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Positive Solutions for Nonlinear Singular Boundary Value Problems on the Half Line

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Abstract

We discuss the existence of positive solutions for singular second-order boundary value problems $x'' = \mu f(t, x, x')$, $ax(0) - bx'(0) = k \geq 0$, $x'(\infty) = 0$, where f may be singular at $x = 0$ and $x' = 0$ and can change sign. Via fixed point theory, we establish the existence of positive solutions under some conditions on f. Our results deal with the situation where the solutions approach the singularities of the equation.

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1 Introduction

Our aim is to establish existence of (nonnegative increasing) solutions to the boundary value problem

$$x''(t) = \mu f(t, x, x'), \qquad t \in (0, +\infty) \tag{1.1}$$

$$ax(0) - b \lim_{t\to 0} x'(t) = k, \qquad \lim_{t\to\infty} x'(t) = 0 \tag{1.2}$$

where $a > 0$, $b \geq 0$, $k \geq 0$, $\mu > 0$, $f(t, x, z)$ is singular at $x = 0$ and $z = 0$ and may change sign. Our result establishes this existence for sufficiently small values of μ and with relatively few conditions on f (a certain bound on $|f|$ and a negativity condition on $f(t, x, z)$ for small z). A parameter such as μ can be called a Thiele modulus, though this terminology is usually applied when the interval is finite [4, 5].

Our basic method is to apply fixed point results for cones in Banach spaces to get existence, but we have to apply this method to a sequence of nonsingular integral equation problems that approximate the problem of interest. The choice of the approximate problems is somewhat delicate as is the process of establishing that the solutions to the approximate problems converge.

There is considerable literature which is relevant to our work. Staněk [14] considers $x'' = \mu q(t) f(t, x, x')$ $(0 < t < T)$ with boundary conditions of the type $ax(0) - bx'(0) = k > 0$, $x(T) = 0$. The methods of [14] are quite similar in outline to ours, despite the finiteness of the interval, but [14] requires the functions q and f to be nonnegative (along with several other conditions).

Via the technique of upper and lower solutions Agarwal and O'Regan [2, 3] discussed the existence of bounded solutions to equations

$$\frac{1}{p(t)}(p(t)x')' = f(t, x, p(t)x') \qquad (0 < t < \infty) \tag{1.3}$$

subject to a boundary condition $ax(0) - b \lim_{t\to 0} x'(t) = c_0$. In the case $p(t) \equiv 1$, A. Constantin [6] established the existence of bounded solutions to equations of the same kind (subject to certain growth conditions on f). In [2, 3, 6] the function $f(t, x, z)$ is always assumed continuous for $(x, z) \in \mathbb{R}^2$. In [15], again with $p(t) \equiv 1$, G. Yang considered positive solutions to (1.3) with boundary conditions $x(0) = x'(\infty) = 0$, allowing $f(t, x, z)$ to be singular at $x = 0$ and $z = 0$ but requiring boundedness of f as $x \to +\infty$. See also the general references [12, 13] and the works [8, 10, 11, 16] but note that [10, 11, 16] do not consider singular equations.

Theorem 1.1. *Assume $f \in C([0, +\infty) \times (0, +\infty) \times (0, +\infty), \mathbb{R})$ and there exist positive $\Phi \in C[0, +\infty)$, $h \in C(0, +\infty)$ and $g \in C(0, +\infty)$ with $\|\Phi\|_\infty = \sup_{t\in[0,+\infty)} |\Phi(t)| < +\infty$, $\int_0^1 h(s)\, ds < +\infty$ and*

$\int_0^\infty \Phi(s) \max_{\frac{1}{c} \le x \le c(1+s)} h(x)\, ds < +\infty$ *for each* $c \ge 1$*. Suppose*

$$|f(t,y,z)| \le \Phi(t)h(y)g(z) \qquad (t > 0, y > 0, z > 0). \tag{1.4}$$

Assume also there is $\beta \in C(0,+\infty)$*,* $\beta(t) < 0$ *and constants* $0 \le \gamma < 1$*,* $\delta > 0$ *so that*

$$f(t,x,z) \le x^\gamma \beta(t) \qquad (t \ge 0,\ x > 0,\ 0 < z \le \delta). \tag{1.5}$$

Then there is $\mu_0 > 0$ *so that for each* μ *in the range* $0 < \mu \le \mu_0$ *(1.1)–(1.2) has a solution* $x = x_0$ *with* $x_0(t) \ge 0$ *and* $x_0'(t) > 0$ *(for all* $t \ge 0$*).*

Our result implies the result of [15] but in contrast to [15], f may be superlinear at $x = +\infty$. In contrast to [3] we allow f to have singularities and our hypotheses admit unbounded solutions. Our result does not generalise several results from the literature where singularities are not considered such as [6, Theorem 1]. In the proof of [6, Theorem 4], a specific singular equation on $[0,\infty)$ is considered where $f(t,x,z)$ is monotone in z and this special feature of the equation is used in establishing the existence of solutions. In addition the solution obtained is bounded below by a positive constant and so does not approach the singularity at $x = 0$.

Some of our methods are inspired by [1] and [15]. As corollaries, we discuss boundedness and unboundedness of positive solutions and we give some concrete examples where our methods allow us to give a specific value for μ_0.

2 Preliminaries

We will consider two spaces of functions in addition to the space $C(0,+\infty)$ of continuous ($\mathbb{R}$-valued) functions on the open interval $(0,+\infty)$ (and similarly for other intervals in $\mathbb{R}$). One is

$$C[0,\infty] = \left\{ x \in C[0,\infty) : x(\infty) = \lim_{t\to+\infty} x(t) \text{ exists} \right\},$$

(the continuous functions on the one point compactification of $[0,\infty)$) normed by the usual supremum $\|x\|_\infty = \sup_{t\ge0} |x(t)|$. The other is $C^1_\infty[0,\infty) = \{x \in C[0,\infty) : x' \in C[0,\infty]\}$ normed by $\|x\| = \max\left(\sup_{t\ge0} \frac{|x(t)|}{1+t}, \sup_{t\ge0}|x'(t)|\right)$. It is easy to verify that the existence of $x'(\infty) = \lim_{t\to\infty} x'(t)$ implies finiteness of $\sup_{t\ge0} \frac{|x(t)|}{1+t}$. See the following elementary lemma (with $\phi(t) = 1+t$).

Lemma 2.1. *If* $\phi(t)$ *is continuous on* $[0,\infty)$ *with* $\inf_{t\ge0}\phi(t) = \delta > 0$ *and* $\lim_{t\to\infty} \frac{\phi(t)}{t} = 1$*, then there is* $K > 0$ *so that for each* $x(t)$ *with* $x' \in C[0,\infty]$

$$\sup_{t\in[0,\infty)} \frac{|x(t)|}{\phi(t)} \le \frac{|x(0)|}{\delta} + K \sup_{t\ge0} |x'(t)|$$

and $\lim\limits_{t\to\infty} \frac{x(t)}{\phi(t)} = x'(\infty)$.

Proof. Let $K_1 = \sup_{t\in[0,\infty]} |x'(t)|$ and $K_2 = \sup_{t\in[0,\infty)} t/\phi(t)$. Then

$$\frac{|x(t)|}{\phi(t)} = \frac{1}{\phi(t)}\left|x(0) + \int_0^t x'(s)\,ds\right| \le \frac{|x(0)|}{\delta} + K_1K_2.$$

Given $\varepsilon > 0$, choose $T > 0$ so that $|x'(t) - x'(\infty)| < \varepsilon/3$ and $|1 - t/\phi(t)| < \varepsilon/(3K_1)$ for all $t \ge T$. Then, for $t > T$

$$\begin{aligned} &\frac{x(t)}{\phi(t)} - x'(\infty) \\ &= \frac{x(T) - Tx'(\infty)}{\phi(t)} + \frac{\int_T^t (x'(s) - x'(\infty))\,ds}{\phi(t)} + \left(\frac{t}{\phi(t)} - 1\right) x'(\infty) \end{aligned}$$

and, from the triangle inequality we deduce

$$\left|\frac{x(t)}{\phi(t)} - x'(\infty)\right| \le \left|\frac{x(T) - Tx'(\infty)}{\phi(t)}\right| + \frac{t-T}{\phi(t)}\frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

for $t > T_1$ and T_1 large enough. □

From [7] and [8] we know that $C[0,\infty]$ and $C^1_\infty[0,\infty)$ are Banach spaces. We will also consider the cone P of non-negative elements in $C^1_\infty[0,\infty)$

$$P = \{x \in C^1_\infty[0,+\infty) : x(t) \ge 0, x'(t) \ge 0 \text{ for } t \in [0,\infty)\}. \tag{2.1}$$

From the Arzela-Ascoli Theorem (or see [7]) we can state a criterion for compactness in $C[0,\infty]$.

Lemma 2.2. *Let $\mathcal{M} \subseteq C[0,+\infty]$. Then $\mathcal{M}$ is relatively compact in $C[0,+\infty]$ if and only if it is (norm) bounded in $C[0,+\infty]$ and equicontinuous at each point of $[0,\infty]$:*

(a) $\sup\{\|x\| : x \in \mathcal{M}\} < \infty$, *and*

(b) for each $t_0 \in [0,\infty]$ and $\varepsilon > 0$, there is a neighbourhood U_0 of $t_0 \in [0,\infty]$ so that $t \in U_0$, $x \in \mathcal{M}$ implies $|x(t) - x(t_0)| < \varepsilon$.

The next lemma follows from [9, Lemma 2.3.1, Theorem 2.3.2].

Lemma 2.3. *Let Ω be a bounded open set in real Banach space E that contains the origin, P be a convex closed cone in E which is proper (that is $P \cap (-P) = \{0\}$) and $A\colon \overline{\Omega} \cap P \to P$ a continuous and completely continuous (that is, maps bounded sets to relatively compact sets) operator. Suppose*

$$\lambda Ax \ne x, \forall x \in \partial\Omega \cap P, \lambda \in (0,1]. \tag{2.2}$$

Then $i(A, \Omega \cap P, P) = 1$ (where $i(A, \Omega \cap P, P)$ is the fixed point index of A on $\Omega \cap P$ with respect to P) and the equation $Ax = x$ has a solution $x \in \Omega \cap P$.

Lemma 2.4. *Let* $\beta \in C((0,\infty))$ *with* $\beta(t) < 0$ *(all* $t \in (0,\infty)$*), and let* $\mu > 0$, $T > 0$ *and* $0 \le \gamma < 1$. *Suppose* $x \in P$ *satisfies* $x(T) > 0$, $\lim_{t\to\infty} x'(t) = 0$ *and*

$$x''(t) \le \mu\beta(t)(x(t))^\gamma \quad (t \in [T,\infty)).$$

Then

$$x'(t) \ge \mu \int_t^\infty (-\beta(s))(\alpha_T(s))^\gamma \, ds \quad \text{for } t \in [T,\infty),$$

where

$$\alpha_T(t) = \left((1-\gamma)\mu \int_T^t (\tau - T)(-\beta(\tau))\, d\tau\right)^{1/(1-\gamma)} \qquad (t \ge T \ge 0).$$

Proof. Note that $\alpha_T(t)$ is positive for $t > T$ and satisfies the integral equation

$$\alpha_T(t) = \mu \int_T^t (\tau - T)(\alpha_T(\tau))^\gamma(-\beta(\tau))\, d\tau \quad (t \in [T,\infty)). \tag{2.3}$$

Integration from t to $+\infty$ yields that

$$x'(t) \ge -\mu \int_t^\infty \beta(s)\,(x(s))^\gamma \; ds, \quad t \in [T,+\infty).$$

Thus, for $t \ge T$,

$$\begin{aligned} x(t) &> x(t) - x(T) \\ &\ge -\mu \int_T^t \int_s^\infty \beta(\tau)\,(x(\tau))^\gamma \; d\tau\, ds \\ &\ge -\mu \int_T^t \int_s^t \beta(\tau)\,(x(\tau))^\gamma \; d\tau\, ds \\ &\ge \mu \int_T^t (\tau - T)(-\beta(\tau))\,(x(\tau))^\gamma \; d\tau. \end{aligned}$$

Note that $\alpha_T(T) = 0 < x(T)$ and hence there is some interval $[T, t_0^*)$ with $t_0^* > 1$ where $x(t) > \alpha_T(t)$ holds. From the inequality

$$x(t) - \alpha_T(t) \ge \mu \int_T^t (\tau - T)\left((x(\tau))^\gamma - (\alpha_T(\tau))^\gamma\right)(-\beta(\tau))\, d\tau$$

(which follows using (2.3)) we can show that $x(t) > \alpha_T(t)$ remains true for all $t \ge T$. This gives the conclusion of the lemma. □

3 Proof of the main result

Lemma 3.1. *Let* $f_0 \in C([0,+\infty) \times [0,+\infty) \times [0,+\infty), \mathbb{R})$*,* $c_0 \geq 0$ *and* $c_1 \geq 0$*. Assume that for each* $M > 0$ *there is* $\Phi_M \in L^1[0,\infty)$ *so that*

$$x \in P, \|x\| \leq M \Rightarrow |f_0(t, x(t), x'(t))| \leq \Phi_M(t) \quad (t \geq 0).$$

For $x \in P$ *(*P *as in (2.1)), we define a function* $(Ax)(t)$ *(on* $t \in [0,\infty)$*) by*

$$(Ax)(t) = c_0 + c_1 x'(0) + \int_0^t \max\left\{0, \int_s^\infty f_0\left(\tau, x(\tau), x'(\tau)\right)\, d\tau\right\} ds.$$

Then $Ax \in P$ *(for all* $x \in P$*) and* $A\colon P \to P$ *is a continuous and completely continuous map.*

Proof. For $x \in P$, let $M = \|x\|$ and then

$$\begin{aligned} 0 &\leq (Ax)(t) \\ &= (Ax)(0) + \int_0^t \max\left\{0, \int_s^\infty f_0\left(\tau, x(\tau), x'(\tau)\right)\right\} d\tau\, ds \\ &\leq (Ax)(0) + \int_0^t \int_0^\infty \Phi_M(\tau)\, d\tau\, ds \\ &\leq (Ax)(0) + t\|\Phi_M\|_1 < +\infty. \end{aligned}$$

(We use $\|\cdot\|_1$ for the integral norm on $L^1[0,\infty)$.) Moreover,

$$\begin{aligned} 0 &\leq (Ax)'(t) = \max\left\{0, \int_t^\infty f_0\left(\tau, x(\tau), x'(\tau)\right)\, d\tau\right\} \\ &\leq \int_t^\infty \Phi_M(\tau)\, d\tau, \quad \forall t \in [0,+\infty), \end{aligned}$$

which yields that $\lim_{t\to\infty}(Ax)'(t) = 0$. Consequently, $A : P \to P$ is well defined.

To show that $A\colon P \to P$ is continuous, we consider a convergent sequence $(x_m)_{m=1}^\infty$ in P converging to $x_0 \in P$. By compactness of the sequence and its limit there is $M > 0$ so that $\|x_m\| \leq M$ for all m (and $\|x_0\| \leq M$). Because of the elementary inequality $|\max\{0,u\} - \max\{0,v\}| \leq |u - v|$ we have

$$\begin{aligned} &|(Ax_m)'(t) - (Ax_0)'(t)| \\ &\quad\leq \int_t^\infty \left|f_0\left(\tau, x_m(\tau), x_m'(\tau)\right) - f_0\left(\tau, x_0(\tau), x_0'(\tau)\right)\right| d\tau \\ &\quad\leq \int_0^\infty \left|f_0\left(\tau, x_m(\tau), x_m'(\tau)\right) - f_0\left(\tau, x_0(\tau), x_0'(\tau)\right)\right| d\tau. \end{aligned} \tag{3.1}$$

Also,

$$|f_0(\tau, x_m(\tau), x_m'(\tau)) - f_0(\tau, x_0(\tau), x_0'(\tau))| \le 2\Phi_M(\tau)$$

and (3.1) with the Lebesgue dominated convergence theorem then implies $(Ax_m)'(t) \to (Ax_0)'(t)$ uniformly for $t \ge 0$. One can show in a similar way that $(Ax_m)(0) \to (Ax_0)(0)$ and then by Lemma 2.1, we have $Ax_m \to Ax_0$.

Finally, let $\mathcal{D} \subseteq P$ be bounded and we claim that $A(\mathcal{D})$ is relatively compact in P (or equivalently in $C^1_\infty[0,\infty)$ since P is closed). Fix $M > 0$ so that $\|x\| \le M$ for $x \in \mathcal{D}$. We have

$$\begin{aligned} |(Ax)'(t)| &\le \int_t^\infty \Phi_M(s)\,ds \qquad (3.2)\\ &\le \int_0^\infty \Phi_M(\tau)\,ds, \end{aligned}$$

which implies that $\{(Ax)' : x \in \mathcal{D}\}$ is bounded. From (3.2) we see that the functions $(Ax)'$ $(x \in \mathcal{D})$ are equicontinuous at ∞. To show equicontinuity at point $t_0 \in [0,\infty)$ observe that

$$\begin{aligned} (Ax)'(t) &= \max\left\{0, \int_t^\infty f_0(\tau, x(\tau), x'(\tau))\,d\tau\right\}\\ &= \max\left\{0, \int_t^{t_0} f_0(\tau, x(\tau), x'(\tau))\,d\tau \right.\\ &\qquad \left. + \int_{t_0}^\infty f_0(\tau, x(\tau), x'(\tau))\,d\tau\right\} \end{aligned}$$

from which we deduce that

$$\begin{aligned} |(Ax)'(t) - (Ax)'(t_0)| &\le \left|\int_t^{t_0} f_0(\tau, x(\tau), x'(\tau))\,d\tau\right|\\ &\le \left|\int_t^{t_0} \Phi_M(\tau)\,d\tau\right| \end{aligned}$$

and so it follows that $\{(Ax)'(t) : x \in \mathcal{D}\}$ is equicontinuous at t_0. From Lemma 2.2 we see that for any $(x_m)_{m=1}^\infty$ in $\mathcal{D}$, there is a subsequence (which we denote again by $(x_m)_{m=1}^\infty$) so that $(Ax_m)'$ converges uniformly on $[0,\infty]$ (to some limit function in $z \in C[0,\infty]$). Passing to a further subsequence we may assume also that $\lim_{m\to\infty}(Ax_m)(0) = x_0(0)$ exists. From Lemma 2.1 we can then conclude that $\lim_{m\to\infty} Ax_m = x_0$ where $x_0(t) = x_0(0) + \int_0^t z(\tau)\,d\tau$. □

Lemma 3.2. *Fix f satisfying the hypothesis (1.4) of Theorem 1.1, $\mu > 0$, $a > 0$, $b \ge 0$, $k \ge 0$ and $n \in \mathbb{N}$. For $x \in P$ (P as in (2.1)), we define a function $(A_n x)(t)$ (on $t \in [0,\infty)$) by*

$$(A_n x)(t) = \frac{k}{a} + \frac{b}{a}x'(0) + \int_0^t \max\left\{0, \int_s^\infty -\mu f_n(\tau, x(\tau), x'(\tau))\,d\tau\right\} ds,$$

where we let $f_n(\tau, x, z) = f\left(\tau, x + \frac{\tau+1}{n}, z + \frac{1}{n}\right)$. *Then* $A_n x \in P$ *(for all* $x \in P$*) and* $A_n \colon P \to P$ *is a continuous and completely continuous map.*

Proof. The idea is to apply Lemma 3.1 with $f_0(t, x, z) = -\mu f_n(\tau, x, z)$. What we need is to establish the existence of $\Phi_M \in L^1[0, \infty)$ for each $M > 0$ to satisfy the assumption in Lemma 3.1.

To compress the notation we introduce

$$H_{\varepsilon,c}(\tau) = \max_{\varepsilon \le x \le c(1+\tau)} h(x), \quad G_{\varepsilon,M} = \max_{\varepsilon \le z \le M} g(z). \tag{3.3}$$

For $x \in P$, it is easy to see that

$$0 \le x(t) = x(0) + \int_0^t x'(s) ds \le x(0) + t\|x\| \le (1+t)\|x\|, \ \ \forall t \in [0, +\infty).$$

Thus

$$\frac{1}{n} \le x(t) + \frac{t+1}{n} \le (1 + \|x\|)(1+t), \ \ \forall t \in [0, +\infty).$$

Fix $M > 0$ and choose a $c > 1 + M$ big enough such that $1/c < 1/n$. Then, for $x \in P$ with $\|x\| \le M$,

$$\frac{1}{c} \le x(t) + \frac{t+1}{n} \le c(1+t), \ \ \forall t \in [0, +\infty). \tag{3.4}$$

Thus, using (1.4),

$$\begin{aligned} |f_n(\tau, x(\tau), x'(\tau))| &= \left| f\left(\tau, x(\tau) + \frac{\tau+1}{n}, x'(\tau) + \frac{1}{n}\right)\right| \\ &\le \Phi(\tau) h\left(x(\tau) + \frac{\tau+1}{n}\right) g\left(x'(\tau) + \frac{1}{n}\right) \\ &\le \Phi(\tau) h\left(x(\tau) + \frac{\tau+1}{n}\right) G_{1/n, \|x\|+1/n}. \end{aligned} \tag{3.5}$$

Together with (3.4) this implies that, for $x \in P$ with $\|x\| \le M$,

$$|f_0(\tau, x(\tau), x'(\tau))| \le \Phi_M(\tau) = \mu \Phi(\tau) H_{1/c, c}(\tau) G_{1/n, M+1/n}.$$

As $\Phi_M \in L^1[0, \infty)$ by the hypotheses, we can apply Lemma 3.1 to get the result. □

Lemma 3.3. *Fix* f *satisfying the hypotheses (1.4) and (1.5) of Theorem 1.1,* $\mu > 0$, $a > 0$, $b \ge 0$, $k \ge 0$ *and* $n \in \mathbb{N}$. *Let* A_n *be as in Lemma 3.2. If* $1/n < \delta$, $0 < \lambda \le 1$ *and* $x \in P$ *satisfies* $x = \lambda A_n x$, *then*

$$x''(t) = \lambda \mu f_n(t, x(t), x'(t)), \qquad t \in (0, +\infty)$$

and $x'(t) > 0$ *for* $t \ge 0$.

Proof. Differentiating the formula defining A_n we have $x'(t) \geq 0$ (for $t \geq 0$), $x(t) \geq x(0) \geq 0$, and $\lim_{t\to\infty} x'(t) = 0$. Because $1/n < \delta$, there is $t_0 > 0$ so that $0 \leq x'(t) + 1/n < \delta$ for all $t \geq t_0$. From (1.5) we then have $f_n(t, x(t), x_0'(t)) = f(t, x(t) + (t+1)/n, x_0'(t) + 1/n) \leq (x(t) + (t+1)/n)^\gamma \beta(t) < 0$ for $t \geq t_0$ and so differentiating the formula defining A_n gives

$$(A_n x)'(t) = -\mu \int_t^\infty f_n\left(\tau, x(\tau), x'(\tau)\right) d\tau \qquad (t \geq t_0).$$

Since $x = \lambda A_n x$ and $f_n(\tau, x(\tau), x_0'(\tau)) < 0$ (for $\tau \geq t \geq t_0$) we deduce $x'(t) > 0$ for $t \geq t_0$.

Thus $\{t \in [0, \infty) : x'(s) > 0 \forall s > t\}$ is nonempty (it contains t_0) and has an infimum t^*. If $t^* > 0$ then we must have $x'(t^*) = 0$. We will show now that $x'(t^*) > 0$ and this will also show that $t^* = 0$, hence $x'(0) > 0$ must hold. If $x'(t^*) = 0$, then by continuity of x' we must then have $t_0^* > t^*$ such that

$$0 < x'(t), \quad x'(t) + \frac{1}{n} < \delta \qquad (t^* < t \leq t_0^*),$$

and hence $f_n(\tau, x(\tau), x_0'(\tau)) \leq (x(\tau) + \frac{1}{n}\tau + \frac{1}{n})^\gamma \beta(\tau) < 0$ for $t^* < \tau \leq t_0^*$.

As $x'(t) > 0$ for $t^* < t \leq t_0^*$, we have

$$x'(t) = \lambda (A_n x)'(t) = -\mu\lambda \int_t^\infty f_n\left(\tau, x(\tau), x'(\tau)\right) d\tau \tag{3.6}$$

in the range $t^* < t \leq t_0^*$. In particular (3.6) holds for $t = t_0^*$ and, by continuity, (3.6) also holds at $t = t^*$. Thus

$$0 = x'(t^*) = -\mu\lambda \int_{t^*}^{t_0^*} f_n\left(\tau, x(\tau), x'(\tau)\right) d\tau + x'(t_0^*) > x'(t_0^*) > 0,$$

a contradiction. We have therefore established $t^* = 0$ (and $x'(0) > 0$).

We can now verify directly that x must satisfy

$$x''(t) = \lambda\mu f_n\left(t, x(t), x'(t)\right), \qquad t \in (0, +\infty). \quad \square$$

□

Notation 3.4. We will continue to use the notation (3.3) with h and g as in the hypothesis (1.4). We also let $I(z) = I_g(z) = \int_0^z \frac{u}{g(u)+1}\, du$. As $g(u) > 0$ we can see that I is continuous and strictly increasing on $[0, \infty)$ with $I(0) = 0$ and $I([0, \infty)) = [0, I_\infty)$ where $I_\infty = \int_0^\infty \frac{u}{g(u)+1}\, du \in [0, \infty]$ may be finite or infinite. Thus there is a monotone increasing continuous inverse function $I^{-1} \colon [0, I_\infty) \to [0, \infty)$ with $I^{-1}(0) = 0$.

Define, for $R > 0$.

$$\Psi_R(s) = \begin{cases} \|\Phi\|_\infty h(s) + R\Phi(s) H_{1,1+R}(s) & s \leq 1 \\ R\Phi(s) H_{1,1+R}(s) & s > 1. \end{cases}$$

For $\mu_0 > 0$ let $J(\mu_0) = \{c > 0 : \mu_0 \|\Psi_c\|_1 < I_\infty\}$.

Lemma 3.5. *Assume that f satisfies the hypotheses of Theorem 1.1. Then, using the notation above, there is $\mu_0 > 0$ so that*

$$\sup_{c\in J(\mu_0)} \frac{c}{\frac{k}{a} + (\frac{b}{a}+1)I^{-1}\left(\mu_0\|\Psi_c\|_1\right)} > 1. \tag{3.7}$$

Moreover, for such μ_0 there is $R_1 > 0$ and $\varepsilon > 0$ so that

$$R_1 > \frac{k}{a} + \left(\frac{b}{a}+1\right) I^{-1}\left(I(\varepsilon) + \mu_0\|\Psi_{R_1+\varepsilon}\|_1\right). \tag{3.8}$$

Proof. Fix $R' > k/a$ and then we can find $\mu_0 > 0$ so that

$$\mu_0\left(\|\Phi\|_\infty \int_0^1 h(s)\,ds + R'\int_0^\infty \Phi(s)H_{1,1+R'}(s)\,ds\right) < I_\infty$$

and

$$\frac{R'}{\frac{k}{a} + (\frac{b}{a}+1)I^{-1}\left(\mu_0\left(\|\Phi\|_\infty \int_0^1 h(s)\,ds + R'\int_0^\infty \Phi(s)H_{1,1+R'}(s)\,ds\right)\right)} > 1.$$

We have μ_0 so that (3.7) holds.

Then there is $R'' > 0$ so that $R'' > (k/a) + (b/a+1)\,I^{-1}\left(\mu_0\|\Psi_{R''}\|_1\right)$. Further, by continuity of I and its inverse, we can choose $\varepsilon > 0$ so that

$$R'' - \varepsilon > \frac{k}{a} + \left(\frac{b}{a}+1\right) I^{-1}\left(I(\varepsilon) + \mu_0\|\Psi_{R''}\|_1\right)$$

and $\varepsilon < R''$. Then let $R_1 = R'' - \varepsilon$. □

Lemma 3.6. *Assume that f satisfies the hypotheses of Theorem 1.1 and f_n is as in Lemma 3.2. Assume also that $0 < \mu \le \mu_0$, $R_1 > 0$ where μ_0, R_1 and $\varepsilon > 0$ satisfy (3.8) and $n \ge n_0 > 1/\min\{\varepsilon, \delta\}$. Then the problem*

$$x''(t) = \mu f_n\left(t, x(t), x'(t)\right), \qquad t \in (0,+\infty) \tag{3.9}$$

$$x(0) = \left(\frac{k}{a} + \frac{b}{a}x'(0)\right), \qquad x'(\infty) = 0 \tag{3.10}$$

has a solution $x = x_n \in P$ with $0 < \|x_n\| < R_1$ and $x_n'(t) > 0$ for all $t \ge 0$.

Proof. We will consider the continuous and completely continuous operator A_n of Lemma 3.2 (for $n \ge n_0$) and we aim to apply Lemma 2.3 to each A_n where $\Omega = \{x \in C^1_\infty[0,+\infty) : \|x\| < R_1\}$. To that end, we claim that there is no $x \in P \cap \partial\Omega = \{x \in P : \|x\| = R_1\}$ with $x = \lambda A_n x$ and $0 < \lambda \le 1$.

Assume on the contrary that the claim is false and $x = x_0 \in P$, $0 < \lambda_0 \leq 1$, $n \geq n_0$ satisfy $\|x_0\| = R_1$ and $x_0 = \lambda_0 A_n x_0$. By Lemma 3.3, $x''(t) = \lambda_0 \mu f_n\left(t, x(t), x'(t)\right)$ (for $t > 0$). Thus, by (1.4),

$$\begin{aligned} -x_0''(t) &\leq \mu \left| f\left(t, x_0(t) + \frac{t+1}{n}, x_0'(t) + \frac{1}{n}\right)\right| \\ &\leq \mu \Phi(t) h\left(x_0(t) + \frac{t+1}{n}\right) g(x_0'(t) + 1/n); \\ \frac{-\left(x_0'(t) + 1/n\right) x_0''(t)}{g(x_0'(t) + 1/n) + 1} &\leq \mu \Phi(t) h\left(x_0(t) + \frac{t+1}{n}\right) (x_0'(t) + 1/n). \end{aligned}$$

Integrating between t and ∞ we get

$$\begin{aligned} & I(x_0'(t) + 1/n) - I(1/n) \\ \leq\ & \mu \int_t^{\infty} \Phi(s) h\left(x_0(s) + \frac{s+1}{n}\right) d\left(x_0(s) + \frac{s+1}{n}\right) \\ \leq\ & \mu \int_0^{\infty} \Phi(s) h\left(x_0(s) + \frac{s+1}{n}\right) d\left(x_0(s) + \frac{s+1}{n}\right). \end{aligned} \tag{3.11}$$

Since $\lim_{t\to+\infty}(x_0(t) + (t+1)/n) = +\infty$ and $x_0(t) + (t+1)/n$ is increasing, there are two situations: either $x_0(t) + \frac{t+1}{n} \geq 1$ for all $t \in [0, +\infty)$; or $x_0(0) + \frac{1}{n} < 1$.

In the second case there is a unique $T > 0$ such that $x_0(T) + \frac{1}{n}T + \frac{1}{n} = 1$ and $1 \leq x_0(t) + \frac{1}{n}t + \frac{1}{n} \leq (R_1 + \varepsilon)(1 + t)$, $\forall t \in [T, +\infty)$. And integrating between t and ∞ we estimate (3.11) from above by

$$\begin{aligned} & I(x_0'(t) + 1/n) - I(1/n) \\ \leq\ & \mu \int_0^{T} \Phi(s) h\left(x_0(s) + \frac{s+1}{n}\right) d\left(x_0(s) + \frac{s+1}{n}\right) \\ & +\mu \int_T^{\infty} \Phi(s) h\left(x_0(s) + \frac{s+1}{n}\right) d\left(x_0(s) + \frac{s+1}{n}\right) \\ \leq\ & \mu \|\Phi\|_\infty \int_0^{T} h\left(x_0(s) + \frac{s+1}{n}\right) d\left(x_0(s) + \frac{s+1}{n}\right) \\ & +\mu \int_T^{\infty} \Phi(s) h\left(x_0(s) + \frac{s+1}{n}\right) \left(x_0'(s) + \frac{1}{n}\right) ds \\ \leq\ & \mu \left(\|\Phi\|_\infty \int_0^1 h(s)\, ds + (R_1 + \varepsilon) \int_0^{\infty} \Phi(s) H_{1,R_1+\varepsilon}(s)\, ds\right) \\ =\ & \mu \|\Psi_{R_1+\varepsilon}\|_1 \end{aligned}$$

($\forall t \in [0, +\infty)$). In the first case we get the same estimate by taking $T = 0$ in the above argument.

Rearranging, and using $I(1/n) < I(\varepsilon)$ together with monotonicity of I^{-1} we get

$$\begin{aligned} I(x_0'(t) + 1/n) &\leq I(\varepsilon) + \mu\|\Psi_{R_1+\varepsilon}\|_1 \\ x_0'(t) &\leq I^{-1}\left(I(\varepsilon) + \mu\|\Psi_{R_1+\varepsilon}\|_1\right) < R_1 \end{aligned} \tag{3.12}$$

by (3.8). Integrating again from 0 to t, we find

$$x_0(t) = x_0(0) + \int_0^t x_0'(s)\,ds \leq x_0(0) + tI^{-1}\left(I(\varepsilon) + \mu\|\Psi_{R_1+\varepsilon}\|_1\right)$$

from which it follows that

$$\begin{aligned} \sup_{t\geq 0} \frac{|x_0(t)|}{1+t} &\leq x_0(0) + I^{-1}\left(I(\varepsilon) + \mu\|\Psi_{R_1+\varepsilon}\|_1\right) \\ &\leq \lambda_0 \frac{k}{a} + \lambda_0 \frac{b}{a} x_0'(0) + I^{-1}\left(I(\varepsilon) + \mu\|\Psi_{R_1+\varepsilon}\|_1\right) \\ &\leq \frac{k}{a} + \left(\frac{b}{a} + 1\right) I^{-1}\left(I(\varepsilon) + \mu\|\Psi_{R_1+\varepsilon}\|_1\right) < R_1 \end{aligned} \tag{3.13}$$

by (3.8). From (3.12), (3.13) and $1/n < \varepsilon$ we get $R_1 = \|x_0\| < R_1$, a contradiction. This establishes the claimed nonexistence of x_0.

We can now apply Lemma 2.3 to A_n to obtain $x_n \in P \cap \Omega$ with $x_n = A_n x_n$. By Lemma 3.3, $x = x_n$ must satisfy (3.9)–(3.10) and $x_n'(t) > 0$ for $t \geq 0$. □

Proof (of Theorem 1.1). Let n_0 be as in Lemma 3.6. For $n \geq n_0$, let $x_n \in P$ satisfy (3.9)–(3.10) and $x_n'(t) > 0$ for all $t \geq 0$. Let $M_n = \sup_{t\in[1,+\infty)} x_n'(t)$ and $\eta = \inf_{n\geq n_0} M_n$. Now we show that $\eta > 0$. In fact, if $\eta = 0$, there is an $\{n_j\}$ such that $M_{n_j} + \frac{1}{n_j} \to 0$ as $j \to +\infty$ and we can assume $M_{n_j} + \frac{1}{n_j} < \delta$ (all j). Our assumption (1.5) implies that

$$f_{n_j}(t, x_{n_j}(t), x_{n_j}'(t)) \leq \beta(t)\left(x_{n_j}(t) + \frac{t+1}{n_j}\right)^\gamma, \quad t \in [1, +\infty).$$

Thus,

$$x_{n_j}''(t) \leq \mu\beta(t)\left(x_{n_j}(t) + \frac{t+1}{n_j}\right)^\gamma, \quad t \in [1, +\infty)$$

and hence

$$x_{n_j}''(t) \leq \mu\beta(t)(x_{n_j}(t))^\gamma \quad t \in [1, +\infty).$$

By Lemma 2.4

$$x_{n_j}'(t) \geq \mu \int_t^\infty (-\beta(s))(\alpha_1(s))^\gamma\,ds, \quad t \in [1, +\infty)$$

and this contradicts $\eta = 0$.

Let $\zeta = \min\{\delta/2, \eta\}$. Since $\lim_{t\to+\infty} x_n'(t) = 0$, set $x_n'(t_n) = \zeta$, $t_n \in [1, +\infty)$. It is easy to see that $x_n'(t)$ is decreasing on $[t_n, +\infty)$. Now we show that

$$x_n'(t) \geq \zeta, \quad t \in [0, t_n], n \geq n_0. \tag{3.14}$$

In fact, if there is a $t_0 \in [0, t_n]$ with $x_n'(t_0) < \zeta$, set $t^* = \sup\{t | x_n'(s) < \zeta$ for all $s \in [t_0, t]\}$. Obviously, $t^* \leq t_n$ with $x_n'(t) < \zeta$ for all $t \in [t_0, t^*)$ and $x_n'(t^*) = \zeta$. Now (1.5) implies that

$$x_n''(t) \leq \mu\beta(t)\left(x_n(t) + \frac{t+1}{n}\right)^\gamma, \quad t \in [t_0, t^*],$$

which yields that $x_n'(t)$ is decreasing on $[t_0, t^*]$. This is a contradiction to $x_n'(t^*) = \zeta > x_n'(t_0)$ and establishes (3.14).

Since $x_n'(t) \geq \zeta$ for all $t \in [0, t_n] \supseteq [0, 1]$, $x_n(t) = x_n(0) + \int_0^t x_n'(s)\,ds \geq \zeta t$ for all $t \in [0, 1]$. Since $x_n(t)$ is increasing on $[0, +\infty)$, we know that $x_n(t) \geq \zeta$ for all $t \in [1, +\infty)$, $n \in N_0$. Then,

$$\zeta \leq x_n(t) + \frac{t+1}{n} \leq (R_1 + \varepsilon)(1 + t), \quad t \in [1, +\infty).$$

From (3.9) and (1.4) we get

$$\begin{aligned} \frac{-(x_n'(t) + 1/n)x_n''(t)}{g(x_n'(t) + 1/n) + 1} &\leq \frac{(x_n'(t) + 1/n)|x_n''(t)|}{g(x_n'(t) + 1/n) + 1} \\ &\leq \mu\Phi(t)(x_n'(t) + 1/n)h(x_n(t) + (t+1)/n) \quad (t > 0) \end{aligned}$$

and integrating the inequality gives

$$\begin{aligned} &|I(x_n'(t_1) + 1/n) - I(x_n'(t_2) + 1/n)| \\ &\leq \mu\left|\int_{t_1}^{t_2} \Phi(t)h\left(x_n(t) + \frac{t+1}{n}\right)\left(x_n'(t) + \frac{1}{n}\right)dt\right|. \end{aligned} \tag{3.15}$$

(for $0 \leq t_1, t_2 < \infty$). Then we have for $t_1 \geq 1$ and $t_2 \geq 1$

$$|I(x_n'(t_1) + 1/n) - I(x_n'(t_2) + 1/n)| \leq \mu\left|\int_{t_1}^{t_2} (R_1 + \varepsilon)\Phi(t)H_{\zeta, R_1+\varepsilon}(t)\,dt\right|. \tag{3.16}$$

If $t_1 \leq 1$ and $t_2 \leq 1$

$$\begin{aligned} &|I(x_n'(t_1) + 1/n) - I(x_n'(t_2) + 1/n)| \\ &\leq \mu\left|\int_{t_1}^{t_2} \Phi(t)h\left(x_n(t) + \frac{t+1}{n}\right)d\left(x_n(t) + \frac{t+1}{n}\right)dt\right| \\ &\leq \mu\left|\|\Phi\|_\infty \int_{x_n(t_1)+(t_1+1)/n}^{x_n(t_2)+(t_2+1)/n} h(s)\,ds\right| \end{aligned} \tag{3.17}$$

(3.16) and finiteness of $\int_1^\infty \Phi(s)H_{1/c,c}(s)\,ds$ guarantee that the functions $I(x_n'(t)+1/n)$ $(n \geq n_0)$ are equicontinuous at each point of $[1,\infty)$. Since $\{x_n'(t)+1/n : n \geq n_0\}$ is bounded on $[0,1]$ by $\sup_n \|x_n\| + 1/n_0 \leq R_1+1$, the functions belonging to $\{x_n(t)+(t+1)/n : n \geq n_0\}$ are equicontinuous on $[0,1]$. Together with (3.17) and the finiteness of $\int_0^T h(s)\,ds$ (for $T = 2R_1+2 > 0$) this implies that the functions $I(x_n'(t)+1/n)$ $(n \geq n_0)$ are equicontinuous at each point of $[0,1]$. Combining with equicontinuity on $[1,\infty)$ we get equicontinuity at each point of $[0,\infty)$. Since I^{-1} is uniformly continuous on $[0, I(R_1+1/n_0)]$, we can deduce that the functions $x_n'(t)$ $(n \geq n_0)$ are equicontinuous at each point of $[0,\infty)$. A similar argument, using $x_n'(\infty) = 0$ shows, via

$$|I(x_n'(t_1)+1/n) - I(1/n)| \leq \mu \left| \int_{t_1}^{+\infty} \Phi(t)H_{\zeta,R_1+\varepsilon}(t)\,dt \right| \quad (t_1 \geq 1),$$

that the functions are also equicontinuous at ∞.

We can therefore apply Lemma 2.2 to conclude that there is a subsequence $(x_{n_j}')_{j=1}^\infty$ of $(x_n')_{n=n_0}^\infty$ that converges uniformly on $[0,\infty]$ to some limit function $y \in C[0,\infty]$. Passing to a further subsequence, and using compactness of $[0,R_1]$ we can also assume that $\lim_{j\to\infty} x_{n_j}(0) = \xi_0$ exists. We can then define $x_0 \in C_\infty^1[0,+\infty)$ by

$$x_0(t) = \xi_0 + \int_0^t y(\tau)\,d\tau$$

and we can see from the fact that x_n satisfies (3.10) that $x = x_0$ must satisfy the boundary conditions (1.2). As $x_{n_j} \in P$ we have $x_0(0) = \xi_0 \geq 0$. As $x_{n_j}'(t) > 0$ for all t we have $x_0'(t) = y(t) = \lim_{j\to\infty} x_{n_j}'(t) \geq 0$.

By continuity at ∞ and $x_0'(\infty) = 0$, there is $t_0 > 0$ so that $x_0'(t) < \delta/3$ for $t \in [t_0,\infty)$. By uniform convergence, there is $j_0 > 3/\delta$ so that $|x_{n_j}'(t) - x_0'(t)| < \delta/3$ for all $j \geq j_0$ and $t \in [t_0,\infty)$. Hence $0 < x_{n_j}'(t) + 1/n_j < \delta$ for all $t \geq t_0$, $j \geq j_0$. From (1.5) we have therefore

$$f_{n_j}(\tau, x_{n_j}(\tau), x_{n_j}'(\tau)) \leq \left(x_{n_j}(\tau) + \frac{\tau+1}{n_j} \right)^\gamma \beta(\tau) \quad (\tau \geq t_0, j \geq j_0).$$

Since x_n satisfies (3.9), Lemma 2.4 gives

$$x_{n_j}'(t) \geq \mu \int_t^\infty (\alpha_{t_0}(\tau))^\gamma (-\beta(\tau))\,d\tau \quad (t \geq t_0, j \geq j_0).$$

Taking limits as $j \to \infty$ we see that $x_0'(t) > 0$ holds for all $t \in (t_0,\infty)$. We claim that in fact $x_0'(t) > 0$ for all $t \geq 0$. To see the claim, consider the set $\{t \geq 0 : x_0'(\tau) > 0 \forall \tau \in (t,\infty)\} \subseteq [0,\infty)$, a nonempty set since it contains t_0. Denote the infimum of this set by t^*. If $t^* > 0$ then necessarily $x_0'(t^*) = 0$.

We show that $x_0'(t^*) > 0$ (so that $t^* = 0$ and $x_0'(0) > 0$). If $x_0'(t^*) = 0$, then there is $t_1 > t^*$ so that $x_0'(t) < \delta/3$ for $t^* \le t \le t_1$. By unform convergence there is $j_0 \ge 3/\delta$ such that $|x'_{n_j}(t) - x_0'(t)| < \delta/3$ for $j \ge j_0$ and all t. Hence $x'_{n_j}(t) + 1/n_j < \delta$ for $t \in [t^*, t_1)$ and $j \ge j_0$. For $j \ge j_0$ fixed, consider $t_j^* = \sup\{t_j : x'_{n_j}(t) + 1/n_j < \delta \forall t \in [t^*, t_j)\} \ge t_1 > t^*$. Note that

$$\begin{aligned} x''_{n_j}(t) &= \mu f_{n_j}(t, x_{n_j}(t), x'_{n_j}(t)) \\ &< \mu \left(x_{n_j}(t) + \frac{t+1}{n_j} \right)^{\gamma} \beta(t) < 0 \quad (t^* \le t < t_j^*) \end{aligned} \tag{3.18}$$

by (1.5). If $t_j^* < \infty$, then $x'_{n_j}(t_j^*) = \delta + 1/n_j$, but this is impossible because (3.18) implies $\delta > x'_{n_j}(t^*) + 1/n_j > x'_{n_j}(t_j^*) + 1/n_j = \delta$. Thus $t_j^* = \infty$ for $j \ge j_0$.

From Lemma 2.4 we have

$$x'_{n_j}(t) \ge \mu \int_{t^*}^{\infty} (\alpha_{t^*}(\tau))^{\gamma} (-\beta(\tau))\, d\tau \quad (t \ge t^*, j \ge j_0).$$

Taking the limit as $j \to \infty$ shows $x_0'(t^*) > 0$. We have therefore shown that $t^* = 0$ and $x_0'(t) > 0$ for $t \ge 0$, which implies that $x_0(t) > 0$ for all $t \in (0, +\infty)$. Consequently, since $\lim_{j \to +\infty} \|x_{n_j} - x_0\| = 0$, we have

$$\inf_{j \ge 1} \left\{ \min_{s \in [t,1]} x_{n_j}(s), \min_{s \in [t,1]} x'_{n_j}(s) \right\} > 0, \ \ t < 1$$

and

$$\inf_{j \ge 1} \left\{ \min_{s \in [1,t]} x_{n_j}(s), \min_{s \in [1,t]} x'_{n_j}(s) \right\} > 0, \ \ t > 1.$$

From

$$x'_{n_j}(t) - x'_{n_j}(1) = \mu \int_1^t f_{n_j}(\tau, x_{n_j}(\tau), x'_{n_j}(\tau))\, d\tau, \ \ t \in (0, +\infty),$$

letting $j \to +\infty$, the Lebesgue Dominated Convergence Theorem and (1.4) guarantee that

$$x_0'(t) - x_0'(1) = \mu \int_1^t f(\tau, x_0(\tau), x_0'(\tau))\, d\tau, \ \ t \in (0, +\infty).$$

By direct differentiation, we have

$$x_0''(t) = \mu f(t, x_0(t), x_0'(t)), \ \ t \in (0, +\infty),$$

which is (1.1). We already established that $x = x_0$ satisfies (1.2). □

4 Examples and consequences

Corollary 4.1. *Assume that the hypotheses of Theorem 1.1 hold together with one of the following conditions*

(i) $\lim_{t\to\infty} t\int_t^\infty \beta(\tau)\,d\tau = -\infty$

(ii) $\int_{t_0}^\infty (\tau - t_0)\beta(\tau)\,d\tau = -\infty$ *for some* $t_0 \geq 0$.

Then all solutions of (1.1)–(1.2) with $x(t) \geq 0$ *and* $x'(t) > 0$ *for* $t \in (0, \infty)$ *are unbounded.*

Proof. Let x_0 be a nonnegative and strictly monotone increasing solution of (1.1)–(1.2). Since $\lim_{t\to\infty} x_0'(t) = 0$, there is $t' \geq t_0$ so that $0 < x'(t) < \delta$ for all $t \in [t', \infty)$. From (1.5) and the integral form of (1.1) we have

$$x_0'(t) \geq \mu \int_t^\infty (x_0(\tau))^\gamma(-\beta(\tau))\,d\tau \qquad (t \geq t').$$

Hence, for $t > t'$,

$$\begin{aligned} x_0(t) - x_0(t') &\geq \mu \int_{t'}^t \int_s^\infty (x_0(\tau))^\gamma(-\beta(\tau))\,d\tau\,ds \\ &= \mu \int_{t'}^t \int_s^t (x_0(\tau))^\gamma(-\beta(\tau))\,d\tau\,ds \\ &\qquad + \mu \int_{t'}^t \int_t^\infty (x_0(\tau))^\gamma(-\beta(\tau))\,d\tau\,ds \\ &= \mu \int_{t'}^t (\tau - t')(x_0(\tau))^\gamma(-\beta(\tau))\,d\tau \\ &\qquad + \mu(t - t') \int_t^\infty (x_0(\tau))^\gamma(-\beta(\tau))\,d\tau. \end{aligned}$$

Given either of the two conditions we conclude $x(t) \to \infty$ as $t \to \infty$. □

Corollary 4.2. *Assume that the hypotheses of Theorem 1.1 hold and that for each* $c' > 0$, $c \geq 1$ *there is* $T = T(c', c) \in [1, \infty)$ *satisfying*

$$\int_T^\infty I^{-1}\left(c' \int_s^\infty \Phi(\tau) \max_{\frac{1}{c} \leq x \leq c(1+\tau)} h(x)\,d\tau\right) ds < \infty$$

(with $I(z) = \int_0^z u/(g(u)+1)\,du$*). Then all solutions of (1.1)–(1.2) with* $x(t) \geq 0$ *and* $x'(t) > 0$ *for* $t \in (0, \infty)$ *are bounded.*

Proof. Let x_0 be a nonnegative and strictly monotone increasing solution of (1.1)–(1.2). From the assumptions $\lim_{t\to 0^+} x_0'(t)$ exists and $\lim_{t\to\infty} x_0'(t) = 0$. It follows easily that $x_0 \in C^1_\infty[0,\infty)$. As in (3.4) there is a $c > 0$ such that $\frac{1}{c} \le x_0(t) \le c(1+t)$ for all $t \in [1,+\infty)$. Then, we can see from (1.1) and (1.4) that

$$\begin{aligned} |x_0''(t)| &\le \mu\Phi(t)h(x_0(t))g(x_0'(t)) \\ -x_0'(t)x_0''(t)/(g(x_0'(t))+1) &\le \mu\Phi(t) \max_{\frac{1}{c}\le x\le c(1+t)} h(x)\|x_0\|, \;\; t\in[1,+\infty). \end{aligned}$$

Integrating we get

$$I(x_0'(t)) - I(0) \le \mu\|x_0\| \int_t^\infty \Phi(\tau) \max_{\frac{1}{c}\le x\le c(1+\tau)} h(x)\, d\tau, \;\; t\in[1,+\infty),$$

and hence (since $I(0) = 0$ and I^{-1} is monotone) if $t \in [t_0,\infty)$ for $t_0 \ge 1$ sufficiently large

$$x_0'(t) \le I^{-1}\left(\mu\|x_0\| \int_t^\infty \Phi(\tau) \max_{\frac{1}{c}\le x\le c(1+\tau)} h(x)\, d\tau\right).$$

Integrating both sides and using the hypothesis we get x_0 bounded. □

Example 4.3. Consider the boundary value problem

$$\begin{cases} x'' = \mu e^{-t}\left(x^{b_1} + x^{b_2} + x^{-b_3}\right)\left(\cos t + (x')^{a_1} - (x')^{-a_2}\right) \quad (t\in(0,\infty)) \\ x(0) - x'(0) = 0, \qquad \lim_{t\to\infty} x'(t) = 0 \end{cases} \tag{4.1}$$

where $a_1 \ge 0$, $a_2 > 0$, $b_2 \ge b_1 > 0$, $0 < b_3 < 1$, $b_1 < 1$ and $\mu > 0$. Then there is $\mu_0 > 0$ so that, for $0 < \mu < \mu_0$, the problem has a bounded nonnegative strictly monotone increasing solution. If $b_2 + 1 < 2 - a_1$, $\mu_0 = +\infty$.

To see that this is so we apply Theorem 1.1 and Corollary 4.2, where $k = 0$, $a = b = 1$, $\Phi(t) = e^{-t}$, $h(x) = x^{b_1} + x^{b_2} + x^{-b_3}$, $g(z) = 1 + z^{a_1} + z^{-a_2}$, $\delta = (1/3)^{1/a_2}$, $\gamma = b_1$ and $\beta(t) = -e^{-t}$. It is easy to see that (1.4) and (1.5) hold. Theorem 1.1 guarantees that there is $\mu_0 > 0$ so that for $0 < \mu < \mu_0$ the problem has a nonnegative strictly monotone increasing solution.

Moreover, since $\lim_{u\to 0^+} \frac{1}{1+u^{a_2}+a^{a_1+a_2}} = 1$, there is a δ_0 with $1 \ge \delta_0 > 0$ such that for $0 \le u \le \delta_0$, $\frac{u^{1+a_2}}{1+u^{a_2}+u^{a_1+a_2}} \ge \frac{1}{2}u^{1+a_2}$. Thus

$$\begin{aligned} I(z) &= \int_0^z \frac{u}{1+g(u)}\, du = \int_0^z \frac{u^{1+a_2}}{1+2u^{a_2}+u^{a_1+a_2}}\, du \\ &\ge \int_0^z \frac{1}{2}u^{1+a_2}\, du = \frac{1}{2(2+a_2)} z^{2+a_2} \quad (0\le z\le \delta_0), \end{aligned}$$

which implies

$$I^{-1}(y) \le (2(2+a_2)y)^{\frac{1}{2+a_2}}, \quad 0 \le y \le I(\delta_0). \tag{4.2}$$

Since $\lim_{s\to+\infty}\int_s^\infty \Phi(\tau)H_{1/c,c}(\tau)\,d\tau = 0$ (with the notation (3.3)), there is a $T = T(c', c) > 0$ such that $\max(c', 1)\int_T^\infty \Phi(\tau)H_{1/c,c}(\tau)\,d\tau < I(\delta_0)$. Then (4.2) yields that

$$\begin{aligned} &\int_T^\infty I^{-1}\left(c'\int_s^\infty \Phi(\tau)H_{1/c,c}(\tau)\,d\tau\right) ds \\ \le\ & \int_T^\infty \left(2(2+a_2)c'\int_s^\infty \Phi(\tau)H_{1/c,c}(\tau)\,d\tau\right)^{\frac{1}{2+a_2}} ds < +\infty. \end{aligned}$$

As a result Corollary 4.2 guarantees that the solutions to (4.1) are bounded.

Assume now $b_2 + 1 < 2 - a_1$. For $z \ge 1$,

$$\begin{aligned} I(z) &= \int_0^z \frac{u}{1+g(u)}\,du \\ &= \int_0^1 \frac{u}{1+g(u)}\,du + \int_1^z \frac{u}{1+g(u)}\,du \\ &\ge I(1) + \frac{1}{3(2-a_1)}[z^{2-a_1} - 1], \end{aligned}$$

we have $I([0,\infty)) = [0,\infty)$ and

$$z \ge I^{-1}\left(I(1) + \frac{1}{3(2-a_1)}[z^{2-a_1} - 1]\right), \quad z \ge 1.$$

Let $y = I(1) + \frac{1}{3(2-a_1)}[z^{2-a_1} - 1]$, so that $y \ge I(1)$ can be arbitrary with $z \ge 1$. Hence

$$I^{-1}(y) \le (3(2-a_1)(y - I(1)) + 1)^{\frac{1}{2-a_1}} \quad (y \ge I(1)). \tag{4.3}$$

Finally, for any $\mu_0 > 0$, consider (3.7). We have already noted $I_\infty = \infty$ and so $J(\mu_0) = [0,\infty)$. It is easy to see that, with the constant $c_0 = \|\Phi\|_\infty \int_0^1 h(s)\,ds$,

$$\begin{aligned} &\frac{k}{a} + \left(\frac{b}{a} + 1\right) I^{-1}\left(\mu_0\left(c_0 + c\int_0^\infty \Phi(s) \max_{1\le x\le(1+c)(1+s)} h(x)\,ds\right)\right) \\ \le\ & 2I^{-1}\left(\mu_0\left(c_0 + 3c(1+c)^{b_2}\int_0^\infty e^{-s}(1+s)^{b_2}\,ds\right)\right). \end{aligned}$$

From (4.3), it is easy to see that if $b_2 + 1 < 2 - a_1$, then the left hand side of (3.7) is ∞ for all μ_0. Thus (4.1) has solutions for all μ (if $b_2 + 1 < 2 - a_1$).

Example 4.4. Consider the boundary value problem (on $t \in (0, \infty)$)

$$\begin{cases} x'' = \mu(1+t)^{-2}\left(x^{b_1} + x^{b_2} + x^{-b_3}\right)(1 + (x')^{a_1} - (x')^{-a_2}) \\ x(0) = 0, \qquad \lim_{t\to\infty} x'(t) = 0, \end{cases} \tag{4.4}$$

where $1 \geq a_1 \geq 0$, $a_2 > 0$, $1 > b_1 \geq 0$, $1 > b_2 > 0$, $1 > b_3 \geq 0$ and $\mu > 0$. Then there is a $\mu_0 > 0$ such that the problem (4.4) has an unbounded nonnegative strictly monotone increasing solution for all $0 < \mu \leq \mu_0$.

To see that this is so we apply Theorem 1.1, Corollary 4.1, where $k = 0$, $a = 1$, $b = 0$, $\Phi(t) = (1+t)^{-2}$, $h(x) = x^{b_1} + x^{b_2} + x^{-b_3}$, $g(z) = 1 + z^{a_1} + z^{-a_2}$, $\delta = (1/3)^{1/a_2}$, $\gamma = b_1$ and $\beta(t) = -(1+t)^{-2}$.

To apply Corollary 4.1 note that $\int_1^\infty (\tau - 1)\beta(\tau)\, d\tau = -\infty$.

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