0 acting freely and transitively on M. In particular, M has the structure of a group with the product $(a,b,c) \odot (z,w,v) \colon= (\mathbf{z},\mathbf{w},\mathbf{v})$ and the unit $o \colon= (0,0,0)$. There is a unique G^0-invariant Riemannian metric on M whose restriction to the tangent space $T_oM = \mathbb{C} \oplus \mathbb{R}^2 \subset \mathbb{C}^3$ is the one inherited from $\mathbb{C}^3$. The transformation $s_o : (z,w,v) \mapsto (-z,w,-v)$ is a symmetry at o by Lemma 3.10. Hence M is a symmetric CR-manifold. It can be verified that $I_M = G_M = G^0 \cup s_o G^0$ is the group of all CR-isometries. The action of $\mathsf{Aut}_{\mathrm{CR}}(M)$ is not proper since this group contains all transformations of the form $(z,w,v) \mapsto (tz, t^2 w, t^3 v)$, $t \in \mathbb{R}^*$.
The group Z of all translations $(z,w,v) \mapsto (z,w,v+c)$, $c \in \mathbb{R}$, is in the center of G^0 and M/Z is CR-isomorphic to the classical Heisenberg group $\{(z,w) \in \mathbb{C}^2 : \mathrm{Im}(w) = z\overline{z}\}$ that already occurred in Example 3.11 in a slightly different form. □

4. Some examples derived from the sphere

We start with the inclusion of the complex manifolds

$$\mathbb{B}_n \subset \mathbb{C}^n \subset \mathbb{P}_n\,, \tag{4.1}$$

where $\mathbb{B}_n := \mathbb{B} := \{z \in \mathbb{C}^n : (z|z) < 1\}$ is the euclidean ball and $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$ is the complex projective space with homogeneous coordinates $[z_0, z_1, \ldots, z_n]$. We identify every $z \in \mathbb{C}^n$ with the point $[1, z] \in \mathbb{P}_n$. It is known that the group $\mathsf{Aut}(\mathbb{P}_n)$ of all biholomorphic automorphisms of $\mathbb{P}_n$ coincides with the group of all projective linear transformations $\mathsf{PSL}(n{+}1, \mathbb{C})$. Furthermore,

$$\mathsf{Aut}(\mathbb{B}) \;=\; \{g \in \mathsf{Aut}(\mathbb{P}_n) : g(\mathbb{B}) = \mathbb{B}\} \;=\; \mathsf{PSU}(1, n)\,.$$

The group $\mathsf{Aut}(\mathbb{B}) \subset \mathsf{Aut}(\mathbb{P}_n)$ has three orbits in $\mathbb{P}_n$ – the ball $\mathbb{B}$, the unit sphere $S := \partial\mathbb{B}$ and the outer domain $\mathbb{D} := \mathbb{P}_n \backslash \overline{\mathbb{B}}$. Actually, it is known that again

$$\mathsf{Aut}(\mathbb{D}) \;=\; \{g \in \mathsf{Aut}(\mathbb{P}_n) : g(\mathbb{D}) = \mathbb{D}\} \;=\; \mathsf{Aut}(\mathbb{B})$$

holds. The spaces $\mathbb{B}_n$, $\mathbb{C}^n$ and $\mathbb{P}_n$ are symmetric Hermitian manifolds with constant holomorphic sectional curvature < 0, $= 0$ and > 0 respectively. Also, $\mathbb{B}_n$ and $\mathbb{P}_n$ are dual to each other in the sense of symmetric Hermitian manifolds.

4.2 Example. The unit sphere $S = \partial\mathbb{B} = \{z \in \mathbb{C}^n : (z|z) = 1\}$ is a CR-submanifold of $\mathbb{C}^n$, whose holomorphic tangent space at $a \in S$ is $H_a = \{v \in \mathbb{C}^n : (a|v) = 0\}$. To avoid the totally real case $n = 1$ let us assume for the rest of the section that always $n > 1$ holds. Then S is a minimal CR-manifold. It is known (compare f.i. [27]) that for every pair U, V of domains in S and every CR-diffeomorphism $\varphi{:}\, U \to V$ there exists a biholomorphic transformation $g \in \mathsf{Aut}(\mathbb{P}_n)$ with $\varphi = g|U$. In particular this implies

$$\mathsf{Aut}_{\mathrm{CR}}(S) \;=\; \{g \in \mathsf{Aut}(\mathbb{P}_n) : g(S) = S\} \;=\; \mathsf{Aut}(\mathbb{B})$$

and the maximal compact subgroups of $\mathsf{Aut}_{\mathrm{CR}}(S)$ are in one-to-one correspondence to the points of $\mathbb{B}$.
By restricting the flat Hermitian metric of $\mathbb{C}^n$, the sphere S becomes an Hermitian CR-manifold. The unitary group $\mathsf{U}(n)$ coincides with $\{g \in \mathsf{Aut}_{\mathrm{CR}}(S) : g(0) = 0\}$, acts transitively on S by isometric CR-diffeomorphisms and it is easy to see that actually $I_M = \mathsf{U}(n)$ holds. Moreover, $z \mapsto 2(a|z)a - z$ defines a symmetry at $a \in S$ and $G_M = \{g \in \mathsf{U}(n) : \det(g) = (\pm 1)^{n-1}\}$. The group $G^0 = \mathsf{SU}(n)$ is simple whereas $I_M = \mathsf{U}(n)$ has center $Z \cong \mathbb{T}$. For every closed subgroup $A \subset Z$ also S/A is an SCR-space with CR-dimension $n - 1$ in a natural way - for instance for $A = \{\pm\,\mathsf{id}\}$ we get the real projective space $\mathbb{P}_{2n-1}(\mathbb{R})$ and for $A = Z$ the complex projective space $\mathbb{P}_{n-1}(\mathbb{C})$. □

The space of (projective) hyperplanes $L \subset \mathbb{P}_n$ is again a complex projective space of dimension n on which the group $\mathsf{Aut}_{\mathrm{CR}}(S) \subset \mathsf{Aut}(\mathbb{P}_n)$ acts with three orbits. These consist of all L meeting S in no, in precisely one and in more than one point respectively. Assume $L \cap S \neq \emptyset$ in the following and consider the domain $W := S\backslash L$ in S. Let $o \in W$ be the point with the maximal distance from the hyperplane $L \cap \mathbb{C}^n$. We will see that W has the structure of a symmetric CR-manifold. We claim that there exists a CR-isomorphic model $Q \subset \mathbb{C}^n$ of W in such a way that Q is closed in $\mathbb{C}^n$ and such that every CR-diffeomorphism of Q is the restriction of a complex affine transformation of $\mathbb{C}^n$, that is $\mathsf{Aut}_{\mathrm{CR}}(Q) = \mathsf{Aff}(Q)$. Indeed, choose a transformation $g \in \mathsf{Aut}(\mathbb{P}_n)$ with $g(L) \cap \mathbb{C}^n = \emptyset$ and put $Q := g(W)$. We call Q an *affine model* of W. Let us consider the two cases $L \cap \mathbb{B} = \emptyset$ and $L \cap \mathbb{B} \neq \emptyset$ separately.

4.3 Example. Let $U := S\backslash L$ for a hyperplane L with $L \cap \mathbb{B} \neq \emptyset$, say $U = \{z \in S : z_1 \neq 0\}$ and $o = (1, 0, \ldots, 0)$. The group $\mathsf{Aut}_{\mathrm{CR}}(U)$ acts transitively on U and has compact isotropy subgroup $\mathsf{U}(n{-}1)$ at o. Therefore, U is a symmetric CR-manifold with

$$I_U = \mathsf{Aut}_{\mathrm{CR}}(U) \;\cong\; \mathsf{U}(1, n{-}1)$$

and the symmetry $\rho = s_o$ at o is given by $\rho(t,v) = (t,-v)$ for all $(t,v) \in \mathbb{C}^{1+(n-1)}$. An affine model is

$$R{:}= \{z \in \mathbb{C}^n : (\rho z|z) = 1\} \;=\; \{(t,v) \in \mathbb{C}^{1+(n-1)} : t\overline{t} - (v|v) = 1\} \tag{4.4}$$

with $(t,v) \mapsto (1/t, v/t)$ a CR-diffeomorphism $U \to R$. The universal covering $\widetilde{R}$ of R is again a symmetric CR-manifold and can be realized via $(s,v) \mapsto (\exp(s), v)$ as (see also Example 3.12)

$$\widetilde{R} = \{(s,v) \in \mathbb{C}^{1+(n-1)} : \exp(s+\overline{s}) - (v|v) = 1\}\,. \qquad \square$$

4.5 Example. Let $V{:}= S\backslash\{a\}$ for some point $a \in S$, say $a{:}= (0,\dots,0,1)$ and hence $o = -a$. Then V is a cell in S and the Cayley transform $(v,t) \mapsto \big(\sqrt{2}\,(v/(1-t), (1+t)/(1-t)\big)$ for all $(v,t) \in \mathbb{C}^{(n-1)+1}$, $t \neq 1$, defines a CR-diffeomorphism of V onto the affine model

$$N{:}= \big\{(v,t) \in \mathbb{C}^{(n-1)+1} : t + \overline{t} = (v|v)\big\}\,.$$

This SCR-space occurs already in Example 3.11 and as a consequence of 3.12 we have

$$I_N \;=\; G_N^0 \rtimes \mathsf{U}(n{-}1) \quad\text{and}\quad \mathsf{Aut}_{\mathrm{CR}}(N) \;=\; \mathsf{Aff}(N) \;=\; G_N^0 \rtimes \big(\mathbb{R}^+ \times \mathsf{U}(n{-}1)\big)\,,$$

where the groups $\mathsf{U}(n–1)$ and $\mathbb{R}^+$ act on N by $(v,t) \mapsto (\varepsilon v, t)$ and $(v,t) \mapsto (sv, s^2 t)$ respectively. Consider for every $s \in \mathbb{R}^+$, the central subgroup $\Gamma_s{:}= \{(v,t) \mapsto (v, t+ins) : n \in \mathbb{Z}\}$ of G_N. Then the quotient manifold $N_s{:}= N/\Gamma_s$ is an SCR-space diffeomorphic to $\mathbb{C}^n \times \mathbb{T}$. But since the groups Γ_s and $\Gamma_{\tilde{s}}$ are not conjugate in $\mathsf{Aut}_{\mathrm{CR}}(N)$ for $s \neq \tilde{s}$ the manifolds N_s and $N_{\tilde{s}}$ are not isomorphic as CR-manifolds. We obtain a continuous family of symmetric CR-manifolds that are pairwise non-isomorphic even in the category of CR-manifolds. $\square$

In analogy to (4.1) we have inclusions

$$U \subset V \subset S\,. \tag{4.6}$$

But in contrast to (4.1) U is not ‘a bounded domain’ (meaning relatively compact) in the cell V. In all three cases $M = U, V, S$, the center Z of I_M is either $\mathbb{T}$ or $\mathbb{R}$ and the quotient CR-manifold M/Z is $\mathbb{B}_{n-1}$, $\mathbb{C}^{n-1}$ and $\mathbb{P}_{n-1}$ respectively. We call the symmetric CR-manifold $R \cong U$ the *dual unit sphere* in $\mathbb{C}^n$ (compare also the discussion at the end of section 7). We remark that the action of $\mathbb{T}$ on U given by scalar multiplication has as quotient the complex manifold $\mathbb{C}^{n-1}$ which is not biholomorphically equivalent to the bounded domain $\mathbb{B}_{n-1} = U/Z$.

The group G_M is simple in case $M = U, S$ and is nilpotent in case $M = V$. Also, the isotropy subgroup K at a point $o \in M$ acts irreducibly on the tangent space $T_a M$ if $M = U, S$ and K is a finite group in the third case. In all three cases the group I_M acts transitively on the subbundle $\{v \in H_a M : a \in M,\ \|v\| = 1\}$ of the tangent bundle TM. In particular, M is an SCR-space of constant holomorphic sectional curvature.

5. A canonical fibration

Let again M be an SCR-space and let $o \in M$ be a fixed point, called base point in the following. Let $G = G_M$ and $K = \{g \in G : g(o) = o\}$ be as before. Then s_o is in the center of K and $\sigma(g){:}= s_o g s_o$ defines an involutive group automorphism σ of G. Therefore K is contained in the closed subgroup $\mathsf{Fix}(\sigma) \subset G$. Let L be the smallest *open* subgroup of $\mathsf{Fix}(\sigma)$ containing K. Then s_o is contained in the center of L and $s_a = s_o$ for all $a \in L(o)$. Identify as before M with the homogeneous space G/K and put $N{:}= G/L$. Then we have canonical fibre bundles

$$G \xrightarrow{\;\mu\;} M \xrightarrow{\;\nu\;} N$$

defined by $g \mapsto gK \mapsto gL$. The typical fibres are K for μ and the connected homogeneous space L/K for ν. The following statement follows directly from the definition of σ.

5.1 Lemma. *The fibration μ satisfies $\mu\circ\sigma = s_o\circ\mu$.*

Because of Lemma 5.1, σ can be seen as a lifting of s_o via μ. On the other hand, s_o can be pushed forward via ν:

5.2 Proposition. *For every $c\in N$, there exists a unique involutive diffeomorphism $s_c\colon N\to N$ such that $\nu\circ s_a = s_c\circ\nu$ for all $a\in M$ with $\nu(a)=c$. The differential $d_a\nu$ has kernel T^+_aM in T_aM and hence induces a linear isomorphism from T^-_aM onto T_cN. Every s_c has c as an isolated fixed point. Furthermore, N is simply connected if M is simply connected.*

Proof. Since the fibration ν is G-equivariant we have to establish the map s_c only for the base point $c:=\nu(o)$ of N. But then s_c is given by $gL\mapsto\sigma(g)L$ since s_o can be identified with the map $gK\mapsto\sigma(g)K$ of M. The tangent space T_oF of the fiber $F\colon=\nu^{-1}(c)=L(o)$ at o is T^+_oM and hence is the kernel of the differential $d_o\nu$. Consequently, the differential d_cs_c is the negative identity on T_cN and hence c is an isolated fixed point of s_c. Now suppose that M is simply connected and denote by $\alpha\colon H\to G^0$ the universal covering group of G^0. Then the subgroup $\alpha^{-1}(K\cap G^0)$ of H is connected and hence by the construction of L also the subgroup $\alpha^{-1}(L\cap G^0)$ is connected, i.e. $N=G^0/(L\cap G^0)$ is simply connected. □

The manifold N in Proposition 5.2 together with all involutive diffeomorphisms s_c, $c\in N$, is a symmetric space in the sense of [20]. An interesting case occurs when N has the structure of a symmetric CR-manifold in such a way that

(i) ν is a CR-map and M, N have the same CR-dimension.
(ii) ν is a partial isometry, i.e. the restriction of $d_a\nu$ to T^-_aM is an isometry for every $a\in M$.
(iii) s_c is a symmetry at c for every $c\in N$.

We say that the SCR-space M has *symmetric reduction* N if the properties (i) – (iii) are satisfied. For every $g\in L$ and $a=g(o)$ we have the commutative diagram

$$\begin{array}{ccc} T^-_oM & \xrightarrow{d_o\Phi_g} & T^-_aM \\ {\scriptstyle d_o\nu}\downarrow & & \downarrow{\scriptstyle d_a\nu} \\ T_cN & \xrightarrow{d_c\Psi_g} & T_cN \end{array}$$

where for better distinction we denote by Φ_g the diffeomorphism of M given by g and by Ψ_g the corresponding diffeomorphism of N (that is, $\Phi_g(hK)=ghK$ and $\Psi_g(hL)=ghL$). Put $H_cN\colon=d_o\nu(H_oM)$ and give it the complex structure for which $d_o\nu\colon H_oM\to H_cN$ is a complex linear ismorphism. Furthermore, endow T_cN with the Riemannian metric for which $d_o\nu\colon T^-_oM\to T_cN$ is an isometry. Then the existence of a G-invariant almost CR-structure on N with property (i) is equivalent to the condition that all operators $d_c\Psi_g$, $g\in L$, leave the subspace H_cN invariant and are complex linear there. In the same way, a G-invariant Riemannian metric on N with property (ii) exists if and only every $d_c\Psi_g$, $g\in L$, is an isometry of T_cN. This happens for instance (after possibly changing the metric of M) if the group L is compact, or more generally, if the linear group $\{d_c\Psi_g : g\in L\}$ is compact.

We notice that as a consequence of Lemma 3.10, condition (iii) is automatically satisfied if (i), (ii) hold. Also, in case that for the fibration $\nu\colon M\to N$ there exists a Riemannian metric on N with the property (ii) and such that in addition every ν-fibre is a symmetric Riemannian manifold, the space M is a bisymmetric space in the sense of [16].

The following sufficient condition for the existence of a symmetric reduction is easily seen, we leave the proof to the reader.

5.3 Lemma. *Suppose that the subgroup $\{g\in I_M : \nu\circ g=\nu\}$ acts transitively on some ν-fibre. Then this group acts transitively on every ν-fibre and M has a symmetric reduction.* □

Not every symmetric CR-space has a symmetric reduction. Consider for instance Example 3.13. Then L^0 is the subgroup of all $(0,b,0)\in M$ with $b\in\mathbb{R}$, and $N\colon=G/L$ can be identified with $\mathbb{C}\times\mathbb{R}$ in such a way that $\nu(z,w,v)=\big(z,\mathrm{Re}(v)-3\mathrm{Re}(w)\mathrm{Re}(z)\big)$. Furthermore, the action of L^0 on $\mathbb{C} \times \mathbb{R}$ is given by $(z,t) \mapsto \big(z, t - 4b\mathrm{Re}(z)\big)$, $b \in \mathbb{R}$. This implies that there cannot exist any G-invariant Riemannian metric on N. Also, there is no CR-structure on N satisfying property (i).

6. A construction principle

In this section we give a Lie theoretical construction of symmetric CR-spaces such that every SCR-space can be obtained is this way. We start with an arbitrary connected Lie group G^0 together with an involutive group automorphism σ of G^0. Then there is a Lie group G with connected identity component G^0 and an element $s \in G$ with $G = G^0 \cup sG^0$ and $\sigma(g) = sgs$ for all g. Let $\mathfrak{g}$ be the Lie algebra of G and denote by $\tau := \mathsf{Ad}(s)$ the Lie algebra automorphism of $\mathfrak{g}$ induced by σ (here and in the following Ad always refers to the group G). Put

$$\mathfrak{l} := \mathsf{Fix}(\tau) \quad \text{and} \quad \mathfrak{m} := \mathsf{Fix}(-\tau) \tag{6.1}$$

Then $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{m}$ is a Lie triple system, see [20]. For every $g \in \mathsf{Fix}(\sigma)$, the decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ is invariant under $\mathsf{Ad}(g)$.

Now choose a compact subgroup $K \subset \mathsf{Fix}(\sigma)$, an $\mathsf{Ad}(K)$-invariant Riemannian metric on $\mathfrak{g}$ and a linear subspace $\mathfrak{h} \subset \mathfrak{m}$ together with a complex structure J on $\mathfrak{h}$ satisfying the following properties:

(i) $\|Jx\| = \|x\|$ for all $x \in \mathfrak{h}$.
(ii) K contains the element s.
(iii) $\mathsf{Ad}(g)$ leaves the subspace $\mathfrak{h}$ invariant and commutes there with J for every $g \in K$.

Notice that $K = \{s, e\}$ with arbitrary $\mathfrak{h} \subset \mathfrak{m}$ and arbitrary J always is an admissible choice. Also, if the compact group K has been chosen, every closed subgroup of K containing s is again an admissible choice.

Since K is compact there exists an $\mathsf{Ad}(K)$-invariant decomposition $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of K. With $\mathfrak{p} := \mathfrak{n} \oplus \mathfrak{m}$ therefore we get the $\mathsf{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Consider the connected homogeneous G-manifold $M := G/K$ and declare $o := K \in M$ as base point. In the following we identify the tangent space T_oM in the canonical way with the Hilbert space $\mathfrak{p}$. Denote by Φ_g the diffeomorphism of M induced by g , that is

$$\Phi_g : M \to M, \quad hK \longmapsto ghK\,,$$

(we do not require here that the G-action is effective, this could be easily achieved by reducing out the kernel of ineffectivity from the beginning). Then for every $g \in K$, the differential $d_o\Phi_g$ is nothing but the restriction of $\mathsf{Ad}(g)$ to $\mathfrak{p}$ and hence there exists a unique G-invariant almost CR-structure with $H_oM = \mathfrak{h}$ and also a unique G-invariant Riemannian metric on M extending the given Hilbert norm of $T_oM = \mathfrak{p}$. In particular, $s_o := \Phi_s$ is an involutive isometric diffeomorphism of M with fixed point o. Clearly, s_o is a symmetry of M at o if and only if $H_o^{-1}M \subset \mathfrak{m}$, where $H_o^{-1}M \subset T_oM$ is the subspace defined in section 2. A more convenient condition for this is given by the following statement.

6.2 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be the Lie subalgebras of $\mathfrak{g}$ generated by $\mathfrak{m}$ and $\mathfrak{h}$ respectively. Then $\mathfrak{a} = [\mathfrak{m},\mathfrak{m}] \oplus \mathfrak{m}$ holds, $\mathfrak{a}$ is an $\mathsf{Ad}(K)$-invariant ideal of $\mathfrak{g}$ and*

(i) *M is a minimal symmetric CR-manifold with symmetry s_o at o if and only if $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$.*
(ii) *In case $s_o := \Phi_s$ is a symmetry of M at o, the weaker condition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$ holds.*

Proof. First notice that $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{l}$, $[\mathfrak{l},\mathfrak{m}] \subset \mathfrak{m}$ by (6.1) and hence that $\mathfrak{a} = [\mathfrak{m},\mathfrak{m}] \oplus \mathfrak{m}$ holds. Obviously, $\mathfrak{a}$ is invariant under $\mathsf{ad}(\mathfrak{l})$ as well as $\mathsf{ad}(\mathfrak{m})$, i.e. $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. Now suppose that s_o is a symmetry of M at o. Then M is a symmetric CR-manifold by the transitivity of the group G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$ follows as in the proof of Proposition 3.6. Now suppose that in addition that M is minimal as an almost CR-manifold. Define inductively $\mathfrak{h}^k := \mathfrak{h}^{k-1} + [\mathfrak{h}, \mathfrak{h}^{k-1}]$ and $\mathfrak{h}^0 = 0$. Then $\mathfrak{h}^k/\mathfrak{h}^{k-1}$ is isomorphic to H_o^kM and $\mathfrak{g} = \mathfrak{k} + \mathfrak{h}^k$ for k sufficiently large, i.e. $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$. Conversely, suppose that $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$ holds. Then M is minimal and the differential of s_0 is the negative identity on $\mathfrak{h} = H_oM$, i.e. s_o is a symmetry at o. $\square$

As an illustration of the construction principle fix integers $p, q \geq 0$ with $n := p + q \geq 2$ and set $G := \mathsf{SU}(n)$. Then the corresponding Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$ is a real Hilbert subspace of $\mathbb{C}^{n\times n}$. Write every $g \in \mathbb{C}^{n\times n}$ in the form $\binom{ab}{cd}$ with a, b, c, d matrices of sizes $p \times q, p \times q, q \times p$, $q \times q$ respectively and denote by σ the automorphism of G defined by $\binom{ab}{cd} \mapsto \binom{a\ -b}{-c\ \ d}$. Fix a closed subgroup $K \subset F := \mathsf{Fix}(\sigma)$ and put $M := G/K$ with base point $o := K \in M$. Identify T_oM with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ and put $H_oM := \mathfrak{m} := \{\binom{0b}{c0} \in \mathfrak{p}\} \approx \mathbb{C}^{p\times q}$ with complex structure defined by $\binom{0\,b}{c\,0} \mapsto \binom{0\ \ \ ib}{-ic\ 0}$. These data give a unique G-invariant Hermitian metric and a unique G-invariant almost CR-structure on M with $gK \mapsto \sigma(g)K$ a symmetry at $o \in M$. It is easily seen that $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{g}$ as well as the integrability condition (2.2) hold, i.e. M is a compact minimal symmetric CR-manifold with symmetric reduction $N := G/L$. Here N is the Grassmannian $\mathbb{G}_{p,q}$ of all linear subspaces of dimension p in $\mathbb{C}^n$ and in particular is a symmetric Hermitian space. If we replace $G = \mathsf{SU}(n)$ by the group $G^d := \mathsf{SU}(p, q)$ and define σ by the same formula, then $L = \mathsf{Fix}(\sigma)$ remains unchanged and for every compact subgroup $K \subset L$ we get the two minimal symmetric CR-manifolds $M = G/K$ and $M^d := G^d/K$ which we call dual to each other. In particular, $N^d := G^d/L$ is a bounded symmetric domain and is the dual of the Grassmannian $\mathbb{G}_{p,q}$ in the sense of symmetric Hermitian spaces.

7. Integrability and complexifications

Assume that M is an arbitrary CR-space with base point $o \in M$, not necessarily symmetric to begin with. Assume that G is a Lie group acting smoothly and transitively on M by CR-diffeomorphisms. Let $K := \{g \in G : g(o) = o\}$ be the isotropy subgroup at o and denote by $\mathfrak{k} \subset \mathfrak{g}$ the corresponding Lie algebras. Then the canonical map $\theta: \mathfrak{g} \to T_oM$ is surjective and has $\mathfrak{k}$ as kernel. Choose a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $\theta: \mathfrak{h} \to H_oM$ is a linear isomorphism. Then there is a unique complex structure J on $\mathfrak{h}$ making $\theta|_{\mathfrak{h}}$ complex linear. It is clear that $\mathfrak{k} \oplus \mathfrak{h} = \theta^{-1}(H_oM)$ does not depend on the choice of $\mathfrak{h}$.

Let $\boldsymbol{\mathfrak{g}} := \mathfrak{g} \oplus i\mathfrak{g}$ be the complexification of $\mathfrak{g}$ and denote for linear subspaces of $\mathfrak{g}$ its complex linear span by the corresponding *boldface letter*, that is for instance $\boldsymbol{\mathfrak{a}} = \mathfrak{a} \oplus i\mathfrak{a}$ in case of $\mathfrak{a}$. The complex structure J of $\mathfrak{h}$ extends in a unique way to a complex linear endomorphism $\boldsymbol{J}$ of $\boldsymbol{\mathfrak{h}}$. Denote by $\boldsymbol{\mathfrak{h}}^{\pm}$ the eigenspaces of $\boldsymbol{J}$ in $\boldsymbol{\mathfrak{h}}$ to the eigenvalues $\pm i$: $\boldsymbol{\mathfrak{h}}^{\pm} = \{Jx \pm ix : x \in \mathfrak{h}\}$. Put $\boldsymbol{\mathfrak{l}} := \boldsymbol{\mathfrak{k}} \oplus \boldsymbol{\mathfrak{h}}^{-}$ for the following. This space does not depend on the choice of $\mathfrak{h}$. The composition of the canonical maps $\mathfrak{h} \hookrightarrow \boldsymbol{\mathfrak{h}} \to \boldsymbol{\mathfrak{h}}/\boldsymbol{\mathfrak{h}}^{-}$ induces a complex linear isomorphism $\mathfrak{h} \cong \boldsymbol{\mathfrak{h}}/\boldsymbol{\mathfrak{h}}^{-} \cong \boldsymbol{\mathfrak{h}}^{+}$. As a consequence, $\mathfrak{g}/\mathfrak{k} \hookrightarrow \boldsymbol{\mathfrak{g}}/\boldsymbol{\mathfrak{l}}$ realizes $T_oM = \mathfrak{g}/\mathfrak{k}$ as a linear subspace of the complex vector space $\boldsymbol{\mathfrak{g}}/\boldsymbol{\mathfrak{l}}$ in such a way that there $H_oM = T_oM \cap iT_oM$ holds. This property will be the key in the proof of Proposition 7.3.

7.1 Proposition. *The CR-structure of M is integrable if and only if $\boldsymbol{\mathfrak{l}}$ is a Lie algebra.*

Proof. Let $\boldsymbol{T}M$ be the complexified tangent bundle of M and denote by $\boldsymbol{\mathfrak{V}} = \mathfrak{V} \oplus i\mathfrak{V}$ the complex Lie algebra of all smooth complexified vector fields on M, i.e. of all smooth sections $M \to \boldsymbol{T}M$. Then $\boldsymbol{\mathfrak{K}} := \{X \in \boldsymbol{\mathfrak{V}} : X_o = 0\}$ is a complex Lie subalgebra of $\boldsymbol{\mathfrak{V}}$. The almost CR-structure of M gives a complex subbundle $\boldsymbol{H}^{0,1}M \subset \boldsymbol{T}M$. Denote by $\boldsymbol{\mathfrak{H}}^{0,1} \subset \boldsymbol{\mathfrak{V}}$ the linear subspace of all vector fields of type $(0,1)$, i.e. $X_a \in \boldsymbol{H}^{0,1}_aM$ for all $a \in M$. The integrability conditon for M is equivalent to $\boldsymbol{\mathfrak{H}}^{0,1}$ being a Lie subalgebra of $\boldsymbol{\mathfrak{V}}$. The action of G on M induces a Lie homomorphism $\mathfrak{g} \to \mathfrak{V}$ that uniquely extends to a complex linear homomorphism $\varphi: \boldsymbol{\mathfrak{g}} \to \boldsymbol{\mathfrak{V}}$. The assumption that G acts by CR-diffeomorphisms implies $\big[\varphi(\boldsymbol{\mathfrak{g}}), \boldsymbol{\mathfrak{H}}^{0,1}\big] \subset \boldsymbol{\mathfrak{H}}^{0,1}$. For $\boldsymbol{\mathfrak{L}} := \boldsymbol{\mathfrak{K}} + \boldsymbol{\mathfrak{H}}^{0,1}$ we have $\boldsymbol{\mathfrak{l}} = \varphi^{-1}(\boldsymbol{\mathfrak{L}})$.

Now suppose that M is integrable. We claim that $\boldsymbol{\mathfrak{l}}$ is a Lie algebra and consider arbitrary vector fields $X, Y \in \varphi(\boldsymbol{\mathfrak{l}})$. It is enough to show for $Z := [X, Y]$ that $Z_o \in \boldsymbol{H}^{0,1}_oM$ holds, i.e. that Z is contained in $\boldsymbol{\mathfrak{L}}$. Write $X = X' + X''$, $Y = Y' + Y''$ with $X', Y' \in \boldsymbol{\mathfrak{K}}$ and $X'', Y'' \in \boldsymbol{\mathfrak{H}}^{0,1}$. Then we have

$$[X', Y''] = [X - X'', Y''] \equiv [X, Y''] \quad \text{and hence}$$

$$(*) \qquad Z \equiv [X', Y''] + [X'', Y'] \equiv [X, Y''] + [X'', Y] \equiv 0 \mod \boldsymbol{\mathfrak{L}}\,.$$

Conversely, suppose that $\mathfrak{l}$ is a Lie algebra. Since G acts transitively on M we have $\boldsymbol{\mathfrak{L}} = \boldsymbol{\mathfrak{K}}+\varphi(\mathfrak{l})$. We claim that M is integrable. Consider arbitrary vector fields $X, Y \in \boldsymbol{\mathfrak{H}}^{0,1}$ and write $X = X' + X''$, $Y = Y' + Y''$ with $X', Y' \in \boldsymbol{\mathfrak{K}}$ and $X'', Y'' \in \varphi(\mathfrak{l})$. We have to show that $Z{:}= [X, Y]$ is contained in $\boldsymbol{\mathfrak{H}}^{0,1}$. Since G acts transitively on M and leaves $\boldsymbol{\mathfrak{H}}^{0,1}$ invariant it is enough to show that $Z_o \in \boldsymbol{H}^{0,1}_o M$ holds, i.e. that $Z \in \boldsymbol{\mathfrak{L}}$. But this follows as in $(*)$. □

7.2 Corollary. *Suppose that M in 7.1 is symmetric with $G{:}= G_M$. Then, if $\mathfrak{h} \subset \mathfrak{g}$ is chosen to be $\mathsf{Ad}(K)$-invariant, M is integrable if and only if $[\boldsymbol{\mathfrak{h}}^-, \boldsymbol{\mathfrak{h}}^-] \subset \boldsymbol{\mathfrak{k}}$.*

Proof. By the choice of $\mathfrak{h}$ we have $[\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{h}$ and hence $[\boldsymbol{\mathfrak{k}}, \boldsymbol{\mathfrak{h}}^-] \subset \boldsymbol{\mathfrak{h}}^-$. The involution $\mathsf{Ad}(s_o)$ of $\mathfrak{g}$ extends to a complex linear involution τ of $\boldsymbol{\mathfrak{g}}$ with $\boldsymbol{\mathfrak{h}} \subset \mathsf{Fix}(-\tau)$. Therefore, $\mathfrak{l}$ is a Lie algebra if and only if the inclusion $[\boldsymbol{\mathfrak{h}}^-, \boldsymbol{\mathfrak{h}}^-] \subset \mathfrak{l}$ holds, that is $[\boldsymbol{\mathfrak{h}}^-, \boldsymbol{\mathfrak{h}}^-] \subset \mathfrak{l} \cap \mathsf{Fix}(\tau) = \boldsymbol{\mathfrak{k}}$. □

We remark that 7.1 and 7.2 remain valid for $\boldsymbol{\mathfrak{h}}^+$ in place of $\boldsymbol{\mathfrak{h}}^-$.

7.3 Proposition. *Let $M = G/K$ be a homogeneous CR-manifold as in Proposition 7.1. Suppose, there exist complex Lie groups $\boldsymbol{L} \subset \boldsymbol{G}$ with Lie algebras $\mathfrak{l} \subset \boldsymbol{\mathfrak{g}}$, where $\mathfrak{l}$ and $\boldsymbol{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{g}$ are as before. Suppose furthermore that G can be realized as real Lie subgroup $G \subset \boldsymbol{G}$ in such a way that the corresponding injection $\mathfrak{g} \to \boldsymbol{\mathfrak{g}}$ is the canonical one and such that $G\boldsymbol{L}$ is locally closed in $\boldsymbol{G}$. Then, if $\boldsymbol{L} \cap G = K$ holds and if $\boldsymbol{L}$ is closed in $\boldsymbol{G}$, $gK \mapsto g\boldsymbol{L}$ realizes M as a locally closed generic CR-submanifold of the homogeneous complex manifold $\boldsymbol{M}{:}= \boldsymbol{G}/\boldsymbol{L}$.*

Proof. The assumptions guarantee that M is imbedded in $\boldsymbol{M}$ as a locally closed real-analytic submanifold with $H_oM = T_oM \cap iT_oM$ in $T_o\boldsymbol{M}$. The result follows since M is a G-orbit in $\boldsymbol{M}$. □

In general, the CR-submanifold M is not closed in $\boldsymbol{M}$. For instance, if M is a bounded symmetric domain and $G = \mathsf{Aut}(M)$ is the biholomorphic automorphism group of M then $\boldsymbol{M}$ can be chosen to be the corresponding compact dual symmetric Hermitian manifold which contains M as an open subset. For the sphere $M{:}= \partial\mathbb{B}_n \subset \mathbb{C}^n$ and $G = \mathsf{Aut}_{\mathrm{CR}}(M)$ we may chose $\boldsymbol{M} = \mathbb{P}_n$. On the other hand, for the same sphere $M = \partial\mathbb{B}_n$ but $G = \mathsf{U}(n)$ we may obtain for $\boldsymbol{M}$ the domain $\mathbb{C}^n\backslash\{0\}$ in $\mathbb{C}^n$ – but also the Hopf manifold $\mathbb{C}^n\backslash\{0\}/\alpha^{\mathbb{Z}}$ for some complex number α with $|\alpha| > 1$.

In the following we illustrate the statements 7.1 – 7.3 by various examples. Denote by σ the inner automorphism of $\mathbb{C}^{n\times n}$ given by $(a_{ij}) \mapsto ((-1)^{i+j}a_{ij})$.

7.4 Example. On the contrary to symmetric Hermitian spaces, the CR-structure of a symmetric CR-space does not need to be integrable. For $n \ge 3$ let $M \subset \mathbb{C}^{n\times n}$ be the nilpotent subgroup of all unipotent lower triangular matrices, i.e. of all $a = (a_{ij})$ with $a_{ii} = 1$ and $a_{ij} = 0$ if $i < j$. Then for the identity $e \in M$, the tangent space T_eM will be identified with the nilpotent algebra $\mathfrak{g}$ of all strictly lower triangular matrices. Denote by G the group generated by all left multiplications with elements from M and denote the restriction of σ to M by the same symbol. Then $G = G^0 \cup \sigma G^0$ acts transitively on M and there exists a unique G-invariant Riemannian metric on N which coincides on T_eM with the one inherited from $\mathbb{C}^{n\times n}$. Also there is a unique G-invariant almost CR-structure on M with

$$H_eM = \mathfrak{h}{:}= \{a \in \mathfrak{g} : a_{ij} = 0 \ \text{ if } j \ne i+1\}$$

and complex structure on $\boldsymbol{\mathfrak{h}}$ inherited from $\mathbb{C}^{n\times n}$. With this structure M is symmetric and minimal. Because of $[\boldsymbol{\mathfrak{h}}^-, \boldsymbol{\mathfrak{h}}^-] \ne 0$ the CR-structure is not integrable. □

7.5 Example. Let $n > d \ge 1$ be fixed integers with $d \le n/2$ and denote by $\mathfrak{g}$ the space of all matrices in $\mathbb{C}^{n\times n}$ having the form (7.6). Also, denote by $\mathfrak{h} \subset \mathfrak{g}$ the subspace of all matrices with $z_k = \alpha_j = 0$ for all $k > d$ and all j. A simple calculation shows that $\mathfrak{g}$ is a real Lie subalgebra of $\mathbb{C}^{n\times n}$ and that $\mathfrak{h}$ generates $\mathfrak{g}$ as Lie algebra. Identifying $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ in the obvious way with the corresponding matrix in $\mathfrak{h}$ we get a complex structure J on $\mathfrak{h} \subset \mathsf{Fix}(-\sigma)$. For all $x, y \in \mathfrak{h}$ the identity $[Jx, y] + [x, Jy] = 0$ is easily verified. $M{:}= \exp(\mathfrak{g})$ is a closed nilpotent subgroup of $\mathsf{GL}(n, \mathbb{C})$ invariant under σ. Precisely as in Example 7.4 M becomes a symmetric

$$
\begin{pmatrix}
0 \\
z_1 & 0 & & & & & \\
\alpha_1 & \overline{z}_1 & 0 & & 0 & & \\
z_2 & -\alpha_1 & z_1 & 0 & & & \\
\alpha_2 & \overline{z}_2 & \alpha_1 & \overline{z}_1 & 0 & & \\
z_3 & -\alpha_2 & z_2 & -\alpha_1 & z_1 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{pmatrix}
\tag{7.6}
$$

Example 7.5**:** All z_k arbitrary complex numbers, all α_k purely imaginary.

minimal CR-manifold. But this time $[\mathfrak{h}^-, \mathfrak{h}^-] = 0$ holds, that is, M is integrable. Furthermore, $\kappa(M) = [(n-1)/(2d-1)]$ and M has CR-dimension d.

Proposition 7.3 gives a prescription for a generic embedding of M. Let $\boldsymbol{G}$ be the connected, simply connected complex Lie group with Lie algebra $\boldsymbol{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{g}$. Then $\exp\colon \boldsymbol{\mathfrak{g}} \to \boldsymbol{G}$ is biholomorphic and in particular, $\boldsymbol{L} := \exp(\mathfrak{h}^-)$ is a closed abelian complex subgroup of $\boldsymbol{G}$ – notice that we have $\mathfrak{k} = 0$ in this case. Now, M embeds in the canonical way into the complex manifold $\boldsymbol{M} = \boldsymbol{G}/\boldsymbol{L}$ which is biholomorphic to the complex vector space $\boldsymbol{\mathfrak{g}}/\mathfrak{h}^-$. Easy for explicit calculations is the case n odd – then $\mathfrak{g} \cap i\mathfrak{g} = 0$ holds in $\mathbb{C}^{n\times n}$ and we can realize the complexification $\boldsymbol{\mathfrak{g}}$ within the complex Lie algebra $\mathbb{C}^{n\times n}$. The commutative subalgebra $\mathfrak{h}^-$ then consists of all matrices obtained from 7.6 by keeping all $\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_d$ and replacing all other entries (including all z_k) by 0. For $n = 3$, $d = 1$ we find the realization

$$M \cong \{(z,w) \in \mathbb{C}^2 : w + \overline{w} = z\overline{z}\}$$

which is the classical Heisenberg group, compare Example 3.11. For $n = 5$, $d = 1$ one can show

$$
\begin{aligned}
M \cong \Big\{(z,w,v_1,v_2,u) \in \mathbb{C}^5 : w + \overline{w} = z\overline{z},\ \ v_1 - \overline{v}_2 = z\overline{z}(z-\overline{z})/6 + \overline{w}z,& \\
u + \overline{u} = w\overline{w} + (z\overline{v}_1 + \overline{z}v_1) + z\overline{z}z\overline{z}/4\Big\}&
\end{aligned}
$$

which is a symmetric CR-manifold with CR-dimension 1, CR-codimension 4 and $\kappa(M) = 4$, compare also Example 3.13. The symmetry at the origin is given by $(z,w,v_1,v_2,u) \mapsto (-z,w,-v_1,-v_2,u)$. □

7.7 Example. Fix an integer $k > 1$ and consider in $\mathbb{C}^2$ the connected CR-submanifold

$$M := \big\{(s,v) \in \mathbb{C}^2 : |s|^{2k} - |v|^2 = 1\big\}$$

which is a k-fold cover of the symmetric CR-manifold R in Example 4.3 via the map $(s,v) \mapsto (s^k, v)$. Therefore also M is a symmetric CR-manifold and G^0_M is a k-fold covering group of $G^0_R = SU(1,1) \cong \mathsf{SL}(2,\mathbb{R})$. Denote by $\mathfrak{g}$ the Lie algebra of G_M. Then it is known that for $\boldsymbol{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{g}$ there does not exist any complex Lie group $\boldsymbol{G}$ into which G admits an embedding induced by the canonical injection $\mathfrak{g} \hookrightarrow \boldsymbol{\mathfrak{g}}$. Therefore the conclusion of Proposition 7.3 cannot hold for this example.
The group G^0_M consists of all transformations

$$(s,v) \mapsto \big((as^k + bv)^{1/k}, \overline{b}s + \overline{a}v\big)\,,$$

where $a, b \in \mathbb{C}$ satisfy $a\overline{a} - b\overline{b} = 1$. It follows that the action of G_M does not extend to all of $\mathbb{C}^2$. But it extends to the domain

$$D := \big\{(s,v) \in \mathbb{C}^2 : |v| < |s|^k\big\} \;=\; \mathbb{R}^+ M$$

on which the group $\mathbb{R}^+ \times G^0_M$ acts transitively and freely. □

8. CR-manifolds derived from bounded symmetric domains

Suppose that E is a complex vector space of dimension n and that $D \subset E$ is a bounded symmetric domain. Then $\mathsf{Aut}(D)$ is a semi-simple Lie group and at every point of D the corresponding isotropy subgroup is a maximal compact subgroup (see f.i. [14]). It is known that there exists a complex norm $\|\cdot\|_\infty$ on E such that D can biholomorphically be realized as the open unit ball

$$D = \{z \in E : \|z\|_\infty < 1\} \tag{8.1}$$

with respect to this norm and that any two realizations of this type are linearly equivalent. In this realization the isotropy subgroup at the origin is linear, i.e.

$$\{g \in \mathsf{Aut}(D) : g(0) = 0\} \;=\; \mathsf{GL}(D)\,.$$

Moreover, $\mathsf{GL}(D)$ is compact. Therefore there exists a $\mathsf{GL}(D)$-invariant complex Hilbert norm $\|\cdot\|$ on E that we fix for the sequel and hence consider E as a complex Hilbert space in the following. We will also always assume that D is given in the form (8.1). For shorter notation we use for the whole section the abbreviation

$$\Gamma\!:= \mathsf{Aut}(D)^0 \quad \text{and} \quad K\!:= \mathsf{GL}(D)^0\,.$$

As a generalization of 4.1 there exists a compact complex manifold Q with

$$D \subset E \subset Q \quad \text{and} \quad \mathsf{Aut}(D) = \{g \in \mathsf{Aut}(Q) : g(D) = D\}\,. \tag{8.2}$$

Q is the dual of D in the sense of symmetric Hermitian manifolds and $\mathsf{Aut}(Q)$ is a complex Lie group acting holomorphically and transitively on Q. The domain E is open and dense in Q and the set $Q\backslash E$ of the 'points at infinity' is an analytic subset of Q, but not a complex submanifold in general.

The boundary ∂D of D is smooth only in the very special case, where also $\|\cdot\|_\infty$ is a Hilbert norm. Nevertheless, ∂D is a finite union of Γ-orbits, which are locally closed CR-submanifolds of E. Every K-orbit M in ∂D is an Hermitian CR-submanifold of E with respect to the metric induced from E, where K acts by CR-isometries. We start with an orbit of a special nature: Denote by $S = S(D)$ the set of all extreme points of the closed convex set $\overline{D}$. The following two statements are well known, but will also be obvious from our discussion below.

8.3 Lemma. *S is a connected generic CR-submanifold of E. Moreover, S is the only compact Γ-orbit in $\overline{D}$, consists of all $e \in \overline{D}$ with $K(e) = \Gamma(e)$ and coincides with the Shilov boundary of D.* □

8.4 Lemma. *The CR-submanifold S is totally real if and only if D is biholomorphically equivalent to a 'tube domain' $\{z \in \mathbb{C}^n : \mathrm{Re}(z) \in \Omega\}$, where $\Omega \subset \mathbb{R}^n$ is an open convex cone. In this case D is said to be of tube type.* □

A bounded symmetric domain D is called *irreducible* if it is not biholomorphically equivalent to a direct product of complex manifolds of lower dimensions. This is known to be equivalent to $K \subset \mathsf{GL}(E)$ acting irreducibly on E. There exists (up to order) a unique representation of D as direct product $D = D_1 \times \cdots \times D_k$, where all D_j are irreducible bounded symmetric domains and are of the form $D_j = E_j \cap D$ for linear subspaces $E_j \subset E$ with $E = E_1 \oplus \cdots \oplus E_k$. Also, there exist direct product representations $S(D) = S(D_1) \times \cdots \times S(D_k)$ for the Shilov boundaries and $K = \mathsf{GL}(D_1)^0 \times \cdots \times \mathsf{GL}(D_k)^0$. We call the D_j the *factors* of D.

We are now able to formulate the main result of this section.

8.5 Theorem. *Let D be a bounded symmetric domain. Then the Shilov boundary S of D is a symmetric CR-manifold and the following conditions are equivalent.*

(i) *The Levi cone of S has non-empty interior at every point.*

(ii) *S is a minimal CR-manifold.*
(iii) *Every smooth CR-function f on S has a unique holomorphic extension to D that has the same smoothness degree on $\overline{D}$ as f.*
(iv) $\mathsf{Aut}(D) = \mathsf{Aut}_{\mathrm{CR}}(S)$.
(v) *D does not have a factor of tube type.*

For the proof (8.16) we use the Jordan theoretic approach to bounded symmetric domains as originated by Koecher [17], for details in the following always compare [21]: There exists a Jordan triple product $E^3 \to E$, $(x, y, z) \mapsto \{xyz\}$, that contains the full structural information of D. This triple product is symmetric bilinear in the outer variables (x, z), conjugate linear in the inner variable y and satisfies certain algebraic and spectral properties. The group $\mathsf{GL}(D)$ of all linear $\|\cdot\|_\infty$-isometries of E coincides with the group of all linear triple automorphisms, more precisely

$$\mathsf{GL}(D) \;=\; \big\{g \in \mathsf{GL}(E) : g\{xyz\} = \{gx\, gy\, gz\} \quad \text{for all} \quad x, y, z \in E\big\}\,.$$

An element $e \in E$ is called a *tripotent* if $\{eee\} = e$ holds. The set $\mathsf{Tri}(E)$ of all tripotents in E is a compact real-analytic submanifold of E and the group K acts transitively on every connected component of $\mathsf{Tri}(E)$. Except for $\{0\}$ every other connected component of $\mathsf{Tri}(E)$ has positive dimension and is contained in ∂D. Tripotents may also be characterized geometrically as 'affine symmetry points' of $\overline{D}$ in the following sense.

8.6 Proposition. *The element $a \in \overline{D}$ is a tripotent if and only if there exists an operator $\sigma \in \mathsf{GL}(D)$ with*
(i) $\sigma(a) = a$,
(ii) *$\sigma(v) = -v$ for all $v \in E$ with $\|a + tv\| \le 1$ for all $t \in \mathbb{T}$.*

We will postpone the proof of this criterion and fix a tripotent $e \in E$ for a moment. The triple multiplication operator $\mu = \mu_e \in \mathcal{L}(E)$ defined by $z \mapsto \{eez\}$ is Hermitian and splits E into an orthogonal sum $E = E_1 \oplus E_{1/2} \oplus E_0$ of eigenspaces to the eigenvalues $1, 1/2, 0$, called the *Peirce spaces* of the tripotent e. The canonical projection $P_k \colon E \to E_k$ maps D into itself and clearly is a polynomial in μ, more precisely

$$(8.7) \qquad P_1 = \mu(2\mu - 1)\,, \qquad P_{1/2} = 4\mu(1 - \mu)\,, \qquad P_0 = (1 - \mu)(1 - 2\mu)\,.$$

The 'Peirce reflection' $\rho := \exp(2\pi i\mu) = P_1 - P_{1/2} + P_0$ is contained in K, fixes e and leaves $\mathsf{Tri}(E)$ invariant. In particular, also the projection $P_1 + P_0$ maps D into itself.

The tripotent $e \neq 0$ is called *minimal* if $E_1 = \mathbb{C}e$ holds and is called *maximal* if $E_0 = 0$ holds. For instance, the Shilov boundary S of D is just the set of all maximal tripotents. E becomes a complex *Jordan algebra* (depending on the tripotent e) with respect to the commutative product $a \circ b := \{aeb\}$, and $e^2 := e \circ e = e$ is an idempotent in E. The Peirce space E_1 is a unital complex Jordan subalgebra with identity element e and conjugate linear algebra involution $z \mapsto z^* := \{eze\}$. For every $a \in E_1$ and powers inductively defined by $a^{k+1} := a^k \circ a$, $a^0 := e$, the linear subspace $\mathbb{C}[a] \subset E_1$ is a commutative, associative subalgebra (notice that the Jordan algebra E_1 is not associative in general). The element $a \in E_1$ is called *invertible* if a has an inverse $a^{-1} \in \mathbb{C}[a]$. The selfadjoint part $A := \{z \in E_1 : z^* = z\}$ of E_1 is a *formally real Jordan algebra*, i.e. a real Jordan algebra such that $x^2 + y^2 = 0$ implies $x = y = 0$ for all $x, y \in A$. Clearly, $E_1 = A \oplus iA$ holds since the involution is conjugate linear. For all $z \in E_1$ we denote by $\mathrm{Re}(z) := (z + z^*)/2 \in A$ the real part of z. The set $Y := \{a^2 : a \in A\}$ of all squares in A is a closed convex cone with $A = Y - Y$ and $Y \cap -Y = \{0\}$. The interior

$$\Omega := \text{Interior of } Y$$

coincides with $\exp(A) \subset A$ and also with the set of all $a \in Y$ that are invertible in A. Ω is an open convex linearly-homogeneous cone in A. The sesqui-linear mapping $\Phi \colon E_{1/2} \oplus E_{1/2} \to E$ defined by $\Phi(u, v) = 2\{euv\}$ takes values in E_1 and satisfies $\Phi(z, z) \in \overline{\Omega}$ for all $z \in E_{1/2}$, and

$\Phi(u,u)=0$ if and only if $u=0$. To indicate the dependence on the tripotent $e\in E$ we also write $E_k(e)$, $k=1,1/2,0$, for the Peirce spaces as well as $A(e)$, $Y(e)$, $\Omega(e)$, ρ_e and Φ_e. Let us illustrate these objects by a typical example.

8.8 Example. Fix arbitrary integers $p\ge q\ge 1$ and consider the complex Hilbert space $E:=\mathbb{C}^{p\times q}$ of dimension $n=pq$. Then $D:=\{z\in E:\mathbb{1}-z^*z>0\}$ is a bounded symmetric domain in E, where $\mathbb{1}=\mathbb{1}_q$ is the $q\times q$-unit matrix. $\|z\|_\infty^2$ is the largest eigenvalue of the Hermitian matrix z^*z, i.e. $\|z\|_\infty$ may be considered as the operator norm of z if considered as operator $\mathbb{C}^q\to\mathbb{C}^p$. The triple product is given by $\{xyz\}=(xy^*z+zy^*x)/2$ and $K\subset\mathsf{GL}(E)$ is the subgroup of all transformations $z\mapsto uzv$ with $u\in\mathsf{U}(p)$ and $v\in\mathsf{U}(q)$. The Hilbert norm on E given by $\|z\|^2=\mathsf{tr}(z^*z)$ is K-invariant. $\mathsf{Tri}(E)$ is the disjoint union of the K-orbits $S_0,S_1,\dots,S_q$, where S_k is the set of all tripotents $e\in E$ that have matrix rank k. In particular, if we write every $z\in E$ as block matrix $\binom{ab}{cd}$ with $a\in\mathbb{C}^{k\times k}$ and matrices b,c,d of suitable sizes, then $e=\binom{\mathbb{1}_q 0}{0\;0}$ is a tripotent in S_k. The corresponding Peirce spaces E_1, $E_{1/2}$ and E_0 consist of all matrices of the forms $\binom{a0}{00}$, $\binom{0b}{c0}$ and $\binom{00}{0d}$ respectively. Furthermore, A is the real subspace of all Hermitian matrices in E_1 and $\Omega\subset A$ is the convex cone of all matrices $\binom{a0}{00}$ with $a\in\mathbb{C}^{k\times k}$ positive definite Hermitian. For every $u=\binom{0b}{c0}\in H_eS_k$ we have $\Phi_e(u,u)=2\{euu\}=\binom{a0}{00}$ with $a=bb^*+c^*c$. Finally, $S=S_q$ is the Shilov boundary of D. S consists of all matrices in E whose column vectors are orthogonal in $\mathbb{C}^p$, or equivalently, which represent isometries $\mathbb{C}^q\to\mathbb{C}^p$. The group Γ is the set of all transformations

$$z\longmapsto(\alpha z+\beta)(\gamma z+\delta)^{-1}\qquad\text{with}\qquad\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\mathsf{SU}(p,q)$$

and $\alpha,\beta,\gamma,\delta$ matrices of sizes $p\times p$, $p\times q$, $q\times p$ and $q\times q$ respectively. In case $p=q>1$ the groups $\mathsf{GL}(D)$ and $\mathsf{Aut}(D)$ have two connected components, in all other cases these groups are connected. – For the special case $q=1$ we get for D the euclidean ball $\mathbb{B}$ in $E=\mathbb{C}^p$ with Shilov boundary the unit sphere $S=S_1=\partial D$ as studied in Example 4.2. For every $e\in S$ then $E_1(e)=\mathbb{C}e$ holds and $E_{1/2}(e)$ is the orthogonal complement of e in the Hilbert space E. □

Two tripotents $e,c\in E$ are called (triple) *orthogonal* if $c\in E_0(e)$ holds. Then also $e\in E_0(c)$ is true and $e\pm c$ are tripotents. An ordered tuple $(e_1,e_2,\dots,e_r)$ of pairwise orthogonal minimal tripotents in E is called a *frame* in E if there does not exist a minimal tripotent $e\in E$ that is orthogonal to all e_j in the triple sense. All frames in E have the same length r, which is called the *rank* of the bounded symmetric domain. Every element $a\in E$ has a representation

$$(8.9)\qquad a=\lambda_1e_1+\lambda_2e_2+\cdots+\lambda_re_r,\qquad \|a\|_\infty=\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_r\ge 0\,,$$

where $(e_1,e_2,\dots,e_r)$ is a frame depending on a. The real numbers $\lambda_j=\lambda_j(a)$ are uniquely determined by a and are called the *singular values* of a. In general, the frame $(e_1,e_2,\dots,e_r)$ is not uniquely determined by a. For every $a\in\overline{D}$ there is a unique representation

$$(8.10)\qquad a=e+u\qquad\text{with}\qquad e=:\varepsilon(a)\in\mathsf{Tri}(E)\quad\text{and}\quad u\in D\cap E_0(e)\,.$$

The Shilov-boundary of D is given by

$$(8.11)\qquad S=\{a\in E:\lambda_1(a)=\lambda_2(a)=\cdots=\lambda_r(a)=1\}\,.$$

In case D is irreducible, the compact group K acts transitively on the set of all frames and hence any two elements $a,b\in E$ are in the same K-orbit if and only if $\lambda_j(a)=\lambda_j(b)$ holds for all j.

These considerations can be used to prove the following property.

8.12 Proposition. *If S is totally real, it is rationally convex.*

Proof. For every $e \in S$, the Jordan algebra $E = E_1(e)$ has e as the unit element. It is known (compare for instance [11] or [22]) that there exists a unique homogeneous polynomial function $N: E \to \mathbb{C}$ of degree r such that the following is satisfied:

(i) $z \in E$ is invertible if and only if $N(z) \neq 0$ and (ii) $N(e) = 1$.

N is called the (generic) norm of the unital Jordan algebra E. It is known that there exists a character $\chi: K \to \mathbb{T}$ such that $N(gz) = \chi(g)N(z)$ holds for all $g \in K$ and all $z \in E$. On the other hand, for every frame $(e_1, \ldots, e_r)$ in E with $e_1 + \cdots + e_r = e$ and every complex linear combination $z = z_1 e_1 + \cdots + z_r e_r$ we have $N(z) = z_1 z_2 \cdots z_r$. This implies the following characterization of the Shilov boundary in the tube type case.

$$S = \{z \in \overline{D} : |N(z)| = 1\}.$$

In particular, for every $a \in \overline{D} \backslash S$, the rational function $(N - N(a))^{-1}$ is holomorphic in a neighbourhood of S and has no holomorphic extension to a, i.e. the rational convex hull of S in E coincides with S. □

The Shilov boundary S of D in Example 8.8 is totally real if and only if $p = q$ holds, and then $S = \mathsf{U}(q)$ is the unitary group. For the unit matrix $e \in E = \mathbb{C}^{q \times q}$ the Jordan product on E is given by $a \circ b = (ab + ba)/2$ and invertibility in the Jordan sense is the same as in the associative sense. In particular, $N(z) = \det(z)$ is the norm of E.

8.13 Proof of Proposition 8.6. In case a is a tripotent, every $v \in E$ with $\|a + tv\| \leq 1$ for all $t \in \mathbb{T}$ is contained in $E_0(a)$ and we may take $\sigma := -\exp(\pi i \mu_a) = P_1 - iP_{1/2} - P_0 \in K$, where μ_a is the triple multiplication operator $z \mapsto \{aaz\}$ on E. Conversely, suppose that a satisfies 8.6.i-ii and write $a = e + u$ as in 8.10. Then $\sigma(u) = -u$ follows from the assumptions. For every $t > 1$ with $tu \in \overline{D}$ we have $a - (1+t)u = e - tu \in \overline{D}$ and hence $\sigma(a - (1+t)u) = e + (2+t)u \in \overline{D}$, i.e $(t+2)u \in \overline{D}$ and hence $u = 0$. Therefore, $a = e$ is a tripotent. □

Fix a frame $(e_1, e_2, \ldots, e_r)$ in E and consider for all integers $0 \leq i, j \leq r$, the *refined Peirce spaces*:

$$E_{ij} := \big\{z \in E \ : \ 2\{e_k e_k z\} = (\delta_{ik} + \delta_{kj})z \quad \text{for} \quad 1 \leq k \leq r\big\}.$$

Then, if we put $e_0 := 0$,

$$E = \bigoplus_{0 \leq i \leq j \leq r} E_{ij}, \qquad E_{ii} = \mathbb{C}\, e_i \qquad \text{and} \qquad \{E_{ij} E_{jk} E_{kl}\} \subset E_{il}$$

hold for all $0 \leq i, j, k, l \leq r$. Also, $\{E_{ij} E_{kl} E\} = 0$ if the index sets $\{i, j\}$ and $\{k, l\}$ are disjoint. Furthermore, D has no tube type factor if and only if $E_{i0} \neq 0$ for $1 \leq i \leq r$. To indicate the dependence of E_{ij} on the given frame we also write $E_{ij}(e_1, e_2, \ldots, e_r)$.

Now consider a Γ-orbit $\Sigma \subset \overline{D}$. Then it is known that there is a tripotent e in E with $\Sigma = \Gamma(e)$ and that $T_e\Sigma = iA \oplus E_{1/2} \oplus E_0$ is the tangent space at $e \in \Sigma$, where the Peirce spaces refer to the tripotent e. This implies that Σ is a homogeneous generic locally-closed CR-submanifold of E with holomorphic tangent space $H_e\Sigma = E_{1/2} \oplus E_0$. The orbit $M := K(e)$ is a compact submanifold of Σ with tangent space $T_e M = iA \oplus E_{1/2}$ and holomorphic tangent space $H_e M = E_{1/2}$. Furthermore, $M = \Sigma \cap \mathsf{Tri}(E)$ and $\varepsilon: \Sigma \to M$ (compare 8.10) is a fibre bundle with typical fibre $D \cap E_0$.

8.14 Lemma. *Every connected component M of $\mathsf{Tri}(E)$ is a symmetric CR-manifold.*

Proof. Fix an arbitrary element $e \in M$. For the decomposition of D into a direct product $D_1 \times \cdots \times D_k$ of irreducible factors we get a decomposition $e = e_1 + \cdots + e_k$ with tripotents $e_j \in E_j$ and a decomposition $M = M_1 \times \cdots \times M_k$ with $M_j = \mathsf{GL}(D_j)^0(e_j)$, that is, we may assume without loss of generality that D is irreducible. To begin with, suppose that M is totally real, i.e. $E_{1/2} = 0$. Then, by irreducibility, also $E_0 = 0$ holds and $M = \exp(iA)$ is the 'generalized unit circle' in $E_1 = E$. Furthermore, $s_e(z) = z^*$ leaves M invariant and hence is a symmetry of M at e, i.e. M is symmetric in this case. Now suppose, that M is not totally real, i.e. $H_e M = E_{1/2} \neq 0$.

Then the Peirce reflection ρ_e maps M into itself and satisfies $H_eM \subset \mathsf{Fix}(-\rho_e)$. Therefore, as soon as we know that M is a minimal CR-manifold we know that ρ_e is a symmetry of M at e and hence that M is symmetric. For the minimality of M it is enough to show that $H_e^2M = iA$ holds, where H_e^2M is as in section 2. But this is a consequence of the following Proposition 8.15. $\square$

8.15 Proposition. *Let e be a tripotent in E and denote by M the connected e-component of $\mathsf{Tri}(E)$. Then $H_e^2M \subset iA(e)$ and the Levi form $E_{1/2}(e) \times E_{1/2}(e) \to E_1(e)$ of M at e is given by $(u,v) \mapsto -2\{euv\}$, i.e. $L - e = -\Phi_e$. In case D has no tube type factor, the convex hull of $\{\Phi_e(u,u) : u \in E_{1/2}(e)\}$ in $A(e)$ has the cone $-\Omega(e)$ as interior and then, in particular, $H_e^2M = iA(e)$ holds.*

Proof. For every $u \in H_eM = E_{1/2}(e)$ define the vector field X^u on E by $X_a^u = 4\{aau\} - 4\{aa\{aau\}\}$ for all $a \in M$. Then $X_e^u = u$ and $X_a^u \in H_aM$ for all $a \in M$ by (8.7). A simple calculation gives $[X^u, X^v]_e = 2\{evu\} - 2\{euv\} \in iA(e)$. This shows that $-\Phi_e$ is the Levi form at $e \in M$. Let C be the convex hull of $\{\Phi_e(u,u) : u \in E_{1/2}(e)\}$. Then $C \subset \overline{\Omega}(e)$ is clear. For the proof of the opposite inclusion fix an arbitrary element $a \in \overline{\Omega}(e)$. Then there exists an integer $k \le r$ and a representation $a = \lambda_1 e_1 + \cdots + \lambda_k e_k$, where $(e_1, \ldots, e_k)$ is a family of pairwise orthogonal minimal idempotents in the formally real Jordan algebra $A(e)$ summing up to e and where all coefficients λ_j are ≥ 0. This means that we only need to show that $e_j \in C$ for $1 \le j \le k$. For this we extend $(e_1, \ldots, e_k)$ to a frame $(e_1, \ldots, e_r)$ of E and fix $j \le k$. Since by assumption D has no tube type factor we have $E_{j0} \ne 0$. But then $\Phi_e(u,v) = 2\{e_j uu\}$ cannot vanish for all $u, v \in E_{j0}$ since otherwise there would exist a tripotent $c \ne 0$ in E_{j0} that is orthogonal to all e_i, $1 \le i \le r$. This implies $e_j \in C$. $\square$

8.16 PROOF OF THEOREM 8.5.

Proof. S is symmetric by Lemma 8.14 since S is a connected component of $\mathsf{Tri}(E)$. Fix an element $e \in S$. Then with Q as in 8.2 there exists an automorphism $\gamma \in \mathsf{Aut}(Q)$, called *Cayley transformation*, mapping D biholomorphically onto the Siegel domain

$$\mathbb{H} := \big\{(t,v) \in E_1 \oplus E_{1/2} : t + \overline{t} - \Phi(v,v) \in \Omega\big\}$$

in $E = E_1 \oplus E_{1/2}$, where the Peirce spaces E_k, the cone $\Omega \subset A$ and the Hermitian map $\Phi\colon E_{1/2} \times E_{1/2} \to E_1$ refer to the tripotent e. The transformation γ satisfies $\gamma^4 = \mathsf{id}$,
$S \cap \mathsf{Fix}(\gamma) = \{\pm ie\}$, $\gamma(-e) = 0$ and is given by

$$\gamma(t,v) = \big((e-t)^{-1} \circ (e+t), \sqrt{2}\,(e-t)^{-1} \circ v\big)\,,$$

where $(e-t)^{-1}$ is the inverse in the unital Jordan algebra E_1. The domain

$$V := \big\{(t,v) \in S : (e-t) \quad \text{is invertible in} \quad E_1\big\}$$

is dense in S and γ defines a CR-diffeomorphism from V onto the CR-submanifold

$$N := \big\{(t,v) \in E_1 \oplus E_{1/2} : t + \overline{t} = \Phi(v,v)\big\} \ \subset \ \partial\mathbb{H}$$

of E, compare also Example 4.5.
(i) $\Longrightarrow$ (ii) is an immediate consequence of the definitions (see 2.3) and holds for every CR-manifold.
(ii) $\Longrightarrow$ (iii). Suppose, S is minimal. Since the Shilov boundary of a bounded symmetric domain of tube type is totally real, D cannot have a factor of tube type. Then by Proposition 8.15, the interior of $\{\Phi(v,v) : v \in E_{1/2}\} = \{L(v,v) : v \in E_{1/2}\}$ coincides with the cone Ω, where L denotes the Levi form of N at 0 with respect to the obvious identification $(T_0N/H_0N) \otimes \mathbb{C} \cong E_1$. Let $v \in \Omega$ be an arbitrary vector. By the extension result of [10], every CR-function f on N extends holomorphically to a small wedge in the direction v, in particular, to a neighbourhood of a subset

of the type $(N+\mathbb{R}_+v)\cap U$, where U is a neighbourhood of 0 in $E_1\oplus E_{1/2}$. Furthermore, the wedge extension is of the same smoothness degree as f (due to [4], see also [5], Theorem 7.5.1, since f is of slow growth by the Cauchy estimates). Using the transformations $(t,v)\mapsto(st,s^2v)$, $s>0$, we see that f automatically extends to a neighbourhood of $(N+\mathbb{R}_+v)$. Since $v\in\Omega$ is arbitrary, f extends holomorphically to the whole of $\mathbb{H}$. This implies via the Cayley transformation that every CR-function f on S has a smooth extension to $D\cap S$ which is holomorphic on D. It remains to prove that the extension of f is of the same smoothness degree on the boundary ∂D. For every $0<r<1$, define the continuous function f_r on $\overline{D}$ by $f_r(z):=f(rz)$. Then for $r\to 1$, the functions f_r converge uniformly on S to f. Since S is the Shilov boundary of D the convergence is also uniform on $\overline{D}$, i.e. f extends continuously to $\overline{D}$. The smoothness is obtained by the same argument applied to the partial derivatives.
(iii) $\Longrightarrow$ (iv). $\mathsf{Aut}(D)\subset\mathsf{Aut}_{\mathrm{CR}}(S)$ follows from (8.2). Assume (ii) and consider a transformation $g\in\mathsf{Aut}_{\mathrm{CR}}(S)$. Then g extends to a continuous mapping $g\colon\overline{D}\to E$ which is holomorphic on D. As a consequence of the maximum principle, $g(\overline{D})$ is contained in the closed convex hull of $g(S)=S$, which is $\overline{D}$. By the same argument, $h:=g^{-1}$ extends to a continuous map $h\colon\overline{D}\to\overline{D}$ which is holomorphic on D. Then $h\circ g=g\circ h=\mathsf{id}$ shows $g\in\mathsf{Aut}(D)$.
(iv) $\Longrightarrow$ (v). Suppose D has a factor of tube type. Then S is a direct product of a CR-manifold with a totally real CR-manifold of positive dimension. In particular, $\mathsf{Aut}_{\mathrm{CR}}(S)$ cannot be a Lie group of finite dimension like $\mathsf{Aut}(D)$.
(v) $\Longrightarrow$ (i) follows from Proposition 8.15. □

Theorem 8.5 together with Proposition 8.12 can be used to calculate both polynomial and rational convex hulls of S explicitly. In particular, they are finite unions of disjoint connected real-analytic CR-submanifolds (forming a stratification in the sense of Whitney). We call a smooth function on such a union a CR-function if it is CR on each single CR-submanifold (this notion is independent of the partition into CR-submanifolds).

8.17 Corollary. *Let $E=E_1\oplus E_2$ be the canonical splitting such that $D_1:=D\cap E_1$ is of tube type and $D_2:=D\cap E_2$ has no tube type factor. Denote by $S_1\subset\partial D_1$ and $S_2\subset\partial D_2$ the corresponding Shilov boundaries. Then the following holds.*

(i) *Both convex and polynomial convex hulls of S coincide with $\overline{D}$.*
(ii) *The rational convex hull $\widehat{S}$ of S is given by $\widehat{S}=S_1\times\overline{D}_2$.*
(iii) *Every smooth CR-function f on S extends uniquely to a CR-function on $\widehat{S}$ of the same smoothness degree.*

Proof. Since S is the Shilov boundary of D, $|P(z)|\le\|P\|_S$ holds for every holomorphic polynomial P and every $z\in\overline{D}$. Hence $\overline{D}$ is contained in the polynomial convex hull of S. The latter is always contained in the convex hull of S, which is $\overline{D}$. This proves (i) (the statement about the convex hull also follows from the classical Krein-Milman theorem).

For the rational convex hull, we obtain $\widehat{S}_1=S_1$ by Proposition 8.12. This shows $\widehat{S}\subset S_1\times\overline{D}_2$. On the other hand, every rational function on E_2, holomorphic in a neighbourhood of S_2, is continuous on $\overline{D}_2$ by Theorem 8.5. This implies the opposite inclusion $\widehat{S}\supset S_1\times\overline{D}_2$ and therefore (ii).

Finally, let f be a CR-function on S. Then, for every $z_1\in S_1$, Theorem 8.5 guarantees that f has a unique smooth extension $\widehat{f}$ to $\{z_1\}\times\overline{D}_2$ which is holomorphic on $\{z_1\}\times D_2$. By the smoothness, $\widehat{f}$ is CR on each CR-submanifold of the boundary $\{z_1\}\times\partial D_2$. To prove the smoothness of $\widehat{f}$ on $\widehat{S}$, we fix a convergent sequence $z_1^m\to z_1^0$. Then $\widehat{f}(z_1^m,\cdot)$ converges to $\widehat{f}(z_1^0,\cdot)$ uniformly on S_2 and therefore on ∂D_2, because S_2 is the Shilov boundary. This shows that $\widehat{f}$ is continuous. The same argument applied to the partial derivatives of $\widehat{f}$ shows that $\widehat{f}$ is of the same smoothness degree as f. Since S_1 is totally real, the holomorphic tangent spaces to every CR-submanifold of $\widehat{S}$ are contained in E_2. This shows that $\widehat{f}$ is CR and finishes the proof of (iii). □

From the classification of all irreducible bounded symmetric domains into the 6 types **I, II,...,VI** (compare f.i. [21] p. 4.11) it follows that there are precisely the following 3 types of irreducible non-tube domains:

$\mathbf{I}_{q,p}$ with $p > q \geq 1$ arbitrary integers. Then, as in Example 8.8, $E = \mathbb{C}^{p\times q}$ and $D = \{z \in E : \mathbb{1} - z^*z > 0\}$ is the bounded symmetric domain of rank $r = q$, where $\mathbb{1}$ is the $q \times q$-unit matrix. The Shilov boundary of D is the set $S{:=}\ S_q$ of all matrices in E whose column vectors are orthogonal in $\mathbb{C}^p$, i.e.

$$S = \{z \in \mathbb{C}^{p\times q} : z^*z = \mathbb{1}\}\,. \tag{8.18}$$

On S the group $\mathsf{SU}(p)$ acts transitively by matrix multiplication from the left with isotropy subgroup $\mathbb{1} \times \mathsf{SU}(p{-}q)$ at $e{:=}\binom{\mathbf{1}}{0} \in S$, i.e. $S = \mathsf{SU}(p)/(\mathbb{1} \times \mathsf{SU}(p{-}q))$ is simply connected, has CR-dimension $(p-q)q$ and CR-codimension q^2. Also, $\mathsf{Aut}_{\mathrm{CR}}(S) \approx \mathsf{U}(p,q)/\mathbb{T}$ is connected. Every closed subgroup $L \subset \mathsf{U}(q)$ acts freely on S by matrix multiplication from the right and S/L again is a symmetric CR-manifold of the same CR-dimension in a natural way. For $L = U(q)$ we get the Grassmannian of all q-planes in $\mathbb{C}^p$ which is the reduction of S as defined in section 5. The typical fibre of the reduction map is the group $\mathsf{U}(q)$. □

$\mathbf{II}_p$ with $p = 2q+1$ an arbitrary odd integer > 3. Let $E{:=}\ \{z \in \mathbb{C}^{p\times p} : z' = -z\}$ and define the bounded domain $D \subset E$ as well as the Jordan triple product by the same formulae as for $\mathbf{I}_{p,q}$. Then again D is a bounded symmetric domain of rank $r = q$ and $\Gamma \subset \mathsf{GL}(E)$ is the group of all transformations $z \mapsto uzu'$ with $u \in \mathsf{U}(p)$. For $j{:=}\ \left(\begin{smallmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{smallmatrix}\right) \in \mathbb{C}^{2q\times 2q}$ the matrix $e{:=}\ \left(\begin{smallmatrix}j & 0\\ 0 & 0\end{smallmatrix}\right) \in \mathbb{C}^{p\times p}$ is in $S{:=}\ S_q$ and Γ has isotropy subgroup $\mathsf{Sp}(q) \times \mathbb{T}$ at e. Therefore $S = SU(p)/(\mathsf{Sp}(q) \times 1)$ is simply connected. The holomorphic tangent space H_eS is the space of all $z = \left(\begin{smallmatrix}0 & u\\ v & 0\end{smallmatrix}\right) \in E$ with $u{=}{-v'} \in \mathbb{C}^{2q}$, i.e. S has CR-dimension $2q$ and CR-codimension $q(2q-1)$. The reduction is the projective space $\mathbb{P}_{2q}(\mathbb{C}) = \mathsf{SU}(p)/S(\mathsf{U}(2q) \times \mathbb{T})$ and $\mathsf{SU}(2q)/\,\mathsf{Sp}(q)$ is the typical reduction fibre. The group $\mathsf{Aut}_{\mathrm{CR}}(S) \;\approx\; \mathsf{SO}^*(2p)/\{\pm 1\}$ is connected (compare [14] p. 451 and p. 518 for the non-compact type D III). □

V Here D is the exceptional bounded symmetric domain of dimension 16 (non-compact type E III on p. 518 of [14]). D has rank 2 and the Shilov boundary $S{:=}\ S_2$ has CR-dimension 8 and CR-codimension 8. On S the group $\mathsf{Spin}(10)$ acts transitively and the reduction $\widetilde{S}$ of S is the symmetric Hermitian manifold $\mathsf{SO}(10)/(\mathsf{SO}(2) \times \mathsf{SO}(8))$, the complex nonsingular quadric of dimension 8. The group $\mathsf{Aut}_{\mathrm{CR}}(S)$ is a non-compact simple exceptional real Lie group of type E_6 and has dimension 78. □

As a generalization of Example 4.3 also the dual of (8.18) (compare (4.4) and section 6) can be described explicitely. Fix $e = \binom{\mathbf{1}}{0} \in S$ and denote by $\rho = \rho_e$ the corresponding Peirce reflection of $E = \mathbb{C}^{p\times q}$. Then $\mathsf{Fix}(\rho) = \mathbb{C}^{q\times q}$ and $\mathsf{Fix}(-\rho) = \mathbb{C}^{(p-q)\times q}$. On

$$R = \{z \in \mathbb{C}^{p\times q} : \rho(z)^*z = \mathbb{1}\}$$

the group $\mathsf{U}(q, p{-}q)$ acts transitively from the left with compact isotropy subgroup $\mathbb{1} \times \mathsf{U}(p{-}q)$ at e. Therefore there is a unique $\mathsf{U}(q, p{-}q)$-invariant Riemannian metric on R which coincides on T_eR with the one induced from E. The restriction of ρ to R is a symmetry of R at e, i.e. is a symmetric CR-manifold. Again, every closed subgroup $L \subset \mathsf{U}(q)$ acts freely on R from the right and S/L is a symmetric CR-manifold of the same CR-dimension. For $L = U(q)$ we get the bounded symmetric domain of type $\mathbf{I}_{q,p-q}$, the reduction of R.

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