Change in a product between two states as the symmetrical sum of changes in each of its factors

R.C. Geary

November 1982

Confidential: Not to be quoted until the permission of the Author and the Institute is obtained.
Change in a product between two states as the symmetrical sum of changes in each of its factors

R.C. Geary

The two states being 1 and 2 (indicated by superscripts) and variables $x_i$ ($i = 1, 2, \ldots, k$) the problem is to find the coefficients $f_i$ so that the identity obtains

$$
\prod_{i=1}^{k} x_i^2 - \prod_{i=1}^{k} x_i^1 = \sum_{i=1}^{k} f_i (x_i^2 - x_i^1),
$$

It is obvious that (i) $f_i$ must be a homogenous sum product of all the variables $x_i^2$ and $x_i^1$ of dimension $(k - 1)$, (ii) that the sum of the coefficients in each $f_i$ must be 1, (iii) the solution is not unique, without the imposition of conditions. As to what these conditions should be, consider the case of two variables

$$
x_1^2 x_2 - x_1^1 x_2 = f_1 (x_1^2 - x_1^1) + f_2 (x_2^2 - x_2^1)
$$

It turns out that one fascinating set of values of $f_1$ and $f_2$ is

$$
(3) \quad f_1 = \frac{x_1^1 + x_2^2}{2}, \quad f_2 = \frac{x_1^2 + x_2^1}{2}.
$$

It will be seen that with these values in (2) the terms on the right, namely $x_1^2 x_2$ and $x_1^1 x_2$, mixed in states, cancel out. But this property also obtains in, say

$$
(4) \quad f_1 = x_2^2, \quad f_2 = x_1^1.
$$

Showing that the solution is not unique. But (3) is the more attractive and useful in the two variable case in its index number application, presently to be discussed. Its main characteristic is its symmetry, with the same coefficients within $f_1$ and $f_2$ and between $f_1$ and $f_2$. This property of symmetry will be adopted for the solution of (1) which will be found to be unique.

In an index number application $x_1$ might be price and $x_2$ quantity, the states time or place, the product value, both for a particular commodity. For many commodities we simply introduce a $\Sigma$ on both sides of (2), and we have divided the sum product of a set of commodities symmetrically and consistent into the sum of contributions due to price and quantity. One is reminded immediately of I. Fisher's Ideal index number treatment in logarithmic
form, except that the present is far simpler and is recommended for trial: the unitary changes in price of quantity between states 1 and 2 are the \( \Sigma \)'s of the expressions on the right of (2) divided by the \( \Sigma \) value at state 1.

It is hard to envisage economic application for \( k \) greater than 2. The problem may be envisaged as a generalisation and without inmodesty described as elegant because, as will appear, it is not my own.

In an earlier draft of this paper I gave the solutions for products 2, 3 and 4 and announced that there was no symmetrical solution for \( k = 5 \) or more! I submitted the paper to an economics journal which reasonably hesitated about acceptance because it was not an economics paper. But, from my point of view more importantly, a Reader stated that my conclusion about \( k = 5 \) or over was wrong - an error in algebra, and very ingeniously suggested a unique general symmetrical solution, i.e. for any number \( k \) variables. He emphasized that he did not supply a solution. This is provided here.

In addition to the three properties of \( f_{1} \) announced at the outset, obviously we have (iv) that the coefficients of the products in \( f_{1} \) in \( (k-1) \) variables all of the same state must be \( 1/k \) (e.g. with \( k = 3 \) the coefficient of \( x_{2}^{2} x_{3}^{2} \) and \( x_{2}^{1} x_{3}^{1} \) in \( f_{1} \) must be \( 1/3 \)). Other products will be termed mixed (in states, e.g. \( x_{2}^{2} x_{3}^{1} \), which must cancel out). In each \( f_{1} \) there will \( 2^{k-1} \) product terms but these can be divided binomially into \( k \) terms:

\[
2^{k-1} = \sum_{n=0}^{k-1} \frac{(k-1)!}{n! (k-n-1)!}
\]

\( n \) may be regarded as the number of state 1 and \( (k-n-1) \) the number of state 2 of which there will indeed be

\[
\frac{(k-1)!}{n! (k-n-1)!}
\]

in every \( f_{1} \). From symmetry, each of these products must have the same coefficients, say \( c_{n} \), and also from symmetry the set of coefficients must be the same in all \( f_{1} \).
As the sum of coefficients in all $f_i$ is unity and the first and last of these is $1/k$ the Reader had the ingenious idea of also equating the sum of coefficients of the $n$-set to $1/k$, i.e.

$$c_n \frac{(k-1)!}{n! (k-n-1)!} = \frac{1}{k^n} \quad n = 0, 1, \ldots, (k-1)$$

or

$$c_n = n! \frac{(k-n-1)!}{k!} \quad n = 0, 1, \ldots, (k-1)$$

The reader disclaimed proving this; actually proof is easy. From (7), $c_0$ and $c_{k-1}$ are $1/k$ as we already know. On the right side of (1) there will be equal number of $+$ and $-$ terms. Every term will be in all the variables.

$$a_1 x_1 a_2 x_2 \ldots x_k,$$

the superscripts $a_i$ being all 1 or 2, permuted in all possible ways. With all the $a_i$ 1 or 2 the coefficients $c_0$ and $c_{k-1}$ are $1/k$, as we have seen.

On the right of (1) the positive coefficient of any term (8) with $n$ of state 1 will be $(k-n) c_n$ and the negative coefficient of the same term is $-c_{n-1}$. Hence we must show that

$$c_n = n! \frac{(k-n-1)!}{(n-1)! (k-n)!} \quad n = 0, 1, \ldots, (k-1)$$

so that (9) is true.

Following are the values of $c_n$ for number of variables $k = 2 - 6$ and, in brackets, the number of terms which symmetrically have the same coefficient:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1/2(1)$</td>
<td>$1/2(1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$1/3(1)$</td>
<td>$1/6(2)$</td>
<td>$1(3)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$1/4(1)$</td>
<td>$1/12(3)$</td>
<td>$1/12(3)$</td>
<td>$1/4(1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$1/5(1)$</td>
<td>$1/20(4)$</td>
<td>$1/30(6)$</td>
<td>$1/20(4)$</td>
<td>$1/5(1)$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$1/6(1)$</td>
<td>$1/30(5)$</td>
<td>$1/60(20)$</td>
<td>$1/60(20)$</td>
<td>$1/30(5)$</td>
<td>$1/6(1)$</td>
</tr>
</tbody>
</table>

The binominal character of the solution is evident, also that, as symmetrical, is is unique.

8 November 1982 R.C. Cearý.