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Actions of groups of birationally extendible automorphisms

ALAN HUCKLEBERRY AND DMITRI ZAITSEV

Fakultät für Mathematik
Ruhr-Universität Bochum
D-44780 Bochum
Germany

1. Introduction

The origin of this work is found in the study of automorphisms of domains D in $\mathbf{C}^n$, $n > 1$. For example, suppose for the moment that D is relatively compact and recall that in this case the group $Aut(D)$ of all holomorphic automorphisms is a Lie group acting properly on D in the compact-open topology ([6], see also [19]). It is important to underline the fact that this group is totally real so that, compared to holomorphic actions of complex Lie groups, there is a lack of naturality.

The actions of compact subgroups $K \subset Aut(D)$ extend to the holomorphic actions of their complexifications. For example, consider the action of $K = S^1$ on an annulus D in the complex plane. A holomorphic function $f \in O(D)$ has a Fourier (Laurent) series expansion with respect to this action. This can be regarded as a formal series on $D^{\mathbf{C}} = \mathbf{C}^*$, where the complexification $K^{\mathbf{C}} = \mathbf{C}^*$ acts holomorphically. In fact, as a special case of Heinzner's Complexification Theorem ([12],see also [13]), any domain D equipped with a compact group K of holomorphic transformations is naturally contained as a K-stable domain in a Stein manifold $D^{\mathbf{C}}$ where the reductive group $K^{\mathbf{C}}$ acts holomorphically. If D is Stein, then it is just a domain of convergence for some "Fourier series" in

$D^{\mathbf{C}}$. Thus, except for convergence questions, in the case of compact groups we are really confronted with actions of reductive groups. In this case the theory of algebraic transformation groups provides us with very stong tools.

For a non-compact subgroup $G \subset Aut(D)$ the situation is substantially different. First of all, as is seen in the simplest example of the disk D in the complex plane, it is *rational* functions which play an important role. Secondly, since orbits are non-compact, one is led to study the action near the boundary. Without loss of generality we may assume that G is closed in $Aut(D)$ so that it acts properly and let $p \in \partial D$ be in the closure of some orbit $z \in D$. The geometry of the action near p is extremely rich. In fact, under reasonable regularity assumptions, it might happen that knowledge of the local action near p determines D itself and general classification results can be proved. There are numerous indications of this (see e.g. [8,17,21,22,31,32]) with Rosay's Theorem being the easiest to state: *If p is a strongly pseudoconvex boundary point, then D is biholomorphically equivalent to the unit ball* $\mathbf{B}_n := \{\sum |z_i|^2 < 1\}$. Under far weaker assumptions scaling methods yield a local description of D near p as also being defined by polynomial inequalities.

It is therefore reasonable to begin the study of G-actions on domains by considering the case were D is defined by polynomial inequalities. In this case, if there is a smooth boundary point where the Levi form is non-degenerate, by combining results of Diederich-Pinchuk ([7]) with those in [33], it follows that $G = Aut(D)$ is a Nash group and its action on D is compatible with the Nash structure. Thus we find ourselves in the setting of real algebraic geometry.

Our main results are stated in sections $2-3$. However, before going to this, we would like to underline some essential points. In general, suppose that G acts effectively by holomorphic transformations on D which extend to rational transformations of the ambient projective variety $V \subset \mathbf{P}^n$ (e.g., if D satisfies Webster's condition (W) below). The graph of every such transformation defines an n-dimensional cycle in $\mathbf{P}_n \times \mathbf{P}_n$ for some integer N. In this way we obtain a set-theoretic embedding of G in the Chow scheme C_n of n-dimensional cycles in $\mathbf{P}_n \times \mathbf{P}_n$. Under certain conditions, which are made precise in the sequel, we show that G lies in finitely many components of C_n. The group operation on G extends rationally to its Zariski closure Q in C_n and endows Q with a structure of a pre-group in sense of A. Weil, which is not a group in general. The action $G \times D \to D$ extends also to a rational action $Q \times V \to V$. Again, this is a pre-transformation space in sense of Weil which, in general, is not a transformation space.

Using basic techniques of Weil, we regularize the "action" $Q \times V \to V$, i.e. construct an algebraic group $\tilde{G}$ and an algebraic variety X, birationally equivalent to Q and V respectively, such that the induced action $\tilde{G} \times X \to X$ is regular (Theorems 1.1, 1.3 and 1.4). As a consequence the action of G extends to a global holomorphic action of the universal complexification $G^{\mathbf{C}}$ on X.

Further, we employ a "lifting procedure" to show that $\tilde{G}$ is a linear algebraic group. Then a result of Sumihiro ([16,27]) yields an equivariant embedding of D in a projective space with a linear action of $\tilde{G}$ (Theorem 2). In case of Siegel domains, such equivariant embeddings were obtained by W. Kaup, Y. Matsushima and T. Ochiai ([14], Theorem 9).

In the case D is *contractible and homogeneous* under the real analytic action of a connected Lie group G of birationally extedible automorphisms, R. Penny ([20]) has shown that the G-action extends to a rational action of a real algebraic group on the ambient space $\mathbf{C}^n$. This is a special case of Theorem 1 below.

We would like to conclude this introduction with an application concerning G-invariant meromorphic functions on D. For $x \in D$, let $d(x)$ denote the codimension of $T_xGx + iT_xGx$ in T_xD and $d := \max_{x \in D} d(x)$. If $f_1, \ldots, f_m$ are G-invariant analytically independent meromorphic functions, then clearly $m \leq d$. Now if $\tilde{G}$ exists as above and G is Nash, e.g. under the conditions of Corollary 2, then the bound d is realized. This follows by applying Rosenlicht's quotient Theorem ([23]) to the $\tilde{G}$-action on X.

As indicated above , we draw our methods from cycle space theory and algebraic group actions. On the other hand, our main motivation is of a complex analytic or representation theoretic nature. Thus we have included details of results which might be standard in one subject and not so well-known in the other.

2. Algebraic extensions

Here we establish conditions for the existence of the above extensions which are birationally equivalent to the ambient space $V \supset D$. A topological group will always assumed to have a countable basis at every point.

Definition 2.1 An **algebraic extension** of a topological group G of holomorphic transformations of a domain $D \subset V$ consists of a homomorphism from G into a complex algebraic group $\tilde{G}$, an algebraic variety X birationally equivalent to V via $\psi: V \to X$, such that $\psi|_D$ is a biregular embedding, and the extension of the action of G to a regular action $\tilde{G} \times X \to X$.

The existence of algebraic extensions implies, in particular, that the automorphisms of D which are elements of G extend to birational mappings from V into itself. In this case we say that G is a **group of birationally extendible automorphisms**.

One condition for the existence of algebraic extensions is given by the following result.

Theorem 1. *Let V be a projective variety, $D \subset V$ an open set and G a Lie group of birationally extendible automorphisms of D. Suppose that G has finitely many connected components. Then there exists an algebraic extension of G.*

The existence of an algebraic extension is also equivalent to the existence of a projective linearization in the following sense.

Definition 2.2 A **projective linearization** of a topological group G of holomorphic transformations of an open set $D \subset V$ consists of a (continuous) linear representation of G on some $\mathbf{C}^{N+1}$ and a birational (onto the image) mapping $i: V \to \mathbf{P}_N$ such that the restriction $i|_D$ is biholomorphic and G-equivariant.

Remark. By a rational mapping between two algebraic varieties V_1 and V_2 we mean a morphism from a Zariski open dense subset $U \subset V_1$ into V_2. The image is defined to be the (Zariski) closure of the image of U. In general, a point $x \in V_1 \setminus U$ may not correspond to a point of V_2.

Theorem 2. *Let V be a rational (i.e. birationally equivalent to $\mathbf{P}_n$) projective variety, D an open subset of the regular locus of V and G a topological group of birationally extendible automorphisms of D. Then G has an algebraic extension if and only if it has a projective linearization.*

Remark. The condition of rationality of V is perhaps too strong. However, some condition is needed. If e.g. $D = V = G$ are elliptic curves and G acts on V by translations, this action coincides with its algebraic extension but has no projective linearization, because G is a compact complex group.

Another sufficient condition for the existence of algebraic extensions is the boundness of the **degree** of the automorphisms defined by elements of G, i.e. the degree of the graphs $Z_f \subset V \times V$ of the corresponding birational automorphisms with respect to fixed embedding $\nu: V \times V \hookrightarrow \mathbf{P}_k$. We also identify $V \times V$ with its image in $\mathbf{P}_k$.

The boundness of the degree means that the graphs lie as cycles in finitely many components of the cycle space $C(V \times V)$. Therefore, the condition of boundness is independent of the choice of the embedding ν.

Theorem 3. *Let V be a projective variety, $D \subset V$ an open subset and G a topological group of birationally extendible automorphisms of D. Then G has an algebraic extension if and only if the degree of the automorphisms $\phi_g : D \to D$ defined by $g \in G$ is bounded.*

In the proof we proceed as follows. Theorem 2 is proven in section 4. If $\tilde{G}$ is the algebraic extension, the rationality of V is used to show that $\tilde{G}$ is a linear algebraic group. Then the linearization follows from a theorem of Sumihiro ([16]). The converse in Theorem 2 is straightforward.

Section 5 is devoted to the proof of Theorem 3. There we exploit the idea that an action of $\tilde{G}$ by rational automorphisms on $D \subset V$ induces an (almost everywhere defined) mapping $\phi_{\tilde{G}}$ from $\tilde{G}$ into the cycle space $C(V \times V)$ which can be regarded as a subvariety of the Chow scheme of an ambient projective space $\mathbf{P}_k$. Here we use the universal property of the cycle space ([1], see also [5], Proposition 2.20). The mapping $\phi_{\tilde{G}} : \tilde{G} \to C_n(\mathbf{P}_k)$ is rational and the boundness of the degree follows from the local constancy of it on the Chow scheme.

The induced mapping ϕ_G from G into the cycle space $C(V \times V)$ is continuous only on an open dense subset $U \subset G$, but the group operation of G extends to a rational "group operation" on the Zariski closure of $\phi_G(U)$ in $C(V \times V)$. This operation is defined via composition of graphs. The objects with rational "group operations" were introduced by Weil ([30]) and called **pre-groups**. The main property is the existence of regularizations of pre-groups, i.e. algebraic groups which are birationally equivalent to given pre-groups and the "group operations" are compatible with the equivalences. This property is used to obtain the algebraic group $\tilde{G}$ for the algebraic extension.

The next step is to prove that the composition of $\phi : U \to C(V \times V)$ and the birational equivalence with $\tilde{G}$ extends to a continuous homomorphism from G into $\tilde{G}$.

The induced "action" of $\tilde{G}$ on V is in general also rational. Such objects were also introduced by Weil ([30]) and called **pre-transformation spaces**. Also in this case he proves the existence of regularizations, i.e. the (regular) actions of the same group on algebraic varieties which are birationally equivalent to the original pre-transformation spaces such that the actions are compatible with the equivalences. In our case, such regularizations X yield the required algebraic extensions.

An exposition for pre-groups and pre-transformation spaces (also not irreducible) is given in [34]. There, one also studies the points where the above regularizations are biregular. This helps in proving that D is embedded biholomorphically in the context of Definitions 1.1 and 1.2.

Theorem 1 is proven in section 6. For this we use a result of Kazaryan ([15]) to show that the action $G \times D \to D$ extends to a meromorphic mapping $\tilde{G} \times V \to V$, where $\tilde{G}$ is a complex manifold with G totally really embedded. Then we prove the boundness of the degree using Proposition 5.1 and the lower semi-continuity of the degree (Lemma 5.1). Finally, the statement follows from Theorem 3.

3. Semialgebraic domains and Nash automorphisms

In general, a domain $D \subset \mathbf{C}^n$ may have no non-trivial holomorphic automorphisms. On the other hand, in many interesting cases the automorphism group is very large. The classical examples are bounded homogeneous domains. Vinberg, Gindikin and Piatetski-Shapiro ([28]) classified them and found their canonical realizations as Siegel domains of II kind. Rothaus ([24]) proved that such realizations are given by (real) polynomial inequalities. For such reasons, as well as those mentioned in the introduction, we are intereseted in studiying domains defined in this way. In fact, we consider more general case of a projective variety V and an open set $D \subset V$ which is a finite union of the domains given by finitely many homogeneous polynomial inequalities. Such set are considered in real algebraic geometry and are called **semialgebraic** (see e.g. [2] for the elementary introduction to the theory of semialgebraic sets).

The following Proposition shows that, for D semialgebraic, the condition given in Theorem 1 is in some sense also necessary .

Proposition 3.1. *Let V be a projective variety, $D \subset V$ a semialgebraic open subset and G a topological group of birationally extendible automorphisms of D. Suppose that there exists an algebraic extension of the action of G. Then G is a subgroup of a Lie group $\tilde{G}$ of birationally extendible automorphisms of D which extends the action of G to a real analytic action $\tilde{G} \times D \to D$ and has finitely many connected components.*

Semialgebraic sets are closely related to the **Nash manifolds** and *Nash groups*. The Nash category is obtained from the real analytic when we assume all mappings are Nash. A **Nash mapping** $f: D \to D$ is a real analytic mapping, such that the graph $\Gamma \subset D \times D$ of f is semialgebraic or, equivalently, Zariski closure Z_f of Γ in $V \times V$ has dimension $n = \dim V$. The reader is refered to [18] and [26] for the precise definitions.

For a semialgebraic subset $D \subset V$ we prove also the following criterion.

Theorem 4. *Let V be a projective variety, $D \subset V$ a semialgebraic open subset and G a topological group of birationally extendible automorphisms of D. The following properties are equivalent:*

1) *G is a subgroup of a Nash group $\tilde{G}$ of* **birationally extendible automorphisms** *of D which extends the action of G to a Nash action $\tilde{G} \times D \to D$;*
2) *G is a subgroup of a Nash group $\tilde{G}$ such that the action $G \times D \to D$ extends to a Nash action $\tilde{G} \times D \to D$;*
3) *G has an algebraic extension.*

Remarks.

1) In condition 2 the automorphisms defined by elements of $\tilde{G} \setminus G$ are not necessarily birationally extendible.
2) Since every Nash group is a Lie group with finitely many components, Proposition 3.1 is a Corollary of Theorem 4.

The proof of Theorem 4 is given in section 7.

In the remainder of the present paragraph we mention several applications of Theorem 4 for the bounded semialgebraic domains. In [33] we gave sufficient conditions on D and G such that G is a Nash group and the action $G \times D \to D$ is Nash. The domain D is assumed to satisfy the following nondegeneracy condition:

Definition 3.1 A boundary of a domain $D \subset \mathbf{C}^n$ is called **Levi nondegenerate** if it contains a smooth point where the Levi form is nondegenerate.

The group G is taken to be the group $Aut_a(D)$ of all holomorphic Nash (algebraic) automorphisms of D. It was proven in [33] that, if D is a semialgebraic bounded domain with Levi nondegenerate boundary, the group $Aut_a(D)$ is closed in the group $Aut(D)$ of all holomorphic automorphisms and carries a unique structure of a Nash group such that the action $Aut_a(D) \times D \to D$ is Nash with respect to this structure.

Now let $G = Aut_r(D) \subset Aut_a(D)$ be the group of all birationally extendible automorphisms of D. Then G satisfies the property 2 in Theorem 4 with $\tilde{G} = Aut_a(D)$. By property 1, G is a subgroup of a Nash group of birationally extendible automorphisms of D. Since G contains all birationally extendible automorphisms of D, G is itself a Nash group with the Nash action on D. We therefore obtain the following corollary.

Corollary 1. *Let $D \subset \mathbf{C}^n$ be a bounded Nash domain with Levi nondegenerate boundary. Then the group $Aut_r(D)$ possesses an algebraic extension.*

We now explain sufficient conditions (due to Webster [29]) such that all algebraic automorphisms of D are birationally extendible. Let D be as in Corollary 1. The existence of finite stratifications for semialgebraic sets (see [2], (2.4.4)) implies that the boundary ∂D is contained in finitely many irreducible real hypersurfaces. Several of them, let us say $M_1, \dots, M_k$, have generically nondegenerate Levi forms. If ∂D is nondegenerate in the sense of Definition 3.1, such hypersurfaces exist. The complexification $M_i^{\mathbf{C}}$ of M_i is defined to be the complex Zariski closures of M_i in $\mathbf{C}^n \times \overline{\mathbf{C}^n}$ where M_i is embedded as a totally real subvariety via the diagonal map $z \mapsto (z, \bar{z})$. It follows that $M_i^{\mathbf{C}}$ is an irreducible complex hypersurface. The Segre varieties $Q_{iw}, w \in \mathbf{C}^n$, associated to M_i are defined by

$$Q_{iw} := \{z \in \mathbf{C}^n \mid (z, \bar{w}) \in M_i^{\mathbf{C}}\}.$$

These complexifications and Segre varieties are important biholomorphic invariants of D and play a decisive role in the reflection principle which can be used to obtain birational extensions.

Definition 3.2 A semialgebraic domain is said to satisfy the condition (W) if, for all i, the Segre varieties Q_{iw} uniquely determine $z \in \mathbf{C}^n$ and Q_{iw} is an irreducible hypersurface in $\mathbf{C}^n$ for all z in the complement of a proper subvariety $V_i \subset \mathbf{C}^n$.

A result of Webster ([29], Theorem 3.5) can be formulated in the following form:

Theorem 5. *Let $D \subset \mathbf{C}^n$ be a semialgebraic domain with Levi nondegenerate boundary which satisfies the condition (W). Further, let $f \in Aut(D)$ be an automorphism which is holomorphically extendible to a smooth boundary point with nondegenerate Levi form. Then f is birationally extendible to $\mathbf{C}^n$.*

Remark. The mentioned statement of Webster assumes that f extends biholomorphically to a smooth boundary point where the Levi form is nondegenerate. By a result of Diederich and Pinchuk ([7]), this holds for all automorphisms.

Corollary 2. *Let $D \subset \mathbf{C}^n$ be a bounded semialgebraic domain which satisfies condition (W). Then the whole group $Aut(D)$ possesses an algebraic extension.*

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4. Linearization

In the present paragraph we prove Theorem 2. Assume we are given a projective linearization $i: V \to \mathbf{P}_N$. Let X denote the (Zariski) closure of the (constructible) image $i(V)$. The subgroup $\tilde{G} \subset GL_N(\mathbf{C})$ of all linear automorphisms of $\mathbf{P}_N$ which preserve X is a complex algebraic subgroup. Then the pair $(\tilde{G}, X)$ yields the required algebraic extension.

The other direction is less trivial. If G has an algebraic extension, we can assume without loss of generality that G coincides with the complex algebraic group $\tilde{G}$. For the convenience of reader we reformulate here the conclusion we need to prove.

Theorem 2'. *Let G be an complex algebraic group operating regularly on a rational algebraic variety X. Let D be an open set contained in the regular locus of a quasi-projective subvariety $U \subset X$. Then there exists a projective linearization.*

Theorem $2'$ will follow from Lemma 4.1., Proposition 4.2. and Sumihiro's Theorem (see below). Since the regular locus of X is G-invariant, we can replace X with this locus and Proposition 4.2 can be applied.

Definition 4.1. *A line bundle L on an algebraic variety X is called* **birationally very ample** *if there exists a finite-dimensional subspace $W \subset \Gamma(X, L)$ which yields a birational mapping i_W from X into the corresponding projective space.*

Lemma 4.1. *Let G be a (complex) algebraic group with a regular action $\rho: G \times X \to X$ on a nonsingular (not necessarily projective) algebraic variety X. Then there exists a birationally very ample line bundle L on X such that, for every $g \in G$, $\rho_g^* L \cong L$. If $U \subset X$ is an open quasi-projective subvariety, the bundle L and subspace $W \subset \Gamma(X, L)$ can be chosen such that i_W is regular on U.*

Proof. Without loss of generality, U is an open dense quasi-projective subvariety of X. Then the inclusion $\varphi: U \to X$ is birational. Let C be a very ample divisor on U and $v_0, \dots, v_N$ a collection of rational functions on U which yields a basis of $O_U(C)$. The rational functions $v_0 \circ \varphi^{-1}, \dots, v_N \circ \varphi^{-1}$ define a birational (onto the image) mapping from X into $\mathbf{P}_N$. Let C' be the union of polar divisors of all $[\tilde{v}_i]$ and $L_{C'} \in Pic(X)$ be the corresponding line bundle. Then $\tilde{v}_i$'s can be regarded as sections in $L_{C'}$ which is therefore birationally very ample.

It remains to obtain the property $\rho_g^* L \cong L$. The birational mapping $\varphi\colon \mathbf{P}_n \to X$ is, by definition, a biregular mapping between Zariski open subsets $U \subset \mathbf{P}_n$ and $U' \subset X$. Set $E := \mathbf{P}_n \setminus U$ and $E' := X \setminus U'$ and let $E_1, \dots, E_k$ and $E'_1, \dots, E'_l$ be the irreducible components of E and E' respectively. One has the following exact sequences:

$$\oplus_i \mathbf{Z}[E_i] \to Pic(\mathbf{P}_n) \to Pic(U) \to 0,$$

$$\oplus_i \mathbf{Z}[E'_i] \to Pic(X) \to Pic(U') \to 0.$$

Since $Pic(\mathbf{P}_n) \cong \mathbf{Z}$, it follows that $Pic(U) \cong Pic(U')$ is discrete. This implies that $Pic(X)$ is discrete. The algebraic group G has finitely many connected components. Therefore, its orbits in $Pic(X)$ are finite. Thus $G(L_{C'}) = \{L_1, \dots, L_s\}$ as an orbit in $Pic(X)$. Since the L_j's are birationally very ample, their tensor product $L := \otimes_j L_j$ is also birationally very ample and satisfy the property $\rho_g^* L \cong L$. **QED**

We now state and prove a sequence of Lemmas which will yield the proof of Proposition 4.1.

Lemma 4.2 . *Let G and X be arbitrary nonsingular algebraic varieties and X be birationally equivalent to $\mathbf{C}^n$. Let $L_{G\times X}$ be a line bundle on $G \times X$. Then there exist line bundles L_G on G and L_X on X such that $L_{G\times X} \cong \pi_G^* L_G \otimes \pi_X^* L_X$.*

Proof. The special case $X = \mathbf{C}$ is contained in Proposition 6.6. of Chapter 2 in [11]. By the induction, we obtain the Lemma for $X = \mathbf{C}^n$. In the general case one has isomorphic Zariski open subsets $U \subset \mathbf{C}^n$ and $U' \subset X$. Set $E := \mathbf{C}^n \setminus U$ and $E' := X \setminus U'$ and let $E_1, \dots, E_k$ and $E'_1, \dots, E'_l$ be the irreducible components of E and E' respectively. Let $L_{G\times X}|G \times U'$ be the restriction and $L_{G\times U}$ its pullback on $G \times U \cong G \times U'$. Since $L_{G\times U}$ corresponds to a divisor C on $G \times U$, it is a restriction of a line bundle $L_{G\times \mathbf{C}^n}$ on $G \times \mathbf{C}^n$ which corresponds to the closure of C in $G \times \mathbf{C}^n$. Applying the Lemma to $L_{G\times \mathbf{C}^n}$, we obtain its splitting which yields a splitting $L_{G\times U'} \cong \pi_G^* L_G \otimes \pi_X^* L_{U'}$. The required splitting for $L_{G\times X}$ is implied now by the surjectivity of the following map:

$$\oplus_i \mathbf{Z}[G \times E_i'] \oplus Pic(G \times U') \to Pic(G \times X).$$

QED

Lemma 4.3. *Let $1 \to G_1 \to G \to G_2 \to 1$ be an exact sequence of algebraic groups. Let either G or both G_1 and G_2 be linear. Then all of groups are linear.*

See e.g. [23] for the proof.

Lemma 4.4. *Let G be an algebraic group with an effective algebraic action $\rho: G \times X \to X$, where X is a nonsingular algebraic variety with $O^*(X) \cong \mathbf{C}^*$. Let L be a birationally very ample line bundle on X such that, for every $g \in G$, $\rho_g^* L \cong L$. Then there exists an algebraic group $\tilde{G}$ with a surjective homomorphism $\pi: \tilde{G} \to G$ such that*

1) *the action $\tilde{G} \times X \to X$ defined by π is lifted to an action $\tilde{G} \times L \to L$, which preserves the fibres and is linear there;*
2) *the kernel of π acts effectively on L.*

Proof. Let $\phi: \rho^* L \to \pi_G^* L_G \otimes \pi_X^* L_X$ be the isomorphism in Lemma 4.2. Since $\rho_g^* L \cong L$, one has $L_X \cong L$.

Let $\tilde{G} \subset L_G$ be the complement of the zero section. Our goal now is to define an algebraic group structure on $\tilde{G}$ and to construct an algebraic action $\tilde{G} \times L \to L$. The action $\tilde{\rho}: \tilde{G} \times L \to L$ is defined as the composition

$$L_G \times L \to \pi_G^* L_G \otimes \pi_X^* L \to \rho^* L \to L, \tag{4.1}$$

where the first mapping is given by two isomorphisms $L_G \times X \to \pi_G^* L_G$ and $G \times L \to \pi_X^* L$. The composition (4.1) makes the following diagram commutative:

$$\begin{array}{ccc} \tilde{G} \times L & \to & L \\ \downarrow & & \downarrow \\ G \times X & \to & X \end{array}$$

Let $g \in G$ be fixed. Then the fibre $(L_G^*)_g (\cong \mathbf{C}^*)$ of $\tilde{G}$ over g defines a 1-dimensional family of automorphisms of L which lift the automorphism $\rho_g: X \to X$ defined by the action of G. (By an automorphism of L we mean an algebraic isomorphism of L onto itself which takes fibres in fibres and is linear on them.) Since $O^*(X) \cong \mathbf{C}^*$, such automorphisms of L form a 1-dimensional family which coincides therefore with the family defined by $(L^*_G)_g$. We obtain a one-to-one correspondence between the elements of $\tilde{G}$ and the liftings of the automorphisms $\rho_g: X \to X$ for $g \in G$.

The set of all automorphisms of L which lift ρ_g for some $g \in G$ forms a group in a natural way. The above one-to-one correspondence transfers this group structure to $\tilde{G}$. The regular mapping in (4.1) defines a group action $\tilde{G} \times L \to L$ with respect to this structure.

Now we wish to prove that the group operation $\tilde{G} \times \tilde{G} \to \tilde{G}$, $(g,h) \to gh$ is algebraic. Since the action $\tilde{\rho}: \tilde{G} \times L \to L$ is algebraic, the map

$$\alpha: \tilde{G} \times \tilde{G} \times L \to L, (g,h,l) \mapsto \tilde{\rho}(g, \tilde{\rho}(h,l))$$

is also algebraic. We find the product $t := gh \in \tilde{G}$ from the relation

$$\alpha(g,h,l) = \rho(t,l). \tag{4.2}$$

For a fixed arbitrary point $l_0 \in \tilde{G}$, the mapping

$$\imath := \pi \times \rho(\cdot, l_0): \tilde{G} \to G \times L$$

is a regular embedding. By (4.2), $t = gh$ can be expressed as follows:

$$t(g,h) = \imath^{-1} \circ (\pi(g)\pi(h), \alpha(g,h,l_0)).$$

This proves the algebraicity of the group operation on $\tilde{G}$. It remains to prove that the inverse map $\tilde{G} \to \tilde{G}, g \mapsto g^{-1}$ is also regular. For this consider

$$\Gamma := \{(g,h) \in \tilde{G} \times \tilde{G} \mid \tilde{\rho}(g, \tilde{\rho}(h,l)) = l \text{ for all } l \in L\}.$$

This is the graph of $g \mapsto g^{-1}$ which projects bijectively on both factors $\tilde{G}$. Since Γ is an algebraic subset and $\tilde{G}$ is nonsingular, the inverse mapping is regular. **QED**

The following is a foundational result for algebraic group actions.

Lemma 4.5. *Let $G \times X \to X$ be an algebraic action of an algebraic group G on an algebraic variety X which lifts to an action on a line bundle L on X. Then the induced action on the space of sections $\Gamma(X,L)$ is rational and locally finite.*

The proof coincides with the proof of Lemma 2.5. in [16], where G is regarded as an arbitrary algebraic group.

Proposition 4.1. *Let G be an algebraic group with an effective algebraic action $\rho: G \times X \to X$, where X is a rational nonsingular algebraic variety with $O^*(X) \cong \mathbf{C}^*$. Let L be a birationally very ample line bundle on X such that, for every $g \in G$, $\rho_g^* L \cong L$. Then G is linear algebraic.*

Proof. Let $W \subset \Gamma(X, L)$ be the finite dimensional subspace in Definition 4.1. By Lemma 4.5. applied to the group $\tilde{G}$, W generates a finite dimensional invariant subspace $\tilde{W} \subset \Gamma(X, L)$ which also yields a birational mapping $i_{\tilde{W}}: X \to \mathbf{P}(\tilde{W}^*)$. We obtain a representation of $\tilde{G}$ in $\tilde{W}^*$. Let $K \subset \tilde{G}$ be its kernel. An element $k \in K$ acts trivially on $i_{\tilde{W}}(X)$ and therefore on X. Since the action $G \times X \to X$ is effective, this implies that $K \subset Ker\,\pi$. But the kernel of π acts effectively on $\Gamma(X, L)$ which implies $K = \{e\}$.

Thus, $\tilde{G}$ is a linear algebraic group. Since G is a homomorphic image of $\tilde{G}$, it is also linear algebraic. **QED**

Lemma 4.6. *Let X be a rational nonsingular algebraic variety and $\rho: G \times X \to X$ be an algebraic action of an algebraic group G which satisfies the property $O^*(G) \cong \mathbf{C}^*$. Let L be a birationally very ample line bundle on X such that for every $g \in G$, $\rho_g^* L \cong L$. Then the action of G on X is trivial.*

Proof. We prove the Lemma by induction on $\dim X$. The condition $O^*(G) \cong \mathbf{C}^*$ implies the connectedness and irreducibility of G. Let $\dim X = 0$. Then X is discrete and the action is trivial.

Now assume $\dim X \geq 1$. Let $x_0 \in X$ be an arbitrary point and define

$$F(x_0) := \{x \in X \mid \forall f \in O^*(X), f(x) = f(x_0)\} \subset X.$$

Since $O^*(G) \cong \mathbf{C}^*$, the orbit Gx_0 lies in $F(x_0)$. This is true for any orbit Gx with $x \in F(x_0)$ and therefore $F(x_0)$ is G-invariant. Let $X_0 \subset X$ be an irreducible component with $x_0 \in X_0$. Since G is irreducible, X_0 and $F'(x_0) := X_0 \cap F(x_0)$ are also G-invariant.

Now two cases are possible. If $\dim F'(x_0) < \dim X$, the action on $F'(x_0)$ is trivial by induction. If $\dim F'(x_0) = \dim X$, then $F'(x_0) = X_0$ and $O^*(X_0) \cong \mathbf{C}^*$. By Proposition 4.1, $\hat{G} := G/Ker(\rho|_{X_0})$ is a linear algebraic group. The condition $O^*(G) \cong \mathbf{C}^*$ for G implies the same condition for $\hat{G}$. Since $\hat{G}$ is linear algebraic, it is trivial. Thus, $Ker(\rho_{X_0}) = G$ which means that the action on X_0 is trivial.

In summary we obtain that, for every $x_0 \in X$ and $g \in G$, $gx_0 = x_0$. This means that G acts trivially. **QED**

The following is straightforward.

Lemma 4.7. *Let G be an algebraic group and $e \in G$ the unit. Then the subvariety*

$$F(e) := \{g \in G \mid \forall f \in O^*(G), f(g) = f(e)\}$$

is an algebraic subgroup.

Lemma 4.8. *Let G be an algebraic group such that the global invertible regular functions separate its points. Then G is linear algebraic.*

This is a corollary of the following lemma:

Lemma 4.9. *Let G be an algebraic group such that the global regular functions separate points of it. Then G is linear algebraic.*

Proof. By Corollary 3. in [23], page 431, there exists an algebraic subgroup $D \subset G$ such that the quotient G/D is linear and such that the kernel of any algebraic homomorphism from G into a linear group contains D. It is enough to prove that $D = \{e\}$.

Assume the contrary. Let $g \neq e$ be an arbitrary point in D. Since the points of G are separated by global regular functions, there exists a function $f \in O(G)$ such that $f(g) \neq f(e)$. By Lemma 4.5, f generates a finite dimensional G-invariant subspace $W \subset O(G)$. The canonical representation of G in W is a homomorphism from G into a linear group such that its kernel does not contain g. This contradicts to the property of D and the fact that $g \in D$. **QED**

Now we drop the assumption $O^*(X) \cong \mathbf{C}^*$ in Proposition 4.1.

Proposition 4.2. *Let G be an algebraic group with an effective algebraic action $\rho: G \times X \to X$ on an algebraic variety X. Let L be a birationally very ample line bundle on X such that, for every $g \in G$, $\rho_g^* L \cong L$. Then G is linear algebraic.*

Proof. We proceed by induction on $\dim G$. The Proposition is trivial for $\dim G = 0$.

By Lemma 4.3, we can assume G to be connected. Let $F(e)$ be the algebraic subgroup defined in Lemma 4.7. If $\dim F(e) = \dim G$, it follows that $F(e) = G$ which implies $O^*(G) \cong \mathbf{C}^*$. Then, by Lemma 4.6, G acts trivially. Since it acts also effectively it is trivial (and of course linear algebraic).

If $\dim F(e) < \dim G$, the subgroup $F(e)$ is linear by the induction. The group $G/F(e)$ satisfies conditions of Lemma 4.8 and is also linear algebraic. Then, by Lemma 4.3, G itself is linear. **QED**

Sumihiro' Theorem. *Let G be a* **linear** *algebraic group operating regularly on a rational algebraic variety X. Let D be an open set contained in the regular locus of a quasi-projective subvariety $U \subset X$. Then there exists a projective linearization.*

The original proof ([16,27]) is given for the case D is an orbit or U is G-invariant. In the general case we take a birationally very ample bundle L given by Lemma 4.1 and follow the proof in [16].

5. Algebraic extensions for bounded degree

The goal of this section is to prove Theorem 3.

Recall that by the *degree* of ϕ_g with respect to a fixed (biregular) embedding $\nu: V \times V \to \mathbf{P}_k$ we mean the degree of the closed graph $Z_g \subset V \times V$ of ρ_g embedded in $\mathbf{P}_k$ via ν.

We use the following universal property of the cycle space ([1], see also [5], Proposition 2.20):

Proposition 5.1 . *Let X and S be irreducible complex spaces. There exist a natural identification between:*

1) meromorphic maps $\phi: S \to C_n(X)$, and

2) S-proper pure $(d+n)$-dimensional cycles F of $S \times X$ $(d = \dim S)$.

Let $\tilde{G}$ be an algebraic extension of G as in Definition 2.1 and $\Gamma \subset \tilde{G} \times V \times V$ be the graph of the rational "action" of the algebraic group $\tilde{G}$. By Proposition 5.1, this action induces a rational mapping $\mu: \tilde{G} \to C_n(\mathbf{P}_k)$. The finiteness of the number of irreducible components of $\tilde{G}$ implies the boundness of the degree of ϕ_g for all g in an open dense subset of $\tilde{G}$. The global boundness is obtained by the following lemma.

Lemma 5.1. *The degree is a lower-semicontinuous function on G.*

Proof. Let $g_0 \in G$ be an arbitrary point and $g_m, m \in \mathbf{N}$ an arbitrary sequence with $g_m \to g_0$. It is enough to prove that $deg(Z_{g_m}) \geq deg(Z_{g_0})$ up to finite set of $m \in \mathbf{N}$. Assume on the contrary that $deg(Z_{g_m}) < deg(Z_{g_0})$ for a subsequence which is again denoted by g_m. By a theorem of Bishop ([3]), Z_{g_m} can be assumed to converge to some cycle Z_0 with $deg(Z_0) < deg(Z_{g_0})$. By the continuity of the action ρ, one has $Z_{g_0} \cap D \times D \subset Z$, which implies $Z_{g_0} \subset Z_0$. On the other hand, by the continuity of degree (which is equivalent to the continuity of the volume), $deg(Z_0) < deg(Z_{g_0})$, which contradicts the above inclusion. **QED**

1. Formulation.

The other less trivial direction in Theorem 3 will be a corollary of the following statement:

Theorem 3′. *Let $D \subset V$ be an open subset in a projective variety V, G a topological group and $\rho: G \times D \to D$ a continuous action such that, for every $g \in G$, the homeomorphism $\rho_g: D \to D$ extends to a birational mapping from V into itself (which we also denote by ρ_g). Assume that the set of degrees of all $\rho_g, g \in G$ is bounded. Then there exist:*
1) an algebraic group $\tilde{G}$,
2) a continuous homomorphism $\phi: G \to \tilde{G}$,
3) an algebraic variety X,
4) an algebraic action $\tilde{G} \times X \to X$,
5) a birational mapping $\psi: V \to X$ such that $\psi|_D$ is biholomorphic and G-equivariant.

2. Properties of the group G.

We begin by noting an elementary basic fact.

Lemma 5.2. *Let G be a topological space and $f: G \to \mathbf{Z}$ a lower-semicontinuous function which is bounded from above. Then the set $U \subset G$ of all local maximums of f is open and dense in G. Moreover, $f|_U$ is locally constant.*

Let $Z \subset G \times V \times V$ and $\nu(Z) \subset G \times \mathbf{P}_k$ be the families of all Z_g and $\nu(Z_g)$, $g \in G$, respectively. We denote by $U \subset G$ the set of all local maxima of the degree, which is open dense by Lemma 5.2.

Lemma 5.3. *Let U be a topological space and $\{\phi_g\}_{g \in U}$ a continuous family of automorphisms of D which exted to birational mappings from V to V with (closed) graphs Z_g. Assume that the degree of Z_g is locally constant on U. Let the automorphisms depend continuously on $u \in U$. Then the family Z is closed in $U \times V \times V$.*

Proof. Let $(g_0, z_0) \in U \times V \times V$ be a point and $(g_m, z_m) \to (g_0, z_0)$ a sequence with $z_m \in Z_{g_m}$. By a theorem of Bishop ([3]), the sequence of cycles Z_{g_m} can be assumed to converge to some Z_0. By the continuity of the automorphisms, one has $Z_{g_0} \subset Z_0$. Since $g_0 \in G$ is a local maximum of the degree and the degree is a continuous function on cycles, one obtains $deg Z_0 \leq deg Z_{g_0}$. This means $Z_{g_0} = Z_0$ and $(g_0, z_0) \in Z$. **QED**

Using the family $Z \cap (U \times V \times V) \subset U \times \mathbf{P}_k$ we define a continuous mapping ϕ from U into the Chow scheme C of cycles in $\mathbf{P}_k$ (see [10,25]). We recall briefly the construction of the components of C. Let $Z_g, g \in P$ be an arbitrary family of irreducible subvarieties of $\mathbf{P}_k$ of fixed dimension n and degree d, parameterized by a set U. The $n+1$-tuples $(H_0, \dots, H_k)$ of hyperplanes in $\mathbf{P}_k$ are parameterized by $S := (\mathbf{P}_k^*)^{n+1}$. We define $V_g \subset \mathbf{P}_k \times S$ by

$$V_g := \{(z, H_0, \dots, H_n \mid z \in Z_g \cap H_0 \cap \dots \cap H_n,$$

and denote by $\pi(V_g) \subset S$ its projection. Then all V_g's, $g \in U$ and, therefore, all $\pi(V_g)$'s are irreducible subvarieties. Moreover, $\pi(V_g)$'s are of codimension 1 and of multidegree $(d, \dots, d)$. They are given uniquely up to multiplications by constants by multihomogeneous polynomials $R_g \subset \mathbf{C}[S]_{d,\dots,d}$ of multidegree $(d, \dots, d)$.

Let $\mathbf{P}_N = \mathbf{P}(\mathbf{C}[S]_{d,\dots,d})$ denote the projectivization of the space of such polynomials and $N_R \subset S, R \in \mathbf{P}_N$ the family of zero sets of them. Therefore we obtain a mapping $\phi: U \to \mathbf{P}_N$ which associates to every $g \in U$ the Chow coordinates $[R] = [R_g] \in \mathbf{P}_N$ of Z_g such that $\pi(V_g) = N_R$.

We utilize the following topological universal property of the Chow scheme:

Proposition 5.2. *Let U be a topological space and $Z_g \in \mathbf{P}_k, g \in U$, a closed family, i.e. the subset*

$$Z = \{(g, z) \mid z \in Z_g\} \subset U \times \mathbf{P}_k$$

is closed. Suppose that the dimension and degree of Z_g are constant. Then $\phi: U \to C$ is a continuous mapping.

Proof. The closedness of $Z_g, g \in U$, implies the closedness of $V_g, g \in U$, because the latter is defined by a closed condition. Since the projective space $\mathbf{P}_k$ is compact, the family of projections $\pi(V_g)$ is also closed. The graph $\Gamma \subset U \times \mathbf{P}_N$ of the mapping ϕ is defined by the condition

$$\Gamma = \{(g, [R]) \mid Z_g \subset N_R\}.$$

It is sufficient to prove that $\Gamma \subset U \times \mathbf{P}_N$ is closed.

Let $(g_0, [R_0])$ be a point in the complement of Γ. This means that $R_0(z_0) \neq 0$ for some $z_0 \in Z_{g_0}$. Then there exist a neighborhood $U(z_0) \subset \mathbf{P}_k$ of z_0 and a neighborhood $U(R_0) \subset \mathbf{P}_N$ of $[R_0]$ such that $R(z) \neq 0$ for all $z \in U(z_0)$ and $[R] \in U(R_0)$. We claim that there exists a neighborhood $U(g_0) \subset U$ such that $U(z_0) \cap Z_g \neq \emptyset$ for all $g \in U(z_0)$. Indeed, otherwise there would be a sequence $g_m \to g_0$ without this property for Z_{g_m}. By a theorem of Bishop ([3]), one has, passing if necessary to a subsequence, $Z_{g_m} \to Z_{g_0}$, which is a contradiction.

Therefore, the whole neighborhood $U(p_0) \times U(R_0)$ of $(g_0, [R_0])$ belongs to the complement of Γ. This proves the closedness of the graph Γ which means the continuity of the mapping ϕ. **QED**

The Chow scheme C is a collection of projective varieties parameterized by the dimension and degree of cycles. In Theorem 5.4 we assume that the set of all degrees of $Z_g, g \in G$ is bounded. Therefore, the image $\phi(U)$ is contained in finitely many components of the Chow scheme. Let Q denote the Zariski closure of $\phi(U)$ in C. It is a projective variety. Let $F \subset Q \times \mathbf{P}_k$ denote the universal family over Q. Since $F_v \subset V \times V$ for all v from the Zariski dense subset $\phi(U)$, one has $F \subset Q \times V \times V$.

Lemma 5.4. *Let Q be an algebraic variety and $F \subset Q \times V \times V$ a closed algebraic family of subvarieties $F_v \subset V \times V, v \in Q$, of pure dimension n. For every v from a Zariski dense subset $\phi(U) \subset Q$, assume that the fibre F_v is the closed graph of a birational mapping $\rho_v : V \to V$. Then this is true for all v from a Zariski* **open** *dense subset Q' with $\phi(U) \subset Q' \subset Q$. Moreover, there exists a Zariski open subset $F' \subset F$ which intersects every graph $F_v, v \in Q'$, along a Zariski dense graph of a* **biregular** *mapping ϕ'_v.*

Proof. Let $Q_1 \subset F$ be the set of all $(v, x) \in Q \times V$ such that the fibres $F_{(v,x)} \subset V$ are finite. Since the fibre dimension is upper-semicontinuous, Q_1 is a Zariski open subset of $Q \times V$. The family F is a finite ramified covering of Q_1. The set

$$R := \{(v, x) \in Q \times V \mid \rho_v \text{ is biregular at } x \}.$$

is a dense subset of Q_1 and the fibres $F_{(v,x)}$ over R consist of single points. Therefore the covering F has only one sheet and every fibre $F_{(v,x)}$, $(v, x) \in Q_1$, consists of a single point. If, for some $v \in Q$, $(\{v\} \times V) \cap Q_1$ is dense in $(\{v\} \times V)$, this means that $F_v \subset V \times V$ is the graph of a rational mapping $\rho_v : V \to V$. This is true for all v from a Zariski open dense subset $Q'_1, \phi(U) \subset Q'_1 \subset Q$, which can be taken to be the intersection of the projections of irreducible components of Q_1 on Q.

Similarly, using the projection on the product of Q and the other copy of V, we can construct Zariski open dense subsets $Q_2 \subset Q \times V$, $Q_2', \phi(U) \subset Q_2' \subset Q$, such that ϕ_v^{-1} is regular at $x \in V$ for all $(v,x) \in Q_2$ and F_v is a graph of a mapping ϕ_v with rational inverse for all $v \in Q_2'$. Then the intersection $Q' := Q_1' \cap Q_2'$ satisfies the required properties.

The required Zariski open subset $F' \subset F$ can be given by the formula

$$F' := \pi_1^{-1}(Q_1) \cap \pi_2^{-1}(Q_2),$$

where $\pi_1, \pi_2 \colon F \to Q \times V$ denote the projection on the product of Q and the first (resp. the second) copy of V. **QED**

Now we wish to extend the group operation $U \times U \to G$ to a rational mapping $Q \times Q \to Q$. Let $Q' \subset Q$ be given by Lemma 5.4.

Lemma 5.5. *There exist rational mappings $\alpha, \alpha_1, \alpha_2 \colon Q \times Q \to C$ such that for all $(v,w) \in Q' \times Q'$, where α (resp. α_1 and α_2) is defined, the fibre $F_{\alpha(v,w)}$ (resp. $F_{\alpha_1(v,w)}$ and $F_{\alpha_1(v,w)}$) coincides with the closed graph of the birational correspondence $\rho_v \circ \rho_w$ (resp. $\rho_v \circ \rho_w^{-1}$ and $\rho_w^{-1} \circ \rho_v$).*

In the construction of the mappings α, α_1 and α_2 we use the following algebraic universal property of the Chow scheme. Recall that C denotes the Chow scheme of $\mathbf{P}_k$ and $F \subset C \times \mathbf{P}_k$ the universal family over C.

Proposition 5.3. *Let X be a quasi-projective variety, $Z \subset X \times \mathbf{P}_k$ a closed pure-codimensional subvariety. Then there exists a rational mapping $i \colon X \to C$ with $Z_g = F_{i(v)}$ for all $v \in X$, such that i is regular at v.*

This is a consequence of Proposition 5.1 and Chow's theorem ([9], p. 167).

Proof of Lemma 5.5. We construct here the extension α_1 of the mapping $(g,h) \mapsto gh^{-1}$. The construction of α and α_2 is completely analogous.

The idea of construction is to consider the family of graphs of $gh^{-1} \colon V \to V$ over $Q \times Q$ and to utilize the above universal property for it. Let W_1, W_2 and W_3 denote different copies of V and π_1 ,π_2 and π_3 be the projections of $W_1 \times W_2 \times W_3$ onto $W_2 \times W_3$, $W_1 \times W_3$ and $W_1 \times W_2$ respectively. Then, for $g_1, g_2 \in U$, the graph of $\phi_{g_1 g_2^{-1}}$ is equal to the closure of

$$Z'_{g_1 g_2^{-1}} = \pi_2(\pi_3^{-1}(Z'_{g_2}) \cap \pi_1^{-1}(Z'_{g_1})), \tag{5.1}$$

where $Z'_{g_1} \subset W_2 \times W_3$ and $Z'_{g_2} \subset W_2 \times W_1$ are the regular parts of the graphs of $\rho_{g_1} \colon W_2 \to W_3$ and $\rho_{g_2} \colon W_2 \to W_1$ respectively.

Using formula (5.1) we define a constructible family $\tilde{F} \subset Q' \times Q' \times W_1 \times W_3$:

$$\tilde{F} = \pi_2(\pi_3^{-1} F_2' \cap \pi_1^{-1}(F_1')), \tag{5.2}$$

where $F_1' \subset Q \times W_2 \times W_3$ and $F_2' \subset Q \times W_2 \times W_1$ are different copies of $F' \subset Q \times V \times V$. This is given by Lemma 5.4

By the choice of F' and Q', every fibre $\tilde{F}_{v_1,v_2}$, $v_1, v_2 \in Q'$ is purely n-dimensional. Therefore the family $\tilde{F}$ is closed and of locally constant degree in a Zariski open dense subset $Q'' \subset Q' \times Q'$. By Proposition 5.3, there exists a rational mapping $\alpha_1 : Q'' \to C$ with $\tilde{F}_{(v_1,v_2)} = F_{\alpha_1(g)}$. Since Q'' is Zariski open and dense, α_1 extends to a rational mapping $Q \times Q \to C$ which has the required properties. **QED**

Since the maps α and α_1 extend the group operations, we write $\alpha(v,w) = vw$ and $\alpha_1(v,w) = vw^{-1}$ whenever these values are defined.

Lemma 5.6. *The mapping $(v,w) \mapsto (vw, w)$ is injective on $Q' \times Q'$.*

Proof. Let $v, w \in Q'$ be arbitrary points. By Lemma 5.4, the fibres F_v, F_w are closed graphs of birational mappings $\rho_v, \rho_w : V \times V$. By Lemma 5.5, the fibre F_{vw} is the closed graph of the composition $\rho_v \circ \rho_w$. If $v_1 w = v_2 w$, their fibres are also equal which implies the equality $\rho_{v_1} \circ \rho_w = \rho_{v_2} \circ \rho_w$. Since $\rho_w : V \to V$ is birational, we obtain $\rho_{v_1} = \rho_{v_2}$, which means $v_1 = v_2$. **QED**

The following Lemma states the existence of right and left divisions of "generic" elements.

Lemma 5.7. *The mappings $(v,w) \mapsto (vw, w)$ and $(v,w) \mapsto (wv, w)$ are birational mappings from $Q \times Q$ into itself with the inverses $(v,w) \mapsto (vw^{-1}, w)$ and $(v,w) \mapsto (w^{-1}v, w)$. The variety Q is pure-dimensional.*

Proof. We prove the statement for the first mapping. The proof for the second one is completely analogous. We first wish to prove that the closed image of $Q \times Q$ under the mapping $(v,w) \mapsto (vw, w)$ lies in $Q \times Q$. Since $\phi(U)$ is Zariski dense in Q, $\phi(U) \times \phi(U)$ is Zariski dense in $Q \times Q$. Then the subset W of $\phi(U) \times \phi(U)$, where vw is defined, is also Zariski dense. Since the mapping $G \times G \to G \times G$, $(g,h) \to (gh, h)$ is a homeomorphism, the preimage U' of $U \times U$ is open dense in $G \times G$ and therefore $U'' := U' \cap \phi^{-1}(W)$ is open dense in $\phi^{-1}(W)$. This implies that $Q'' := (\phi \times \phi)(U'')$ is Zariski dense in W and thus in $Q \times Q$.

Let $(v,w) = (\phi(g), \phi(h)) \subset Q''$ be an arbitrary point. By Lemma 5.5, the fibre F_{vw} is the closed graph of the composition $\rho_v \circ \rho_w$. The latter birational mapping coincides with the automorphism ρ_{gh} defined by $gh \in U$. This means that $vw \in Q$. Since Q'' is Zariski dense, this inclusion is valid for all $(v,w) \in Q \times Q$ where vw is defined. Thus the mapping $(v,w) \mapsto (vw,w)$ is a rational mapping from $Q \times Q$ into itself.

The projective variety Q has finitely many irreducible components. Let $Q_0 \subset Q$ be a component of maximal dimension and Q_1 an arbitrary component. Then $(vw,w) \in Q_2 \times Q_1$ for $(v,w) \in Q_0 \times Q_1$, where Q_2 is also a component of Q. By the choice of Q_0, $\dim Q_2 \le \dim Q_0$. Since the restriction on $Q_0 \subset Q_1$ of the mapping in Lemma 5.6 is injective on the open subset $(Q_0 \times Q_1) \cap (Q' \times Q')$, its closed image coincides with $Q_2 \times Q_1$. Therefore the composition of $(v,w) \mapsto (vw,w)$ and $(v,w) \mapsto (vw^{-1},w)$ is defined in an open dense subset Q'' of $Q_0 \times Q_1$. It is equal to the identity on the Zariski dense subset $Q'' \times \phi(U)$, i.e. it is the identity. By the injectivity in Lemma 5.6, the mapping $(v,w) \mapsto (vw,w)$ is birational from $Q_0 \times Q_1$ into $Q_2 \times Q_1$ with the inverse $(v,w) \mapsto (vw^{-1},w)$. In particular, $\dim Q_2 = \dim Q_1$.

Now, by Lemma 5.6, the components Q_2 are different for different Q_1 and fixed Q_0. If Q_1 runs through all components, Q_2 also does. This implies that Q is pure-dimensional. Thus, we can take for Q_0 and Q_1 any two components and repeat the above proof. **QED**

Lemma 5.8. *Let $Q' \subset Q$ be as in Lemma 5.4 and vw be defined and in Q' for $v,w \in Q'$. Then $\rho_{vw} = \rho_v \circ \rho_w$.*

Proof. In case $v = \phi(g), w = \phi(h)$ for $g,h,gh \in U$ one has $\rho_{vw} = \rho_{gh} = \rho_g \circ \rho_h = \rho_v \circ \rho_w$. Since the set of above points $(v,w) \in Q' \times Q'$ is Zariski dense, the required relation is valid in general. **QED**

We now establish the associativity of the operation $(v,w) \to vw$.

Lemma 5.9. *Let $u,v,w \in Q$ be arbitrary points. Then $(uv)w = u(vw)$ whenever both expressions are defined.*

Proof. By Lemma 5.7, the above expressions are defined on a Zariski open dense subset $Q'' \subset Q^3$. For $u,v,uv,vw \in Q'$ the fibres of both expressions are the graphs of $\rho_u \circ \rho_v \circ \rho_w$ by Lemma 5.8. The latter set is Zariski dense. **QED**

Lemma 5.10. *The operation $(v,w) \to vw$ induces a group structure on the set S of all irreducible components of Q.*

Proof. The associative property follows from Lemma 5.9. Lemma 5.7 implies the existence of right division in S. The existence of left division is

proved analogously by using the birational correspondence $(v,w) \to (v, v^{-1}w)$.
QED

It follows from Lemmas 5.7 and 5.9 that Q is an **algebraic pre-group** in sense of [34]. Recall that an **algebraic pre-group** is an algebraic variety V with a rational mapping $V \times V \to V$, written as $(v,w) \mapsto vw$, such that:

1) for generic $(u,v,w) \in V \times V \times V$ both expressions $(uv)w$ and $u(vw)$ are defined and equal (generic associativity condition);
2) the mappings $(v,w) \mapsto (v, vw)$ and $(v,w) \mapsto (v, wv)$ from $V \times V$ into itself are birational (generic existence and uniqueness of left and right divisions).

The regularization theorem for the algebraic pre-groups can be stated as follows (see [30]; [34], Theorem 3.1).

Lemma 5.11. *There exists a birational homomorphism τ between Q and an algebraic group $\tilde{G}$.*

Remark. By a **birational homomorphism** we mean a birational correspondence τ such that $\tau(uv) = \tau(u)\tau(v)$ whenever all expressions are defined (cf. [34], Definition 3.2).

3. Properties of the action on V.

Lemma 5.12. *There exists a rational action $\tilde{\rho}\colon Q \times V \to V$, i.e. $\tilde{\rho}(vw,x) = \tilde{\rho}(v, \tilde{\rho}(w,x))$ for generic choice of $(v,w,x) \in Q \times Q \times V$ such that the following diagram is commutative whenever the mappings are defined:*

$$
\begin{array}{ccc}
G \times V & \xrightarrow{\rho} & V \\
\downarrow{\scriptstyle \phi \times id} & & \downarrow{\scriptstyle id} \\
Q \times V & \xrightarrow{\tilde{\rho}} & V
\end{array}
$$

Proof. By Lemma 5.4, there exists a Zariski open dense subset $Q' \subset Q$ such that, for every $v \in Q'$, the fibre F_v is the closed graph of a birational mapping $\rho_v\colon V \to V$. These birational mappings together define the action $\tilde{\rho}\colon Q' \times V \to V$ which extends to a rational mapping $\tilde{\rho}\colon Q \times V \to V$. The commutativity of the diagram follows from the coincidence of the closed graph of $\rho_g, g \in U$, with the fibre $F_{\phi(g)}$.

The property $\tilde{\rho}(vw,x) = \tilde{\rho}(v, \tilde{\rho}(w,x))$ is true for $v, w, vw \in \phi(U)$ and $x \in D$. Since the set of such (v,w,x) is Zariski dense in $Q \times Q \times V$, this is true for generic choices of (v,w,x). **QED**

To simplify the notation, we write $\rho\colon Q \times V \to V$ instead of $\tilde{\rho}\colon Q \times V \to V$. Then the property $\tilde{\rho}(vw, x) = \tilde{\rho}(v, \tilde{\rho}(w, x))$ can be written as the associativity condition $(vw)x = v(wx)$.

The "action" $Q \times V \to V$ is rational. Furthermore, it may happen that an element $v \in Q$ does not define a birational automorphism of V. This is not the case, however, if we replace Q by $\tilde{G}$.

Lemma 5.13. *Let $\tilde{G} \times V \to V$ be the rational action which is induced by the action $\rho\colon Q \times V \times V$ via the birational homomorphism $\tau\colon Q \to \tilde{G}$. Then, for every element $v \in \tilde{G}$, the restriction $\rho_v\colon V \to V$ is a birational automorphism of V and the product vw in $\tilde{G}$ corresponds to the composition $\rho_v \circ \rho_w$ of these automorphisms. Moreover, the action of $\tilde{G}$ is effective.*

Proof. Let $Q' \subset Q$ be the open dense subset given by Lemma 5.4 $Q'' \subset Q'$ an open dense subset where the birational homomorphism $\tau\colon Q \to \tilde{G}$ is biregular. We can regard Q'' as a Zariski open dense subset of $\tilde{G}$.

Let $v \in \tilde{G}$ be arbitrary and $w \in vQ'' \cap Q''$. Then $v = wu^{-1}$ for $w, u \in Q''$. By Lemma 5.4, the fibres F_w and F_u coincide with closed graphs of ρ_w and ρ_u. Therefore there exist points in $w \times V$ and $u \times V$ where ρ is defined. By Lemma 5.8, $\rho_v = \rho_w \circ \rho_u^{-1}$, which is also a birational automorphism of V.

By Lemma 5.8, one has $\rho(vw, x) = \rho(v, \rho(w, x))$ for all (v, w, x) in a Zariski dense subset of $\tilde{G} \times \tilde{G} \times V$. Therefore this is true for all values of (v, w, x) whenever the expressions are defined. This implies $\rho_{vw} = \rho_v \circ \rho_w$ for all $v, w \in \tilde{G}$.

Assume that $\rho\colon \tilde{G} \times V \to V$ has a kernel K and take $k \neq 1 \in K$. Let $Q'' \subset Q$ be a Zariski open dense subset where the birational homomorphism $\tau\colon Q \to \tilde{G}$ is biregular. We can regard Q'' as a Zariski open dense subset of $\tilde{G}$. Let $v = kw \in Q'' \cap kQ''$ be an arbitrary point. Since k is in the kernel, $\rho_{kw} = \rho_w$. On the other hand, w and kw are different points in the Chow scheme $C \supset Q''$ with different fibres. Since the fibres are the closed graphs of corresponding automorphisms, this is a contradiction. **QED**

Recall that V is an algebraic pre-transformation $\tilde{G}$-space ([34], Definition 4.1) if

1) for generic $(v, w, x) \in \tilde{G} \times \tilde{G} \times V$, both expressions $(vw)x$ and $v(wx)$ are defined and equal (generic associativity condition);
2) the mapping $(v, x) \mapsto (v, vx)$ from $Q \times V$ into itself is birational.

Corollary 5.1. *V is an algebraic pre-transformation $\tilde{G}$-space.*

4. The homomorphism from G into $\tilde{G}$.

Up to now we constructed an open dense subset $U \subset G$ and a local homomorphism $\phi: U \to Q$. We wish to extends ϕ to a homomorphism from G into $\tilde{G}$, which is compatible with the action on V.

Lemma 5.14. *Let $Q'' \subset Q$ be a Zariski open dense subset.*
1) For every $g \in G$ there exist two points $v, w \in Q''$ such that $\rho_g = \rho_v \circ \rho_w^{-1}$;
2) The points v, w can be chosen to be in $\phi(U)$;
3) If $g_m \to g_0$ is any convergent sequence in G, the corresponding sequence $v_m, w_m \in Q''$ can be chosen to converge to some v_0, w_0 with $\rho_{g_0} = \rho_{v_0} \circ \rho_{w_0}^{-1}$.

Proof. Let $g \in G$ be fixed and $Q' \subset Q$ be given by Lemma 5.4. We can assume $Q'' \subset Q'$. Let $F \subset Q'' \times V \times V$ be the universal family over Q'', which, by Lemma 5.4, consists of closed graphs of birational automorphisms $\rho_v: V \to V$. Analogous to the formula (5.2) we can consider the family F' of compositions $\rho_g \circ \rho_v, v \in Q''$. Let Q_0 denote an irreducible component of Q. As in the proof of Lemma 5.5, we conclude that the fibre F'_v coincides with the closed graph of $\rho_g \circ \rho_v$ for all v from a Zariski open subset $Q''_g \subset Q'' \cap Q_0$. By Proposition 5.3, this family yields a rational mapping $r_g: Q''_g \to C$.

We wish to prove that $r_g(Q''_g) \subset Q$. For this we return to our group G. Let $U \subset G$ be the chosen open dense subset. Then the translation gU is also an open dense subset of G and so is the intersection $U' := gU \cap U$. This implies that $\phi(U')$ is Zariski dense in Q and therefore $\phi(U') \cap Q''$ is Zariski dense in Q''. Now, for every $v = \phi(h) \in \phi(U') \cap Q'', h \in U'$ the fibre F'_v is the closed graph of ρ_{gh}. Since $gh \in U$, one has $r_g(v) = \phi(gh) \in Q$. By the density of $\phi(U') \cap Q''_g$, the image of $r_g(Q''_g)$ lies in Q.

Since the compositions of ρ_g with different automorphisms of V are different, the mapping $r_g: Q''_g \to Q$ is injective. Therefore the image $r_g(Q''_g)$ intersects the open dense subset Q''. Let $v \in Q'' \cap r_g(Q''_g)$ be an arbitrary point. The fibre F_v is the closed graph of the birational automorphism ρ_v and, at the same time, is the closed graph of $\rho_g \circ \rho_w$, where $w \in Q''$. This means $\rho_g = \rho_v \circ \rho_w^{-1}$ which finishes the proof of the part 1.

The point $v \in Q'' \cap r_g(Q''_g)$ can be chosen to lie in $\phi(U \cap gU)$. Then $v, w \in \phi(U)$ and the part 2 is also proven.

If we are given a convergent sequence $g_m \to g_0$, we can choose a point $v \in Q''$ such that $v \in r_{g_m}(Q''_{g_m})$ for all $m = 0, 1, \ldots$. Then all $w_m \in Q''_{g_m}$ lie in the component Q_0 which is included in a single component of the Chow scheme. This means that the degree of ρ_{w_m} is constant. The convergence $\rho_{g_m} \to \rho_{g_0}$ implies $\rho_{w_m} \to \rho_{w_0}$. By Lemma 5.3, the family of closed graphs of ρ_{w_m} is closed. By Proposition 5.2, $w_m \to w_0, m \to \infty$. This proves the part 3. **QED**

Lemma 5.15. *There exists a continuous homomorphism $\phi: G \to \tilde{G}$, such that $\rho_{\phi(g)} = \rho_g$ for all $g \in G$. The image $\rho(G)$ is Zariski dense in $\tilde{G}$.*

Proof. Let $Q'' \subset Q$ be a Zariski open dense subset where the birational mapping homomorphism $\tilde{G}$ is biregular. We can identify Q'' with a subset of $\tilde{G}$. By Lemma 5.14, applied to the set $Q'' \subset Q$ and an element $g \in G$, one has $\rho_g = \rho_v \circ \rho_w^{-1}$ for $v, w \in \tilde{G}$. By Lemma 5.13, $\rho_g = \rho_{vw^{-1}}$. Then we define $\phi(g) := vw^{-1}$. Since, by Lemma 5.13, the action of $\tilde{G}$ is effective, this definition of $\phi(g)$ is independent of the choices of v and w.

The property $\rho_{\phi(g)} = \rho_g$ is satisfied by construction of ϕ. By Lemma 5.13, ϕ is a homomorphism. The continuity of ϕ follows from Lemma 5.14, part 3.

The image $\phi(G)$ contains the image $\phi(U)$, which is Zariski dense in Q. If $Q'' \subset Q$ is a Zariski open dense subset, where the isomorphism between Q and $\tilde{G}$ is biregular, the intersection $Q'' \cap \phi(U)$ is Zariski dense in Q. The set Q'' can be regarded as a Zariski open dense subset of $\tilde{G}$. This which yields the density of $Q'' \cap \phi(U)$ and therefore of $\phi(G)$. **QED**

5. The regularization of the action $\tilde{G} \times V \to V$.

Let $D \subset V$ be as in Theorem 3′. We noted in Corollary 5.1 that V is an algebraic pre-transformation $\tilde{G}$-space. The theory of A. Weil (see Theorem 4.1 in [34]) gives the existence of the regularizations of algebraic pre-transformation spaces which are regular at the so-called points of regularity. Recall that a point x in an algebraic pre-transformation $\tilde{G}$-space V is called a **point of regularity** if the mapping $x' \mapsto ux'$ from V into itself is biregular at $x' = x$ for generic $u \in \tilde{G}$ (see [34], Definition 4.3).

If $v \in \phi(G), x \in D$, then the mapping $x' \mapsto vx'$ is biregular at x. Since, by Lemma 5.15, $\phi(G)$ is Zariski dense in $\tilde{G}$, D consists of points of regularity. By Theorem 4.1 in [34], there exists a birational regularization $\psi: V \to X$, i.e. $\tilde{G}$ acts regularly on X, the mapping ψ is birational on V, biregular on D and $\tilde{G}$-equivariant. In particular, $\psi|_D$ is G-equivariant. This is exactly the conclusion of Theorem 3′. **QED**

6. Algebraic extensions for the case of finitely many connected components

In this section we prove Theorem 1. Let G be a Lie group of birationally extendible automorphisms of $D \subset V$ with finitely many connected components.

For fixed $g \in G$ we obtain an n-dimensional subvariety $Z_g = \nu(\Gamma_g) \subset \mathbf{P}_k$ which corresponds to a point $\rho(g)$ in the cycle space $C(\mathbf{P}_k)$ ([1,5]). In order to apply the universality of the cycle space we embed our family $Z_g, g \in G$ in a meromorphic family $Z_g, g \in \tilde{G}$.

As a real analytic manifold, G can be embedded totally really and closed into a complex manifold G' with $\dim_{\mathbf{R}} G = \dim_{\mathbf{C}} G'$ ([4]). We wish to extend the action $G \times D \to D$ to a meromorphic mapping $\tilde{G} \times V \to V$, where $\tilde{G}$ is a neighborhood of G in G'. Since G is embedded totally really, the meromorpic extension is unique. Therefore it only must be constructed locally with respect to G. For the proof we utilize the following result of Kazaryan ([15]). A subset $E \subset D'$ is called *nonpluripolar* if there are no plurisubharmonic functions $f\colon D' \to \mathbf{R} \cup \{-\infty\}$ such that $f|_E \equiv -\infty$.

Proposition 6.1. *Let D' be a domain in $\mathbf{C}^n$ and let $E \subset D'$ be a nonpluripolar subset. Let D'' be an open set in a complex manifold X. If f is a meromorphic function on $D' \times D''$ such that $f(g, \cdot)$ extends to a meromorphic function on X for all $g \in E$, then f extends to a meromorphic function in a neighborhood of $E \times X \subset D' \times X$.*

We wish to prove the required extension at a point $g_0 \in G$. For this we fix a coordinate neighborhood $E \in G'$ of g_0 regarded as a neighborhood in $\mathbf{C}^p$, such that $G \cap E = \mathbf{R}^p \cap E =: E_{\mathbf{R}}$. The map $\mu = \nu \circ (id \times \phi)\colon G \times V \to \mathbf{P}^k$ is real analytic on $E_{\mathbf{R}} \times D$ and extends therefore to a holomorphic map in a neighborhood $D' \times D''$ of $E_{\mathbf{R}} \times D \subset \mathbf{C}^p \times V$. (Here we must replace D by a bit smaller neighborhood $D'' \subset D$).

The set $E_{\mathbf{R}}$, being an open subset of $\mathbf{R}^p$, is nonpluripolar. We apply Proposition 6.1 to the coordinates of the map μ in any affine coordinate chart in $\mathbf{P}_k$. We conclude that μ extends to a meromorphic map $\tilde{\mu}$ defined in a neighborhood of $\{x_0\} \times V \subset G' \times V$ into V. Since V is compact, we can choose this neighborhood of the form $\tilde{G} \times V$.

Now we can apply Proposition 5.1 to the meromorphic family $Z_g, g \in \tilde{G}$. We obtain a meromorphic mapping $\phi\colon \tilde{G} \to C_n(\mathbf{P}_k)$. Since the number of components of G is finite, $\tilde{G}$ can be also assumed to possess this property. Then the image $\phi(\tilde{G})$ lies also in finitely many components of $C_n(\mathbf{P}_k)$ which means the boundness of the degree for all Z_g with g in an open dense subset $U \subset \tilde{G}$. By Lemma 5.1, the degree is globally bounded. Now the application of Theorem 3 yields the algebraic extension required by Theorem 1. **QED**

7. The proof of Theorem 4

$1 \Longrightarrow 2$. The proof is trivial. **QED**

$2 \Longrightarrow 3$. By Theorem 3, it is sufficient to prove the boundness of the degree. Let G be a subgroup of a Nash group $\tilde{G}$ such that the action $G \times D \to D$ extends to a Nash action $\tilde{G} \times D \to D$. We prove the statement for arbitrary Nash manifold $\tilde{G}$ and Nash map $\tilde{G} \times D \to D$ (which is holomorphic for every fixed $g \in G$) by induction on $\dim \tilde{G}$. This is obvious for $\dim \tilde{G} = 0$.

Let $U \subset \tilde{G}$ and $W \subset \mathbf{P}_k$ be Nash coordinate charts and $\phi_j(g) \colon D \to \mathbf{C}$ be the jth coordinate in W of $\nu \circ (id \times \phi_g) \colon D \to \mathbf{P}_k$ for $g \in U$ (taken on its set of definition). Since the map $\phi_j \colon U \times D \to \mathbf{C}$ is Nash, it satisfies a polynomial equation $P_j(g, x, \phi_j(g, x)) \equiv 0$ of degree d. This yields nontrivial polynomial equations of degree not larger than d for all $\phi_j(g) \colon D \to \mathbf{C}$, $g \in U$, outside a proper algebraic subvariety N. The calculation of the required degree, i.e. the intersection number with a linear projective subspace L of codimension n, yields additional linear equations for the coordinates in W. For L generic and g in the complement of another proper subvariety N', this intersection number in finite. Since the degrees of polynomial equations for this intersection are bounded, the intersection number is also bounded (Bezout theorem). This proves the statement for $\tilde{G} = U \backslash (N \cup N')$.

The intersection $U \cap (N \cup N')$ admits a finite stratification in lower dimensional Nash manifolds (see e.g. [2]). By induction, the required degree is bounded for every stratum. This proves the boundness of degree for $g \in U$. Since the Nash atlas is finite, we obtain the required boundness for the Nash manifold $\tilde{G}$. **QED**

$3 \Longrightarrow 1$. Let $\rho \colon \tilde{G} \times X \to X$ be an algebraic extension. We identify the open subset D with its embedding in X. Then we define $\tilde{G}'$ to be the subgroup of $\tilde{G}$ which consists of all elements which leave D invariant. In general, this is not an algebraic subgroup. For our statement, it is sufficient to prove that $\tilde{G}'$ is a Nash subgroup.

We utilize the following property of semialgebraic sets ([33], Lemma 6.2).

Lemma 7.1. *Let A, B and $C, C' \subset A \times B$ be semialgebraic sets. Then the set of $a \in A$ such that $C_a \subset C'_a$ is semialgebraic.*

Here C_a and C'_a denote the fibres $\{b \in B \mid (a, b) \in C\}$ and $\{b \in B \mid (a, b) \in C'\}$ respectively. Now we set $A := \tilde{G}$, $B := D$, $C := (pr_{\tilde{G}} \times \rho)(\tilde{G} \times D)$, $C' := \tilde{G} \times D$ in Lemma 7.1. The set C is semialgebraic by the Tarski-Seidenberg Theorem ([2], Theorem 2.7.1). By Lemma 7.1, the set $G_1 := \{g \in \tilde{G} \mid g(D) \subset D\}$ is semialgebraic. Again, Lemma 7.1, applied to $A := G_1$, $B := D$, $C := G_1 \times D$ and $C' := (pr_{G_1} \times \rho)(G_1 \times D)$, shows that the subgroup $\tilde{G}'$ is semialgebraic. Therefore it is a Nash subgroup and the statement is proven. **QED**

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