Understanding the kinetics of catalysed reactions in microheterogeneous thin film electrodes

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1. Introduction

The problem of quantitatively describing the transport and kinetics of reactants within bounded thin polymeric films (aka chemically modified electrodes) is very challenging and various approaches have been developed over the last 30 years [1,2]. The early work of Saveant et al. [3] and Albery and Hillman [4] is seminal. This early work had a focus on describing diffusion coupled with bimolecular reaction or diffusion coupled with substrate pre-activation within a thin film. This topic remains of interest due to the fact that it enables the technologically important areas of chemical/bio-sensing and surface catalysis within fuel cell and electrolysis devices to be described in a mathematically precise manner [5,6]. Using this approach an analytical solution to a well defined reaction/diffusion problem can be developed which will describe how the concentration of the reactant (or substrate) will vary through the catalytic layer as a function of distance and time. It also enables an expression for the net reaction rate or reaction flux to be derived. In electrochemical systems the reaction flux is usually expressed as a current. This can be related in a definite way to real system parameters such as catalyst loading, catalyst concentration, substrate concentration and film thickness. The mathematical model should make simple predictions on how the reaction rate depends on each of the latter parameters and thereby enable the rational design of a modified electrode in which substrate detection/catalysis is optimized. We have recently summarised advances in modelling the mechanism of mediated electron transfer at redox active surfaces where the binding interaction between surface site and substrate can be complex [7].

In the present paper we focus attention on amperometric chemical sensing via surface immobilized redox active catalytic species (such as redox enzymes) which are embedded in a polymeric support matrix. In amperometric detection the target analyte diffuses through the solution phase, partitions into the catalytic film, diffuses within the film and subsequently is oxidized at the catalyst surface within the layer. The oxidized active form of the reduced catalyst is regenerated via application of an oxidizing potential at the support electrode. Communication between the underlying support electrode is accomplished in one of two ways. If the matrix containing the catalytic species is electronically conductive then the active form of the catalyst may be regenerated at a polymer strand which is in direct electronic communication with the support electrode surface and hence responds the applied potential input. Conversely if the polymer matrix in non conducting a small molecule redox mediator may be used to shuttle charge between catalyst and support electrode thus facilitating catalyst regeneration and turnover. The redox mediator (in its reduced form)
Nomenclature

General Michaelis Menten Problem

\[ s \text{ (mol cm}^{-3}\text{)} \] - Substrate concentration in film.
\[ s_i = k_s^{-1} \text{ (mol cm}^{-3}\text{)} \] - Substrate concentration at outer region of layer which in absence of substrate polarization effects is given by the product of the substrate partition coefficient and the bulk substrate concentration \( k_s^{-1} \).
\[ x \text{ (cm)} \] - Distance variable.
\[ L \text{ (cm)} \] - Layer thickness.
\[ \chi = \frac{x}{L} \] - Normalised distance.
\[ u = \frac{x}{k_s^{-1}} \] - Normalised substrate concentration.
\[ \alpha = \frac{k_s}{k_{Ss}} \] - Normalised saturation parameter.
\[ K_M \text{ (mol cm}^{-3}\text{)} \] - Michaelis Constant. Quantifies affinity of catalyst site for substrate. If this parameter is small then the substrate/catalyst binding affinity is low. If it is large the binding affinity will be high.
\[ k_r \text{ (s}^{-1}\text{)} \] - Catalytic rate constant or turnover frequency of catalytic reaction. Measures decomposition rate of ES complex to form products.
\[ \frac{k_R}{K_M} \text{ (cm}^3\text{mol}^{-1}\text{s}^{-1}\text{)} \] - Bimolecular rate constant for catalyst/substrate reaction.
\[ c_s \text{ (mol cm}^{-3}\text{)} \] - Total catalyst concentration. Specifically for enzyme catalysis \( c_s = e_s \) = total enzyme concentration.
\[ k = \frac{k_r c_s}{K_M} \text{ (s}^{-1}\text{)} \] - Pseudo first order rate constant for catalyst/substrate reaction.
\[ \gamma = \frac{L^2}{D_X} = \left( \frac{L}{X_K} \right)^2 = \Phi^2 = \frac{f_R}{f_{DX}} \] - Compares the flux for catalytic reaction with that of diffusion of species X. Can refer either to substrate S or mediator B. Termined reaction/diffusion parameter.
\[ D_X \text{ (cm}^2\text{s}^{-1}\text{)} \] - Diffusion coefficient of species X. X can refer to substrate or charge transfer mediator. Also can refer to inhibitor species if inhibition effects are considered.
\[ \Phi = \sqrt{\beta} = \frac{L}{X_K} \] - Thiele Modulus. Describes ratio between film thickness and reaction layer thickness \( X_K \) which is the distance into the film the reactant species travels before it is destroyed by reaction with catalyst. This will be important when the Thiele modulus is large and catalytic reaction will be much more rapid than reactant diffusion.
\[ X_K = \sqrt{\frac{D_X}{k}} \text{ (cm)} \] - Reaction layer thickness. Can refer either to substrate or mediator depending on system investigated.
\[ \beta = \frac{\gamma}{1 + \alpha} \] - Composite kinetic parameter derived via AGM procedure.
\[ y = \frac{a}{x} \text{ (mol cm}^{-3}\text{)} \] - Normalised current defining sensor response when operating in amperometric detection mode.
\[ n, \, z \] - Number of electrons transferred.
\[ F \text{ (Faraday constant, 96,485C mol}^{-1}\text{)} \] - Current passed during amperometric detection.
\[ A \text{ (cm}^2\text{)} \] - Electrode area.

Concentration polarization effect

\[ D_X, D_P \text{ (cm}^2\text{s}^{-1}\text{)} \] - Substrate diffusion coefficient in the film and the solution phase.
\[ \nu = \frac{D_P}{k_s D_P \gamma / L} \] - Biot number, which compares the rate of substrate diffusion in the solution to that in the polymer film.
\[ X_P \text{ (cm)} \] - Nernst diffusion layer thickness in solution phase adjacent to polymer thin film.

Potentiometric detection with Michaelis-Menten kinetics

\[ \omega = \frac{p}{K_M} \] - Normalised product concentration.
\[ w_0 \text{ normalised product concentration at electrode surface.} \]
\[ p \text{ (mol cm}^{-3}\text{)} \] - Product concentration.
\[ E, \, E^\circ \text{ (V)} \] - Electrode potential and standard electrode potential respectively.

Reaction/diffusion with competitive inhibition

\[ K_i^\prime = K_i \left( 1 + \frac{w}{K_M} \right) \] - Modified Michaelis constant taking competitive inhibition into account.
\[ K_i \text{ Dissociation constant for catalyst/inhibitor complex.} \]
\[ \omega = \frac{w}{K_M} \] - Normalised inhibitor concentration.
\[ \lambda = \frac{k_s}{K_M} \] - Ratio between inhibitor concentration in polymer film to the dissociation constant for catalyst / inhibitor complex.
\[ \gamma = \frac{f_{DX}}{f_R} \] - Modified reaction/diffusion parameter with competitive inhibition present.
\[ \alpha' = \frac{a + \lambda w}{1 + \lambda w} \] - Modified saturation parameter with competitive inhibition present.

Normalised oxidized mediator.

Enzyme catalysis with small molecule mediation

\[ a, \, b \text{ (mol cm}^{-3}\text{)} \] - Reduced and oxidized mediator concentrations.
\[ a^- \text{ (mol cm}^{-3}\text{)} \] - Bulk concentration of reduced mediator.
\[ D_A = D_B = D_M \text{ (cm}^2\text{s}^{-1}\text{)} \] - Diffusion coefficient of mediator species.
\[ \kappa = \frac{f_{SD}}{f_{MC}} \] - Dimensionless kinetic ratio comparing rates of substrate/catalyst reaction (substrate and oxidized redox catalyst/enzyme) and oxidized mediator/reduced catalyst/enzyme reactions.
\[ k \text{ (cm}^3\text{mol}^{-1}\text{s}^{-1}\text{)} \] - Bimolecular rate constant for redox mediator/enzyme reaction.
\[ \gamma = \frac{f_{SC}}{f_{SD}} \] - Substrate reaction/diffusion parameter. Compares substrate/enzyme reaction flux to that for substrate diffusion.
\[ \gamma' = \frac{f_{MD}}{f_{MC}} \] - Mediator reaction/diffusion parameter. Compares substrate/enzyme reaction flux with that of mediator diffusion.
\[ k_M, k_S \] - Partition coefficient of substrate and redox mediator.
\[ \mu = \sqrt{\frac{\gamma}{1 + \alpha + \kappa}} \] - Composite kinetic parameter obtained via AGM technique in redox catalysis problem.
\[ \zeta = \frac{k_M}{D_M / L} \] - Competition parameter relating rate constant for direct oxidized mediator regeneration at electrode surface with that of redox mediator diffusion.
\[ F_r = \zeta^{-1} \] - Loss factor.
The steady state reaction/diffusion equation is given by:

\[ \theta = \frac{\tau_M}{\tau_S} = \frac{fS}{fD} \frac{\kappa S\Delta s^-}{L} = \frac{\kappa S\Delta s^-}{L} \]

Diffusion competition parameter. Compares substrate diffusion rate to that of redox mediator in layer.

\[ \Psi_S = \frac{iL}{nF \Delta DKS^+} = \frac{\psi}{a} \text{ Normalised current response for reaction in a conducting polymer layer and in a non conductive polymer layer respectively.} \]

Diffusion from the solution, partitions into the film, diffuses to the catalytic site, reacts there and subsequently a fraction of the oxidized mediator diffuses to the electrode where it is regenerated via oxidation to its reduced form to continue the catalytic cycle. Some oxidized mediator diffuses in the opposite direction and is lost from the layer. These scenarios are presented in Fig. 1.

Furthermore the substrate and mediator concentration profiles through the catalytic layer depend both on distance and time and on the nature of the chemical rate law operating. If we ignore the time dependence then the steady state reaction/diffusion equation is given by:

\[ \frac{Df}{\varepsilon^{1-1} d\xi} \left( \varepsilon^{1-1} d\xi \right) - R(s) = 0 \]

In this expression we have ignored a complicating factor such as product inhibition. The value of the parameter \( \alpha \) is dependent on the type of modified electrode system studied. For reaction/diffusion within a slab then \( \alpha = 1 \). For reaction/diffusion within a hemispherical film deposited on an inlaid disc microelectrode surface then spherical coordinates are used and \( \alpha = 3 \). Finally for reaction/diffusion within a coated wire electrode we use cylindrical polar coordinates and set \( \alpha = 2 \). For planar diffusion within a slab we set \( \xi = x \) whereas for spherical and diffusion we note that \( \varepsilon = r \) and \( \xi = \rho \) respectively.

Typically for enzymatic and metal oxide catalysts the mechanism involves a binding interaction forming an adduct between the catalytic site and the substrate which subsequently decomposes to product. This is the Michaelis-Menten mechanism. In this case the form of the rate equation is:

\[ R(s) = \frac{k_c c_s s}{K_M + s} \]

This expression is non linear with regard to substrate concentration \( s \). The thermodynamics of binding are expressed by the Michaelis constant \( K_M \) which is a measure of the affinity of the substrate for the catalyst site, and the turnover rate by the catalytic rate constant \( k_c \). The net catalytic efficiency is formally expressed by the ratio \( k_c/K_M \). Typically the reaction rate or current varies with substrate concentration in a non linear manner, usually hyperbolic. Substrate transport follows the Fick diffusion equation. In the following discussion we restrict attention to the case where the catalytic layer is immobilized within a conducting matrix and so mediator transport and reaction kinetics may be neglected. Substrate diffusion in solution will be neglected and only substrate diffusion within the surface layer will be considered and quantified by the diffusion coefficient \( D_S \). Combining eqs. 1 and 2 we obtain the following non linear differential equation for the diffusion and reaction of substrate \( S \) within the thin layer:

\[ \frac{D_S}{\varepsilon^{1-1} d\xi} \left( \varepsilon^{1-1} d\xi \right) - \frac{k_{c_s} c_s s}{K_M + s} = 0 \]

In this paper we discuss analytical solutions of eq. 3 which are valid for all values of substrate concentration and use this solution to derive analytical expressions for the current response under steady state conditions. We consider simple linear reaction/diffusion within a slab, and analytical expressions valid for all values of substrate concentration for the steady state current response for this system will be derived. In the next paper non linear reaction/bounded diffusion within a thin hemispherical film deposited on an inlaid disc electrode and within a film coated microwire electrode will be discussed.

2. Planar reaction diffusion in a bounded slab

We initially consider the simple case of planar diffusion and reaction of substrate \( S \) within a thin slab of thickness \( L \) immobilized on a support electrode containing a homogeneous distribution of catalytic particles of concentration \( c_c \). Hence the governing reaction diffusion equation admits the following form:

\[ \frac{D_S}{\varepsilon^{1-1} d\xi} s \frac{k_{c_s} c_s s}{K_M + s} = 0 \] (4)

In this expression we introduce the pseudo first order rate constant \( k = k_c c_c / K_M \) where \( c_c \) denotes the total catalyst concentration (mol cm\(^{-3}\)) respectively. This equation must be solved subject to the following boundary conditions:

\[ x = 0 \quad \frac{ds}{dx} = 0 \quad x = L \quad s = ks^- \] (5)

Here \( \kappa \) denotes the partition coefficient of substrate and \( s^- \) is the bulk concentration of substrate in solution. Hence the product \( ks^- \) represents the reactant concentration at the layer solution interface. The latter boundary condition implicitly assumes that concentration polarization of substrate in the solution may be neglected.

We introduce the following dimensionless quantities:

\[ u = \frac{s}{ks^-} \quad x = \frac{x}{L} \quad a = \frac{k_s}{K_M} \quad \gamma = \frac{\Phi^2}{D_S} \] (6)

Where \( u, x \) represent the dimensionless concentration and distance parameters respectively. Furthermore \( a \) denotes a saturation parameter and \( \gamma \) defines a reaction/diffusion parameter. The saturation parameter compares the value of the substrate concentration in the layer to the Michaelis constant. When this parameter is small the catalytic kinetics is unsaturated and the rate is first order with respect to substrate concentration. When it is large the kinetics are saturated and zero order kinetics prevail. The reaction/diffusion parameter compares the rate of reaction between substrate and catalyst moiety and the rate of substrate diffusion in the layer and is directly related to the Thiele modulus via the following expression: \( \sqrt{\gamma} = \Phi = L/X_k \) where \( X_k = \sqrt{D_S/k} \) denotes a characteristic reaction layer thickness which is a measure of the distance travelled by the substrate in the film before it reacts with the immobilized catalyst particle.

Hence eq. 4 transforms to:

\[ \frac{d^2 u}{d\xi^2} = \frac{\gamma a}{1 + au} = 0 \] (7)

which must satisfy the following boundary conditions:

\[ x = 0 \quad u = u_0 \quad \frac{du}{d\xi} = 0 \]

\[ x = 1 \quad u = 1 \] (8)

Now the net amperometric current corresponding to the rate of substrate reaction in the layer is given by the following equivalent expressions
relating the reaction flux:

\[
\begin{align*}
    i &= nFAD_x \left( \frac{ds}{dx} \right)_{x=L} - \left( \frac{ds}{dx} \right)_{x=0} = nFAD_x \left( \frac{ds}{dx} \right)_{x=L} \\
    &= nFA \int_0^L k_s(s(x))dx = nFA(k_s/k_M)c_c \int_0^L s(x)dx
\end{align*}
\]

Note that in eq. 9 we have noted that the pseudo first order rate constant is of the Michaelis-Menten form given by: \( k = k_c c_c/(K_M + s^-) \) and that \( (ds/dx)_{x=0} = 0 \). Both definitions of normalised current presented above give rise to the same result. We use the differential expression based on the Fick diffusion expression outlined above in our presentation. It is a characteristic of Michaelis-Menten reaction kinetics that the pseudo rate constant for reaction between catalyst and substrate depends on substrate
concentration and in particular, on the relationship between the bulk substrate concentration and the value of the Michaelis constant for the substrate/catalyst reaction. When \( k_1 \ll K_M \) then the pseudo first order rate constant is given by \( k = \frac{k_1 C_0}{K_M} \).

Consequently we can introduce a normalised steady state current or reaction flux \( y \) as follows:

\[
y = \frac{dL}{nFAK_S D_S} = \alpha \left( \frac{du}{dx} \right)_{x=1} = \frac{a \gamma}{1 + a \alpha} \int_0^1 u(\chi)d\chi
\]

(10)

Hence the problem reduces to evaluating an analytical expression for \( u \) which will be valid for all values of \( \gamma \) and \( \alpha \). Once this is achieved an analytical expression for the normalised flux of the amperometric sensor can be readily derived via eq. 10.

In earlier work [8] we have proposed analytical solutions to eq. 9 which are valid for the limiting cases of low and high saturation parameter values, and have proposed a solution based on the reasonable assumption that the non-linear kinetic term \( \frac{a + u^2}{1 + au} \) can be approximated by the linear expression \( \frac{a + u^2}{1 + au} \). Hence the reaction/diffusion equation transforms to:

\[
\frac{d^2 u}{dx^2} \left( \frac{a + u}{1 + au} \right) = 0
\]

(11)

We have shown that this approximation is valid only for certain values of \( a \) and \( u \). Specifically the approximation pertains for all values of \( u \) where the Michaelis-Menten kinetics are unsaturated (when \( \alpha < 1 \)). For \( \alpha > 1 \) the approximation becomes inaccurate if significant depletion of substrate occurs within the film, if \( u \) falls to less than 0.8 at any point in the film. This will occur when the parameter \( \gamma \) is large. In short our strategy was to transform the non linear reaction/diffusion equation into a linear equation which can be readily integrated.

In recent years Rajendran and co-workers [9,10] have used the variational iteration method (VIM) to model the response of a potentiometric and amperometric enzyme sensor in which linear diffusion is coupled to non linear Michaelis-Menten kinetics. This technique produces solutions to the boundary value problem in terms of convergent series requiring no linearization or small perturbation. The analytical results valid for all saturation parameter values were compared with those earlier limiting cases proposed by Lyons et al. [8] and were found to be in good agreement. More recently Malvandi and Ganji [11] developed a variational iteration method coupled with Padé approximation (VIM-Padé) to obtain analytical expressions involving rational functions for substrate concentration profiles for bounded catalytic systems with non linear Michaelis-Menten Kinetics. Rajendran et al. [12-14] outlined how the method of homotopy perturbation could be used to derive an analytical expression for the substrate concentration profile within a thin layer when the reaction kinetics exhibit Michaelis-Menten kinetics. Finally Dharmalingam and Veeramuni [15] applied the Akbari-Ganj method (AGM) to develop an expression for the amperometric current response to non linear reaction/diffusion in an electroactive polymer film.

In this paper we further develop the AGM to examine steady state non linear reaction diffusion in bounded thin films of planar slab geometry with a particular focus on amperometric detection. Hence we solve eq. 7 subject to the conditions outlined in eq. 8 to obtain approximate closed form analytical expressions for the substrate concentration profile and the normalised reaction flux which are valid for all values of the saturation parameter \( \alpha \) and defined values of the reaction diffusion parameter \( \gamma \). We will compare the approximate solution with the numerical solution obtained using the NDSolve facility in Mathematica 12 to determine the parameter set where goodness of fit between the simulated and closed analytical solution is optimized. We do this for the slab geometry. The cases of both spherical and cylindrical diffusion coupled with non linear Michaelis-Menten reaction kinetics will be discussed in a subsequent paper.

We follow the recent excellent work of Dharmalingam and Veeramuni [15] and assume that a suitable solution for the reaction/diffusion presented in eq. 7 will have the following form:

\[
u(\chi) = A \cosh(\beta \chi) + B \sinh(\beta \chi)
\]

(12)

We can readily show using the boundary conditions presented in eq. 8 that \( A = \sec \beta \) and \( B = 0 \) and we obtain:

\[
u(\chi) = \frac{\cosh(\beta \chi)}{\cosh(\beta)}
\]

(13)

Hence substituting eq. 13 into eq. 14 we obtain:

\[
f(u', u, \chi) = \beta^2 \sec \beta \cosh(\beta \chi) - \frac{\gamma u}{1 + au} = 0
\]

(14)

which will only be true provided \( \chi = 1 \) and so And specifically from eq. 8 we note that:

\[
f(\chi = 1) = \beta^2 - \frac{\gamma}{1 + \alpha} = 0
\]

(16)

Consequently we obtain that:

\[
\beta = \sqrt{\frac{\gamma}{1 + \alpha}}
\]

(17)

Hence the solution to eq. 6 is:

\[
u(\chi) = \frac{\cosh \left[ \sqrt{\frac{\gamma}{1 + \alpha}} \right]}{\cosh \left[ \sqrt{\frac{\gamma}{1 + \alpha}} \right]}
\]

(18)

Furthermore the substrate concentration at \( \chi = 0 \) is given by: \( u_0 = \sech \left[ \sqrt{\gamma} \right] \cosh \left[ \sqrt{\gamma} \right] \). When we have unsaturated catalytic kinetics and \( \alpha << 1 \), then eq. 18 reduces to:

\[
u(\chi) \equiv \sech \left[ \sqrt{\gamma} \right] \cosh \left[ \sqrt{\gamma} \chi \right]
\]

(19)

This is the same as eq. 10 in our initial 1996 paper [8]. Alternatively for saturated catalytic kinetics \( \alpha >> 1 \) and eq. 18 reduces to:

\[
u(\chi) \equiv \sech \left[ \sqrt{\frac{\gamma}{\alpha}} \right] \cosh \left[ \sqrt{\frac{\gamma}{\alpha}} \chi \right]
\]

(20)
Eq. 20 can be simplified further. If the argument in the hyperbolic cosine functions is small then we can Taylor expand the functions to give:
\[
\cosh\left(\frac{\gamma}{1 + \alpha}\right) \approx 1 + \frac{\gamma}{2a} \quad \text{and} \quad \text{sech}\left(\frac{\gamma}{1 + \alpha}\right) \approx 1 - \frac{\gamma}{2a}
\]
and so the substrate concentration profile when the catalytic kinetics are saturated is alternatively given by:
\[
u(\chi) \equiv \left(1 - \frac{\gamma}{2a}\right)\left(1 + \frac{\gamma}{2a}\chi^2\right) = 1 - \frac{\gamma}{2a}\left(1 - \left(1 + \frac{\gamma}{2a}\right)\chi^2\right)
\] (21)

This expression for the concentration profile is valid when \(\frac{\gamma}{2a} < 1\). The expression in eq. 20 is the same as that presented in eq. 11 of our initial 1996 paper [8].

When the reaction/diffusion parameter \(\gamma\) is large then the catalytic reaction kinetics are much faster than substrate diffusion through the film, then we note that
\[
\cosh\left(\frac{\gamma}{1 + \alpha}\right) \equiv \frac{1}{2} \exp\left(-\frac{\gamma}{1 + \alpha}\right) \quad \text{and} \quad \text{sech}\left(\frac{\gamma}{1 + \alpha}\right) \equiv 2 \exp\left(-\frac{\gamma}{1 + \alpha}\right)
\]
and so the normalised substrate concentration profile takes the following form:
\[
u(\chi) \equiv \exp\left[-\frac{\gamma}{1 + \alpha}(1 - \chi)\right]
\] (22)

Physically this expression corresponds to an exponential decay in concentration from an initial value of \(u = 1\) at \(\chi = 1\) with a time constant of \(\sqrt{\frac{\gamma}{1 + \alpha}}\) in a direction going in to the film from the outer surface. Hence there is considerable concentration polarization of substrate in the layer. Alternatively when the reaction/diffusion parameter is small corresponding to the case where catalytic reaction kinetics are more sluggish than substrate diffusion through the layer corresponding to for \(\gamma \ll 1\) we note that:
\[
u(\chi) = \text{sech}\left(\sqrt{\frac{\gamma}{1 + \alpha}}\right) \cosh\left(\frac{\gamma}{1 + \alpha}\chi^2\right) \equiv \left(1 - \frac{\gamma}{2(1 + \alpha)}\right) \times \left(1 + \frac{\gamma}{2(1 + \alpha)}\chi^2\right) \equiv 1
\] (23)

Hence under these circumstances there is little concentration polarization of substrate within the film. The unsaturated catalytic kinetics are much more sluggish than substrate diffusion, and a uniform substrate concentration with little depletion is expected in the layer.

The approximate analytical solution outlined in eq. 18 is directly compared with the numerical integration of the non-linear reaction/diffusion equation presented in eq. 7. This was achieved using the NDSolve capability in Mathematica 12. The results are presented in Fig. 2 for the case of unsaturated catalytic kinetics and in Fig. 3 for saturated catalytic kinetics. We note that the correspondence between simulated and closed form analytical solutions are excellent when the reaction kinetics are unsaturated. However in Fig. 3B which corresponds to the situation where saturated kinetics pertains, the agreement between simulated values and those derived via eq. 21 is very good. The agreement with the more general solution presented in eq. 18 (orange curve) is less good. There is excellent agreement between the simulated profile and eqs. 18 and 21 in the region where \(\chi = 1\). We compute the normalised current from the concentration gradient at that point in the analysis so the expression derived for the normalised current is in excellent agreement with that derived via numerical simulation. When reaction between the substrate and the catalyst is very fast then eq. 18 is seen to underrepresent the substrate concentration through the film when compared with the result of numerical simulation. Deviation from the simulation arise typically when the saturation parameter is large and

![Unsaturated catalytic kinetics](image_url)

**Fig. 2.** The variation of substrate concentration within the layer under conditions of unsaturated catalytic kinetics (\(\alpha \ll 1\)). The numerical solution (blue curve) is compared with eq. 18 (orange curve). The substrate diffusion rate as compared with the rate of catalytic reaction increases from panel A through panel D.
when the reaction/diffusion parameter is large. This is typically in region IV of the kinetic case diagram (see later discussion) when the outer region of the film is partially saturated and the inner region unsaturated. This case pertains for $1 < \alpha < \gamma/2$. Under these circumstances eq. 21 is the more appropriate expression to adopt for the concentration profile of substrate in the layer. In Fig. 4 we examine the general case where the saturation parameter is close to unity. Here we choose $\alpha = 1$ and compare simulation results with the closed form analytical solution under conditions where the balance between catalytic kinetics and substrate diffusion is varied. Again very good agreement is observed. The largest divergence is observed in panel C where substrate depletion is significant which occurs at large $\gamma$ values. These results suggest that the general solution is of most use in the scenario where there is balance between catalytic reaction kinetics and substrate diffusion and where the substrate concentration in the film is close to the Michaelis constant of the catalytic reaction.

The normalised current response $y$ is obtained via eq. (10). We may readily show that:

$$y = a \left( \frac{dy}{dx} \right)_{\gamma = 1} = \alpha \sqrt{\frac{\gamma}{1 + \alpha}} \tanh \left( \sqrt{\frac{\gamma}{1 + \alpha}} r \right)$$

(24)

This expression is termed the general case and will pertain to the situation where the catalytic kinetics are neither unsaturated nor saturated when $\alpha$ is close to unity. In Fig. 5 the variation of the normalised current computed via eq. 24, with saturation parameter $\alpha$, is presented for values of the reaction/diffusion parameter $\gamma$ in the range 0.05 to 15. This is in effect a normalised calibration curve, which depicts the variation of current response with substrate concentration. In Fig. 6 the normalised current is plotted as a function of reaction/diffusion parameter $\gamma$, for various defined values of the saturation parameter ranging from 0.1 to 10. Limiting values for the normalised amperometric current response valid for all $\gamma$ values are readily derived in the limits of: $\alpha < < 1$ and $\alpha >> 1$ respectively. Furthermore other limiting expressions are obtained in the limit of $\gamma < < 1$ and $\gamma >> 1$ for all $\alpha$ values. Eq. 24 defines the general case.

In our earlier 1996 paper [8] we quoted an empirical expression constructed by Albery and co-workers [16] for immobilized enzyme electrodes which could be adapted to describe reaction/diffusion in electroactive thin films. Indeed we fitted our experimental data to this expression. The Albery equation is:

$$y_A = \sqrt{2y(a - \ln(1 + a))} \tanh \left( \frac{a \sqrt{2a}}{(1 + a) \sqrt{2(a - \ln a)}} \right)$$

(25)

In Fig. 7 we compare eqs. 24 and 25 derived using AGM proposed by Dharmalingam and Veeramuni [15] directly for a fixed value of $\gamma = 15$ and for a range of saturation parameter values between 0 and 100. Both normalised current response curves exhibit a similar development but the Albery expression over estimates the normalised flux by a significant amount. In Fig. 8 we present the variation of the steady state flux ratio $Y = y/y_A$ as a function of saturation parameter $\alpha$ values over a wide range from 0 to 1000. Each curve presented corresponds to a set value of

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**Fig. 3.** The variation of substrate concentration within the layer under conditions where the catalytic kinetics are saturated ($\alpha = 20$). In panel A the catalytic kinetics are much slower than substrate diffusion, whereas in panel B substrate diffusion is much slower than the catalytic kinetics and substrate depletion is significant. Here the best fit with the simulation (blue curve) is the profile corresponding to eq. 21 (green curve) rather than that predicted from eq. 18 (orange curve).

**Fig. 4.** The variation of substrate concentration within the layer under conditions where the catalytic kinetics are borderline between unsaturated and saturated ($\alpha = 1$). In panel A the catalytic kinetics are much slower than substrate diffusion, whereas in panel B catalytic kinetics and substrate diffusion are in balance. In panel C substrate diffusion is much slower than the catalytic kinetics and substrate depletion is significant. The numerical solution (blue curve) and the closed form analytical solution (orange curve) are in very good agreement except when substrate depletion within the layer is significant (large $\gamma$ values).
Fig. 5. The variation of normalised current with saturation parameter computed via eq. 24. This defines the calibration curve for an amperometric sensor exhibiting Michaelis-Menten kinetics.

Fig. 6. Variation of normalised current response computed via eq. 24 with reaction/diffusion parameter for various values of saturation parameter.

Fig. 7. The expression derived by Albery (eq. 25) and the expression presented in the current work derived from AGM (eq. 24) for the normalised flux compared for a fixed value of reaction/diffusion parameter $\gamma = 15$.

Fig. 8. The variation of the normalised flux ratio with saturation parameter. Curves have been computed for various values of the reaction/diffusion parameter.
the reaction/diffusion parameter \( \gamma \) ranging from 0.1 to 1000. This figure suggests that when \( \gamma \) values are less than 0.1 the normalised flux ratio is essentially unity over the entire range of \( \alpha \) values. However, as the magnitude of \( \gamma \) increases corresponding to more favourable catalytic reaction kinetics in the layer, the normalised flux ratio \( Y \) initially decreases with increase in \( \alpha \) value, to a broad minimum located within a specific \( \alpha \) value range, and then increases again as the saturation parameter value is increased still further to approach a value of unity in the limit of large saturation parameter values. Furthermore, the location of the flux ratio minimum varies with \( \gamma \) value, being located at increasingly larger \( \alpha \) values as the catalytic kinetics become more rapid. So the prediction of normalised current according to Albery (eq. 25) and the present work (eq. 24) are very similar when the catalytic kinetics are sluggish over a wide range of substrate concentration, but when the kinetics are more rapid the Albery expression over predicts the normalised flux by a factor of 20–35% over a significant range of \( \alpha \) values.

As previously noted [8] we can identify four limiting cases of eq. 24 and also of eq. 25. The behaviour of the system can be well described in terms of a kinetic case diagram which is a plot of \( \log \gamma \) versus \( \log \alpha \). This case diagram is outlined in Fig. 9. One limiting case arises when the catalytic kinetics are saturated. Hence \( \alpha \ll 1 \) the reaction kinetics are rapid and we note that the normalised flux reduces to:

\[
y = \alpha \sqrt{\gamma} \tanh[\gamma]
\] (26)

This expression combines two limiting cases. Case I pertains when the catalytic kinetics are slower than substrate diffusion and \( \gamma < 1 \). Hence \( \tanh \sqrt{\gamma} \equiv \sqrt{\gamma} \) and eq. 26 reduces to:

\[
y = \alpha \gamma
\] (27)

Translating back into dimensioned quantities we obtain:

\[
i = nFA(k_c/K_M)c_2X_Kks^{1/2}
\] (28)

Hence the amperometric current response is first order with respect to substrate concentration, catalyst concentration, and layer thickness. The reaction occurs uniformly throughout the film. On the other hand when \( \gamma > 1 \) the reaction kinetics are rapid and we note that \( \tanh \sqrt{\gamma} \equiv 1 \) and eq. 26 reduces to:

\[
y = \alpha \sqrt{\gamma}
\] (29)

This corresponds to case II. Under these circumstances \( L >> X_K \) the layer thickness is much greater than the kinetic length and so reaction occurs in a thin reaction layer at the outside of the film. This will pertain when the reaction kinetics between substrate and catalytic sites occur rapidly. Hence the current response is given by:

\[
i = nFA(k_c/K_M)c_2X_Kks^{32}
\] (30)

Here the current again is first order in substrate concentration, first order with respect to catalyst concentration, and occurs in a thin reaction layer of thickness \( X_K \). Note that eq. 26 joins the two cases I and II. When \( \alpha \gg 1 \), \( \gamma > 1 \) and the catalytic kinetics are saturated. Hence the normalised current adopts the following form which is valid \( \forall \gamma \):

\[
y \equiv \alpha \sqrt{\gamma} \tanh[\gamma] \equiv \sqrt{\gamma}/[1 + \alpha]
\] (31)

When \( \gamma \) is small then \( \sqrt{\gamma}/[1 + \alpha] \ll 1 \) and \( \tanh[\sqrt{\gamma}/[1 + \alpha]] \equiv \sqrt{\gamma}/[1 + \alpha] \) and so the normalised current reduces to:

\[
y \equiv \gamma
\] (32)

This expression defines case III. Here the catalytic kinetics are more sluggish than substrate diffusion and the substrate concentration is the film is larger than the Michaelis constant for the catalytic reaction. Hence the current response for case III is:

\[
i = nFAk_c c_2 L
\] (33)

Hence the rate determining step involves the decomposition of the catalyst/substrate adduct to form products. This occurs throughout the entire thickness of the immobilized film. The current is zero order in substrate concentration and first order with respect to catalyst and layer thickness. Finally when \( \gamma >> 1 \) and the catalytic kinetics are rapid then eq. 31 reduces to:

\[
y \equiv \sqrt{\gamma}/[1 + \alpha]
\] (34)

This defines case IV where both the catalytic kinetics are rapid and the substrate concentration in the layer is larger than the Michaelis constant. Under these circumstances the current response is given by:

\[
i = nFA(k_c/\sqrt{K_M})c_2X_Kks^{32}
\] (35)

In case IV the current is half order with respect to substrate concentration in the layer, is first order with respect to catalyst and occurs in a thin reaction layer adjacent to the film/solution interface. Hence cases II and IV are connected via eq. 31. In the kinetic case diagram cases II and IV are separated by the line \( \gamma = \alpha \) located in the top right hand quadrant. Case IV pertains when one has thick catalytic layers and when the kinetics are saturated when \( s >> X_K \). Under such circumstances one may expect that the outermost regions of the film will be completely saturated whereas the inner regions of the film are unsaturated.

When the kinetics are sluggish then \( \gamma \) is small then \( \tanh[\sqrt{\gamma}/(1 + \alpha)] \equiv \sqrt{\gamma}/(1 + \alpha) \) and \( \forall \alpha \) the general eq. 24 reduces to:

\[
y \equiv \frac{\alpha \gamma}{1 + \alpha}
\] (36)

This expression can be seen to join cases I and III. This is just a normalised form of the Michaelis Menten equation. It will be valid for thin films where there is very little concentration polarization of the substrate in the film and where the reaction kinetics are rate determining.

Finally, we consider the situation where the catalytic kinetics are rapid compared with substrate diffusion. Hence \( \gamma >> 1 \). Hence \( \forall \alpha \) such that \( \alpha < \gamma \).
In this we differentiate between the second differs from that previously adopted since we have replaced the solution. Under such circumstances the boundary conditions take the following form:

\[ x = 0 \Rightarrow \frac{ds}{dx} = 0 \]
\[ x = L \Rightarrow s = s_L \]
\[ x = L \quad \frac{ds}{dx} \bigg|_{x=L} = \frac{D_s}{X_D} (s^\infty - s_L) \]  

(39)

The first boundary condition is the same as that previously noted. The second differs from that previously adopted since we have replaced \( s^\infty \) the bulk concentration of substrate with \( s_L \) the concentration of the substrate in the film/solution interface. We also need to introduce a third boundary condition that of flux balance at the film/solution interface. In this we differentiate between \( D_s \) the substrate diffusion coefficient in the film and \( D_q \) that in the solution. Furthermore \( X_D \) represents the diffusion layer thickness. We introduce the Biot number which compares the rate of substrate diffusion in the solution with that of substrate diffusion in the film:

\[ u = \frac{D_s/X_D}{kD_f/L} \]  

(40)

We can readily transform the boundary conditions to the following form:

\[ x = 0 \quad \frac{du}{dx} = 0 \]
\[ x = 1 \quad \frac{ds}{dx} \bigg|_{x=1} = \alpha (1 - u_1) \quad u = u_1 \]  

(41)

Assuming that the solution of eq. 6 is of the form outlined in eq. 12 then using the boundary conditions in eq. 41 we can readily show that:

\[ u(x) = \frac{1}{1 + \frac{a}{\beta} \tanh |\beta|} \]  

(42)

Also the normalised current \( y \) is given by:

\[ y = \frac{a \beta u_1 \tanh |\beta|}{1 + \frac{a}{\beta} \tanh |\beta|} \]  

(43)

Note that:

\[ \beta = \sqrt{\frac{y (1 + \frac{a}{\beta} \tanh |\beta|)}{1 + \frac{a}{\beta} \tanh |\beta|}} \]  

(44)

Hence we see that the addition of concentration polarization of substrate in solution makes a clean evaluation of the \( \beta \) parameter difficult. This is because concentration polarization in the layer and in solution are intrinsically coupled. We note that when the Biot number is large corresponding to substrate diffusion in solution being larger than that in the layer \( \nu^{-1} \rightarrow 0 \) and \( \beta = \sqrt{\frac{y}{1 + a}} \) which we obtained previously.

Using the ‘magic’ approximation which leads to eq. 11 and solving using the boundary conditions outlined in eq. 41 we can show that:

\[ u(x) = u_1 \sech \left[ \frac{\sqrt{y}}{1 + \alpha} \right] \cosh \left[ \frac{\sqrt{y}}{1 + \beta} \right] + \alpha \left\{ \sech \left[ \frac{\sqrt{y}}{1 + \alpha} \right] \cosh \left[ \frac{\sqrt{y}}{1 + \beta} \right] - 1 \right\} \]  

(45)

Where:

\[ u_1 = \frac{1 - \nu^{-1} \sqrt{\frac{\sqrt{y}}{\nu^{-1}}} \tanh \left[ \frac{\sqrt{y}}{\nu^{-1}} \right]}{1 + \nu^{-1} \sqrt{\frac{\sqrt{y}}{\nu^{-1}}} \tanh \left[ \frac{\sqrt{y}}{\nu^{-1}} \right]} \]  

(46)

We note that when the Biot number is very large \( u_1 \equiv 1 \) as it should. Finally, the steady state current response is given by:

\[ y_M = \frac{\alpha \sqrt{y} \tanh \left[ \frac{\sqrt{y}}{1 + \alpha} \right]}{1 + \frac{\sqrt{y}}{\alpha (1 + \alpha)} \tanh \left[ \frac{\sqrt{y}}{1 + \alpha} \right]} \]  

(47)

We note that for large Biot numbers eq. 47 reduces to

\[ y_M = \alpha \sqrt{y} \tanh \left[ \frac{\sqrt{y}}{1 + \alpha} \right] \]  

(48)

This expression reduces to eq. 27 for case I, eq. 29 for case II, eq. 32, and for case III provided that \( \sqrt{y}/\alpha << 1 \) and \( \alpha >> 1 \). If both \( y \) and \( \alpha \) are both

### Table 1

<table>
<thead>
<tr>
<th>Limiting kinetic case</th>
<th>Normalised current ( y )</th>
<th>Current ( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( y = ay )</td>
<td>( i = nFA(k_c/k_0)c_0Lsa^{-\alpha} )</td>
</tr>
<tr>
<td>II</td>
<td>( y = a\sqrt{t} )</td>
<td>( i = nFA(k_c/k_0)c_0X_k\sqrt{a} )</td>
</tr>
<tr>
<td>III</td>
<td>( y = y )</td>
<td>( i = nFA(k_c/E_c)L )</td>
</tr>
<tr>
<td>IV</td>
<td>( y = \sqrt{y}/E_c )</td>
<td>( i = nFA(E_c/E_0)c_0X_k\sqrt{a} )</td>
</tr>
<tr>
<td>V</td>
<td>( y = \sqrt{y}/E_c )</td>
<td>( i = nFA(E_c/E_0)c_0X_k\sqrt{a} )</td>
</tr>
<tr>
<td>VI</td>
<td>( y = a\sqrt{y}/E_c )</td>
<td>( i = nFA(E_c/E_0)c_0X_k\sqrt{a} )</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Case</th>
<th>( \alpha )</th>
<th>( \varepsilon )</th>
<th>L</th>
<th>( X_D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>IV</td>
<td>1/2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
large then \( \tanh \left[ \frac{\sqrt{\gamma}}{1 + \alpha} \right] \approx 1 - 2 \exp[-2\sqrt{\gamma}/(1 + \alpha)] \approx 1 \) and

\[
\gamma_M \equiv \alpha \sqrt{\gamma} \tag{49}
\]

Hence for case IV the 'Magic' approximation over-estimates the current response by a factor of typically \( \sqrt{\alpha} \).

3.2. Potentiometric sensor response

We can readily derive an expression for the response of the sensor when it is operating in potentiometric mode. Early work on this topic was reported by Mell and Maloy [17] for amperometric detection and by Carr et al. [18,19] for potentiometric detection. In this case the pertinent differential equations for substrate and product are given by:

\[
\begin{align*}
\frac{d^2 u}{dx^2} + \frac{p u}{1 + au} &= 0 \\
\frac{d^2 w}{dx^2} + \frac{p w}{1 + aw} &= 0 
\end{align*}
\tag{50}
\]

Here we have introduced the normalised product concentration as \( w = \gamma' / \kappa \). The pertinent boundary conditions are:

\[
\begin{align*}
\chi = 0 & \quad \frac{du}{dx} = \frac{dw}{dx} = 0 \\
\chi = 1 & \quad u = 1 \quad w = 0 
\end{align*}
\tag{51}
\]

Furthermore we note that:

\[
\frac{d^2 u}{dx^2} + \frac{d^2 w}{dx^2} = 0
\tag{52}
\]

We have previously indicated that:

\[
u(\chi) = \text{sech} \left[ \frac{\sqrt{\gamma}}{1 + \alpha} \right] \cosh \left[ \frac{\sqrt{\gamma}}{1 + \alpha} \right] \tag{53}
\]

Hence the normalised product concentration is:

\[
w(\chi) = 1 - \text{sech} \left[ \frac{\sqrt{\gamma}}{1 + \alpha} \right] \cosh \left[ \frac{\sqrt{\gamma}}{1 + \alpha} \right] \tag{54}
\]

In particular the potentiometric sensor device measures the product concentration at \( \chi = 0 \) sc:

\[
w_0 = 1 - \text{sech} \left[ \frac{\sqrt{\gamma}}{1 + \alpha} \right] \tag{55}
\]

Again in the low \( \alpha \) limit the product concentration at the electrode surface is

\[
w_0 \equiv 1 - \text{sech} \left[ \frac{\sqrt{\gamma}}{\sqrt{L/X_k}} \right] \tag{55}
\]

Now if the catalytic reaction kinetics are fast then we use:

\[
\text{sech} \left[ \frac{\sqrt{\gamma}}{\sqrt{L/X_k}} \right] \approx 2 \exp \left[-\sqrt{\gamma} \right] = 2 \exp[-L/X_k] \tag{56}
\]

And eq. 54 reduces to:

\[
w_0 \equiv 1 - 2 \exp[-L/X_k] \tag{57}
\]

Hence the surface product concentration (with respect to the bulk substrate concentration) reduces rapidly in an exponential manner as the ratio \( L/X_k \) increases. Whereas in contrast when saturated kinetics pertains:

\[
w_0 \equiv 1 - \text{sech} \left[ \frac{\sqrt{\gamma}}{2a} \right] \tag{58}
\]

Hence the surface product concentration under these conditions of high substrate concentration is given by \( p_0 = k_{cc}L/2D_k/L \). Hence we note that the limiting expressions for the product surface concentration at \( \chi = 0 \) are in excellent agreement with those previously reported by Carr [18,19]. The potential is related to the product concentration by the Nernst equation:

\[
E = E^0 + \frac{RT}{2F} \ln p_0 \tag{59}
\]

The steady state concentration profiles of substrate and product within the immobilized layer are outlined in Fig. 10 for a low value of the saturation parameter when the kinetics are first order and unsaturated and for various values of reaction/diffusion parameter in the range 0.1 to 100.

These results mirror nicely the computed results of Brady and Carr [19]. The variation of \( w_0 \) versus \( \alpha \) is outlined in Fig. 11 for various values of \( \gamma \), whereas a plot of \( w_0 \) versus \( \gamma \) for typical values of \( \alpha \) are presented in Fig. 12. We note that \( w_0 \) decreases rapidly with increasing substrate concentration in the layer, and the rate of decrease is more prevalent the larger
the value of γ assumed. Furthermore w₀ increases rapidly with increasing γ especially when the saturation parameter value is small. For increasing values of α a marked induction region is noted in the w₀ versus γ profile. The product concentration at the electrode surface will only begin to increase significantly with respect to the bulk substrate concentration in the layer only when the rate of the catalytic kinetics becomes quite rapid with respect to substrate diffusion in the film.

3.3. Competitive inhibition effects

Up until now we have neglected that inhibition effects may be important, especially in biocatalytic systems. The effect of substrate inhibition on the amperometric current response has been reported by Kulys and Baronas [20] and by Rajendran et al. [21]. The former utilized a digital simulation to obtain the predicted current response whereas the latter utilized the Adomian Decomposition Method (ADM) to obtain the steady state amperometric current response. In the present communication we discuss the general situation of competitive inhibition where an inhibitor molecule I will compete with the substrate S for a binding site on the catalyst. We assume that the inhibitor binds to the active site of the catalyst and does not affect catalysis since once the catalyst / substrate complex is formed catalysis occurs. The form of the rate equation assuming Michaelis-Menten kinetics and competitive inhibition is given by:

$$ R = \frac{dp}{dt} = \frac{k_c s w}{K_M + \frac{w}{K_I} + s} $$

where w denotes the inhibitor concentration and Kᵢ is the dissociation constant for the inhibitor/catalyst complex. Hence the pertinent reaction/diffusion equation is given by:

$$ D_s \frac{d^2 s}{dx^2} - \frac{k_c s w}{K_M \left(1 + \frac{w}{K_I}\right) + s} = 0 $$

From this we note that we can define an apparent Michaelis constant for the catalytic reaction given by:

$$ K_M' = K_M \left(1 + \frac{w}{K_I}\right) $$

And so eq. 60 can we written as:

$$ D_s \frac{d^2 s}{dx^2} - \frac{k_c s w}{K_M' + s} = 0 $$

We introduce the following normalised variables:

$$ u = \frac{s}{K_M}, \quad w = \frac{w}{K_I}, \quad \alpha = \frac{k_c}{K_M}, \quad \gamma = \frac{\lambda \omega}{K_M}, \quad \omega = \frac{w}{K_I}, \quad \mu = \frac{s}{K_M} $$

Substituting these quantities into eq. 60 we obtain the following reaction/diffusion equation:

$$ \frac{d^2 u}{dx^2} - \alpha' u = 0 $$

This is just the same equation as we have been discussing previously (eq. 7) with modified definitions of the saturation parameter and reaction/diffusion parameter to take inhibitor concentration and binding into account. Hence the boundary conditions remain the same as outlined in eq. 8 with redefined parameters. When \( \lambda \omega < < 1 \) then \( \gamma' \approx \gamma \) \( \alpha' \approx \alpha \) and inhibition effects can be neglected.

We can readily evaluate the concentration profile of substrate by following the procedure as previously outlined. Hence we can show that:

$$ u(\gamma) = u_0 \cosh \left[ \sqrt{\frac{\gamma'}{1 + \alpha'}} \right] = \text{sech} \left[ \sqrt{\frac{\gamma'}{1 + \alpha'}} \right] \cos \left[ \sqrt{\frac{\gamma'}{1 + \alpha'}} \right] $$

Now the normalised flux is given by:

$$ y = \alpha \left( \frac{du}{dx} \right)_1 $$

Hence the normalised current response is given by:

$$ y = \alpha \sqrt{\frac{\gamma'}{1 + \alpha'}} \tanh \left[ \sqrt{\frac{\gamma'}{1 + \alpha'}} \right] = \alpha \sqrt{\frac{\gamma'/(1 + \lambda \omega)}{1 + \alpha'/(1 + \lambda \omega)}} \tanh \left[ \frac{\gamma'/(1 + \lambda \omega)}{1 + \alpha'/(1 + \lambda \omega)} \right] $$

This expression reduces to:

$$ y = \alpha \sqrt{\frac{\gamma}{1 + \lambda \omega + \alpha}} \tanh \left[ \frac{\gamma}{1 + \lambda \omega + \alpha} \right] $$
This defines the normalised current response for an amperometric sensor which is valid for all values of the saturation parameter $\alpha$, the reaction/diffusion parameter $\gamma$ and the inhibition parameter $\lambda\omega$. Clearly when $\lambda\omega \ll 1$ eq. 68 reduces to eq. 24 as indeed it should. The variation of normalised current with fixed value of reaction/diffusion parameter $\gamma = 1.5\alpha$ as a function of saturation parameter $\alpha$ for various values of inhibition parameter $\lambda\omega$ ranging from 0.1 to 100 is outlined in Fig. 13.

Furthermore normalised current as a function of $\gamma$ for a fixed $\alpha$ value (0.1) for various values of the inhibition factor in the range 0–100 is presented in Fig. 14. In both figures the effect of inhibition is very clear. This data refers to the situation where the catalytic kinetics are not too fast and the kinetics are unsaturated. The diminution in the normalised current response as compared with the case where inhibition does not occur with increasing inhibition is clearly evident from these figures.

Again various limiting cases can be identified. Firstly we note that $1 + \alpha + \lambda\omega = 1 + \alpha \left(1 + \frac{\lambda\omega}{\alpha}\right)$. Hence eq. 68 becomes:

$$y = \alpha \sqrt{\frac{y}{1 + \alpha \left(1 + \frac{\lambda\omega}{\alpha}\right)}} \tanh \left[ \sqrt{\frac{y}{1 + \alpha \left(1 + \frac{\lambda\omega}{\alpha}\right)}} \right] \tag{70}$$

Now if $\lambda\omega/\alpha < 1$ then eq. 69 reduces to eq. 24 which pertains when the inhibitor concentration is zero. Then as before when $\alpha < < 1$ and $\lambda\omega < < \alpha$ then we obtain the result presented in eq. 27 which describes kinetic case I. On the other hand for $\alpha >> 1$ and if $\lambda\omega < < \alpha$ then we obtain eq. 34 corresponding to ‘mixed’ kinetic case IV. Here the catalytic kinetics are rapid and the substrate concentration in the layer is larger than the Michaelis constant. The reaction occurs within a distance $X_K$ in the film. As noted previously one might expect the outermost regions of the film to be completely saturated whereas the inner regions are unsaturated.

Alternatively if $\lambda\omega/\alpha >> 1$ then $1 + \alpha \left(1 + \frac{\lambda\omega}{\alpha}\right) \approx 1 + \lambda\omega$ and eq. 70 reduces to:

$$y \approx \alpha \sqrt{\frac{y}{1 + \lambda\omega}} \tanh \left[ \sqrt{\frac{y}{1 + \lambda\omega}} \right] \tag{71}$$

Also if $\lambda\omega < < 1$ then by necessity $\alpha < < 1$ and eq. 70 reduces to eq. 26 derived previously corresponding to unsaturated catalytic kinetics. This defines the case I/II scenario. Alternatively if $\lambda\omega >> 1$ then eq. 70 reduces to:

$$y \approx \alpha \sqrt{\frac{y}{\lambda\omega}} \tanh \left[ \sqrt{\frac{y}{\lambda\omega}} \right] \tag{72}$$

This is a new expression and describes the normalised current response when competitive inhibition pertains. Two new limiting kinetic cases can be presented depending on the magnitude of the quantity $\gamma/\lambda\omega$. First

Fig. 13. Variation of normalised current for an amperometric sensor exhibiting Michaelis-Menten kinetics when competitive inhibition is important. The curves were computed using eq. 69. This expression is valid for all values of the saturation parameter. The reaction/diffusion parameter is fixed and the inhibitor concentration is varied.

Fig. 14. The effect of competitive inhibition on the normalised current response for fixed saturation parameter (0.1) as a function of substrate reaction/diffusion parameter.
when $\gamma < < \lambda \omega$ then $\tanh[\sqrt{\gamma/\lambda \omega}] \approx \sqrt{\gamma/\lambda \omega}$ and eq. 70 reduces to:

$$y \approx \frac{\alpha \gamma}{\lambda \omega}$$

(73)

This is case V. This expression transforms into the following:

$$i = nFA \left\{ \frac{1}{w/K_i} \right\} \frac{k_c}{K_M} \xi^m \chi^s$$

(74)

And we obtain case II. This is very similar to the current expression associated with case I but the current magnitude is decreased by a factor of $w/K_i$ arising from the competitive inhibition. The reaction order with respect to layer thickness, catalyst concentration and substrate concentration is unity. The rate decreases in direct proportion to the inhibitor concentration and increases in proportion to the inhibitor/catalyst dissociation constant. This makes sense since if the affinity between inhibitor and catalyst site is low inhibition effects will not be very marked.

As in case I we have unsaturated catalytic kinetics. Second, when $\gamma >> \lambda \omega$ then $\tanh[\sqrt{\gamma/\lambda \omega}] \approx 1$ and eq. 71 becomes:

$$y \approx \frac{\alpha \gamma}{\lambda \omega}$$

(75)

This is case IV. The relevant expression for the current is:

$$i = nFA \left\{ \frac{1}{\sqrt{w/K_i}} \right\} \frac{k_c}{K_M} \xi^m \chi^s$$

(76)

Again the catalytic kinetics are unsaturated but occur in a thin reaction layer of length $\chi$. Competitive inhibition reduces the observed current by a factor of $\sqrt{w/K_i}$ compared to that pertaining in the absence of inhibition. The current decreases in proportion to the square root of the inhibitor concentration and increases directly with the square root of the dissociation constant of the inhibitor/catalyst complex.

If $\lambda \omega/\alpha < < 1$ then eq. 69 reduces to eq. 31 which defines the joint between limiting kinetic cases III and IV. If $\gamma/\alpha < < 1$ then eq. 30 reduces as previously shown to kinetic case II defined in eq. 31 whereas if the opposite condition pertains simplification of eq. 31 produces eq. 34 characteristic of the mixed kinetic case IV.

Finally we can examine the special case of thin catalytic layers where the concentration polarization of substrate in the layer is minimal. Under such circumstances the differential equation reduces to:

$$\frac{d^2 u}{d x^2} = \frac{\gamma'}{1 + \alpha'} = 0$$

(77)

Under such circumstances the concentration profile is given by the direct integration of eq. 77 to obtain:

$$u = 1 + \frac{\gamma}{2(1 + \alpha')(\chi' - 1)} = 1 - \frac{\gamma'}{2(1 + \alpha') \left( 1 - \chi' \right)}$$

(78)

This situation will be valid provided $\gamma'/2(1 + \alpha') < 1$ or $\gamma/\lambda \omega \alpha$ < 1. Furthermore the normalised current is:

$$y \approx \frac{\alpha \gamma}{\lambda \omega}$$

(79)

This defines the normalised current for a pure kinetic process valid for all values of substrate concentration taking the effect of competitive inhibition into account.

We note that eq. 79 can be cast in a Lineweaver Burk form as follows:

$$\frac{1}{y} = \frac{1 + \frac{1}{\lambda \omega}}{\alpha}$$

(80)

We can readily show that the inverse reaction flux depends in a linear manner on inverse bulk substrate concentration and is given by:

$$\frac{1}{f_{\Sigma}} = \frac{nFA}{\kappa_{catalyst}/K_M} \left\{ \frac{1}{1 + w/K_i} \right\} \frac{1}{\kappa_{catalyst}L}$$

(81)

Hence we expect the slope of the LB plot to vary with inhibitor concentration $w$ and the intercept to be independent of inhibitor concentration. This is a standard result in enzyme kinetics.

4. Redox catalysis in bounded non-conducting thin films

4.1. Statement and solution of boundary value problem

Up until now we have discussed the rather simple situation of catalysis at catalytic particles dispersed in a conducting matrix. When the matrix is not conducting a small molecule redox mediator is often utilized as illustrated in Fig. 1. This situation has been discussed previously by Lyons [20] for mediated catalysis in carbon nanotube films, by Bartlett and Pratt [21,22] and by Rajendran and Kirthiga [23] for mediated enzyme catalysis. Consequently only a brief discussion of the problem will be outlined here. Further details of the analysis can be found in the Lyons paper [20]. Here we consider catalytic particles (either redox enzymes or nanoparticles) immobilized in a highly dispersed mesh of polymer or carbon nanotube, which is in turn immobilized on a support electrode surface. A small molecule redox mediator species is used to both regenerate the reduced catalyst and to transfer electrons either to the polymer matrix chain or to the underlying support electrode surface thereby generating a current which can be measured. In this more complicated situation the pertinent transport and kinetic equations of both substrate and redox mediator must be solved in order to obtain an analytical expression for the amperometric current response.

As before we denote the bulk concentration of substrate $S$ as $S^+$ and oxidized redox mediator $A$ (such as for example molecular oxygen) as $A^-$. We assume that both of these species diffuse through the external bathing solution and rapidly partition into the polymer/carbon nanotube matrix with partition coefficients given by $k_{S,A}$. For ease of analysis we neglect concentration polarization of both substrate and redox mediator in solution and note that when $x = L$, $S_L = k_{S,L}S^+$, $A_L = k_{A,L}A^-$. As before we assume that the reaction between oxidized catalyst $C_0$ and substrate $S$ proceeds via Michaelis-Menten kinetics to generate product $P$ and reduced catalyst $C_r$. The oxidized catalyst is regenerated via reaction of the reduced catalyst with the oxidized mediator $A$ in the film generating the reduced mediator $B$. The reaction is assumed to follow simple bimolecular reaction kinetics. However it is quite possible that the reaction between oxidized mediator and reduced catalyst can involve adduct formation and thus involve a Michaelis-Menten type mechanism. Transduction occurs via one of two routes. If the support matrix is conducting the reduced mediator reacts along the length of the polymer/nanotube strand giving rise to a current. If the polymer is non conducting then the reduced mediator (e.g. H2O2) is required to diffuse to the support electrode surface and react there with an electrochemical rate constant $k'$ to regenerate the oxidized mediator. In this way the catalytic cycle is repeated. When the mediator reacts along the strand we can write that the net reaction flux is:

$$f_S = f_A = \frac{i}{nFA} = D_S \frac{dS}{dx} \mid_{x=L} = \int_0^L k_{catalyst}/K_M \xi(x) \, dx$$

(82)

Alternatively when the mediator reacts at the underlying support electrode surface the net flux differs from that arising from the substrate
reaction $f_S$ and is given by:

$$f_S = \frac{i}{nF_A} = D_b \left( \frac{\partial \chi}{\partial x} \right)_{x=0} = k'b_0$$

Here $b$ represents the concentration of reduced mediator. The following reaction/diffusions for mediator and substrate pertain:

$$\begin{align*}
D_b \frac{\partial^2 b}{\partial x^2} + \frac{ka_k c_S}{ka(s+K_M) + k_s} &= 0 \\
D_S \frac{\partial^2 S}{\partial x^2} + \frac{ka_k c_S}{ka(s+K_M) + k_s} &= 0
\end{align*}$$

Here $k$ denotes the bimolecular rate constant between reduced catalyst and oxidized mediator. These reaction/diffusion equations must be solved subject to the following boundary conditions. First if the reduced mediator species $B$ reacts on the polymer/carbon nanotube fibres throughout the thickness $L$ of the layer then:

$$\begin{align*}
x &= 0 & \frac{\partial S}{\partial x} &= 0 \\
x &= L & a_l &= \kappa a_S \infty \\
f_S &= D_b \left( \frac{\partial \chi}{\partial x} \right)_{x=L}
\end{align*}$$

Alternatively, if the reduced mediator $B$ reacts at the support electrode surface then we solve the reaction/diffusion equation for the substrate concentration, substitute into the reaction/diffusion equation for mediator and integrate the latter subject to the following boundary conditions:

$$\begin{align*}
x &= 0 & b &= b_0 & \frac{\partial b}{\partial x} &= \frac{f_S}{k^*} \\
x &= L & b &= b_L = 0 & a &= a_L = \kappa a_{\infty}
\end{align*}$$

Furthermore we assume that the concentration of the oxidized mediator has a uniform value of $a = \kappa a_{\infty}$ throughout the film.

We have previously shown [20] that the following dimensionless variables and parameters may be introduced:

$$\begin{align*}
\chi &= \frac{x}{L} & u &= \frac{s}{a_{\infty}} & v &= \frac{b}{k_s a_{\infty}} & a &= \frac{a_{\infty}}{K_M} \\
\kappa &= \frac{f_{\kappa}}{f_{SD}} &= \frac{(k_s/K_M) c_L k_S a_{\infty}}{k_c c_L k_s a_{\infty}} \\
\gamma_s &= \frac{f_{\kappa}}{f_{SD}} &= \frac{(k_s/K_M) c_L k_S a_{\infty}}{k_s L a_{\infty}} \\
\gamma_u &= \frac{f_{\kappa}}{f_{SD}} &= \frac{(k_s/K_M) c_L k_S a_{\infty}}{k_s L a_{\infty}}
\end{align*}$$

Note that $\gamma_u$ compares the rate of substrate/catalyst reaction with the rate of substrate diffusion in the layer, $\gamma_s$ compares the rate of substrate/catalyst reaction to that of mediator diffusion and $\kappa$ compares the rate of substrate/catalyst reaction to that of mediator/catalyst reaction. We assume that the mediator diffusion coefficients are equal and so $D_A = D_b = D_S$. It may be shown that:

$$\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\gamma_s u}{1 + (a + \kappa)u} &= 0 \\
\text{Also we note that:}
\frac{\partial^2 v}{\partial x^2} + \frac{\gamma_u u}{1 + (a + \kappa)u} &= 0
\end{align*}$$

The pertinent boundary conditions in dimensionless form for reaction in a conducting matrix are given by:

$$\begin{align*}
\chi &= 0 & \frac{\partial u}{\partial \chi} &= 0 \\
\chi &= 1 & u &= 1
\end{align*}$$

Here the normalised substrate flux (defined slightly differently than previously) is given by:

$$\begin{align*}
\psi_s = \frac{f_s}{f_{SD}} &= \frac{f_s}{\kappa_k D_{S} a_{\infty}/L} = \left( \frac{\partial u}{\partial \chi} \right)_{1} = \int_{0}^{1} \gamma_s u(\chi) \, d\chi = \frac{\gamma_s}{\alpha}
\end{align*}$$

Here we have normalised with respect to the substrate diffusion flux in the layer.

When the polymer/nanotube ensemble is not so conducting then the current flowing measures the reaction of the reduced mediator at the underlying support electrode and the relevant boundary conditions in dimensionless form are:

$$\begin{align*}
\chi &= 0 & v &= v_0 \\
\chi &= 1 & v &= 0
\end{align*}$$

Hence the net observed flux in normalised form is:

$$\begin{align*}
\psi_b = \frac{\gamma_u}{\gamma_s} \psi_s + \left( \frac{\partial v}{\partial \chi} \right)_{1}
\end{align*}$$

This is a parameter which directly compares the heterogeneous electrochemical rate constant for reduced mediator transformation to product at the support electrode to the diffusive rate constant for reduced mediator transport to the support electrode surface. Now there are three ways in which the reduced mediator is lost from the film. First it can be turned over at the electrode. Second, it can escape into the bulk solution. Third, it can be turned over by the substrate. Hence one may write that:

$$\begin{align*}
\psi_b = \frac{\gamma_u}{\gamma_s} \psi_s + \left( \frac{\partial v}{\partial \chi} \right)_{1}
\end{align*}$$

Hence the mediator flux and substrate flux will differ.

We can use the ideas developed earlier on in the present paper to assert that eq. 88 admits the following solution:

$$u = \text{sech}[\mu] \cos[\mu \chi]$$

Here $\mu$ is given by:

$$\mu = \sqrt{\frac{\gamma_s}{1 + \alpha + \kappa}}$$

Furthermore the normalised substrate flux $\psi_s$ is given by:

$$\psi_s = \mu \tanh[\mu] = \sqrt{\frac{\gamma_s}{1 + \alpha + \kappa}} \tanh \sqrt{\frac{\gamma_s}{1 + \alpha + \kappa}}$$

This expression is valid for all values of substrate concentration. The main difference between this normalised flux expression and that derived previously in this paper is the inclusion of the $\kappa$ term in the expression. Previously a redox mediator moiety was not considered.

Note that the normalised flux used in this section of the paper differs from that defined in the initial section via $\psi_s = 2\psi_{S}$. 
We can readily show after some considerable algebra [20] that the normalised reduced mediator concentration in the layer is given by:

$$v(x) = v_0(1 - x) + \frac{\gamma_M}{\mu}(1 - \text{sech}\sqrt{\mu}x - \frac{\gamma_M}{\mu} \text{sech}\sqrt{\mu}(\cosh\sqrt{\mu}x - 1)) \quad (99)$$

Furthermore the mediator concentration at the electrode surface is given by:

$$v_0 = \frac{\theta}{1 + \zeta}\left(1 + \alpha\left(1 + \frac{\kappa}{a}\right)\right) \quad (100)$$

We also note that the mediator flux is given by:

$$\Psi_M = \left(\frac{\partial v}{\partial x}\right)_0 = \frac{\gamma_M/\mu^2}{(1 + \zeta^2)\left(1 - \text{sech}[\mu]\right)} \quad (101)$$

This flux expression is valid for all values of $\alpha, \gamma, \kappa$. It is important to note that the full non-linear differential equations have been solved without any approximation. In this analysis we have noted that the parameter $\theta$ which defines the balance between substrate and mediator diffusion into the film is given by:

$$\theta = \frac{\gamma_M}{\gamma_s} = \frac{\kappa_D s^m / L}{\kappa s D a^m / L} \quad (102)$$

We also have noted that

$$\frac{\gamma_M}{\mu^2} = \theta\left(1 + \alpha\left(1 + \frac{\kappa}{a}\right)\right) \quad (103)$$

The variation of observed normalised flux with the substrate reaction/diffusion parameter $\gamma_s$ when the reaction occurs in a conducting matrix is outlined in Fig. 15 and in a non-conducting matrix in Fig. 16. In Fig. 15 the variation of normalised substrate flux with $\gamma_s$ for various values of $\kappa/a$ and $\alpha$ is outlined. A fixed value of $\kappa = 1$ was adopted in the calculation.

In Fig. 16 the variation of mediator flux with the more general parameter $\mu$ is outlined. In this figure we varied $\gamma_M$ over a wide range, and the loss factor was set at 0.1. In Fig. 17 the variation of mediator flux with $\mu$ is again presented but this time $\gamma_M = 20$ and the loss factor was varied from 0.1 to 20. Clearly when the layer is non-conducting and a redox mediator is used for amperometric detection the normalised flux varies characteristically with increasing value of the parameter $\mu$. As shown in Fig. 13, calculations were performed for a fixed value of loss factor of 0.1. Initially a plateau type response is established for low values of the $\mu$ parameter. This is most marked when $\gamma_M$ is large corresponding to the case where the flux for substrate/catalyst reaction is larger than that for mediator diffusion. The flux then exhibits a significant drop over the region $0.5 < \mu < 10$. The magnitude of the $\Psi_M$ plateau decreases steadily with decreasing value of the parameter $\gamma_M$. For $\mu > 50$ the normalised flux is minimal for all values of the parameter $\gamma_M$. A similar type of general variation of normalised flux with $\mu$ is seen in Fig. 14. In this calculation the parameter $\gamma_M = 20$, and the loss factor set at a given value. Each curve corresponds to a fixed loss factor which ranges from 0.1 to 20. Hence the loss factor value has a significant effect on the normalised current response. The greater the loss factor the smaller the corresponding value for $\Psi_M$ for any value of $\mu$.

4.2. Thin layer limit

Eqs. 98 and 101 may be simplified when the layer is thin. Under such circumstances we set $u \equiv 1$ in eq. 88 and note:

$$\frac{d^2u}{dx^2} - \frac{\gamma_s}{1 + \alpha\left(1 + \frac{\kappa}{a}\right)} = 0 \quad (104)$$
Integrating once and noting the boundary condition at $\chi = 0$, $\frac{du}{d\chi} = 0$ we obtain:

$$\frac{du}{d\chi} = \frac{-\kappa \gamma}{1 + \alpha(1 + \frac{\kappa}{\alpha})}$$

and so the normalised current response is readily obtained as:

$$\Psi_B(\gamma) = \frac{d\gamma}{d\chi} = \frac{-\kappa \gamma}{1 + \alpha(1 + \frac{\kappa}{\alpha})} \tag{105}$$

This expression is valid $\forall \alpha, \kappa$ values.

A general expression for the expected current response when the layer is not conducting can be derived from eq. 89 by setting $u \equiv 1$ to obtain:

$$\frac{d^2\gamma}{d\chi^2} = \frac{-\kappa \gamma}{1 + \alpha(1 + \frac{\kappa}{\alpha})} - \frac{\theta \gamma}{1 + \alpha(1 + \frac{\kappa}{\alpha})} = -\theta \mu^2 \tag{106}$$

Integrating once we note:

$$\frac{d\gamma}{d\chi} = -\theta \mu^2 \gamma + K_1 \tag{107}$$

Integrating again:

$$\gamma(\chi) = -\theta \mu^2 \gamma + K_1 \chi + K_2 \tag{108}$$

The integration constants can readily be evaluated from the boundary conditions as:

$$K_1 = \frac{\theta \mu^2}{2} - \zeta^{-1} \Psi_B \quad K_2 = \nu_0 = \zeta^{-1} \Psi_B$$

Consequently the normalised flux is given by:

$$\Psi_B = \frac{1}{1 + \zeta^{-1}} \left( \frac{\theta \mu^2}{2} \right) = \frac{\theta}{2(1 + \zeta^{-1})} \left( \frac{\kappa \gamma}{1 + \alpha(1 + \frac{\kappa}{\alpha})} \right) \tag{109}$$

This defines the situation for a thin non conducting catalytic layer where the observed current arises via reaction of a redox mediator at the inner support electrode. Again this is valid $\forall \alpha, \kappa$ values when diffusive effects of substrate can be neglected.

The full solution to the boundary value problem and that derived via the thin layer approximation are related via a simple factor which depends on the parameter $\mu$. We may readily show that:

$$\Psi_B, \Psi_{\alpha, TL} = \mu^{-1} \tanh \mu \tag{110}$$

These equations are illustrated in Fig. 18 and Fig. 19.

It is interesting to note that $\Psi_B \equiv \Psi_{\alpha, TL}$ when $\mu$ is small. In contrast when under similar conditions we note that:

$$\frac{\Psi_B}{\Psi_{\alpha, TL}} \equiv 1 + a \left( 1 + \frac{\kappa}{\alpha} \right) \tag{111}$$

And the equivalence between the full solution and the thin film approximation for the current arising for reduced mediator oxidation at the support electrode will depend both on the magnitudes of $\kappa$ and $\alpha$. For instance if the ratio $\kappa/\alpha >> 1$ then $\frac{\Psi_B}{\Psi_{\alpha, TL}} \equiv 1 + \kappa$ and $\Psi_B \equiv \Psi_{\alpha, TL}$ when $\kappa < 1$. This condition is valid when the reaction between substrate and oxidized catalyst is much slower than the reaction between reduced catalyst and oxidized mediator. Alternatively if the ratio $\kappa/\alpha < 1$ then $\frac{\Psi_B}{\Psi_{\alpha, TL}} \equiv 1 + \alpha$ and $\Psi_B \equiv \Psi_{\alpha, TL}$ when $\alpha < 1$. This will happen when the substrate concentration in the layer is much less than the Michellis constant.

Hence the validity of the thin layer approximation depends on the ratio $\frac{k}{\alpha} = \frac{k_c}{k_{sc} \alpha}$ which relates the rate of catalyst/substrate decomposition to that of oxidized mediator/ reduced catalyst reaction (active catalyst regeneration). When active oxidized catalyst regeneration is slower than catalyst/substrate complex dissociation into product and reduced catalyst then the thin layer approximation will be good when the rate of substrate reaction with oxidized catalyst is much slower than the rate of active catalyst regeneration, and $f_{SC} << f_{MC}$. If the opposite pertains and catalyst/substrate complex dissociation into product and reduced catalyst is slow compared with active oxidized catalyst regeneration then the thin film approximation will be good provided the substrate concentration in the layer is low.

4.3. Limiting rate expressions and the kinetic case diagram

The form of the mathematical equations presented for substrate flux and observed flux suggest that the pertinent kinetic case parameters which can be used to geometrically define the problem in a kinetic case diagram are $\mu, \alpha, \kappa, \frac{\kappa}{\alpha}$ and $\gamma_5$. Typical expressions for the substrate flux and observed flux for several approximate kinetic cases are presented in table III. These
are derived by taking suitable limiting values of the key reaction parameters as done in previous sections of this paper.

4.3.1. Limiting kinetic cases in a conducting matrix

We first consider transport and catalytic kinetics in a conducting layer. Here the observed flux or current corresponds to the substrate reaction flux. We note that the normalised substrate flux $\Psi_S$ is defined via eq. 98. In the limit where $\mu < < 1$ then $\gamma_S < < 1 + a \left( \frac{1}{1 + \frac{1}{a}} \right)$. Hence we note that $\tan \mu \equiv \mu$ and the normalised flux reduces to:

$$\Psi_S \equiv \frac{\gamma_S}{1 + a} \left( \frac{1}{1 + a} \right)$$

This general expression is valid for all values of the saturation parameter $a$ and the ratio $\kappa/\alpha$. Further simplification can be obtained by examining the magnitude of the latter factor. First when $\kappa < < 1$ corresponding to the condition that $k_\alpha < < k_{\alpha}a$ we obtain:

$$\Psi_S \equiv \frac{\gamma_S}{1 + a}$$

Eq. 114 can be further simplified depending on the magnitude of $a$. When the substrate concentration in the layer is much less than the Michaelis constant then $a < < 1$ and so the normalised flux is given by:

$$\Psi_S \equiv \frac{\gamma_S}{1 + a} \left( \frac{1}{1 + a} \right)$$

Eq. 116 defines kinetic case I. Here the unsaturated reaction kinetics between oxidized catalyst and substrate is slow and rate determining. Here the reaction flux is first order with respect to catalyst concentration, layer thickness and substrate concentration and is independent of mediator concentration. On the other hand if in eq. 114 we assume that $a >> 1$ then the normalised flux reduces to:

$$\Psi_S \equiv \frac{\gamma_S}{1 + a} \left( \frac{1}{1 + a} \right)$$

This expression defines kinetic case III. Here saturated catalytic kinetics pertains and the decomposition of the catalyst/substrate complex to form reduced catalyst and product is slow and rate determining. Here the reaction order with respect to catalyst concentration and layer thickness is unity. The reaction flux is independent both of substrate and mediator concentration. Hence eq. 114 connects the cases I and III. Returning to eq. 113 we now examine the possibility that $\kappa/\alpha >> 1$, so that $k_\alpha >> k_{\alpha}a$. Under such circumstances we note that eq. 113 reduces to:

$$\Psi_S \equiv \frac{\gamma_S}{1 + a}$$

Again two possibilities must be considered. First when $\kappa < < 1$ case I is regained and the flux is given by eq. 115. Alternatively when $\kappa >> 1$ we note:

$$\Psi_S \equiv \frac{\gamma_S}{1 + a} \left( \frac{1}{1 + a} \right)$$

Here we have kinetic case V where one has rate determining reaction between oxidized mediator and reduced catalyst to regenerate the active oxidized form of the catalyst. Hence eq. 117 defines the join between kinetic cases I and V. Hence kinetic cases I, III and V arise when $\mu < < 1$.

We now examine the situation where $\mu >> 1$. Here $\gamma_S >> 1 + a \left( \frac{1}{1 + \frac{1}{a}} \right)$. In this circumstance we write that $\tan \mu \equiv 1$ and so from eq. 98 we note that the normalised substrate flux reduces to:

$$\Psi_S \equiv \mu = \sqrt{\frac{\gamma_S}{1 + a} \left( \frac{1}{1 + a} \right)}$$

Again we look first at the situation where $\kappa/\alpha << 1$. Hence the normalised flux reduces to:

$$\Psi_S \equiv \sqrt{\frac{\gamma_S}{1 + a}}$$

When the substrate concentration in the film is much less than the Michaelis constant then $a < < 1$ and eq. 120 reduces to:

$$\Psi_S \equiv \sqrt{\frac{\gamma_S}{1 + a}}$$

This defines the kinetic case II. Here the unsaturated catalytic kinetics between oxidized catalyst and substrate are slow and rate determining in a thin reaction layer of thickness $X_p$ at the outer film/solution interface. Here the substrate flux will be first order with respect to substrate concentration and catalyst concentration and zero order with layer thickness and mediator concentration. Alternatively when $a >> 1$ we obtain:

$$\Psi_S \equiv \sqrt{\frac{\gamma_S}{1 + a}}$$

This is an interesting case which we label case IV. Even though the bulk substrate concentration in the film is large with $k_{\alpha}s^* >> K_M$, the local substrate concentration in the reaction layer of dimension $X_p$ will be lower than the Michaelis constant and the reaction flux will vary with the square root of the substrate concentration and the reaction will be unsaturated. Here the substrate reaction flux is first order with respect to catalyst concentration, half order with respect to substrate concentration and zero order with layer thickness and mediator concentration. Note that eq. 120 defines the join between kinetic cases II and IV. If

$$\kappa/\alpha >> 1$$

then the normalised flux in eq. 119 reduces to:

$$\Psi_S \equiv \sqrt{\frac{\gamma_S}{1 + a}}$$

This defines the join between kinetic cases II and VI. If $\kappa << 1$ then eq. 123 reduces to eq. 121 again and we have case II. Alternatively if $\kappa >> 1$ then the normalised flux is

$$\Psi_S \equiv \sqrt{\frac{\gamma_S}{1 + a}}$$

In this situation the substrate flux is the geometric mean of the rates of the oxidized mediator/reduced catalyst reaction and the oxidized catalyst/substrate reaction. This occurs in a thin reaction layer of length $X_p$. Here in case VI we note that the substrate flux varies linearly with catalyst concentration, is independent of layer thickness and is half order with both substrate and mediator concentrations. Hence when $\mu >> 1$ cases II, IV and VI pertain. In total we need to consider six distinct limiting cases when we consider reaction and diffusion in a conducting matrix.

4.3.2. Limiting kinetic cases in a non-conducting matrix

Here the observed flux or current is given by the mediator reaction flux. The expression for the normalised mediator flux at the inner support
electrode for the situation where the catalytic matrix is non conducting is given by:

\[ \Psi_B \equiv \frac{-\gamma_m}{\mu} \left( \frac{1}{1 + F_i} \right) \left( 1 - \text{sech} \mu \right) \tag{125} \]

This expression is valid for all values of \( \mu \) and is quite a general result valid for all values of \( \alpha, \kappa \) and \( \frac{k}{a} \). In eq. 125 we have introduced the loss factor as follows:

\[ F_i = \frac{e^{-1}}{Dm/L} = \frac{Dm/L}{k} \tag{126} \]

Here the loss factor represents the balance between diffusion away from and heterogeneous electrochemical reaction of the reduced mediator at the support electrode surface. When \( \mu \ll 1 \) then \( \text{sech} \mu \equiv 1 - \frac{\mu^2}{2} \) and so

\[ 1 - \text{sech} \mu \equiv \mu^2/2 \]. Hence when \( \gamma_s < 1 + \alpha \left( 1 + \frac{\kappa}{a} \right) \) we obtain that the observed flux is:

\[ \Psi_B \equiv \frac{-\gamma_m}{2} \left( \frac{1}{1 + F_i} \right) \left( 1 - F_i \right) \left( 1 + a \right) \left( 1 + \frac{\kappa}{a} \right) \tag{127} \]

This is a modified case I. Here we have rate determining unsaturated catalytic kinetics modified by the factor \( \phi \equiv \frac{1}{2 \left( 1 + \frac{Dm/L}{k} \right)} \). From this we see that a significant quantity of the reduced mediator generated at the catalytic site is lost to subsequent reaction at the support electrode. At least half is lost to diffusion and leaching from the catalytic film. Half will be lost when the loss factor is very small. Significantly more than 50% will not reach the support surface when the loss factor \( F_i \) is large.

Now turning to the situation where \( \mu >> 1 \) then \( \gamma_s >> 1 + \alpha \left( 1 + \frac{\kappa}{a} \right) \). Hence we note that \( \text{sech} \mu \equiv 2 \exp \left( -\mu \right) \) and so

\[ 1 - \text{sech} \mu \equiv 2 \exp \left( -\mu \right) \equiv 1 \] and so the normalised mediator flux is:

\[ \Psi_B \equiv \left( \frac{1}{1 + F_i} \right) \left( \frac{\gamma_m}{\mu^2} \right) \left( 1 + a \left( 1 + \frac{\kappa}{a} \right) \right) \tag{128} \]

This expression is valid for all values of the parameters \( \kappa/a, \alpha, \kappa, \). Again some limiting kinetic limiting cases can be identified. First, when \( \frac{\kappa}{a} < 1 \) then the mediator flux is given by:

\[ \Psi_B \equiv \frac{\theta}{1 + F_i} \left( 1 + a \right) \tag{129} \]

This expression joins the limiting kinetic cases VII and VIII and is valid for all values of \( a \). In particular when \( \alpha << 1 \) then

\[ \Psi_B \equiv \frac{\theta}{1 + F_i} \frac{k_D \text{ss}^m}{L} \tag{130} \]

Case VII corresponds to slow rate determining substrate diffusion which is modified by the loss factor as indicated. We call this the titration case.

Alternatively when \( \alpha >> 1 \) we note that:

\[ \Psi_B \equiv \frac{\alpha \theta}{1 + F_i} \frac{k_D \text{ss}^m}{L} \tag{131} \]

Again, this case VIII corresponds to modified slow rate determining substrate diffusion modified by the factor \( g' = \frac{\alpha}{1 + F_i} \). Here we see that the if the substrate concentration in the layer is large and much larger than the Michaelis constant, the effect of the mediator loss factor on the net reaction flux can be mitigated.

Alternatively, if \( \frac{\alpha}{a} > 1 \) then

\[ \Psi_B \equiv \frac{\theta}{1 + F_i} \left( 1 + \kappa \right) \tag{132} \]

This expression valid for all values of the parameter \( \kappa \) joins the kinetic cases VII and IX. Case VII, termed the titration case, will be valid when \( \kappa << 1 \). Hence when \( \kappa/\kappa A << 1 \), \( \kappa/\kappa A >> 1 \) will be valid. Under such circumstances eq. 130 is valid. When the opposite pertains \( \kappa >> 1 \) and the flux becomes:

\[ \Psi_B \equiv \frac{\kappa \theta}{1 + F_i} \frac{k_D \text{ss}^m}{L} \tag{133} \]

Here eq. 133 represents case IX which is slow substrate diffusion modified by the factor \( g' = \frac{\alpha}{1 + F_i} \). Here when \( \kappa >> 1 \) such that the substrate/catalyst rate is much greater than the mediator/catalyst rate the diminution of the reaction flux due to the influence of the loss factor can be mitigated.
Since the reaction occurs at $\chi = e$

$$
\frac{dv}{d\xi} \Big|_0 = \frac{v_0}{e} \quad \frac{du}{d\xi} \Big|_1 = \frac{1}{1-e}
$$

(136)

Hence substituting eq. 136 in eq. 135 and simplifying we can show that:

$$
\epsilon = \frac{v_0}{v_0/\theta} = \theta
$$

(137)

Now the mediator flux is given by:

$$
\Psi_B = -\frac{dv}{d\xi} = -\frac{v_0}{e} = \theta - v_0
$$

(138)

We recall that $v_0 = \kappa^{-1}\Psi_B = FL\Psi_B$ and so substituting into eq. 138 we finally note:

$$
\Psi_B = \frac{\theta}{1 + F_L}
$$

(139)

This of course is case VII discussed previously. As noted previously this result is valid for $\alpha < 1$. It is also valid however when $\alpha > 1$. Under such conditions in the substrate limited part of the film the substrate concentration will be considerably less than its bulk value, and will be less than $K_M\alpha$. We see also that cases VIII and IX are titration cases modified by the factors $\alpha$ and $\kappa$ respectively. Case I is a titration case modified by the factor $\Psi_B/2$.

The kinetic case diagrams are presented in Fig. 20 and Fig. 21. Here a plot of logs versus log definition the limiting cases are outlined for small $\mu$ (Fig. 20) and large $\mu$ (Fig. 21). A summary of limiting kinetic cases is presented in Table 3.

We finally can examine the limiting kinetic cases in the thin film approach. First when the layer is conducting we note that the normalised flux is given by eq. 105 which is

$$
\Psi_S = \left(\frac{du}{d\xi}\right)_{\varepsilon=1} = \mu^2 = \frac{\kappa}{1 + \alpha(1 + \frac{\kappa}{\alpha})}
$$

(140)

This is the same as eq. 113 as obtained previously as indeed it should be since it was obtained in the limit of $\mu < 1$. Again when $\kappa/\alpha < 1$ we get either case I or III depending on the magnitude of the substrate concentration. In contrast when $\kappa/\alpha > 1$ we get case I or V depending on whether

<table>
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<td>$f_S = (k_c/K_M)c_2X_0a^{\gamma_S}$</td>
<td>$\Psi_B = \left(\frac{\theta}{2(1 + F_L)}\right)$</td>
<td>$f_B = \frac{(k_c/K_M)c_2X_0a^{\gamma_S}}{2(1 + Du/L)}$</td>
</tr>
<tr>
<td>II</td>
<td>$\Psi_S = \kappa_S$</td>
<td>$f_S = (k_c/K_M)c_2X_0a^{\kappa_S}$</td>
<td>$\Psi_B = \left(\frac{\theta}{2(1 + F_L)}\right)$</td>
<td>$f_B = \frac{k_cL}{2(1 + Du/L)}$</td>
</tr>
<tr>
<td>III</td>
<td>$\Psi_S = \gamma_S$</td>
<td>$f_S = k_cL_a^{\gamma_S}$</td>
<td>$\Psi_B = \left(\frac{\theta}{2(1 + F_L)}\right)$</td>
<td>$f_B = \frac{k_cL}{2(1 + Du/L)}$</td>
</tr>
<tr>
<td>IV</td>
<td>$\Psi_S = \gamma_S$</td>
<td>$f_S = (k_c/K_M)c_2X_0a^{\gamma_S}$</td>
<td>$\Psi_B = \left(\frac{\theta}{2(1 + F_L)}\right)$</td>
<td>$f_B = \frac{k_cL}{2(1 + Du/L)}$</td>
</tr>
<tr>
<td>V</td>
<td>$\Psi_S = \gamma_S$</td>
<td>$f_S = k_cL_a^{\gamma_S}$</td>
<td>$\Psi_B = \left(\frac{\theta}{2(1 + F_L)}\right)$</td>
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</tbody>
</table>
\( \kappa << 1 \text{ or } \kappa >> 1 \). Considering a non-conducting matrix we note that the flux is:

\[
\Psi_n = \frac{\theta}{2(1 + F_1)} \left\{ \frac{\gamma S}{1 + \alpha (1 + \frac{\gamma}{\kappa})} \right\} = \frac{\theta}{2(1 + F_1)} \Psi_S
\]

(141)

When \( \kappa/\alpha >> 1 \) the normalised mediator flux reduces to:

\[
\Psi_n \equiv \frac{\theta}{2(1 + F_1)} \left\{ \frac{1}{1 + \kappa} \right\}
\]

(142)

When \( \kappa << 1 \) we again get case I and when \( \kappa >> 1 \) we get case V. We conclude that only 3 cases I, III and V pertain for thin films regardless of the conductivity of the catalytic matrix.

5. Concluding comments

In this paper we have examined the problem of describing the transport and kinetics of catalytic reactions in which the catalyst is immobilized within a support matrix. We have examined a mathematical procedure based of the AGM recently discussed by Dharmalingam and Veeramuni [15] which enables a closed form approximate analytical solution to the Michaelis-Menten kinetic rate equation when coupled to Fickian diffusion in thin bounded film. We have compared this approximate closed form analytical expression with a numerical simulation obtained using the NDSolve facility in Mathematica 12 and have indicated an excellent agreement for parameter sets of \((\gamma, \alpha)\) which are appropriate. This AGM derived solution for current is accurate for substrate concentration values close to the Michaelis Constant. The analysis was applied to diffusion/reaction in a planar slab, and in a subsequent paper will be extended to reaction/diffusion a hemispherical polymer film coated on an inlaid disc electrode and in a thin layer coated on a wire electrode. Analytical solutions for both the amperometric and potentiometric sensor response were provided. We then extended this useful analysis to consider the effect of concentration polarization in the solution, and to competitive inhibition. Finally, the analysis was extended to a polymer modified electrode when a redox mediator is used in the polymer film. Kinetic case diagrams were developed and nine approximate limiting expressions for the amperometric response at steady state when the catalytic matrix is either conducting or insulating were developed.

Credit author statement

This paper was conceived and written solely by the corresponding author.

Declaration of Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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This paper gathers together ideas and analysis developed over the last 25 years. It is particularly appropriate since the topics of mathematical modelling and electrochemical sensing are topics often studied by my good friend and colleague Richard Compton. I wish Richard every good wish for his 65th birthday. He has been an inspiration for me throughout the years. Long may his scientific output continue. The work has been financially supported by Trinity College Dublin and Science Foundation Ireland.

References