Undrained Triaxial Experimental Investigations and Hyperviscoplastic Modelling of Peat Materials

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PhD Thesis
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Declaration

I, the undersigned, declare that this thesis has not been submitted as an exercise for a degree at this or any other university. I further declare that, except where reference is made in the text, it is entirely my own work. I agree the library may lend or copy this thesis upon request.

Lin Zhang
September 13, 2017
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Abstract

This PhD research includes two parts, viz. experimental investigation on the undrained mechanical properties of undisturbed fibrous peat and a finite strain constitutive model within a thermodynamically consistent framework based on the experimental results of the tested peat.

From the laboratory investigation, nonlinear behaviour with large strain was observed from the loading-unloading tests on peat. The influence of cell pressure, strain rate, stress relaxation on the constitutive behaviour as well as strain recoveries from unloading were investigated. Relaxation tests were carried out for overstress quantification as well as to obtain the equilibrium state. The tested peat was categorised as a rate-dependent material with equilibrium hysteresis. Structural anisotropy was also investigated by testing the undisturbed vertical and horizontal specimens under the same conditions. Although in the proposed model, the structural anisotropy was not included, the experimental data provided the foundation for the future development of anisotropic constitutive models.

The observed material behaviour motivated a rheological model comprising four parallel layers, each consisting of elastic, viscoelastic and elastoplastic elements. The proposed hyperviscoplastic model was derived from the entropy inequality. Each part of the model was verified against their analytical solutions. The model parameter fitting started with the rate-independent equilibrium tests, where the hyperelastoplastic model was fitted to the defined laboratory equilibrium test. Two compression-relaxation tests, carried out at different strain rates, were used for the parameter estimation of the rate-dependent hyperviscoplastic model. The hyperviscoplastic model was validated against five strain rate tests under various load cases as well as an undrained triaxial creep test. The finite strain constitutive model derived within a thermodynamically consistent framework showed its versatility in simulating peat behaviour in various load cases by a good agreement of the experimental results. Also, from the process of the deriving the hyperviscoplastic model, it is perceived that it would be easy to extend the current model with more features in a mathematically and thermomechanically consistent manner.
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Nomenclature

Throughout the thesis bold face symbols denote tensors and vectors. Normal face letters denote scalar quantities.

Abbreviations

\( \mathbf{ep} \)  Elastic part of elastoplastic.
\( \mathbf{eq} \)  Rate-independent equilibrium.
\( \mathbf{ev} \)  Elastic part of viscoelastic.
\( \mathbf{lru} \)  Loading-relaxation-unloading.
\( \mathbf{lur} \)  Loading-unloading-reloading.
\( \mathbf{lu} \)  Loading-unloading.
\( \mathbf{p} \)  Elastoplastic.
\( \mathbf{v} \)  Viscoelastic.

Greek symbols

\( \varepsilon \)  Strain tensor of Green-Lagrange or Almansi type on intermediate configuration.
\( \varepsilon, \varepsilon, \varepsilon \)  Small strain tensor / its Kelvin mapping / its Voigt mapping.
\( \eta \)  Viscosity.
\( d\Gamma \)  Area element.
\( \chi \)  Motion.
\( \lambda \)  Stretch / compression.
\( d\Omega \)  Volume element.
\( \bar{\psi}, \psi \)  Specific Helmholtz free energy function / Helmholtz free energy density function.
\( \psi_i \)  Residual in iteration \( i \).
\( \rho \)  Mass density.
\( \sigma, \sigma, \sigma \)  Cauchy stress tensor / its Kelvin mapping / its Voigt mapping.
\( \tau \)  Kirchhoff stress.
\( \tau_{ov} \)  Overstress on intermediate configuration.
\( \theta \)  Absolute temperature.

Operators
\( \mathbf{a} \cdot \mathbf{b} \) Dot product of \( \mathbf{a} \) and \( \mathbf{b} \).

\( \mathbf{A} : \mathbf{B} \) Double contraction of \( \mathbf{A} \) and \( \mathbf{B} \).

\( \mathbf{a} \otimes \mathbf{b} \) Dyadic product of \( \mathbf{a} \) and \( \mathbf{b} \).

\( \mathbf{A} \otimes \mathbf{B} \) \( \frac{1}{2} \left[ (\mathbf{A} \otimes \mathbf{B})^T + (\mathbf{A} \otimes \mathbf{B})^{2T} \right] \)

\( \chi^* \) Push forward operator.

\( \chi^* \) Pull back operator.

\( \text{div} \) Divergence operator.

\( \text{Grand} \) Material gradient operator.

\( \mathbf{H}(\bullet) \) Heaviside step function.

\( (\bullet)^{-1} \) Inverse operator.

\( \dot{\mathbf{v}} \) Material time derivative of \( \mathbf{v} \).

\( \varepsilon \) Contravariant Oldroyd stress rate.

\( \Delta \varepsilon \) Covariant Oldroyd strain rate.

\( \text{sign}(\bullet) \) Sign function.

\( (\bullet)^T, (\bullet)^{ab} \) Transpose operator; transposition of \( a \)th and \( b \)th base vector.

\( \text{tr}(\mathbf{a}) \) \( \text{tr}(\mathbf{A}) = \mathbf{A} : \mathbf{I} \) Trace operator.

**Roman symbols**

\( \mathbf{b} \) External body force by unit mass / left Cauchy Green deformation tensor in 4.1.2.

\( \mathbf{C}, \mathbf{C}, \mathbf{C} \) Tangent moduli (fourth order tensor) / its Kelvin mapping / its Voigt mapping.

\( \mathbf{C}, \mathbf{C} \) Right Cauchy Green tensor / its Kelvin mapping.

\( \mathbf{C}_p, \mathbf{C}_p \) Plastic Right Cauchy Green type tensor / its Kelvin mapping \( \dot{\mathbf{C}}_p = 2 \mathbf{F}_p \varepsilon^\Delta_p \mathbf{F}_p \).

\( \mathbf{C}_v, \mathbf{C}_v \) Viscous Right Cauchy Green type tensor / its Kelvin mapping \( \dot{\mathbf{C}}_v = 2 \mathbf{F}_v \varepsilon^\Delta_v \mathbf{F}_v \).

\( \mathbf{d} \) \( \mathbf{d} = \text{sym}(\text{grad} \, \mathbf{v}) = \frac{1}{2}(\text{grad} \, \mathbf{v} + \text{grad}^T \, \mathbf{v}) \) rate of deformation tensor.

\( \mathbf{E} \) Green-Lagrangian strain tensor.

\( \mathbf{e} \) Euler-Almansi strain tensor.

\( \mathbf{E} \) Young’s modulus.

\( \mathbf{F} \) Deformation gradient tensor.

\( \mathbf{f} \) Force vector.

\( \mathbf{I}, \mathbf{I} \) Second order identity tensor / its Kelvin mapping.

\( \mathbf{I} \) Fourth order identity tensor in Kelvin mapping.

\( \mathbf{U} \) Internal energy.

\( I_1, I_2, I_3 \) Principal invariants of Right Cauchy Green tensor \( \mathbf{C} \).

\( A_{ij}, A_{ijkl} \) Index notation of second order tensor / fourth order tensor.

\( \dot{\mathbf{C}} \) Strain tensor of Cauchy-Green type on intermediate configuration.

\( J \) \( J = \det(\mathbf{F}) \), volume ratio.

\( J_{ij} \) Index notation of numerical Jacobian in Kelvin mapping.
<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>$E_{\text{kin}}$</td>
<td>Kinetic energy.</td>
</tr>
<tr>
<td>$\mathbf{K}$</td>
<td>Stiffness Matrix in Kelvin mapping in global Newton iteration scheme.</td>
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<tr>
<td>$l$</td>
<td>Spatial velocity gradient.</td>
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<td>$n$</td>
<td>Outward unit normal vector.</td>
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<td>$\nu$</td>
<td>Poisson's ratio.</td>
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<tr>
<td>$\mathbf{P}$</td>
<td>First Piola-Kirchhoff stress.</td>
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<tr>
<td>$P_{\text{ext}}$</td>
<td>External mechanical power.</td>
</tr>
<tr>
<td>$C_1, D_2, \alpha$</td>
<td>Nonlinear material model parameters.</td>
</tr>
<tr>
<td>$\mathbf{q}$</td>
<td>Heat flux vector.</td>
</tr>
<tr>
<td>$Q$</td>
<td>Thermal power.</td>
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<tr>
<td>$\mathbf{r}$</td>
<td>Residual vector in Kelvin mapping.</td>
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<tr>
<td>$r$</td>
<td>Heat source per unit mass and time.</td>
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<tr>
<td>$R%$</td>
<td>Relaxation ratio.</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Material model plastic parameter.</td>
</tr>
<tr>
<td>$S, \dot{S}$</td>
<td>Second Piola-Kirchhoff stress / its Kelvin mapping.</td>
</tr>
<tr>
<td>$\mathbf{S}$</td>
<td>$\mathbf{S} = \dot{S}/C_1$ dimensionless 2nd Piola-Kirchhoff stress quantity.</td>
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<tr>
<td>$s$</td>
<td>Specific entropy.</td>
</tr>
<tr>
<td>$t$</td>
<td>Surface traction.</td>
</tr>
<tr>
<td>$T$</td>
<td>Time.</td>
</tr>
<tr>
<td>$\mathbf{u}$</td>
<td>Displacement vector.</td>
</tr>
<tr>
<td>$\mathbf{U}^{i+1}, \mathbf{U}^t$</td>
<td>Displacement solution at time $t + \Delta t$ for the $i + 1^{th}$ iteration / displacement solution at time $t$.</td>
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<tr>
<td>$v$</td>
<td>Spatial velocity tensor.</td>
</tr>
<tr>
<td>$\mathbf{x}, \mathbf{X}$</td>
<td>Position vector in spacial configuration / position vector in material configuration.</td>
</tr>
<tr>
<td>$z$</td>
<td>State vector in Kelvin mapping.</td>
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Glossary

axial stretch/compression a deformation measure which equals the ratio of the current deformed length to the original length of a specimen in a given axial direction.

current configuration the spatial position of all material particles of a material body at current time $t > t_0$.

hysteresis different paths in state-space when the driving forces of a process are reversed due to path-dependencies.

isochoric constant volume.

material (Lagrangian) description the parameterisation of a motion with respect to the material coordinates in the reference configuration.

mechanical equilibrium the balance of forces and the associated motion.

pull-back the operation that transforms a vector or tensor-valued quantity based on the current configuration to the reference configuration.

push-forward the operation that transforms a vector or tensor-valued quantity based on the reference configuration to the current configuration.

reference configuration the simultaneous position of all particles within a body referred to a fixed reference time $t_0$.

secular effect Dutch expression, linearly increasing deformation with logarithm of time under the constant effective stress.

spatial (Eulerian) description the parameterisation of a motion with respect to the current (or spatial) coordinates in the current configuration.

thermodynamic equilibrium the cessation of all dissipative processes of a system under constant external conditions.
thermodynamically consistent a term used in material theory to indicate the compliance with the first and second laws of thermodynamics.
Chapter 1

Introduction

1.1 Background

Peat has been considered as an “ordinary extraordinary” material by Hobbs (1986) which consists of decomposed fragmented remains of dead vegetation that have accumulated under waterlogged conditions. Current constitutive models for peat are mainly based on theories developed for fine-grained mineral soils. Because of their exceptionally high water content and organic nature, the deformation of peat materials under load is very large, coupled with remarkably high strength at the high water content levels. In order to replicate the observed peat behaviours, soil models developed for inorganic soils are often adapted based on empirical evidence and engineering judgement. There are basically two types of peat models, i.e. empirical models that correlate geotechnical behaviour of peat materials with their physical properties, e.g. water content, void ratio, degree of decomposition, etc. and constitutive models describing stress-strain relationships (Berry and Poskitt, 1972; Den Haan, 1996; Karunawardena, 2007; Madaschi and Gajo, 2015), albeit some empirical relations, such as Darcy's law, are used in the constitutive models.

Peat deposits are formed of partly decomposed fragmented plant remnants, resulting in their fibrous nature. The three distinctive characteristics of fibrous peat summarized by Mesri and Ajlouni (2007) are: i) fibrous peat possesses very high initial permeability; ii) permeability decreases dramatically under loading because of its very large compressibility; iii) fibrous peat has extreme compressibility and secondary compression is often more significant for peat deposits than other geotechnical materials. For convenience of modelling, the structure of peat materials can be treated as isotropic and anisotropic based on degree of decomposition (Hendry et al., 2012). Amorphous peat with a high degree of decomposition (H10 based on the von Post scale) can be taken as isotropic (MacFarlane and Radforth, 1965). For peat material at lower states of decomposition, structural anisotropy should be taken into account in their constitutive models. Yamaguchi et al. (1985) found that natural fibrous peat was cross-anisotropic and Hendry et al. (2012) confirmed the same finding and stated that the extent of cross-anisotropy was found to increase with
an increasing fibre content. For both fibrous and amorphous peats, large deformation is commonly encountered in-situ as well as in the laboratory experiments.

The majority of the current geotechnical models for peat were developed based on the constitutive models for mineral soils, where the understanding of the mechanism of peat geotechnical properties was insufficient, particularly the models developed based on curve-fitting. Some plausible phenomenological models proposed for peat were in small-strain one-dimensional fashion. Due to the lack of an unequivocal classification of the peat geomechanical behaviour, the variations of the laboratory and in-situ findings perplexed the choice of peat models. Therefore, the constitutive models in the literature had limited capabilities in simulating various strain/stress-range, load cases, testing conditions, strain rates, etc. Few constitutive models were proposed for peat in the finite strain or thermodynamically consistent frameworks described in Chapter 2.

1.2 Scope of the study

The undrained constitutive investigation of the peat material simplifies the material behaviour by eliminating the pore water flow, but can be straightforwardly extended into an effective stress formulation with the permeation description of the pore water in the saturated conditions. The study aims to find a rational ingress to the complex mechanical phenomena by setting up clear and plausible theories of the peat behaviour.

This study was carried out for the two main purposes:

1. In the literature, peat material was reported as a rate-dependent elastocplastic material. The first purpose of this study is to experimentally investigate the phenomenological nonlinear, rate-dependent, structurally anisotropic constitutive behaviour of the tested peat from the fundamental mechanisms for modelling purpose.

2. To propose a constitutive model based on the experimental proof within a thermodynamically consistent approach; to validate the model performance in simulating the experimental results in various load cases.

Based on the material categorisation proposed by Haupt (2000), the rate-dependence and equilibrium tests are studied to categorise the tested peat material. The main effort in the laboratory experiments in Chapter 3 focuses on a) the large-strain nonlinear constitutive relationships and b) strain recoveries of peat in undrained loading-unloading triaxial tests at various strain rates. The cell pressure, strain rate, stress relaxation effects on these two aspects are investigated. Structural anisotropy is studied by comparing the undrained triaxial tests and falling head permeability tests on the undisturbed vertical and horizontal peat specimens. Although the structural anisotropy is not included in the constitutive
model, the experimental data provide laboratory foundation for the future development of the constitutive models considering structural anisotropy of peat.

Based on the categorisation and the experimental discoveries, a finite strain constitutive model is proposed in Chapter 4. The constitutive model adopts a hyperviscoelastic model (Görke et al., 2010; Nagel, 2012) and is extended with a hyperplastic component. The constitutive model consists of three layers, i.e. hyperelastic, hyperviscoelastic and hyperelastoplastic components by using the overlay concept. The constitutive relationship of each layer is derived from Clausius-Duhem entropy inequality which guarantees the proposed model produces thermodynamically reasonable results. Two time-scale rate-dependence is considered in the hyperviscoelastic model layer.

Chapter 5 introduces the numerical implementation of the constitutive models. A numerical algorithm is adopted which yields consistent tangent operators for the local stress iterations and the global Newton scheme. The numerical implementation of the hyperelastic, hyperviscoelastic and hyperelastoplastic models are verified with their analytical solutions. The model parameter fittings start with the rate-independent equilibrium tests, whereby the hyperelastoplastic model is fitted against the defined experimental equilibrium state test. Compression-relaxation, carried out at two strain rates, are used for the parameter fittings of the rate-dependent components of the hyperviscoplastic model. The complete model with the fitted material parameters is then validated against undrained triaxial tests carried out at five different strain rates under various load cases as well as an undrained triaxial creep test under comparable testing conditions. Modifications to the loading boundary conditions as well as to the plastic flow rule are discussed on improving the model performance in peat modelling.

Chapter 2 reviews the laboratory investigations as well as the constitutive models for peat materials. A finite strain constitutive model with its versatilities in modelling the complex peat geomechanical behaviours based on the the experimental proof is absent from the literature, which brings the necessity of this study.

The study is summarised and concluded in Chapter 6. The limitations on the experimental tests and the proposed numerical model are elaborated, whereby insights for future peat research both on experimental and numerical aspects are indicated.

Appendix A summarises the development of the rheological models for peat materials and the evolution of the application of time-line theory in peat from the literature. Appendix B provides the explanation of the product of the Right Cauchy Green tensor and elaborates the derivation of an analytical solution for the hyperelastoplastic model as well as the value range of parameter $\alpha$. 
Chapter 2

Literature Review

With the development of society, technology and economy, there has been an increasing need for construction activities, such as transportation infrastructures, wind farms, foundations, civil and industrial constructions, etc., in places that would have been considered unsuitable for construction in the past. In these places, soft soils, such as peat, are one of the most problematic geotechnical issues to be dealt with. The peatland formation has resulted from climatic states where precipitation exceeds evaporation and provides waterlogged conditions for peat accumulation to occur. The peatland coverage of the national area of Ireland, 17 % ca. 13,000 km², ranked the third in the world following Finland and Canada (Hobbs, 1986) in 1983. More recent data on the peatland distribution in Europe has been reported by Montanarella et al. (2006). Engineering problems of peat, such as failure of embankments (Den Haan and Feddema, 2013), containment dykes (McInerney et al., 2006), slope failures (Long and Jennings, 2006) and wind farm foundation failures (Lindsay and Bragg, 2004), have gathered the attention on the research of peat mechanical properties in recent decades especially in the northern countries such as Sweden, the UK, Ireland, the Netherlands, Germany etc. Research on peat geotechnical properties is mostly reported in the Netherlands, Ireland, the UK, Sweden, Canada, the USA and Malaysia. More detailed peatland distribution all over the world can be found in Huat et al. (2014).

Peat material consists of decomposed fragmented remains of dead plant vegetation that has accumulated over time under waterlogged conditions. The classification of peat varies, where in soil science peat is termed as an organic soil with organic content of more than 50 %, and in geotechnical engineering peat includes soils with an organic content of more than 20 % (Kogure, 1999). The standard classification of ASTM (ASTM, 2007) defines peat as an organic soil with its low ash content of less than 25 % ash by dry weight, i.e. an organic content of more than 75 %. Albeit different definitions are proposed, peat materials all over the world share some common properties. The formation process grants peat materials with the inherent structural anisotropy resulted from the prostrate fibres. The degree of this anisotropy is dependent on the level of the bio-degradation of the vegetarian constituents in peat. Due to its exceptionally high water content and organic nature,
the deformation of peat material under loading is very large compared with mineral soils (e.g. clay, silt, etc.). Current geotechnical practice for peat is mainly based on theories developed for fine-grained mineral soils, which are extended to consider the large strain, state-dependent and anisotropic behaviour of peat. This chapter reviews the experimental tests on peat, showing the mechanical properties of peat materials and the constitutive models proposed and adopted for peat.

2.1 Experimental investigation on peat materials

Peat material is considered a challenging natural material in geotechnical engineering practice. Challenges of experimental investigation of peat materials derive from the difficulties handling and preparing peat samples as well as problems achieving the appropriate stress levels and analysing the experimental results in standard laboratory apparatus designed for conventional mineral soil testing (Long, 2005).

In soil mechanics, investigation of the properties of a geomaterial always starts with solving engineering problems. Engineering research practice is to correlate peat strength with its physical properties, such as water content, organic content, Atterberg limits, etc. There is also research into peat’s mechanical properties from its physical fundamentals. The former approach has the advantages of providing direct and fast solutions to complex geomaterial problems and the disadvantage of limited applicability when accounting for material variations. This later approach is normally challenging in quantifying the specific properties of a complex material, yet fundamental in understanding the material.

In the literature, laboratory experimental investigations of peat mechanical behaviour are mainly in two categories, i.e. consolidation and strength. Consolidation is the water expulsion process of the peat material under loading. The strength investigation of peat in the literature focuses on the rate-dependent as well as (both structural and induced) anisotropic pre-failure/failure tests in drained and undrained conditions. This section reviews the mechanical properties found in the various laboratory tests in terms of consolidation and strength.

2.1.1 Peat consolidation

The distinctive characteristics of peat consolidation from those of mineral soils are (1) the primary consolidation with excess pore water pressure dissipation is relatively short and creep is the dominant and fundamental process (Hobbs, 1986); (2) the initial permeability of fibrous peat is very high, typically 100 to 1,000 times the initial permeability of soft clay and silt deposits and upon compression the permeability decreases dramatically (Mesri and Ajlouni, 2007); (3) the consolidation anisotropy is induced by fibres (structural anisotropy) as well as by anisotropic loading (induced anisotropy) (Zwanenburg, 2005); (4)
the organic solid particle in peat is likely to be compressible (Long and Jennings, 2006; Zwanenburg, 2005); (5) biodegradation occurs during the consolidation process (Pichan and O’Kelly, 2012). Regarding the long-term creep, it is pivotal to understand the consolidation hypotheses A and B (Ladd et al., 1977) to be discussed in the next section.

2.1.1.1 Concepts of primary consolidation and secondary compression—Hypotheses A & B

The term primary consolidation is used for describing the soil consolidation phase with excess pore water pressure dissipation (Terzaghi, 1942), whereas the secondary compression refers to the further settlement associated with constant effective stress, first named as “secular effect” by Buisman (1936). The primary consolidation and secondary compression are taken to refer to consecutive phases of the compression process, where secondary compression commences after the primary (hydrodynamic) phase has essentially reached completion.

Primary consolidation is solely a process of the dissipation of excess pore water pressure and effective stress increase, as defined by Terzaghi (1942). There are mainly four conceptual ideas in explaining secondary compression (Bartholomeeuseen, 2003). Terzaghi (1941) made his first attempt to explain that the secondary compression was due to the slow viscous inter-granular movement attributed to the highly viscous absorbed electrical double water layer of the solid particles. Subsequently, Terzaghi (1953) stated that secondary compression was due to the re-adjustment of clay particles to a more stable position, which has been widely accepted as the explanation of secondary compression of granular geomaterials. As the second explanation, Taylor (1942) stated that the magnitude of compression depends on the rate of compression due to viscous effects of the absorbed double layer and introduced “plastic resistance”, based on bond resistance and viscous structural resistance. The structural change during secondary compression is due to the disturbance of structure or the structural readjustment, which is independent of permeability. The third explanation is by the concept of macro- and micro-pores (De Josselin de Jong, 1968), in which the pores of different sizes transmit the expelled water in order to allow settlement to occur. De Josselin de Jong (1968) suggested that during primary consolidation, water is only expelled from the macropores, whereas the excess pore water pressure is measured. After primary consolidation, pore water is gradually expelled from the micropores into the macropores, causing secondary compression to occur. Barden (1968) claimed that a consolidation mechanism relevant to fibrous peat involves the drainage of a system of micropores into a system of coarser channels. The fourth interpretation of secondary compression is by means of rate process theory which describes secondary compression as the displacement of the bonds between molecules or group of molecules at the clay particle contacts (Wu et al., 1966). The basis of rate process theory is that atoms,
molecules, and/or particles participating in a time-dependent flow or deformation process (termed flow units) are constrained from movement relative to each other by energy barriers separating adjacent equilibrated positions (Wu et al., 1966; Mitchell, 1992).

In concert with the mechanism of the time-dependent behaviour of soils, another major question is when creep occurs during the settlement process. Experimental analysis and models developed generally treat consolidation and creep compression as either separate or simultaneous processes. This division leads to two philosophies, called hypotheses A and B (Ladd et al., 1977; Jamiolkowski et al., 1985). Both hypotheses consider primary consolidation as a hydrodynamic phase in the settlement process, and controlled by the dissipation of excess pore water pressure. Hypothesis A separates strains induced from the dissipation of excess pore water pressure from those due to creep and predicts that the relationship between end-of-primary (EOP) void ratio and effective stress is the same for both laboratory and field conditions. Consequently, at the completion of primary consolidation, the stress-strain curve obtained in-situ is the same as that obtained in the laboratory on small test specimens. Hypothesis B assumes that some sort of "structural viscosity" is responsible for creep and that creep occurs during both the consolidation phase and under the final (constant) effective stress. Hence, for hypothesis B, the EOP strain increases with sample thickness (Degago et al., 2011). An implication of the two hypotheses in terms of effective stress-void ratio is presented in Figure 2.1. Although a certain amount of experimental work has been performed in an effort to prove or disprove either of the respective hypotheses (Berre and Iversen, 1972; Leonards, 1972; Mesri and Godlewski, 1977; Mesri and Choi, 1985), much of the evidence is unconvincing (Leroueil et al., 1985b). In the past decade, this topic has been thoroughly discussed and a common view has been obtained that cohesive soils behave in conformity with hypothesis B. The advocates of hypothesis A (Mesri et al., 1973; Mesri and Godlewski, 1977; Mesri and Choi, 1985; Mesri et al., 1997; Feng, 1991; Mesri, 2003; Mesri and Vardhanabhuti, 2005) claimed the existence of the uniqueness of the void ratio-effective stress relationship, however their experimental data and mathematical derivations were re-examined by other researchers (Degago et al., 2009; Yin and Graham, 1990; Degago et al., 2011) and an opposing conclusion was made. The main defects of the proofs supporting hypothesis A lie in the ambiguous determination of EOP as well as the violation of basic axioms of continuum mechanics by claiming a unique EOP effective stress-strain relationship, irrespective of consolidation duration (Degago et al., 2013). Further, some laboratory and field test data imply the validity of hypothesis B, such as Kabbaj et al. (1988) who compared the behaviour of four test embankments with the results of laboratory tests and showed that the laboratory EOP consolidation curve strongly underestimates in-situ settlements. Consolidation models both based on hypothesis A and B are reviewed in Section 2.2.
2.1.1.2 Permeability and consolidation of peat

The permeability of peat decreases with decreasing void ratio and water content (Hobbs, 1986). Carlsten (1991) suggested a vertical permeability in the range $10^{-5} - 10^{-7}$ m/s for virgin peat and showed this value reduced by $10^3$ at a relative compression of 50%. Despite the spatial variations of the peat types, similar trends have been reported by Berry and Vickers (1975) for a fibrous UK peat, Dhowian and Edil (1980) for fibrous Portage peat (Wisconsin, the USA) and Mesri et al. (1997) for a fibrous Middleton peat (Wisconsin). The permeability of peat not only depends on the void ratio, but it is also related to the microstructure of the peat material, and so depends on the degree of humification and the structural anisotropy (MacFarlane and Radforth, 1965). The more humified the peat is, the lower the permeability it possesses. Figure 2.2 illustrates the relationship between the permeability and the degree of humification.

Dhowian and Edil (1980) reported the permeability anisotropy resulted from fibre orientation with a horizontal permeability of about 300-fold higher at a given void ratio than the vertical permeability by the variable hydraulic head method during consolidometer tests. However, much smaller permeability anisotropy was reported by most other researchers. Mesri et al. (1997) reported that the anisotropic permeability with the value $k_{ho}/k_{vo} = 10$ at a in-situ void ratio of 12.0 for Middleton peat. Zhang (2013) calculated the permeability of the vertical and horizontal specimens in oedometer tests and reported that the ratio of the coefficients of permeability anisotropy $k_v/k_h$ increased from 2.37 to 3.19 with the void ratio change from 10 to 2 with a $10^3$ drop in permeability values. Zwanenburg (2005) had a closer examination of the permeability during the different triaxial test stages and found that for the initial phase a considerable change in permeability occurs while for the following phase the permeability is nearly constant. The reported permeability data in the literature shows the inherent variations of peat materials.
In the literature, the terms consolidation and drained compression are often interchanged to describe the same process. This review deliberately separates the two terms to facilitate a clear understanding of the mechanical behaviour of peat, albeit consolidation is a subset of compression. In this review, consolidation of peat refers to the dehydration process under an applied load for a period of time, whereas the compression of peat refers to the rest compressive behaviour, either drained or undrained, under a strain- or stress-controlled compressive load for a period of time. Due to the cellular parental vegetation of peat and the interstices between the partly decomposed plant remains, the two-level structure of macro- and micro-pores is a substantiated interpretation of the long-term creep for peat materials (Edil and Den Haan, 1994). Figure 2.3 shows a scanning electron micrograph that distinctly presents the porous structure of a slightly decomposed (H4 on von Post scale) sphagnum peat.

Adams (1963) described that primary consolidation of fibrous peat was considered to be due to the drainage of water from the macropores and that secondary compression was due to very slow drainage of water from micropores into macropores. Barden (1968) claimed that the dominant viscous mechanism governing the consolidation of amorphous (colloidal) peats could be described in terms of structural viscosity and reversible thixotropy (the time dependent change in particle arrangements, adsorbed water structure and distribution of ions (Den Haan, 1992)). Dhowian and Edil (1980) observed significant compression and effluent flow occurred following the dissipation of measurable excess pore water pressure, which supported the two-level structure possibility, with permeability decreas-
ing drastically during consolidation. From observations of in-situ seepage tube and auger-hole tests, Hemond and Goldman (1985) concluded that the dramatic changes in apparent hydraulic conductivity of peat can be explained by its large elastic storativity of pore water. Hobbs (1986) stated that the consolidation of peat involves the expulsion of pore water accompanied by a structural rearrangement of the solid particles; the two processes occurring simultaneously in the early stages, but following the decline of the excess pore water pressure to a very small value, the structural rearrangement and expulsion of water from micropores continue as a creep-like process. The rate of creep generally increases with increasing organic content due to a higher porosity (Landva and La Rochelle, 1983). Although Hobbs (1986) and Mesri et al. (1997) found that, in most cases, the EOP consolidation can be estimated from Casagrande’s log-time method or Taylor’s square root-time method, Edil and Den Haan (1994) concluded that the only reasonable estimations for the time period $t_p$ required to achieve EOP are those based on pore water pressure measurements. Zhang and O’Kelly (2015) scrutinized the salt addition effect on the consolidation of fibrous peat which experimentally supported the postulation of two-level structure in fibrous peat, where the salt additions accelerated the consolidation process of the fibrous peat due to electro-osmotic effect.

Oedometer tests without pore water pressure measurements are the most commonly used tests for investigation of the one-dimensional (1D) consolidation of peat. A typical incremental load oedometer test result of undisturbed fibrous peat is presented in Figure 2.4, which depicts the 1D loading-unloading void ratio–effective stress curves during both the primary consolidation and secondary compression. The rebound index, $C_s = \Delta e / \Delta \sigma'_v$, increases with overconsolidation ratio, $OCR = \sigma'_{v(max)}/\sigma'_v$, and increases slightly with the decrease in $\sigma'_{v(max)}$ from which unload takes place (Mesri et al., 1997). Mesri and his coworkers (Mesri and Choi, 1985; Mesri and Godlewski, 1977; Mesri and Castro, 1987) found
that there is a unique relationship between the coefficient of secondary compression $C_\alpha$ and the primary compression index $C_c$. The $C_\alpha/C_c$ concept will be elaborated later in the modelling review section.

Figure 2.4: EOP void ratio versus effective stress curves of 24 undisturbed Middleton peat specimens (Mesri et al., 1997).

Over a three year period, Berry (1983) carried out Rowe cell consolidation tests to determine the long-term secondary compression behaviour of a fibrous peat (Figure 2.5). The two-level structural consolidation model was used to predict the field consolidation curve and estimate the required surcharge, and Berry (1983) concluded that although a field pilot scheme is also necessary, the consolidation theory for peat may be used to determine the optimum surcharge load required for the reclamation of a peat deposit. From Figure 2.5, it can be seen that the tangential slopes of the consolidation curves gradually increase after the hydrodynamic phases (curve shown before the vertical arrow in Figure 2.5). Edil and Dhowian (1979) and Dhowian and Edil (1980) term this increase as tertiary compression and interpreted it as decreasing strain rate, changing at an increasing rate. Colleselli et al. (2000) studied the compressibility characteristics of three Italian peats using a 70.5 mm diameter oedometer and a 75.5 mm Rowe cell with pore water measurement at the base of the specimen and found that the higher was the fibre content of peat, the larger was the tertiary compression.

The main characteristics of peat consolidation can be summarized as (1) peat is extremely compressible under loading, experiencing high initial rates of primary consolidation and substantial creep (Landva and La Rochelle, 1983; Santagata et al., 2008; Wong et al., 2008) with the creep component often greater than the hydrodynamic component (Farrell, 2012); (2) tertiary compression usually occurs in fibrous peat consolidation (O’Loughlin, 2001); (3) the consolidation of peat is related to the micro-structure which depends on the
Figure 2.5: Typical experimental results for vertical consolidation tests on 250 mm diameter, by 75 mm thickness undisturbed samples of fibrous peat (Berry, 1983)).

parental vegetation as well as the degree of humification (Hobbs, 1986; Pichan and O’Kelly, 2012; Zhang and O’Kelly, 2015), and is extremely spatially variable (Hobbs, 1986).

An important contribution on the 1D consolidation of peat was made in GeoDelft with their $K_0$-CRS oedometer, which is a constant rate of strain oedometer with measured lateral stress (Den Haan, 2001). Drainage is from the top and pore water pressure is measured at the bottom of the specimen, with a dimension of 63 mm in diameter and an initial height of 20 mm. The apparatus provides continuous readings of strain, vertical and horizontal total and effective stresses, side friction loss and pore pressure. The complete stress condition obtained from the $K_0$-CRS tests makes the test results amenable to the 1D as well as 2D $abc$ isotache model (Den Haan, 1994, 1996), which is one of the most prominent models developed for peats. Figure 2.6 shows the capability of the three testing methods to generate information on creep properties in a $K_0$-CRS test. This test could provide the model parameters $a$, $b$ and $c$ by different test methods (Den Haan, 2001).

Figure 2.6: Three testing methods for creep properties in a $K_0$-CRS test (translated from Den Haan (2001)).

In addition to the 1D consolidation tests especially on the long-term compression of
peat, consolidation tests in triaxial cells were carried out for the investigations of (1) the isotropic and anisotropic consolidation effects on the strength of peat (Adams, 1961) as well as (2) the consolidation of soft peat specimens to target levels of effective stress for the subsequent compression/shear tests (Zhang and O’Kelly, 2014b). The stiffness as well as shear strength of peat can be increased by a large reduction of water content through consolidation (Den Haan, 1997). Triaxial testing is one of the routinely used methods in practice due to its good repeatability. The triaxial apparatus provides close control of the specimen stress and boundary conditions, with pore-water pressure measurement. The pros and cons of effective stress strength testing of peat in triaxial compression has been described by O’Kelly (2015) and O’Kelly and Orr (2014). As reported in O’Kelly and Zhang (2013), an extensive review of the literature produced relatively few studies reporting consolidated-drained (CD) triaxial compression testing of peat (Adams, 1961; Hollingshead and Raymond, 1972; Holubec and Langston, 1972; Tsushima et al., 1977; Marachi et al., 1983; Farrell and Hebib, 1998; Zhang and O’Kelly, 2014b) and a few on consolidated-undrained (CU) triaxial compression (Yamaguchi et al., 1985; Zwanenburg, 2005; Hendry, 2011; Hendry et al., 2014). The characteristics of peat consolidation in triaxial cell are summarized as: (1) The primary consolidation is quick; (2) Specimen size effect exists in the triaxial consolidation tests. Zwanenburg (2005) reported that a clear Mandel-Cryer effect is found in large triaxial samples (0.4 m in diameter and 0.6 m in height) whereas no Mandel-Cryer effect can be distinguished for a conventional triaxial sized sample. The Mandel-Cryer effect is the term used for the inhomogeneous consolidation behaviour in peat where the consolidation of the outer radius of the sample leads to volumetric strain in the outer area and induces a redistribution of stresses in which a part of load carried by the outer radius is transferred to the yet unconsolidated inner core of the sample. (3) Pore pressure measurements may be influenced by the presence of bio-gas trapped within the tested peat specimens (Hobbs, 1986; Zwanenburg, 2005; Couwenberg, 2009).

Regarding laboratory investigation of the time-dependent behaviour of peat, very few relaxation tests (constant strain, zero strain rate) were reported in the literature. Drained relaxations were carried out in the $K_0$-CRS cells by Zwanenburg (2005) and Den Haan and Kruse (2007), which is a special case of the constant strain rate test at zero strain rate. The test data was used for calibrating the consolidation model derived from creep tests.

### 2.1.2 Compression and shear of peat

The two-fold purpose of compression tests, either strain- or stress-controlled, on peat are to obtain (1) the compressive behaviour (quantifying settlements); (2) the drained/undrained strengths (maximum stress at failure). In the early days of soil mechanics theory, laboratory tests on peat focused on the shear strength of peat in drained/undrained compression tests, whereas the compression behaviour was studied separately in consolidation
tests. The following section reviews the conventional Mohr-Coulomb theory on quantifying peat’s shear strength; the distinguished peat behaviours found in the triaxial tests; and the rate-dependent tests carried on peat.

The conventional approach for quantifying the strength of peat materials in the laboratory is using Mohr-Coulomb failure criterion, i.e. deriving the friction angle and the intrinsic cohesion for both drained and undrained triaxial tests. Opinions differ over the nature of peat strength from entirely cohesive to entirely frictional in 1950s and 1960s by Hanrahan (Hanrahan, 1952, 1954a,b; Hanrahan and Walsh, 1965; Hanrahan et al., 1967), the strength of peat was commonly defined by the friction and cohesion. Low effective cohesion values of nearly 0 kPa for peat materials have been reported by many researchers such as Marachi et al. (1983); Farrell and Hebib (1998). Very high friction angles up to 55° were reported for fibrous peats (Adams, 1961; Farrell and Hebib, 1998) due to fibre effects. Hendry (2011) stated that the shear strength of intact peat is made up of inter-particle friction as well as tension in the peat fibres. The triaxial test results were compared by Long (2005) with results obtained by other testing apparatus, such as simple shear test, vane shear test, direct shear box test, etc. Yamaguchi et al. (1985) explain the high friction angle obtained from undrained triaxial tests on fibrous peat as the predominantly horizontal fibres providing additional shearing resistance and an elastic stiffness that is cross-anisotropic. It is anticipated that a higher friction angle could be obtained for more structured or fibrous material. However, O’Kelly and Zhang (2013) obtained similar effective friction angles from consolidated drained triaxial compression tests for the same peat material with three very different fibre contents and Edil and Wang (2000) reported the same finding for fibrous and amorphous peats. It is also interesting to note that the universally reported effective friction angle for peat without fibre enhancement is around 30° defined at failure or 20 % strain in different testing apparatus, including Dutch peat (Den Haan et al., 1995), Irish peat (Farrell and Hebib, 1998; Zhang and O’Kelly, 2014b), Canadian peat (Adams, 1963; Hollingshead and Raymond, 1972; Landva and La Rochelle, 1983). The summary of effective friction angle $\phi’$ from the drained triaxial tests reported in the literature is presented in Figure 2.7. The central problem in obtaining the cohesion and friction parameters in drained triaxial tests lies in that peak values of stress are usually not achieved even for shear strain larger than 20 %. As reported by O’Kelly (2015), unlike shearing at frictional contacts in uncemented mineral soils, the strength of peat is derived from the connectivity between elementary structures; that is, failure mostly involves tearing of elementary structures, entangled fibres and cellular connections (Boylan et al., 2008; Landva et al., 1986). The physical meaning of the friction angle and cohesion in peat may not be the same as in mineral soils since the particle concept in fibrous peat is ambiguous. O’Kelly (2015) argued that the measured value of $\phi’$ was also likely to be strain-rate dependent.
Adams (1961, 1965) claimed that it is impossible to secure undisturbed samples for laboratory testing. He carried out a series of drained and undrained triaxial tests on undisturbed peat samples of water content ranging 200% - 600% and reported a small value of the coefficient of lateral earth pressure $K_0 = 0.18$, which confirmed the behaviour of a small lateral strain during the drained compression tests and found that preconsolidation and anisotropic consolidation had little effect on the strength parameters of peat in triaxial compression. The consolidated drained compression tests on the undisturbed peat specimens lasted for three months until sign of failure showed and very large axial strain (up to 50%) and deviator stress (up to almost 900 kPa) were recorded. The drained triaxial compression of peat is found to be nearly 1D without bulging (O’Kelly and Zhang, 2013; Zhang and O’Kelly, 2014b). Hanrahan (1954a) reported that during undrained triaxial compression of an Irish fibrous peat, the horizontal effective stress reached zero at failure. A major technical hurdle in evaluating peat strength is that, due to the low bulk unit weight of peat of typically 9.5 - 11.5 kN/m$^3$, the effective stresses occurring in the field are lower than the lower bound stress-level capabilities of most testing equipment (O’Kelly, 2015). Drained triaxial compression tests are seldom performed because of gross change in specimen shape and dimensions that occur during the course of the tests (Edil and Wang, 2000), and also presumably because the compression stage necessitates significantly slower rates of axial strain in order to allow direct measurement of the effective stress shear responses. However, for undrained triaxial tests on peat, the relatively large testing strain rate may magnify the eccentric placement of the specimen under loading, thus introducing specimen distortion failure (Hendry, 2011). A typical undrained triaxial compression test on undisturbed fibrous peat is illustrated in Figure 2.8. Hendry et al. (2012) define the continuing compression as a continual yielding and state that the geometrical eccentricity during testing leads to the development of a shear plane, thus a peak deviator.

Figure 2.7: Values of effective friction angle $\phi'$ deduced from drained triaxial compression of peat (adapted from O’Kelly and Zhang (2013)).
stress value can be obtained. For both undrained and drained triaxial compression tests, a failure envelope is usually not developed (Farrell and Hebib, 1998; Ajlouni, 2000; Hendry, 2011; O’Kelly and Zhang, 2013; Zhang and O’Kelly, 2014b). The role of peat fibres in its shear strength is claimed to be only in providing elastic tensile resistance (Hendry, 2011) but Zhang (2013) found the fibres can also resist compressive stress from oedometer tests on undisturbed vertical and horizontal peat specimens.

Figure 2.8: Stress strain response of consolidated undrained triaxial testing of two Canadian undisturbed peat samples (Hendry, 2011).

The fibrous character of peat is one of its most distinctive properties. Farrell and Hebib (1998) investigated the mechanical behaviour of a fibric peat and found that peat specimens tested in drained triaxial compression deformed approximately one-dimensionally under loading without achieving failure (as defined by peak deviatoric stress) for up to 35 % axial strain. In undrained triaxial compression, the pore water pressure may rapidly build up, almost reaching the applied cell pressure, thereby producing an effective lateral stress approaching zero on account of the low Poisson’s ratio of fibric peat (Farrell, 2012). O’Kelly and Zhang (2013) and Zhang and O’Kelly (2014b) performed drained triaxial compression tests on peat specimens prepared from the same parent material and found that the lowest Poisson’s ratio (0.02 - 0.03) was achieved for undisturbed peat, a slightly larger Poisson’s ratio (0.04 - 0.05) for reconstituted peat and a significantly larger Poisson’s ratio (0.13 - 0.16) for blended peat at up to 20 % axial strain. The low Poisson’s ratio for undisturbed and reconstituted peat material was attributed to the lateral resistance induced by the fibres and also to the tensile strength of the fibres themselves.

It is a universally accepted fact that the mechanical behaviour of peat is time-dependent (Hobbs, 1986; Farrell, 2012; Degago et al., 2009; Mesri and Ajlouni, 2007; Den Haan and Kruse, 2007), and thus rate-dependent. The testing of the strain rate effect of soil is mostly studied in clay (Leroueil et al., 1985a; Hight and Leroueil, 2003) showing that the yield stress, the entire 1D compression curve and the undrained shear strength are strain-rate dependent for clay soils. Little similar work for peat is reported in the literature (Long and Boylan, 2013). Ajlouni (2000) found that differences in the shearing rate did not affect the shear strength envelope for Middleton peat in consolidated undrained triaxial tests.
However, the deficit in the strain rate effect test in the undrained triaxial compression by Ajlouni (2000) was that the strain rate in individual test was changing irregularly. The rate-dependent behaviour of peat in 1D CRS cell with open drainage has been studied in Dutch peat (Den Haan and Kruse, 2007) and Irish peat (Long and Boylan, 2013). However, there are very few studies on the rate-dependent behaviour of peat in undrained tests. O’Kelly and Zhang (2013) carried out drained triaxial compression tests on six reconstituted peat specimens at axial strain rates ranging from 0.07 %/h to 0.417 %/h and found the specimens compressed at different strain rates mobilized similar values of deviator stress and principal effective stress ratios. Similarly to the rate-dependent behaviour of clays, a higher strain rate results in a higher deviator stress in undrained triaxial tests on peat (Zhang et al., 2015).

Other than the reviewed consolidation, compression, fibre-effect and strain-rate effect laboratory tests on peat, thermal conductivity and the approximate damping envelope were studied by Hanson et al. (2000) for Middleton fibrous peat. Results indicate that the thermal conductivity of peat is lower than the thermal conductivity of typical sand and silt soils. The large void ratios of the peats (10±2) prevent the effective transfer of heat through the soil. The heat capacity of peat is higher than that of sands and silts. The high water content of peats controls the thermal parameters. It is innovative to investigate the thermal properties of peat, however, they only reported the thermal conductivity of peat without the thermal effect on the constitutive relation of peats, such as the thermal effect on the viscosity, which is of more a geotechnical interest.

The laboratory tests on peat have been comprehensively reviewed. Although studies on peat have been carried out for decades (since 1940s), the characteristic experiments for understanding the geomechanical properties of the complex peat materials are still inadequate. There are two reasons for this. Firstly, effort in the early decades of peat research was following the conventional soil mechanics theory, which brought confusion and dead-ends, worsened by the extreme spatial variety of peats, e.g. the testing strain rate in drained triaxial compression which is determined from triaxial consolidation based on the Terzaghi consolidation theory (Terzaghi et al., 1996) after the full dissipation of the measurable excess pore water pressure. The second reason is that, from an engineering point of view, most of the laboratory test methods where the integrated peat properties (anisotropy, rate-dependency, elasticity, yielding etc.) and testing conditions (drained, undrained, 1D, triaxial, simple shear, etc.) perplex the understanding of the test results, e.g. testing on the "elastic" rebound in the oedometer test unloadings. A third reason could be the time-consuming tests for peats. The long-term consolidation/compression tests for peat could take from days to years or even decades. It is crucial to understand the basic comprehensive physical mechanism of the complex behaviour of peat, so that the myriad observations in the laboratory experiments, as well as on site, could be correctly and ef-
iciently explained and analysed. Intuition and postulations are certainly unavoidable at the current development of analytical methods.

2.2 Constitutive models for peat materials

There are two types of peat models in the literature, namely empirical models that correlate the geotechnical behaviours of peat materials with their physical properties, e.g. water content, void ratio, degree of decomposition, fiberosity, etc. and constitutive models describing stress-strain relationships. Empirical correlations are beyond the scope of this review. Constitutive models developed for peat were derived from the laboratory experimental results. The review of the constitutive models for peat follows the corresponding order of the review on the laboratory experiments.

2.2.1 One-dimensional consolidation/compression models

Most of the 1D models developed for peat are consolidation models which only take peat volumetric behaviour into account (using material quantities such as void ratio \( e \) and specific volume \( 1 + e \)). Models based on hypothesis A which consider that creep begins from the end of "primary" consolidation, determined by full dissipation of excess pore water pressure. These models generally assume a Terzaghi-style solution for the consolidation process, employing a creep coefficient such as \( C_\alpha \) (coefficient of secondary compression) to model continuing settlement under constant effective stress. Early work by researchers studying creep (Šuklje, 1957; Bjerrum, 1967; Janbu, 1969; Šuklje, 1969) assumed that the creep rate was determined by the current state of effective stress and current void ratio, thus invoking hypothesis B. These formulations can be classified as isotache models (Garlanger, 1972; Den Haan, 1996) and further research approaches developed on such isotache theory naturally consider creep occurs during primary consolidation.

2.2.1.1 \( C_\alpha/C_c \) concept

In developing the \( C_\alpha/C_c \) concept, Mesri and co-workers (e.g. refer to Mesri and Godlewski (1977)) proposed that a unique relationship exists between the secondary compression coefficient \( (C_\alpha = \partial e / \partial \log t) \) and the compression index \( (C_c = \partial e / \partial \log \sigma') \) that holds true at all combinations of elapsed time, effective stress and void ratio for a variety of natural materials, including peats. Figure 2.9 shows the procedure used to compute \( C_\alpha/C_c \) values, i.e. obtaining parameter \( C_\alpha \) from the \( e - \log t \) curves and the corresponding \( C_c \) from the \( e - \log \sigma' \) curves. Mesri and Castro (1987) reported that \( C_\alpha/C_c \) ranges 0.04–0.06 for highly organic plastic clays and 0.02–0.10 for peats. For any given soil, three or four pairs of \( C_\alpha \) and \( C_c \) values are generally sufficient for evaluating \( C_\alpha/C_c \).
Mesri and Castro (1987) stated that the values of \( C'_a/C_c \) for peats are in the remarkably narrow range of 0.02-0.10. Karunawardena (2007) reported that the \( C'_a/C_c \) relationship observed for a Sri Lankan peaty clay, namely \( C'_a = 0.0341C_c \), was in good agreement with the \( C'_a = 0.035 \) relationship reported for amorphous peaty clay by Mesri et al. (1997). Adachi et al. (1984) and Oka (2005) mathematically derived the interrelationship between the viscoplastic parameter \( m' \) (introduced in the later EVP model section) with the \( C'_a/C_c \) and Leroueil and Hight (2003) verified this relationship experimentally and concluded that the two approaches used for describing the viscous behaviour of soils during secondary consolidation are equivalent. Around 1990s, researchers had fierce arguments on the concept of \( C'_a/C_c \) applied to the compression of peat. Fox et al. (1992, 1994) concluded that the \( C'_a/C_c \) concept is difficult to apply on account of tertiary compression (the increase in slope of the secondary compression settlement-log \( t \) curve) which may be induced by further decomposition of the peat organic structure in laboratory conditions. By comparing their calculated results (based on the procedure presented in Figure 2.9) with those obtained by Mesri and Castro (1987) for the same test data (Figure 2.10a), Fox et al. (1992, 1994) showed that the determination of the \( C'_a/C_c \) value is subjective and they also claimed that the concept is not applicable to peat compression. Lefebvre et al. (1984) reported considerable scatter in calculating the \( C'_a/C_c \) ratio (Figure 2.10b), although the mean value of 0.06 agrees well with the value of 0.052 suggested for peat soils by Mesri et al. (1997). Graham et al. (1983) suggest that the \( C'_a/C_c \) ratio is strain dependent, rather than constant, for a given clay.

The \( C'_a/C_c \) concept provides a 1D consolidation model merely by curve-fitting, disregarding the micro-mechanism of soil compression. The great scatter of the \( C'_a/C_c \) values for peats indicates the limitation of its implementation in predicting peat mechanical behaviour.
2.2.1.2 One-dimensional rheological models

The 1D rheological models developed for peat have followed the Terzaghi (1942) piston and spring analogy, with the three main models developed for peat summarized in Appendix A.

Firstly, Barden (1968) proposed a 1D rheological model for clay and amorphous peat based on the model presented by Gibson and Lo (1961). This rheological model did not follow the convention of separating a stress system into its volumetric and deviatoric components because the separate investigations of volumetric and deviatoric components may have less fundamental significance in studies of soil deformation when considering that a soil can dilate strongly under shear stress and that the basic mechanism of volume change is one of the local shear. Instead, the rheological model treats the volumetric and deviatoric stresses as a single process by generalization in terms of the effective stress ratio. The model is basically a Kelvin element, placed inside the Terzaghi pot (Figure 2.11). The outer container is filled with water and the perforated piston represents the pore-water drainage properties of the element. In the Kelvin model, the nonlinear spring structural viscosity and thixotropy (aging hardening) are described by means of a power law for clay. However, Barden and Berry (1965) have indicated that the nonlinearity of the spring is relatively unimportant. To minimize the number of parameters involved in the mathematical treatment of the model, it is assumed that the spring is linear. Hence the improvement of Barden’s model compared with the model of Gibson and Lo’s lies in the nonlinear dashpot, which gives the characteristic linear secondary consolidation against log time and also takes account of the dominant effect of load-increment ratio.
Similar to the time factor \((T_v)\) derived from Terzaghi (1942)'s consolidation continuity equation, which governs the rate of pore water pressure dissipation during primary consolidation, Barden (1968) defined a time factor \((T_s)\) from his proposed rheological model which governs the structural viscosity of the soil skeleton during secondary consolidation. The time factor \(T_s\) is independent of the thickness of the soil layer considered in this model (i.e. soil structural viscosity is independent of soil layer thickness), although this can be seriously questioned. Furthermore, this model is not able to consider the moving drainage boundary caused by large strains and greatly decreasing permeability as the macropores are compressed, both of which are important considerations for modelling peat. Thus the solution applies only as a first approximation, but nevertheless provides a useful theoretical framework for investigating peat behaviour.

Edil (1982) and Edil and Mochtar (1984) used the rheological model developed by Gibson and Lo (1961) to predict the settlement response of Middleton fibrous peat in the laboratory and in-situ. The rheological model can be represented by a top spring connected to a Kelvin element in series (Figure 2.12) with the top spring representing primary consolidation and the compression of the Kelvin element representing secondary compression (creep); thereby indicating hypothesis B. The time-dependent strain, \(\epsilon(t)\), is expressed in Equation (2.1) (Edil and Mochtar, 1984) for large values of elapsed time: where \(\Delta\sigma\) is the applied stress increment; \(a\), \(b\), and \(\lambda/b\) are empirical parameters that implicitly govern consolidation, creep and creep-rate respectively; \(t_a\) is the time period following completion of primary consolidation (i.e. applied stress increment has produced a corresponding increase in effective stress). Edil and Dhowian (1979) demonstrated that the values of these parameters could be conveniently determined from field or oedometer test data. As the model is only applicable under constant effective stress \(t > t_a\), it cannot reliably predict the consolidation phase (Lan, 1992). In addition, Edil and Mochtar (1984) found that the creep parameters \(b\) and \(\lambda/b\) were highly nonlinear, with significantly different creep rates occurring between field and laboratory conditions.

\[
\epsilon(t) = \Delta\sigma \left[ a + b \left(1 - e^{-\lambda/b t} \right) \right] \quad \text{for} \quad t > t_a
\]  

(2.1)
Some proposed 1D rheological models based on the micromechanical consolidation concepts for peat shed light on a promising two-level structure concept in peat consolidation. Berry and Poskitt (1972) developed separate rheological models for amorphous and fibrous peats, as classified by MacFarlane and Radforth (1965). According to MacFarlane and Radforth (1965), solid particles in amorphous peat are mainly colloidal in size, with the majority of the pore water adsorbed around the grain structure. In contrast, fibrous peat essentially has an open structure, with enclosed secondary structures of mainly non-woody fine fibrous materials. Berry and Poskitt (1972) suggested that the physical mechanisms controlling creep for amorphous peat are similar to those for clay, i.e. creep occurs due to the gradual readjustment of the solid particles into a more stable arrangement, following disruption of the soil skeleton, associated with compression due to consolidation. The Gibson and Lo (1961) model was employed for amorphous peat but the springs used were nonlinear in order to simulate the soil’s nonlinear compressibility. Based on rate process theory (Christensen et al., 1964; Wu et al., 1966), the strain rate of amorphous peat, $\dot{\varepsilon}_1$, was in the form given by Equation (2.2): where $\alpha(e)$ and $\beta(e)$ are rheological parameters that depend on the current value of the void ratio, $\sigma'$ is the vertical effective stress in the dashpot. Fibrous peat was modelled on the macro- and micro-pore network concept advanced by Adams (1963), which was simulated using a double Terzaghi pot (Figure 2.13). The mathematical treatment of this model requires that only the macro-pore pressure in the element be integrated over the thickness of the soil layer in order to predict the 1D consolidation behaviour of fibrous peat. The axial strain rate of the micropores, $\dot{\varepsilon}_2$, is based on rate process theory and takes the form given by Equation (2.3): where $A(e_2)$ and $B(e_2)$ are rheological parameters that depend on the current value of the void ratio and $u_2$ is the equivalent average pore pressure in the micropores. A flow law based purely on physical intuition was postulated to describe water flow from micropores to macropores, with consideration of Darcy’s law. These models have been verified by Berry and
Poskitt (1972) to some extent from laboratory testing of amorphous and fibrous peats. Their study demonstrated the concept of constitutive modelling by considering the distinctive micro-mechanical characteristics of amorphous and fibrous peats. However, the lack of proper testing techniques for the mechanical properties of micropores meant that the corresponding parameters were merely based on physical intuition. Furthermore, the consolidation equations are highly nonlinear and suffer from mathematical complexity when extending to multi-dimensional situations.

\[
\dot{\varepsilon}_1 = \beta(e)\sinh(a(e)\sigma')
\]

(2.2)

\[
\dot{\varepsilon}_2 = A(e_2)\Delta p \sinh \left[ \frac{B(e_2)}{\Delta p} u_2 \right]
\]

(2.3)

Figure 2.13: Rheological representation of Berry and Poskitt (1972) model for fibrous peat.

2.2.1.3 Time-line theory and isotache models

The evolution of time-line theory for consideration of time-dependent behaviour in soil models is summarized in Appendix A.

Bjerrum (1967) proposed the time-line theory to describe the creep of clay by a series of parallel time lines in \( e - \log \sigma' \) space, wherein each line represents equal periods of sustained loading. For any given value of the effective overburden pressure and void ratio, there corresponds an equivalent time of sustained loading and a certain rate of delayed consolidation, independent of the manner in which the clay has reached these values. The unique relationship between void ratio, effective overburden pressure and time is presented in Figure 2.14. Instead of primary and secondary compression, Bjerrum (1967) used the terms instant (immediate) and delayed compression to describe the void ratio reductions occurring simultaneously with the increase in effective stress and at unchanged effective stress, respectively. The definitions of these two compression classifications are illustrated in Figure 2.15. Some researchers (Wahls, 1962; Bjerrum, 1967; Yin and Graham, 1989) claimed that the separation of the compression response into primary and secondary contributions is rather arbitrary and, furthermore, that this division is unsuited
to describe the behaviour of the soil structure with respect to effective stress. Bjerrum's work is, however, recognised as a major contribution towards a thorough understanding of the time dependence of soil compression and is the basis for the majority of current 1D constitutive models (Imai, 1995). Nevertheless, many of Bjerrum's concepts, particularly the instant time line, are vague and are either difficult to understand (Imai, 1995) or implement (Christie and Tonks, 1985). More recently, the time-line theory has tended to be superseded by the isotache principle.

Šuklje (1957) proposed the isotache concept in order to describe rate effects on the compressibility of clayey soils. This approach introduces a unique relationship between the strain and the pre-consolidation pressure corresponding to strain rate in association with the viscosity. In contrast to Bjerrum (1967)'s time-line theory, each creep isotache corresponds to a constant void ratio rate. This means that any combination of void ratio, vertical effective stress and rate of change in void ratio is unique, and this remains valid during the entire soil compression process: i.e. for both hydrodynamic and constant effective-stress processes. The isotaches illustrate satisfactorily the difference between the faster consolidation for a sudden increase in effective stress and the slower consolidation attributed to its creep. They also give a clear interpretation of the influence of the previous secondary compression on the decrease in consolidation rate associated with the additional loading. An important implication of the isotache principle is that the apparent pre-consolidation pressure is strain-rate dependent. Further, assuming Darcy's law and the parabolic form of isochrones, the corresponding calculated pore pressure values do not agree closely with the measured values; i.e. when the plastic resistance at the beginning of consolidation is small, the calculated initial values of the pore pressure exceed the

Figure 2.14: Compressibility and shear strength of clay exhibiting delayed consolidation (Bjerrum, 1967).
measured ones. The isotache model is developed on the basis of the logarithmic-time law of secondary compression (i.e. a linear relationship exists between secondary compression (void ratio and $\log t$) which has been supported by a number of field and laboratory observations (Bjerrum, 1967; Garlanger, 1972). However Šuklje (1957) himself claimed the validity of this linear relationship assumption is limited. Studies (Mesri and Godlewski, 1977; Berry and Poskitt, 1972; Edil and Dhowian, 1979; Dhowian and Edil, 1980; Den Haan, 1994) have revealed that, particularly for peat, the slope of the secondary compression settlement–$\log t$ curve often increases (i.e. tertiary compression).

Based on Bjerrum's time-line theory, Garlanger (1972) presented a 1D consolidation model for clay soils by including the effect of water in the voids exhibiting creep under constant effective stress, along with a numerical solution for the mathematical equations. Here, the creep volumetric change is a function of effective stress and time. By employing a linear $\log e - \log \sigma'$ relationship, the following description of creep was obtained ($t_i$ is the time given to the instant line, $\sigma'_c$ is the effective preconsolidation pressure, $\sigma'_0$ is the effective initial pressure, $\sigma'_f$ is the effective final pressure).
\[-\Delta \log e = a \log \frac{\sigma'_c}{\sigma'_0} + b \log \frac{\sigma'_f}{\sigma'_c} + c \log \frac{t_i + t}{t}\]

(2.4)

Building on Garlanger (1972) model, Den Haan (1994) proposed an improved \textit{abc} consolidation model for 1D consolidation of soft clay and peat. Den Haan (1996)’s \textit{abc} model differed from Garlanger (1972)’s model in adopting Šuklje (1957) isotache concept. In other words, Den Haan (1996)’s model essentially assumes a unique relationship between strain, effective stress and creep strain rate. Natural strain (Hencky strain) is used instead of linear engineering strain. A linear relationship is employed between \(\ln \sigma'_v\) and Hencky strain, which has been found valid for the virgin compression behaviour of a wide range of soils (Den Haan, 1992), resulting in the compression curves with constant load increment ratio (LIR), being equidistant in the virgin stress range. In \(\ln \sigma'_v\)-Hencky strain space, a system of parallel lines called isotaches is assumed to exist, with the secular strain rate constant for a given isotache (see Figure 2.16).

![Figure 2.16: Illustration of isotaches of the \textit{abc} model (Den Haan, 1996).](image-url)

The advantage of using the isotache concept is that the strain rate is used instead of compression time to describe the viscous behaviour of soil in developing compression isotaches. In this manner, the difficulty in relating the elapsed time since the start of loading to the intrinsically time-dependent (i.e. viscous or rate-dependent) behaviour of soil is avoided. In the \textit{abc} consolidation model, the total strain rate \(\dot{\varepsilon}^H\) is the combination of direct strain rate \(\dot{\varepsilon}^H_d\) and creep strain rate \(\dot{\varepsilon}^H_s\) (Equation (2.5)), after Buisman (1936)’s concepts
of ’direct’ and ’secular’ effects in soil compression, with superscript $H$ indicating Hencky strain, i.e. $\dot{\varepsilon}^H = \ln\left(\frac{1+e}{1+e_0}\right)$. This additive decomposition of strain rate is derived from the additive decomposition of total strain into direct and secular strains for small strains.

$$\dot{\varepsilon}^H = \dot{\varepsilon}_{d}^H + \dot{\varepsilon}_{s}^H$$  \hspace{1cm} (2.5)

The direct strain rate $\dot{\varepsilon}_{d}^H$ is determined from the rate of increase in effective stress and is described arbitrarily by the compressibility parameter $a$, referred to as the natural swelling index (Equation (2.6), where $\sigma'_v$ is the vertical effective stress mobilised during primary consolidation and $t$ is the elapsed time period since the start of loading). The creep strain rate $\dot{\varepsilon}_{s}^H$ (uniquely defined by the present vertical effective stress and total strain) is described in the equation of isotaches (intrinsic time lines) by the two parameters $b$ and $c$ (Equation (2.6)), where $\nu$ is specific volume and $V_1$ is the specific volume at unit effective stress on the creep isotache at a creep strain rate $\dot{\varepsilon}_{s,0}^H$ (see Figure 2.16). Parameters $b$ and $c$ are referred to as the natural compression index and natural secondary compression index, respectively. The isotache equation (with parameters $b$ and $c$) only pertains to virgin compression under constant or monotonically increasing effective stress. On introducing intrinsic time $\tau$, the creep strain rate can be interchangeably related by $\dot{\varepsilon}_{s}^H = c/\tau$. The parameters $a$, $b$ and $c$ can be derived from the swelling ($C_s$) and compression ($C_c$) indices and creep coefficient ($C_\alpha$) (Equation (2.7)). The isotaches described in Equation (2.6) for the direct ($\dot{\varepsilon}_{d}^H$) and secular ($\dot{\varepsilon}_{s}^H$) phases of consolidation are essentially bilinear in a double natural logarithmic plot. Den Haan (1994) showed that for cases involving large strain, the logarithmic small strain exceeds engineering strain and the $abc$ model fitted the measured incremental-oedometric compression response of a soft clay and peat satisfactorily. O’Loughlin (2001) adopted the $abc$ model in predicting the 1D creep behaviour of a bog peat and concluded that the isotache concept is valid for peat if Hencky strain and natural strain rate are used. However, the $abc$ model is not suitable for overconsolidated behaviour since the isotaches are neither parallel nor linear (distorted) in the overconsolidated region. Also the stress increase during unloading relaxation cannot be modelled by the $abc$ model (Den Haan, 2001). The $abc$ model (Den Haan, 1996) developed for peat has its provision for coping with the highly nonlinear behaviour encountered in peat soils, although structural anisotropy of fibrous peat is not included.

$$\dot{\varepsilon}_{d}^H = a \frac{d\sigma'_v}{d\tau}; \quad \dot{\varepsilon}_{s}^H = \dot{\varepsilon}_{s,0}^H \left[ \frac{\nu}{V_1} \left(\frac{\sigma'_v}{\sigma'_v^{(b)}}\right)\right]^{1/b}$$  \hspace{1cm} (2.6)

$$a = \frac{C_s}{(1 + e_0)\ln 10}; \quad b = \frac{C_c - C_s}{(1 + e_0)\ln 10}; \quad c = \frac{C_\alpha}{(1 + e_0)\ln 10}$$  \hspace{1cm} (2.7)
2.2.1.4 Elastic visco-plastic (EVP) models

The elastic visco-plastic (EVP) model proposed by Yin and Graham (1989, 1990, 1994, 1996) for clay soils is essentially an extension of Bjerrum (1967)’s time-line concept. However, the EVP model employs the isotache concept (i.e. unique relationship assumed between strain, effective stress, strain rate) along with a more rigorous treatment of Bjerrum (1967)’s instant time line. An important implication of the isotache concept is that the apparent preconsolidation stress is strain-rate dependent. The EVP model is one of the first works in developing an alternative approach using commonly accepted logarithmic relationships for elastic, plastic and time-dependent straining, rather than fitting simple functions (usually graphical) to experimental data. The EVP model assumes that in the normally consolidated region, the creep strain rate for a given effective stress and strain state is unique. Furthermore, at a given creep strain rate, linear relationships exist between strain and log $\sigma'_v$, and also between strain and logarithm of creep strain rate. Engineering strain is used for small strain conditions, with the total vertical strain ($\varepsilon_z$) Equation (2.8)) comprised of three components, where $\varepsilon^e_z$ is the elastic (recoverable) strain, $\varepsilon^{sp}_z$ the time-independent plastic strain induced only by the applied stresses and $\varepsilon^{tp}_z$ the (time-dependent) viscous-plastic strain. In the elastic stress range, the strain is composed of $\varepsilon^e_z$ and $\varepsilon^{tp}_z$ (i.e. viscoelastic). In the plastic stress range, stress increases are accompanied by both elastic and plastic strains (latter given by sum of $\varepsilon^{sp}_z$ and $\varepsilon^{tp}_z$). By extending the $\lambda - \kappa$ model used in critical-state soil mechanics to include time and strain-rate effects, the time-independent elastic strain $\varepsilon^e_z$, time-independent plastic strain $\varepsilon^{sp}_z$ and creep strain $\varepsilon^{tp}_z$ are defined by Equation (2.9). In these equations, $e$ is void ratio, $\varepsilon_{z0}$ corresponds to the strain at the start of loading; $\sigma'_v$ is the vertical effective stress on the normally consolidated $\lambda$-line at strain $\varepsilon_{z0}$; $\kappa$ (reloading index) and $\lambda$ (compression index) are the gradients of the unload-reload and normally-consolidated line, respectively, in the critical-state soil mechanics framework and $\psi$ is a creep coefficient.

$$\varepsilon_z = \varepsilon^e_z + \varepsilon^{sp}_z + \varepsilon^{tp}_z$$  \hspace{1cm} (2.8)

$$\varepsilon^e_z = \varepsilon_{z0}^e + \frac{\kappa}{1 + e} \ln \left( \frac{\sigma'_v}{\sigma'_{v0}} \right) \hspace{1cm} \varepsilon^{sp}_z = \varepsilon_{z0}^{sp} + \frac{\lambda - \kappa}{1 + e} \ln \left( \frac{\sigma'_v}{\sigma'_{v0}} \right) \hspace{1cm} \varepsilon^{tp}_z = \varepsilon_{z0}^{tp} + \frac{\psi}{1 + e} \ln \left( \frac{t}{t_0} \right)$$  \hspace{1cm} (2.9)

The total strain ($\varepsilon_z$) starting at the ‘instantaneous’ loading can be written as Equation (2.10):

$$\varepsilon_z = \varepsilon_{z0} + \frac{\lambda}{1 + e} \ln \left( \frac{\sigma'_v}{\sigma'_{v0}} \right) + \frac{\psi}{1 + e} \ln \left( \frac{t - t_{i-1} + t_0}{t_0} \right)$$  \hspace{1cm} (2.10)

where $\varepsilon_{z0}$ ($= \varepsilon^{sp}_z + \varepsilon^{tp}_z$) is a reference strain and $\sigma'_{v0}$ a reference stress that correspond respectively to the strain at the beginning of loading and to the value of $\sigma'_v$ on the normally
consolidated $\lambda$-line at strain $\epsilon_{z0}$. The time $t$ is the total elapsed time since the beginning of the test up to the present time with the current stress $\sigma'_{z}$, and $t_{t-1}$ is the total elapsed time until the end of the previous loading increment. Originally proposed by Yin and Graham (1989, 1994, 1996) for clay soils, O’Loughlin et al. (2001) have found that when applied to fibrous peat, the EVP model grossly overpredicts settlements.

### 2.2.1.5 Viscoelastic viscoplastic model

A recent plausible 1D rheological model was proposed by Madaschi and Gajo (2015) by identifying both the short- and long-term mechanisms of the 1D visco-elasto-plastic behaviour of inorganic/organic geomaterials. The short-term mechanism of the consolidation is assumed to be instantaneous (strain measure indicated by superscript $i$) which is the same concept of immediate compression presented in Figure 2.15, where the long-term mechanism of the consolidation is assumed to be due to viscosity (strain measure indicated by superscript $v$). The small strain 1D phenomenological model proposed by Madaschi and Gajo (2015) is illustrated in Figure 2.17. Similar to the EVP model, hypothetically the total strain ($\epsilon$) is additively decomposed into elastic ($\epsilon_e$) and plastic ($\epsilon_p$) fractions from experimental ideas (Equation (2.11)). A linear stress-strain relationship is adopted for the final elastic and plastic strain responses with the usual compressibility indices $\kappa$ and $\lambda$ (Equation (2.12)).

![Figure 2.17: Viscoelastic viscoplastic model proposed by Madaschi and Gajo (2015).](image)

From Equation (2.12), the constitutive relations for the elastic and plastic parts are derived in Equation (2.13) and Equation (2.14), respectively, where $\psi_e(\dot{\epsilon}_e)$ and $\psi_p(\dot{\epsilon}_p)$ are suitable viscosity functions that have been selected from the experimental evidence. By using two partition coefficients $\alpha_e$ and $\alpha_p$, ranging between 0 and 1, the elastic and plastic strains are partitioned into two parts, i.e. an instantaneous part and a delayed part.
\[ \sigma_{ei} = \sigma_0 \exp \left( \frac{\epsilon_{ei} - \epsilon_0}{\alpha_e \kappa} \right), \quad \sigma_{ve} = \sigma_0 \exp \left( \frac{\epsilon_{ve} - \epsilon_0}{(1 - \alpha_e) \kappa} \right) \left[ 1 + \psi_e (\dot{\epsilon}_{ve}) \right] \] \tag{2.13}

\[ \sigma_{pi} = \sigma_0 \exp \left( \frac{\epsilon_{pi} - \epsilon_0}{\alpha_p (\lambda - \kappa)} \right), \quad \sigma_{vp} = \sigma_0 \exp \left( \frac{\epsilon_{vp} - \epsilon_0}{(1 - \alpha_p) (\lambda - \kappa)} \right) \left[ 1 + \psi_e (\dot{\epsilon}_{vp}) \right] \] \tag{2.14}

The viscoelastic viscoplastic model is a simple rheological model which considers the consolidation mechanisms of the inorganic/organic soils. The models shows fairly good simulation results of the oedometer tests on Kevico peaty silt (Madaschi and Gajo, 2015). The linear constitutive relationship between the small strain and logarithmic stress constrains its application in the large-strain peat materials. From Figure 2.17, it can be seen that the elastic and plastic strains are independent, i.e. the elastic recovery strain is independent on the plastic strain history. However, for peat materials, the elasticity is strain-history dependent, which will be experimentally presented in Chapter 3. Therefore based on the viscoelastic viscoplastic model, a further rearrangement of the model components could be improved to better capture the consolidation behaviour of peat.

### 2.2.2 3D consolidation/compression models

The 1D consolidation models are extended into three-dimensional (3D) consolidation / compression models. Some important issues should be taken into account when extending 1D consolidation/compression models to 3D. Model parameters obtained from 1D experimental curve-fitting analysis disregard the mechanism of the consolidation/compression behaviour, thus the concepts of the 1D parameters are difficult to extend into corresponding 3D concepts. In soil mechanics, based on the theory of elasticity, it has been common practice to study volumetric and deviatoric deformation behaviour separately. However this division is artificial and disregards the complete load-deformation response of a soil. Furthermore, settlement based on the separation of volumetric and deviatoric deformations, and analysis of the settlement rate based on the elastic soil skeleton may well be in error for situations where the loading produces significant local yielding. Elastic perfectly-plastic models which consider nonlinear material behaviour have been applied to predict the consolidation behaviour of peat (Brinkgreve et al., 1994). Elasto-plastic models formulated using critical state theory, which considers hardening plasticity by incorporating an expanding yield stress represented by a non-fixed yield surface, have shown acceptable results for the primary loading phase. However, as expected, their performance in predicting long-term behaviour was not as satisfactory: a potentially significant drawback for modelling peat for which the contribution of creep to the overall settlement is significant.
2.2.2.1 Soft Soil Creep model

Since existing 1D isotache models have not been formulated as differential equations, the 3D creep model could not be straight-forwardly obtained by extension from its 1D format. The Soft Soil (SS) and Soft Soil Creep (SSC) models (Stolle et al., 1999; Vermeer and Neher, 1999) implemented in the PLAXIS finite element code are proposed to deal with soft mineral soils and peat. These models formulate a constitutive law in differential form to solve transient and/or maintained loading problems. In essence, the SSC model is an isotache model having the same stress-strain-creep strain rate relationships given by Equations (2.5) and (2.6). By using the critical-state soil mechanics framework, the parameters \(a, b\) and \(c\) of the 1D model have been converted to material parameters \(\kappa^*, \lambda^*\) and \(\nu^*\) in Equation (2.15). On extending the 1D model to general states of stresses and strains, the effective stress invariants for pressure \(p' = 1/3(\sigma'_1 + \sigma'_2 + \sigma'_3)\) and deviatoric stress \(q = 1/\sqrt{2}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}\) are adopted. Having determined the ellipses in the effective volumetric-deviatoric \((p' - q)\) stress plane from the modified Cam Clay model (Roscoe and Burland, 1968), an equivalent pressure \(p'_\text{eq}\), in the form of (2.16), has been defined: where \(M\) is the slope of stress path in \(p' - q\) stress plane; \(c'\) is the apparent cohesion and \(\phi'\) is the effective friction angle.

\[
\begin{align*}
\kappa^* &\approx 2a \\
\lambda^* &\equiv \kappa^* + b \\
\nu^* &\equiv c
\end{align*}
\] (2.15)

\[
p'_\text{eq} = p' - \frac{q^2}{M^2(p' - c'\cot(\phi'))}
\] (2.16)

The elastic part of the total strain rate employs Hooke’s law. The equivalent pressure \(p'_\text{eq}\) is taken as a plastic potential function for deriving individual creep strain-rate components. The parameters of the SS and SSC models can be determined either from isotropic consolidation tests or oedometer testing (Den Haan, 2001). However the approximation of the relationship between the isotropic recompression index \(\kappa^*\) and the 1D swelling indices \(a\) and \(C_s\) can only provide a rough estimation of the model parameter, since the horizontal to vertical stress ratio changes during 1D unloading (Den Haan, 2009). Using the SS and SSC models incorporated in PLAXIS, Osorio-Salas (2012) back analysed vacuum-consolidation test data obtained for a fibrous peat deposit, reporting acceptable results for predicted vertical displacements. However these models were found to overestimate the ground heave that occurred following removal of the vacuum pressure (unloading phase), presumably for the aforementioned reason. O’Kelly and Orr (2014) and O’Kelly (2015) concluded that consolidated undrained and consolidated drained standard triaxial compression testing are not particularly useful in determining the effective stress strength parameter values for peat. O’Kelly and Orr (2014) and O’Kelly (2015) postulated that the apparent cohesion \((c')\) mobilised due to the presence of entangled fibres and cellular connections in
peats manifests as an increase in the effective friction angle producing non-conservative values. Hence the numerical modelling of peat materials based on the effective stress parameters obtained from these test methods should be used with care.

### 2.2.2.2 3D EVP large strain model

Using the EVP model proposed by Kimoto and Oka (2005), Karunawardena (2007) performed an extensive study on the consolidation behaviour of a Sri Lankan peaty clay (amorphous peat with initial void ratio of 2.0-8.0 and organic content of 20 % - 50 %) by considering creep settlement, finite strain and changes in permeability and compressibility occurring during the peat consolidation process. The model is based on Perzyna (1963)’s elasto-viscoplastic theory, Cam Clay and empirical evidence. The elastic strain rate tensor $\dot{\varepsilon}_e$ is given by a generalized Hooke’s law and is computed by the summation of the deviatoric and volumetric components. The model hardening rule is based on the overconsolidation (OC) boundary surface in Equation (2.17), defined by the modified Cam Clay model (Roscoe and Burland, 1968), which delineates the OC region from the normally consolidated (NC) region (Figure 2.18). In Equation (2.17), anisotropic consolidation history is considered by introducing the pressure-normalized deviatoric stress tensor $\eta^*_{ij}$, where the stress parameter that represents the anisotropic consolidation history $\bar{\eta}^* = \sqrt{(\eta^*_{ij} - \eta^*_{ij(0)}) (\eta^*_{ij} - \eta^*_{ij(0)})}$ with $\eta^*_{ij(0)}$ is the initial value of $\eta^*_{ij}$; and $M^*_m$ is the value of $\sqrt{\eta^*_{ij} \eta^*_{ij}}$ determined when the volumetric strain increment changes from compression to swelling; $\sigma^*_m$ denotes the mean effective stress and $\sigma^*_mb$ is the isotropic consolidation yield stress; $\sigma^*_mc$ and $\sigma^*_mp$ are the mean effective stress at the intersections of the OC boundary surface and the viscoplastic potential function at the $\sigma^*_m$ axis, respectively. The Cam Clay static yield function is used to describe the yield surface for the peaty clay at its static equilibrium state. The viscoplastic potential function is given by Equation (2.18), where $\bar{M}^*_m$ is defined on the basis of the yield surface boundary.

$$f_b = \bar{\eta}^* + M^*_m \ln \left( \frac{\sigma^*_m}{\sigma^*_mb} \right)$$

$$f_p = \bar{\eta}^* + \bar{M}^*_m \ln \left( \frac{\sigma^*_m}{\sigma^*_mp} \right)$$

The credible feature of Karunawardena (2007)’s EVP model is that the effect of structural degradation has been taken into consideration by introducing viscoplastic strain softening in addition to strain hardening with viscoplastic volumetric strain. Degradation of the soil structure leads to contraction of the OC boundary surface with viscoplastic deformation (Karunawardena, 2007). The viscoplastic strain rate tensor is calculated using Perzyna (1963)’s viscoplastic theory, of which the rate-independent yield function used in describing viscoplastic strain can exceed zero - an effect known as ‘overstress’. The
Karunawardena (2007) model also considers variable compressibility, as was the case in the analysis of stage-constructed embankments which applied high stress levels on the Sri Lankan peaty clay deposit. The compressibility parameters were calculated based on the stress levels and proportioned to changes in specific volume of the peaty clay. The constitutive model and analysis methods adopted successfully predicted the consolidation behaviour of the amorphous peat (Karunawardena, 2007). Since structural anisotropy was not considered, the proposed EVP model is not suitable for fibrous peats.

### 2.2.2.3 Hyperelastic/hyperplastic models

Thermodynamically consistent approaches for peat modelling have been proposed. There are two main branches, one is Zhang and O’Kelly (2014a) and the other is Boumezerane et al. (2015). Both of the researches only proposed the thermodynamically consistent framework for peat models, neither of the model were validated against experimental results. The detailed extension of the Zhang and O’Kelly (2014a) model will be elaborated in Chapters 4 and 5. Boumezerane et al. (2015) proposed a rate-dependent hyperplastic model for peat by taking rate-dependent behaviour, anisotropy, large-strain, compressible solid phase in drained conditions. The constitutive relations are derived using Gibbs free energy and force potential (dissipation) functions originally proposed by Houlsby and Puzrin (2007). The model is relatively simple. The hyperplastic model serves as a good implementation of the hyperplastic theories (Houlsby and Puzrin, 2007) in peat modelling within the thermodynamically consistent framework. However, the material constitutive relationship derived from small strain kinematics is not capable of describing the large strain experienced in peat.
2.2.3 Anisotropic consolidation/compression models

The anisotropic models were developed for the cross-anisotropic behaviour of fibrous peat by mainly three approaches, i.e. rotations of the yielding and hardening, an overlay addition of fibre reinforcements and modelling fibres individually. The details of the approaches are elaborated as follows.

Based on the isotropic creep model (Soft Soil Creep model), Leoni et al. (2008) developed a new anisotropic creep model (ACM) by assuming rotated Modified Cam Clay ellipses as contours of volumetric creep strain rates, and a rotational hardening law to account for changes in anisotropy due to viscous strains. The ACM introduces three additional material parameters to account for cross-anisotropy. Firstly, a scalar quantity $\alpha$ is used to describe the orientation of the normal consolidation surface (Figure 2.19), which constrained the anisotropic model for triaxial conditions only. Thus the equivalent pressure in Equation (2.16) becomes (2.19). $\alpha$ acts like a rotational hardening parameter of which the evolution is governed by creep strains according to the rotational hardening in Equation (2.20), where two soil constants $\omega$ and $\omega_d$ are introduced. $\omega$ and $\omega_d$ control the rate of rotation which can be determined by the critical state angle and $\lambda^*$ in Equation (2.15), respectively. This anisotropic model provides the simple formulation to account strain-induced anisotropy. Den Haan (2014) evaluated the ACM for peat and found that all three anisotropy parameters of ACM reverse sign when applied to peat. The model parameters can be obtained from the triaxial compression and simple shear tests but not triaxial extension test. Den Haan (2014) claimed that a better anisotropic model should be developed by directly accounting for the fibre effects.

\[ p_{eq}' = p' + \frac{(q - \alpha p')^2}{(M^2 - \alpha^2)p'} \]  

(2.19)

Figure 2.19: Anisotropic creep model: current state surface (CSS) and normal consolidation surface (NCS) in triaxial stress space (Leoni et al., 2008).
\[
\dot{\alpha} = \omega \left[ \left( \frac{3q}{4p'} - \alpha \right) \dot{\varepsilon}_{\text{vol}} + \omega_{\alpha} \left( \frac{q}{3p'} - \alpha \right) \dot{\gamma} \right]
\]  

(2.20)

Similar to the bilinear criterion of fibrous peat failure used by Cola and Cortellazzo (2005), Hendry (2011) developed a conceptual model for the undrained response of peat based on the idea that its undrained behaviour is defined by the cross-anisotropic stiffness and strength produced by the development of tension within the predominantly horizontally-oriented peat fibres. Hendry (2011) and Hendry et al. (2012) considered the undrained fibre effect as cross-anisotropic elastic behaviour by adopting uniform stiffness within the horizontal plane, with different stiffness adopted for the vertical direction. In this case, the constitutive relationship considering fibre effects requires five elastic constants to describe the soil behaviour. In order to simplify the cross-anisotropic relationship, Graham and Houlsby (1983)’s simplification was adopted which allows the description of the anisotropic response in triaxial compression with the coincident vertical axis of the soil and the testing apparatus. Hence the number of material elastic constants can be reduced to three. The anisotropic elastic model of fibrous peat adopts a decoupled relationship between deviatoric and volumetric responses. The pore water pressure response in undrained condition was considered by coupling between the shear and volumetric effects. A consequence of both changes in the mean total and deviatoric stresses is that the elastic stress path no longer follows a path of constant mean effective stress.

Fox and Edil (1993) developed a discrete element model (DEM) ‘FIBER’ to account for the contribution of the peat fibres in peat mechanical behaviour. The FIBER model was the first research on the micromechanical modelling of peat using DEM which investigates the relationship between the macro-continuum characteristics and micro-scale characteristic of granular soil particle and contacts. In this model, the peat microstructure is considered as an assemblage of deformable cellular fibres in which each fibre is composed of an irregular arrangement of bar elements. These bars have the capacity for both elastic and creep deformation. Fibres can become completely separated and undergo rigid body motion. Contacts between the fibres are ‘hard’ such that no interpenetration of bodies can occur. Hence settlement results from the deformation of the fibres themselves. FIBER gives qualitatively good simulations of stress and temperature effects on creep, tertiary compression, thermal precompression and the unload-reload preconsolidation effect, even for a relatively simple mesh. The behaviour of all solid particles in the soil system is completely known in the DEM approach (Fox and Edil, 1993). Intraparticle stresses, contact forces, damping rotation, translation and sliding are known throughout the solution. Further, initial and boundary conditions can be completely defined and sample reproducibility is guaranteed. The main disadvantage of the DEM is its complexity and, to a lesser extent, computational expense.
Teunissen and Zwanenburg (2015) proposed a small strain overlay model for peat to add the fibre enhancement separately to the isotropic stiffness matrix. The whole matrix of a peat was hypothetically decomposed into two layers, i.e. the isotropic material matrix layer and the fibre layer. Both layers undergo the same strain rate but have their own effective stress. The model is capable of simulating the fibre orientation effect on the compression behaviour and strength of peats. This approach has provided some insights into the anisotropic modelling of peat.

2.3 Summary

The experimental work on peat reveals that the laboratory geomechanical investigations on peat are challenging, from sampling to testing to analysing. A wide variation of consolidation/compression behaviours has been reported. Four main reported features of peat material can be found in the literature, i.e. large strain, rate-dependency, irrecoverable deformation, fibres. However, a systematic investigation to describe the comprehensive constitutive behaviours of peats is absent from the literature. For constitutive models proposed for peat, few finite strain models are reported in the literature. The majority of the existing models are limited to capturing certain behaviours, such as consolidation, creep, relaxation etc., rather than being able to describe peat from its physical mechanisms, particularly the models developed from conventional curve-fittings. Therefore, a systematic laboratory study on peat undrained behaviour has been carried out by the author to categorise the material in order to choose the most appropriate constitutive models. Then a finite strain model accounting for the laboratory experimental findings is adopted within a thermodynamically consistent framework.
Chapter 3
Laboratory Experiments

This chapter elaborates the work programme of the laboratory experiments for the constitutive modelling purpose. The study aims to find a rational ingress to the complex mechanical phenomena by setting up clear and plausible theories of the peat behaviour. As reported in the literature, peat material exhibits complex geomechanical phenomena, including large strain, time-dependence, rate-dependence, irrecoverable strain, etc., the material categorisation proposed by Haupt (2000) was adopted to account for the integrated geomechanical behaviour of peat in the proposed constitutive model. Laboratory experiments performed in the conventional triaxial testing apparatus focused on the nonlinear stress-strain relationship, rate-dependence, strain irrecoverability, and structural anisotropy of an undisturbed peat in undrained conditions. Undrained tests were performed for calibrating and validating the constitutive model with simplified material behaviour by eliminating the pore water dissipation.

The Haupt (2000) material theory was based on the rate-dependence and the stress-strain hysteresis of the thermodynamic equilibrium test, where a thermodynamic equilibrium refers to the cessation of all dissipative processes of a system under constant external conditions. The objectives of the experimental work are:

- To investigate the rate-dependent and strain irrecoverable properties of the peat material in compression and unloading, relaxation and creep tests in undrained conditions. The experimental results will categorise the tested peat material and provide insights on the structure of the constitutive model to be proposed.

- To obtain experimental data for model parameter calibration as well as model validation.

- Anisotropic permeability and stress-strain relationships of vertical and horizontal specimens to be investigated for future extension of the constitutive model.

The following laboratory tests follow the sequence of recovered strain range of undisturbed vertical specimens in loading-unloading-relaxation tests, rate-dependent behaviour
in peat and structural anisotropy. Triaxial testing apparatus have been used for investigat-
ing the above mentioned undrained mechanical properties.

3.1 Experiment methodology

The methodology of the laboratory experiments on categorizing peat behaviour is based on Haupt (2000). The observable behaviour tendencies of materials are divided into 4 cat-

ergories:

1. rate-independent without hysteresis
2. rate-independent with hysteresis
3. rate-dependent without equilibrium hysteresis
4. rate-dependent with equilibrium hysteresis

If the stress-strain curves depend on the strain rate at which the experiments were car-
ried out, the material is rate-dependent. If the material responses are rate-independent, then the hysteresis effect can be checked (Figure 3.1(b)), where the hysteresis refers to the irrecoverability of the stress-strain relationship subjected to loading-unloading pro-
ces due to energy dissipation. Should the experimental data indicate a significant rate-
dependence of the material response, attention is turned to the equilibrium curves. Com-
paring to the mechanical equilibrium, the thermodynamic equilibrium refers to the ces-
sation of all dissipative processes under constant external conditions. The equilibrium curve may indicate the existence of hysteresis between the stress-strain relationship (Fig-
ure 3.1(d)). Corresponding to four categories of material experimental behaviours, there are four categories of mathematical models, *viz.* the theories of elasticity, plasticity, vis-
coelasticity and viscoplasticity.

Undrained triaxial tests on undisturbed peat are carried out at different strain rates and the equilibrium state behaviour is investigated, of which the experimental observations serve for the evidence of the constitutive models to be proposed. The methodology of the experimental work is shown in the following Table 3.1:

<table>
<thead>
<tr>
<th>Test type</th>
<th>Experimental tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test condition determination tests</td>
<td>Cell pressure effect tests</td>
</tr>
<tr>
<td></td>
<td>Proving ring determination tests</td>
</tr>
<tr>
<td>Characterisation tests</td>
<td>Rate-dependence tests</td>
</tr>
<tr>
<td></td>
<td>Thermodynamic equilibrium tests.</td>
</tr>
</tbody>
</table>

Table 3.1: Methodology of the experimental work.
Based on the experimental methodology, the experimental work plan was improved and completed along the course of the experimental investigations serving for the core testing on the rate-dependence and the equilibrium hysteresis. The finalised experimental work plan is presented in Table 3.2. The detailed experimental tests are presented in the following sections of this chapter.

<table>
<thead>
<tr>
<th>Experimental Test</th>
<th>Purpose</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undrained triaxial loading-unloading tests</td>
<td>Cell pressure effect on the stress-strain relationship</td>
<td>Undrained loading-unloading triaxial tests up to 5% axial strain with cell pressures of 0 kPa, 20 kPa, 40 kPa and 120 kPa, respectively, on an undisturbed vertical peat specimen.</td>
</tr>
<tr>
<td></td>
<td>Strain rate effect on the stress-strain relationship</td>
<td>Undrained loading-unloading triaxial tests up to 5% axial strain with 0 kPa cell pressure on three undisturbed peat specimens at axial strains of 16.0%/h, 1.60%/h and 0.16%/h.</td>
</tr>
<tr>
<td>Undrained relaxation tests</td>
<td>Relaxation effect on the strain recovery</td>
<td>Four undrained loading-unloading tests with step relaxations at axial strain rates of 16.0%/h, 1.60%/h and 0.16%/h, respectively.</td>
</tr>
<tr>
<td></td>
<td>Stress relaxation and logarithm of time relationship</td>
<td>Two undrained triaxial relaxation tests at axial strain rates of 16.0%/h and 1.60%/h with pore water pressure measurement.</td>
</tr>
</tbody>
</table>
Stress relaxation ratio
Three undrained triaxial loading-unloading tests with step relaxations at axial strain rates of 160 %/h, 16.0 %/h and relaxation durations of 16 and 20 mins.

Temperature effect on the stress relaxation
Two undrained triaxial relaxation tests with relaxation duration of 1700 min and 1 week, respectively.

Thermodynamic equilibrium test
Undrained compression tests with 24 hour step relaxations at an axial strain rate of 16.0 %/h to axial strains of 2 %, 4 % and 6 %. Comparison between undrained triaxial loading-unloading-reloading test with step relaxations at 160 %/h with undrained triaxial loading-unloading-reloading test at 4.81 %/h. Undrained triaxial loading-unloading tests at 0.16 %/h.

Consolidation and undrained triaxial loading-unloading tests
Strain history effect on the stress-strain relationship
Three undrained triaxial loading-unloading-reloading tests at 16.0 %/h of strain cycles of [5 %, 10 %, 15 %, 20 %], [10 %, 15 %, 20 %] and [15 %, 20 %], respectively. An undrained triaxial loading-unloading test up to 5 % axial strain at an axial strain rate of 0.16 %/h on consolidated vertical specimen (void ratio changed from 10.4 to 8.4).

Undrained triaxial "creep" test
Validation of the rate-dependence of the proposed model
An undrained triaxial "creep" test on an undisturbed vertical specimen with two load increments (10 kPa for 7 days and 20 kPa for 52 days).

Undrained triaxial loading-unloading tests on undisturbed horizontal specimens
Anisotropic stress-strain relationship
Three undrained triaxial loading-unloading-reloading tests on undisturbed horizontal specimens at axial strain rates of 16.0 %/h, 1.60 %/h and 0.16 %/h, respectively.

Anisotropic permeability
Two falling head permeability tests on undisturbed vertical and horizontal specimens with filtered bog water and distilled water, respectively.

Table 3.2: Experimental work plan.

The reasons for using undrained triaxial tests to investigate the rate-dependence of peat material are (1) the boundary conditions of triaxial tests are controllable; (2) undrained rate-dependence of peat results from the readjustment of pore water and the macro- and micro-porous structure, which is an intrinsic property of undrained effective stress constitutive behaviour. Quantifying and modelling the rate-dependence of fluid-filled porous media in peat is complicated by the superposition of several non-linear effects. For example, the rate-dependence of such materials is due to both flow-dependent features caused
by the momentum interaction between the solid matrix and the mobile pore fluid, and flow independent phenomena associated with dissipative mechanisms in the solid matrix itself. An independent and physically based quantification of both contributions is practically only possible if first the flow-dependent features are removed from the picture by performing isochoric (constant volume) tests (fluid flow is primarily caused by volume changes of the pore space in a hydro-mechanical setting). Based on the resulting quantification of solid matrix rate dependence, non-isochoric tests such as consolidated drained triaxial tests can then be performed and modelled with a biphasic theory based on the effective stress concept to account for flow-dependent phenomena.

3.2 Characteristics of tested peat sample

The undisturbed peat blocks were collected from Clara bog, County Offaly in Ireland. The physical properties of the tested peat sample have been reported in O’Kelly and Zhang (2013) and Zhang and O’Kelly (2014b). For the integrity of the research, some selected properties of the peat sample are re-stated in this section. The location of the collected peat sample is shown in Figure 3.2.

Peat blocks were extracted from a freshly cut face bank under water at about 2.5 m below the ground surface. The retrieved peat blocks were sealed in sampling boxes and stored in laboratory environment. Some selected properties of the peat sample are listed in Table 3.3. The tested peat material was taken as fully saturated for being retrieved under ground water level. Practically it is unlikely for a natural peat material to be fully saturated due to the existence of occluded air bubbles generated by bio-degradation (Hobbs, 1986). In some of the undrained triaxial tests, the Skempton B value, around 0.95, was obtained at the end of tests for the undisturbed peat specimens. The degree of saturation of each specimen was calculated from the specimen initial dimension, mass, water content and particle density, of which the value ranges from 92 % to 96 %. In general, the undisturbed peat specimens tested in the undrained triaxial tests without a consolidation stage are not fully saturated. The peat material is classified as SCN-H4-B3-F3(S)-R1(N)-W1(PN) based on the extended von Post peat classification (Hobbs, 1986).

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>water content, %</td>
<td>689</td>
</tr>
<tr>
<td>Particle density, Mg/m³</td>
<td>1.42</td>
</tr>
<tr>
<td>Loss in dry mass on ignition, %</td>
<td>98.6</td>
</tr>
<tr>
<td>Fibre content retained on 63 μm sieve, %</td>
<td>74.2</td>
</tr>
<tr>
<td>Fibre content retained on 150 μm sieve, %</td>
<td>63.5</td>
</tr>
<tr>
<td>pH</td>
<td>3.8</td>
</tr>
</tbody>
</table>

Table 3.3: Some properties of Clara bog peat (O’Kelly and Zhang, 2013; Zhang and O’Kelly, 2014b).
3.3 Testing apparatus

The standard triaxial testing apparatus was slightly modified to accommodate the unloading and relaxation tests so that the self-weight of the load measurement (load cell and proving ring) does not lead to over unloading and stress relaxation. The load measurement of the triaxial set-up was tied to the triaxial frame at the start of the compression tests so that it remained at its original position to measure the force against the specimen during the unloading and relaxation stages.

Two types of triaxial set-ups were used in this study. Triaxial1 consists of load cell, electronic linear variable differential transformer (LVDT) with a resolution of 0.01 mm/mV, pore water pressure transducer (500 kPa with a resolution of 0.01 kPa), cell pressure and back pressure Global Digital Systems (GDS) hydraulic controlling systems (resolution of 1 kPa) and temperature recorders (resolution of 0.01 °C) to record the temperature both in the surrounding atmosphere and on the triaxial cell wall. All tests in Triaxial1 have been carried out with computer controlled GDS software. Triaxial2 consists of a load proving ring (resolution of 0.43 N/mV), electronic LVDT (resolution of 0.005 mm/mV), pore water pressure transducer (500 kPa, resolution of 0.01 kPa), and temperature recorder. All the measurement devices were calibrated before tests.

The two types of load measurements are illustrated in Figure 3.3. In the standard triaxial set-up, the load cell is free to move in the vertical direction. Due to its large self-weight relative to the estimated peat specimen strength, the load cell with freedom in the vertical direction impairs the unloading and relaxation results of a peat specimen. The advantage of a load cell is that the deformation required to measure the changing load sustained by peat is negligible. Whereas, the proving ring deforms to measure the changing load, it provides unreal force values as it pushes the specimen downward during the unloading and relaxation stages. This can be mitigated by choosing the proving ring with proper thickness to ensure balance between the measurement resolution and the ring deformation.
this study, a 5 kN proving ring was chosen based on a few trial tests to eliminate the proving ring effect.

### 3.4 Undrained triaxial loading-unloading tests on vertical specimens

The loading-unloading \((lu)\) behaviour of undisturbed vertical peat specimens has been studied in undrained triaxial tests at axial strain rates of 16.0 \%/h, 1.6 \%/h and 0.16 \%/h, respectively, reaching axial strains around 5 \%, 10 \%, 15 \% and 20 \%. The elastic range of undisturbed peat specimens can be obtained via the loading-unloading-reloading \((lur)\) cycles in undrained triaxial test conditions. The main purposes of \(lu\) tests were to obtain:

- Strain recovery and the nonlinearity of stress-strain relationship;
- Strain rate effect on the peat constitutive relation;
- Strain history effect on strain recovery.

Additionally, the cell pressure effect on the stress-strain behaviour within 5 \% axial strain in undrained triaxial \(lu\) tests was investigated.

A typical deviator stress-axial strain \(lu\) relationship of the undrained triaxial \(lu\) tests on an undisturbed vertical peat specimen at an axial strain rate of 16.0 \%/h is presented in Figure 3.4. There are negative deviator stress values at the end of the unloading. This could have resulted from the extra unloading from the expulsion of the triaxial cell water (including the changes of hydrostatic pressure on the specimen and buoyancy on the load measurement) as well as temperature fluctuations. In the triaxial tests, the triaxial cell was filled with water and freely connected to the atmosphere in order to be comparable.
with tests carried out with higher cell pressures and to create a zero cell pressure, respectively. When the specimen was compressed, the load measurement was pushed into the cell and thus expelled the cell water through the atmosphere connection. The expelled water was not refilled during the unloading stage, leaving a smaller hydrostatic pressure on the specimen. Also, the buoyancy on the load measurement increased during loading and decreased during unloading due to its changing volume submerged in the cell water. For the Triaxial1 set-up with a load cell, the absolute value of the negative deviator stresses are smaller than those of Triaxial2 with a proving ring as the proving ring deforms while measuring load, with 0.0472 mm/kPa for the 38 mm initial diameter specimen.

From Figure 3.4, it can be seen that the strain recovery range in undrained tests should not be determined based on the shape of the compression curves. The strain recovered from the unloading tests is considered as elastic strain. Two approaches to determine the elastic strain range, i.e. from the lu test stress-strain curve as well as from the lur tests. For the presented lur test, Table 3.4 lists the values of the strains, from which the discrepancies between the elastic strains obtained from the strain at the detaching point between load measurement and the specimen during unloading and from the strain at the start of reloading were not significant. The estimated elastic strains from unloading and the calculated elastic strains from reloading tests of the vertical undisturbed peat specimens with available unloading and reloading data (including tests at different strain rates as well as tests on specimens of different initial void ratios) are plotted in Figure 3.5. It can be concluded that the estimated elastic strain from the unloading curve provided slightly smaller but a fair estimation for the actual elastic recovery strain. From Figure 3.5, it can also be seen that the recovered strain from the total strain up to 20 % in undrained triaxial tests was large, for about 14 % axial strain recovered from a 20 % total strain, which deviated from the values reported in the literature which was commonly taken as 5 % (Hendry, 2011) judged by the shape of stress-strain curve. Since all the tests were carried out with the computer controlled system, the difference between the designed axial strain and the actual strain was due to the initial correction made for the contact between the load cell and the tested specimen.

3.4.1 **Cell pressure effect on loading-unloading tests**

The cell pressure effect was investigated by undrained lu tests up to a 5 % axial strain with cell pressures of 0 kPa, 20 kPa, 40 kPa, 120 kPa, respectively, on an undisturbed vertical peat specimen at the axial strain rate of 16.0 %/h. In order to eliminate the specimen variation in the test, the same specimen was used, where 5 % axial strain was adopted to keep the strain-history effect on the constitutive relationships within a small range. The test results presented in Figure 3.6 demonstrates that the cell pressure has negligible effect on the strain recovery as well as on the deviator stress-axial strain behaviour of the lu tests.
Figure 3.4: Deviator stress-axial strain relationship of an undisturbed vertical peat specimen tested at $\dot{\varepsilon}_a = -16.0\%/h$ in undrained triaxial $lu$ tests.

<table>
<thead>
<tr>
<th>Designed axial strain (%)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial strain (%)</td>
<td>4.63</td>
<td>9.61</td>
<td>13.90</td>
<td>18.71</td>
</tr>
<tr>
<td>Residual strain from reloading (%)</td>
<td>0.439</td>
<td>0.994</td>
<td>1.167</td>
<td>-</td>
</tr>
<tr>
<td>Elastic strain from reloading (%)</td>
<td>4.57</td>
<td>9.06</td>
<td>13.17</td>
<td>-</td>
</tr>
<tr>
<td>Strain at 0 kPa during unloading (%)</td>
<td>0.62</td>
<td>1.70</td>
<td>1.86</td>
<td>3.15</td>
</tr>
<tr>
<td>Elastic strain estimated from unloading (%)</td>
<td>4.38</td>
<td>8.35</td>
<td>13.03</td>
<td>16.73</td>
</tr>
</tbody>
</table>

Table 3.4: Strains in $lu$ tests.

Figure 3.5: Summary of the estimated and measured elastic strains of the $lu$ tests.

within an axial strain range of 5%. The slight difference of the residual strains from the elastic recovery was due to the different maximum axial strains that have been reached. A zero confining pressure was taken for all the following undrained triaxial tests that will be
elaborated in the next sections.

![Figure 3.6: Undrained triaxial \(lu\) tests of an undisturbed vertical peat specimen with cell pressures of 0 kPa, 20 kPa, 40 kPa and 120 kPa at an axial strain rate of 16.0 %/h.](image)

### 3.4.2 Strain rate effect on strain recovery in loading-unloading tests

The axial strain rate effect on the deviator stress-axial strain relationship of undisturbed vertical peat specimens in the undrained triaxial \(lu\) tests is presented in Figure 3.7. The deviator stress dropped more than 5 kPa when the axial strain rate was ten-fold smaller for the tested undisturbed peat specimens reaching an axial strain of 5 %. It is evident that the constitutive behaviour of peat is rate-dependent which applies to all soils. The 0.16 %/h test has a relaxation between the loading and unloading stages and the early part was influenced by temperature fluctuation, which will be explained further in Section 3.5. Comparing the loading stages of the three strain rate tests, a higher strain rate generates a higher stiffness.

In order to corroborate the strain rate effect on the elastic strain recovery of undisturbed vertical peat specimens in undrained triaxial \(lu\) tests, the estimated elastic strains calculated based on unloading curves were plotted in Figure 3.8. The 1.60 %/h test specimen presented in Figure 3.7 was included in Figure 3.8. The deviation of the elastic strains was due to specimen variation as well as asymmetrical loading shown in Figure 3.9. The elastic strain recoveries for the three strain rates provided a consistent relationship between the total axial strains and the estimated elastic strains. The elastic strain scatters at a total strain of 20 % due to specimen variations as well as possible asymmetrical deformations. Considering specimen variations, the testing strain rate effect on the strain recovery was insignificant.

In the unloading tests of undisturbed peat specimens, it was observed that the elastic strain recovery was more delayed in the tests at a higher unloading strain rate. As the axial strain was unloaded to a zero value, it provided time for the delayed elasticity to recover.
Figure 3.7: Strain rate effect on the deviator stress-axial strain relationship in the undrained \textit{lu} tests of the undisturbed vertical peat specimens.

Figure 3.8: Strain rate effect on the elastic recovery in the undrained \textit{lu} tests of the undisturbed vertical peat specimens.

Therefore, the retardation in elastic recovery at small strain levels (anelasticity, cf. Zener (1948)) was not, and should not be, taken into account in the above reported data.

3.5 \textbf{Undrained triaxial relaxation tests}

The thermodynamic equilibrium state of the peat materials was investigated in the undrained triaxial relaxation tests. Relaxation tests were carried out by compressing the peat specimen to an axial strain level at a pre-set axial strain rate and recording the stress dissipation at the constant axial strain and, ideally, constant temperature. The unconsolidated undrained triaxial tests were used to test the stress relaxations in order to get rid of the non-uniformity induced by consolidation, where Alexandre and Martins (2012) claimed that after the "end" of primary consolidation when the drainage is closed, there will be a re-distribution of void ratio across the sample, where smaller void ratios are expected near
the drainage frontier and greater void ratios are expected at the "undrained" axis in the case of the cylindrical specimens. The purposes of the undrained relaxation tests were as follows:

- To prove and quantify the overstress in peat compression and unloading;
- To find or define the thermodynamical equilibrium state of peat in undrained \( lu \) tests.

According to Perzyna (1966), the "distance" between the current stress and that on the boundary of a "quasi-static" elastic domain is termed as \textit{overstress} for viscoelastic materials. The triaxial test set-up with the load cell (Triaxial1) for the relaxation tests was the same as for the \( lu \) tests, i.e. the freedom of the load cell in the vertical direction was constrained so that it would not drop during the relaxation test. Caution should be taken in choosing the proper proving ring for load measurement in the second triaxial set-up (Triaxial2) for relaxation tests. Figure 3.10 presents relaxation test results with an over-sensitive proving ring, from which the axial strain was still increasing during relaxation due to the shape expansion of the proving ring. Lade and Karimpour (2015) investigated the effect of load cell expansion on the relaxation tests of sand specimens and proposed a correction for the load cell expansion. However, the load cell expansion introduces experimental errors in relaxation tests which should be avoided before the start of the tests.

In this study, a 5 kN proving ring was chosen based on a few trial tests to eliminate the proving ring effect. The expansion ratio (ratio between change in vertical expansion and change in load measured) of the 5 kN proving ring is 0.0272 mm/N which is equivalent to 0.472 mm proving ring expansion for a 10 kPa stress relaxation for the standard triaxial specimen. The testing programme of relaxation was as follows:
1. Axial strains were maintained at the end of the loading stage in lu tests at different strain rates, followed by complete unloadings. The loading-relaxation-unloading tests were to testify the relaxation effect on the elastic strain recoveries.

2. A preliminary study was carried out to investigate the stress relaxation regarding to loading strain rates. Undrained triaxial relaxation tests were carried out with two axial strain rates, namely 160 %/h and 16.0 %/h, with zero cell pressure and pore water pressure measurement. The 160 %/h test was allowed a 10 hour relaxation period after reaching an axial strain of 5 % but the relaxation termination was not reached. Therefore the 16.0 %/h test was allowed a 24 hour relaxation at an axial strain of 5 %.

3. Three undrained triaxial lu tests with step relaxations were carried out to investigate the stress dissipation with respect to loading strain rates and the duration of relaxations.

4. In order to obtain the thermodynamic equilibrium state of undrained peat compression, relaxation tests were carried out at the strain rates of 160 %/h (24 hours), 16.0 %/h (1 week) and 0.16 %/h (2 days). The temperature effect was recorded during the stress relaxations.

3.5.1 Relaxation effect on elastic recovery in loading-unloading tests

Rate-dependence and relaxation were the two main tests for the time-dependent properties of the peat material. From the data reported in Section 3.4.2, the testing strain rate had an insignificant influence on the strain recovery of the undisturbed vertical peat specimens in the undrained triaxial lu tests. It is reasonable to predict that relaxation also has negligible impact on strain recovery during unloading since the rate-dependent and relaxation behaviours are rooted in the same viscous mechanism of peat geomechanical properties. Strain recovery data in Figure 3.8 is further plotted with the elastic strains obtained from
the loading-relaxation-unloading (lru) tests in Figure 3.11. As expected, the elastic strains from the lru tests merge into the same total strain-elastic strain relationship.

![Graph showing relaxation effect on the elastic recovery in the undrained lru tests of the undisturbed vertical peat specimens.](image)

Figure 3.11: Relaxation effect on the elastic recovery in the undrained lru tests of the undisturbed vertical peat specimens.

### 3.5.2 Relationship between stress relaxation and logarithm of time

The relationship between stress relaxation and time was preliminarily studied by undrained triaxial compression tests to an axial strain of 5% at strain rates of 160%/h followed by a 10 hour (600 min) relaxation and 16.0%/h followed by a 24 hour (1440 min) relaxation. Pore water pressure was measured during the tests. The deviator stresses are plotted against logarithm of time in Figure 3.12. The pore water pressure increase was low during undrained compression as the specimens were not fully saturated, where the Skempton B was around 0.95 after the test. The pore water pressure variations during relaxation were very small, the noise recorded was rooted in the 1 kPa accuracy of the pore pressure transducer. The specimen tested at 160%/h had a more obvious pore water pressure increase during the compression than the specimen tested at 16.0%/h. The tenfold increase of strain rate resulted in a deviator stress increase of 20.1% to reach an axial strain of 5%. A linear relationship can be found between deviator stress and logarithm of time after an initial time period, which is consistent with undrained relaxation tests on clays (Murayama and Shibata, 1961; Hicher, 1988) and on sand (Lade and Karimpour, 2015). Relaxation did not seem to cease after the 10 hour and 24 hour periods of time for the two tests from Figure 3.12. Further study on the stress relaxation ratio was carried out with triaxial undrained lu tests with step relaxations.

### 3.5.3 Stress relaxation ratio

Undrained triaxial lu tests with step relaxations were used to investigate the stress relaxation ratio (defined as the ratio between the relaxed stress and the initial stress) with re-
gards to the testing axial strain rates as well as the time period of relaxations. Three undrained triaxial $lu$ tests with step relaxations were carried out, two of which were tested at an axial strain rate of 160 $\%$/h with relaxation durations for 160 $\%$/h and 20 min, respectively. A third test was carried out at an axial strain rate of 16.0 $\%$/h with relaxation time for 16 min.

The deviator stress-axial strain relations are plotted in Figure 3.13. At the very start of the compressions, the three compression curves had similar initial stiffness. Higher deviator stress was achieved with the higher strain rate at the same level of axial strain. The deviator stress relaxations during loading stage decayed by 14.6 $\%$ ± 0.5 $\%$, 8.9 $\%$ ± 0.9 $\%$ and 18.7 $\%$± 0.1$\%$ of their original values for 160 $\%$/h 16 min, 16.0 $\%$/h 16 min and 160 $\%$/h 20 min tests, respectively. The deviator stress relaxation ratio (defined in Equation (3.1), where $q$ is the deviator stress after relaxation; $q_0$ is the deviator stress before relaxation; max function uses the larger value of $q$ and $q_0$) is presented in Figure 3.14.

$$R\% = \frac{q_0 - q}{\max(q_0, q)} \times 100\% \quad (3.1)$$

It can be seen that the relaxation ratio of deviator stress during the loading stage is nearly constant for the same duration whereas the relaxation ratio during unloading stage approaches 100 $\%$ due to the zero value of $q$. In order to better interpret the relaxations during unloading, the virtual elastic strain calculated using the stress relaxation increase during unloading relaxations divided by the tangent modulus of the unloading stress-strain curve is plotted in Figure 3.15. The virtual elastic strain indicates the virtual strain recovery from the overstress estimated by a tangent elastic modulus. A linear relationship can be found between the virtual elastic strain and the total axial strain during the unloading stage. This is consistent with the linear relationship between the recovered elastic strain and total strain in the undrained triaxial $lu$ tests in Figure 3.11, which indicates the mechanism of undrained relaxation is viscoelastic. The unloading relaxation proves that the strain recovery during unloading is a hybrid behaviour of unloading and relaxation.
The above analysis of the stress relaxation ratio presents the proof of the existence of overstress in the undrained peat compression and unloading. The "quasi-static" state is
to be investigated for the rate-independent constitutive relation by leaving the relaxation for sufficiently long time period until the dissipation of the overstress terminated.

### 3.5.4 Temperature effect on stress relaxation

To obtain the constitutive relationship for a rate-dependent material with equilibrium hysteresis, "quasi-static" state stress-strain relationship forms the basis for the viscoplastic material model. Haupt (2000) claimed that the "quasi-static" state is defined as an *equilibrium state* which can be asymptotically approximated by the termination points of relaxations. Undrained triaxial relaxation tests were carried out with up to a one week relaxation period.

From Figure 3.12, the stress relaxation appears to have a constant rate subject to the logarithm of time, i.e. the stress dissipation during relaxation was exponentially decreasing with time. The stress relaxations would cease when the overstress fully dissipated and the pore water as well as the solid porous structure reached an equilibrium state as an entire system. It was found that triaxial relaxation tests were very sensitive to the surrounding temperature in the stress range of interest in peat. Figure 3.16 shows a noticeable impact on the relaxation behaviour for 1700 min by the surrounding temperature in a preliminary test. The temperature controlling system in the laboratory testing condition provided a testing environment with a temperature fluctuation of ±3 °C, which introduced a refractory difficulty in the determination of the end of stress relaxations. Due to temperature fluctuations, it was difficult to judge when the stress relaxation ceased from Figure 3.16.

![Figure 3.16: Temperature effect on relaxation behaviour.](image)

Further test with one week relaxation was carried out for the interest of the stress relaxation trend in a relatively longer term for a compression at 16.0 %/h. The deviator stress relaxation was plotted with time along with temperature recorded in Figure 3.17, it can be seen that the temperature fluctuations impaired the determination of the end of relaxation by 2.1 kPa/°C and the average rate of stress relaxation after 6000 min was $-5.2 \times 10^{-6}$ kPa/min.
which could be taken as the end of relaxation as the stress dissipation at this rate is much smaller than the temperature influence. Since the stress relaxation was sensitive to the temperature fluctuation, it was difficult to determine the duration to attain an acceptable stress relaxation rate for the termination of relaxation. An alternative approach for obtaining the equilibrium state of the rate-dependent material was preferable.

\[ \text{Deviator stress /kPa} \]
\[ \text{Temperature /\degree C} \]

\[ 0 \quad 2000 \quad 4000 \quad 6000 \quad 8000 \quad 10000 \]
\[ 0 \quad 5 \quad 10 \quad 15 \]

**Figure 3.17:** Relaxation test with one week duration with normal time scale.

### 3.5.5 Thermodynamic equilibrium state

Undrained triaxial compression tests on an undisturbed peat specimen at an axial strain rate of 16.0 %/h to axial strains of 2 %, 4 % and 6 % with relaxations for 24 hours were carried out to determine the position of the thermodynamic equilibrium compression curve. Figure 3.18 presents the stress relaxation curves plotted against normal time scale for the three axial strains. The deviator stress increases were due to the temperature rise in daily cycles. By defining the non-decreasing deviator stress as the termination of stress relaxation, an equilibrium curve was plotted by connecting the relaxation termination points in Figure 3.19 as defined by Haupt (2000). It is noteworthy that the equilibrium curve obtained from relaxation tests was not efficient in the case of peat materials as the viscoelastic property constitutes the bulk amount of its total mechanical behaviour. More relaxation steps were required to obtain a smooth thermodynamic equilibrium curve, contradicting the time efficiency of implementing relaxation tests. The test result illustrated in Figure 3.19 provided a conception of the equilibrium curve of the undrained triaxial peat compression in the strain range of 0—6%. Alternatively, sufficient slow tests were carried out for scrutinizing equilibrium states.

According to Haupt (2000), a slower test could be found in the close vicinity of the termination points of relaxations of a quicker test. An undrained triaxial compression test at an axial strain rate of 4.81 %/h was compared with the undrained triaxial compression test at an axial strain rate of 160 %/h with 16 min relaxation. Figure 3.20 provides experimental
Figure 3.18: Deviator stress relaxations of undrained triaxial compressions to 2 %, 4 % and 6 % axial strains.

Figure 3.19: Undrained triaxial compression test with relaxations: equilibrium curve.

proof of the Haupt (2000)’s statement in peat material. From the stress rise recorded at the end of the unloading of the 4.81 %/h test, it was observed that the testing strain rate was not sufficiently slow to attain the equilibrium state.

Undrained triaxial lrut tests were carried out at an axial strain rate of 0.16 %/h on undisturbed peat specimen to axial strains of 5 %, 10 % and 20 %. The deviator stress-axial strain of the 5 % test was presented in Figure 3.21. The surrounding temperature variation instigated the deviator stress changes, illustrated in Figure 3.22. Following the compression to an axial strain of 5 %, the axial strain was retained for further relaxation for approximate two days (2970 min). The stress relaxation is presented in Figure 3.23. The initial deviator stress drop during the relaxation started after 100 min and ceased before 1000 min. The average peak deviator stress dissipation rate (slope of the steepest stress drop in Figure 3.23) was 0.0038 kPa/min. Comparing with the temperature effect on the deviator stress measured, which was 2.1 kPa/°C, the amount of stress relaxation was at the same magnitude as
the stress variation instigated by an average temperature change of 1.0 °C during a 9 hour relaxation period. Therefore, the undrained triaxial compression test at the axial strain rate of 0.16 %/h can be taken as the equilibrium state test. It is also of interest to compare the defined equilibrium state test at an axial strain rate of 0.16 %/h with the equilibrium curve by connecting the relaxation termination points. Figure 3.24 demonstrates the agreement of the two equilibrium curves, which provided further proof of the rationality of using the 0.16 %/h test as the equilibrium state test.

The aim of defining the equilibrium state test is to facilitate categorising the tested peat material as well as choosing specific material models. The conventional way of proposing a constitutive model for peat was rather subjective, for instance to interpret as plasticity any deviation of the stress–strain curve from linearity. The rate-dependence and relaxation tests indicated that peat is a rate-dependent material. Thus by defining the equilibrium state test, the equilibrium hysteresis then can be investigated. Figure 3.25 demonstrates
Figure 3.22: Deviator stress oscillation due to temperature fluctuation.

Figure 3.23: Deviator stress relaxation of the 0.16 %/h test.

Figure 3.24: Comparison of the 0.16 %/h test with equilibrium curve from the relaxation tests.
an equilibrium hysteresis of the constitutive behaviour of the tested peat, thus indicating a plastic component in the constitutive model.

![Figure 3.25: Undrained triaxial lu test at axial strain rate of 0.16 %/h (equilibrium state).](image)

### 3.6 Strain history effects

Three undisturbed vertical peat specimens were tested at an axial strain rate of 16.0 %/h. The first specimen experienced the strain cycles of 5 %, 10 %, 15 % and 20 %; whereas the second specimen experienced the strain cycles of 10 %, 15 % and 20 %; and the third specimen experienced strain cycles of 15 % and 20 %. Figure 3.26 presents the overlapping of the axial strain-deviator stress curves for the three specimens to an axial strain of approximately 20 %. Figure 3.26 shows a strain history effect on the deviator stress-axial strain relationship. The tangent stiffness becomes higher with a specimen with more strain history.

![Figure 3.26: Overlapping of the deviator stress-axial strain relationships of the 16.0%/h test of three undisturbed vertical specimens with different strain histories.](image)
In order to further explore the strain history effect on the elastic strain recovery of the undrained triaxial \textit{lu} tests, large consolidometer tests were used for compressing and consolidating the undisturbed peat columns with a dimension of 150 mm in diameter and 140 mm in height. Compressed peat samples to a smaller void ratio were trimmed for the undrained triaxial tests. The compression/consolidation incited an average water content drop of 140\%, equivalent to a decrease of 2.0 in void ratio on condition of full saturation, where the void ratio changed from 10.4 to 8.4. Undrained triaxial \textit{lu} tests with the same procedures have been carried out with the compressed peat samples (labelled with $e_1$ in figure illustrations). By combining the elastic recovery strains of the $e_1$ specimens with Figure 3.11, Figure 3.27 displays that the same peat material with lower void ratio possessed smaller strain recoveries from unloading. Comparing the two equilibrium state tests of the undisturbed peat specimen (labelled as $e_0$) with the compressed peat specimen to an axial strain of 5\% in Figure 3.28, the $e_1$ specimen had a higher stiffness as expected with lower water content and a slightly smaller strain recovery (also indicated in Figure 3.27).

![Figure 3.27: Strain history effect on the elastic recovery of the undrained triaxial compression tests.](image)

**3.7 Undrained triaxial creep tests**

Two undrained triaxial creep tests were carried out additionally for calibrating the time-dependent properties of the undisturbed peat and to obtain the equilibrium state. The test set-up was realised by using the light-weight triaxial cells with a plunger of known mass. The calculated weights for the determined stress were directly applied at the top of the plunger. Similar to the triaxial compression tests, the triaxial cell for creep tests was filled with water and connected freely to the atmosphere, thus providing a zero confining pressure. The room temperature for the creep tests was $20 \pm 3 ^\circ C$. As the friction between
the plunger and the triaxial cell was not measurable and hence unknown, the actual load applied to the specimen in the creep test should be smaller than the calculated value.

The dimensions of the undisturbed peat specimens tested in the undrained triaxial creep tests (height of 76 mm and diameter of 38 mm) were the same as those of the specimens tested in the undrained triaxial relaxation tests. The first specimen was allowed to undrained creep under 10 kPa for 25 days followed by an undrained creep under 20 kPa for 96 days, where the stress was calculated based on the initial cross-sectional area of the specimen. As the cross-sectional area at both ends of the specimen was constrained by the top cap and the base, area change at both ends was negligible. The second creep test was directly applied with a calculated 20 kPa and allowed to creep for 80 days. The stress applied to the undrained specimen was decreasing during the creep process.

The creep result of the first specimen is presented in Figure 3.29, of which the initial 30 min compression was not recorded for the 20 kPa creep test. From the figure, the axial creep strain has a nearly linear relationship with logarithm of time until approximately 10000 min (7 days) when it reached equilibrium and ceased creeping. For the 20 kPa load increment, the creep strain presented an S shape curve on the semi-log plot, and reached equilibrium in approximately 75000 min (52 days) after the start of the test.

The creep strain of the second undrained triaxial creep test is presented in Figure 3.30, which also shows a nearly linear creep curve on a semi-log plot. Different from the first creep tests, the creep curve plotted on the semi-log scale of the second test did not demonstrate a termination of the creep, indicating that no equilibrium state was reached. The increase of the creep curve slope plotted on the semi-log scale could be the tertiary compression which is commonly reported in one-dimensional oedometer creep tests of peat materials. Comparing to the undrained triaxial relaxation tests, it took a lot longer to arrive at the equilibrium state in the undrained triaxial creep tests even with a smaller applied
Figure 3.29: Undrained triaxial creep test on undisturbed peat specimen at two load increments of 10 kPa and 20 kPa.

stress.

Figure 3.30: Undrained triaxial creep test on undisturbed peat specimen at 20 kPa for 80 days.

3.8 Tests on structural anisotropy

The formation process destines the structural cross-anisotropy of peat materials. The degree of the structural anisotropy is dependent on the degree of humification (Hendry et al., 2012). In this study, the sphagnum peat was classified as lightly degraded with a degree of humification of H4 on von Post scale. The laminated layers of the peat sample can be discernible by naked eyes (Figure 3.31). In testing on the structural cross-anisotropy, undrained triaxial lu tests were carried out on horizontal specimens (with vertical fibre alignment) with the same testing conditions on vertical specimens (with horizontal fibre alignment).
3.8.1 Anisotropic permeability

The falling head permeability test was carried out following the test procedure in Head (1994). A permeameter cell with dimensions of 100 mm in diameter and 98 mm in length was used for the falling head permeability test. The undisturbed peat block was trimmed into the permeability mould using a sharp knife. Undisturbed peat block was tested in both vertical (horizontal fibres) and horizontal (vertical fibres) directions. The permeameter cell set-up is presented in Figure 3.32. Fresh bog water (pH = 3.8) as well as distilled water were used in the permeability tests. The permeability test with the falling head standpipe of an inner diameter of 19 mm was firstly carried out with filtered bog water (through 63 \( \mu m \) sieve) to determine the coefficient of permeability to prevent the potential physical or chemical influence of the water flow on the natural peat (Huat et al., 2014). Subsequently, distilled water was used to test the permeability of the same sample to check the test result difference between these two standpipe liquids. The coefficient of permeability, \( k \), is calculated using Equation (3.2), where \( L \) (mm) is the length of sample, \( A \) (mm\(^2\)) is cross-section area of the specimen, \( a \) (mm\(^2\)) is the cross-section area of standpipe tube, \( h_1 \) (mm) and \( h_2 \) (mm) are the standpipe water heads above the water level of the tested sample; \( t \) (min) is the time period for water head drop from \( h_1 \) to \( h_2 \). The testing temperature was 19 °C.

\[
k = 3.84 \frac{aL}{At} \log_{10} \left( \frac{h_1}{h_2} \right) \times 10^{-5} \text{ m/s (3.2)}
\]

For the undisturbed vertical specimen, four tests were carried out with filtered bog water and 2 with distilled water. For the undisturbed horizontal specimen, four tests were carried out with filtered bog water and four with distilled water. The average coefficient of permeability is presented in Table 3.5. The anisotropy of permeability of undisturbed peat was not significant, i.e. the coefficient of permeability of horizontal specimen was marginally lower than that of the vertical specimen where both were at the same order of magnitude. The test results obtained using filtered bog water gave a higher measured value for the falling head coefficient of permeability than using distilled water, which is
Table 3.5: Coefficient of permeability of undisturbed vertical and horizontal specimens.

<table>
<thead>
<tr>
<th>Specimen tested</th>
<th>Void ratio</th>
<th>Coefficient of permeability (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Filtered bog water</td>
</tr>
<tr>
<td>Undisturbed vertical specimen</td>
<td>11.2</td>
<td>$3.82 \times 10^{-6}$</td>
</tr>
<tr>
<td>Undisturbed horizontal specimen</td>
<td>11.0</td>
<td>$2.85 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

consistent with the relationship between the coefficient of permeability and the pH of the pore water of tropical peaty soils reported by Asadi et al. (2011). However the small difference between the test results indicated distilled water could be used to obtain acceptable test results. The finding of a slightly higher vertical permeability than the horizontal permeability by the falling head tests is consistent with the permeabilities calculated from the 1D oedometer tests reported in Zhang (2013) and the constant head permeability test carried out on a slightly biodegraded peat material by O’Kelly (2007).

### 3.8.2 Undrained triaxial tests on horizontal peat specimens

Undrained triaxial $lu$ tests were carried out at axial strain rates of 16.0 %/h, 1.60 %/h and 0.16 %/h on undisturbed horizontal specimens. The test conditions were the same as those for testing vertical specimens. The horizontal specimens possessed fibres in the vertical direction.

To compare the undisturbed horizontal specimen with the undisturbed vertical specimen tested at 16.0 %/h reaching axial strains of 5 %, 10 %, 15 % and 20 %, the same tests were carried out on the undisturbed horizontal specimen. Analogous to Figure 3.4, the test results for the horizontal specimen are presented in Figure 3.33. For the undrained triaxial $lu$ tests, the horizontal specimen reached higher deviator stresses at 5 % (by ca. 30 kPa -
20 kPa) and 10 % axial strains (by ca. 39 kPa - 30 kPa), and the stress-strain relationship showed a pronounced strain softening after 10 % axial strain. The estimated elastic strains based on the unloading curves are listed in Table 3.6. Comparing to the values in Table 3.4, the elastic strain recovered from the horizontal specimen was smaller than from the vertical specimen. It is interesting to note that the elastic strain recoveries after axial strains of 15 % and 20 % have similar values, which indicates the axial strain within the 15 % and 20 % was basically inelastic. To better illustrate the strain softening of the horizontal specimen failure, the test at strain rate of 1.60 %/h is presented in Figure 3.34 with the specimen failure mode illustrated in Figure 3.35.

![Deviant stress-axial strain relationship of an undisturbed horizontal peat specimen tested at an axial strain rate of 0.16 %/h.](image)

Figure 3.33: Deviator stress-axial strain of an undisturbed horizontal peat specimen tested at an axial strain rate of 16.0 %/h.

<table>
<thead>
<tr>
<th>Axial strain (%)</th>
<th>5.06</th>
<th>11.58</th>
<th>19.80</th>
<th>28.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strain at 0 kPa during unloading (%)</td>
<td>1.86</td>
<td>6.22</td>
<td>9.51</td>
<td>19.64</td>
</tr>
<tr>
<td>Elastic strain estimated from unloading (%)</td>
<td>3.2</td>
<td>5.36</td>
<td>10.29</td>
<td>9.31</td>
</tr>
</tbody>
</table>

Table 3.6: Strains in the undrained triaxial test of a horizontal specimen at an axial strain rate of 16.0 %/h.

Using the definition of the equilibrium state for horizontal specimen tests, the deviator stress-axial strain relationship tested at 0.16 %/h is presented in Figure 3.36. It also shows a strain softening from the axial strain of about 9 % with an equilibrium hysteresis. The difference of the stress-strain relationships between the vertical and horizontal specimens derives in the fibre orientations, where in vertical specimens the fibres sustained tension and in horizontal specimens the fibres sustained compression. The "failure" modes of the tested vertical and horizontal specimens are illustrated in Figure 3.37. The vertical specimen kept nearly constant cross-section area after the triaxial test while through-cracks took place in the horizontal specimen.
3.9 Summary

The experimental work investigated the elastic and inelastic behaviours including plasticity, rate-dependence, relaxation as well as creep, and structural anisotropy of the undisturbed peat in undrained triaxial test conditions. The following conclusions can be made:

1. Large strain recovery was experienced in undrained testing of the undisturbed vertical peat specimens, where around 15% out of 20% total strain was reversible with a notable scatter due to the specimen variations as well as the loading conditions. The estimated elastic strain from the unloading curve provided slightly smaller but a fair estimation of the actual elastic strain from the \textit{lur} tests;

2. The cell pressure had negligible effect on the elastic recovery and the deviator stress-axial strain relationship of the undrained triaxial \textit{lu} tests on vertical undisturbed peat
Figure 3.36: Deviator stress-axial strain of an undisturbed horizontal peat specimen tested at an axial strain rate of 0.16 %/h.

Figure 3.37: Specimens after undrained triaxial $lu$ tests.

3. The constitutive behaviour of peat was highly rate-dependent, with higher deviator stress reached at a higher axial strain rate for the same level of axial strain. The elastic strain recovery was independent of the axial strain testing rates as well as the relaxations for the vertical specimens;

4. A linear relationship was found between the deviator stress and the logarithm of time after an initial time period for the vertical specimen relaxation tests. The pore water pressure variations were very small during the stress relaxation;

5. The existence of overstress was proved by the nearly constant deviator stress relaxation ratio during the loading stage of vertical undisturbed peat specimens for the same relaxation duration. A linear relationship was found between the virtual elastic strain and the total strain during the unloading stage of the $lu$ tests with step relaxations of vertical undisturbed peat;
6. It was difficult to determine the duration of the relaxation test corresponding to an acceptably low stress relaxation rate in order to terminate the relaxation tests in the present investigation since the relaxation tests on vertical peat specimens were sensitive to temperature fluctuations;

7. Equilibrium state could be determined either by connecting the termination points of stress-strain curves with long enough relaxations, or from states obtained in the slow rate test at an axial strain of 0.16 %/h for the tested vertical undisturbed peat specimen in undrained conditions;

8. The vertical peat material with a smaller initial void ratio had a higher stiffness and a slightly smaller elastic recovery strain for an axial strain of 5 %. The proportion of elastic strain to total strain was smaller for the vertical peat specimen with a smaller initial void ratio;

9. Compared to the undrained triaxial relaxation tests, the undrained triaxial creep tests took a lot longer to reach the equilibrium state under comparable testing conditions;

10. The anisotropy of permeability of undisturbed peat was not significant, i.e. the coefficient of permeability of horizontal specimen was marginally lower than that of the vertical specimen measured in falling head permeability tests. The coefficient of permeability tested in falling head method with filtered bog water and distilled water both gave acceptable values, nevertheless the coefficient of permeability tested using distilled water attained a smaller value than using filtered bog water;

11. In the undrained triaxial \( lu \) tests, the horizontal specimens reached higher deviator stresses at 5 % and 10 % axial strains, and showed a pronounced strain softening after about 10 % axial strain compared to vertical specimens which did not reach failure even at the axial strain of 20 %. The elastic strain proportion to the total strain of the horizontal specimens was smaller than that of the vertical specimens. The vertical specimen kept nearly constant cross-section area after the undrained triaxial \( lu \) tests while through-cracks took place in horizontal specimens.

Based on the aforementioned experimental findings, a comprehensive constitutive model accounting for nonlinear large strain and rate-dependent as well as equilibrium hysteresis within a thermodynamically consistent framework is proposed in Chapter 4.
Chapter 4

Constitutive Modelling

A constitutive model was proposed based on the experimental work elaborated in Chapter 3 on the undisturbed peat in undrained conditions. The experimental results indicated the tested peat can be categorized as a rate-dependent material with equilibrium hysteresis subjected to large strains. Thus the proposed model should take the nonlinear large strain, rate-dependence and equilibrium hysteresis into account. To consider the large strain, a hyperviscoplastic model within a thermodynamically consistent framework was adopted. The principal advantage of a thermodynamically consistent framework is to guarantee a priori that the proposed models produce thermodynamically reasonable results. This chapter introduces a constitutive model of peat in the context of modern mechanics.

4.1 Some basic continuum mechanics

The proposed constitutive equations were formulated in the framework of a finite strain constitutive theory. Thus some basics of the continuum mechanics are briefly introduced as the preliminaries for the proposed constitutive models. More details can be found in textbooks on continuum mechanics such as Holzapfel (2000) and Bonet and Wood (1997).

4.1.1 Configurations

A collection of material points or particles is termed a material body which can be characterised by macroscopic quantities. A configuration refers to the simultaneous position of all particles within the body. The geometrical arrangement of these particles is determined uniquely at any instant of time. Figure 4.1 illustrates that the reference configuration refers to a fixed reference time \( t_0 \) and the current configuration refers to a subsequent time \( t > t_0 \). When \( t_0 = 0 \) the reference configuration is the initial configuration, whereby the reference configuration needs not necessarily to be the initial configuration. Let \( X \) be the position vector of material point \( X \) in the reference configuration and after time \( t - t_0 \) it moves as \( x \) with a current position of \( x \) in Euclidean 3D-space (capital letters denoting quantities in the
reference configuration and small letters for quantities in the current configuration). The mapping of a material point from position \( X \) to \( x \) at any time \( t \) is called a motion, \( \chi(X, t) \), of a body in Equation (4.1). Thus the displacement is \( u = x - X \). The characterisation of the motion with respect to the material coordinates in the reference configuration and time, given by Equation (4.1), is called material (Lagrangian) description. The characterisation of the motion with respect to the current (or spatial) coordinates in the current configuration and time \( t \) refers to spatial (Eulerian) description presented in Equation (4.2). In the sequel, Cartesian coordinate systems will be inferred without loss of generality.

\[
X = \chi^{-1}(x, t)
\]

4.1.2 Strain and stress measurements and work conjugates

There are manifold measures of strains and stresses for quantifying the mechanical behaviour of a continuum body. For finite deformation problems, the deformation gradient tensor is a crucial primary measure of deformation as a two-point tensor linearly mapping a vector \( dX \) to \( dx \), defined in Equation (4.3). The determinant of the deformation gradient tensor, \( J = \det(F) \), is the volume ratio. The right Cauchy-Green deformation tensor \( C \) is defined as a rotation-independent measure of deformation in Equation (4.4) in the reference configuration and the Green-Lagrangian strain tensor \( E \) is defined in Equation (4.5), where \( I \) is the second-order identity tensor. Equivalently, in the spatial description the left

Figure 4.1: Configuration and motion of a continuum body (adapted from Holzapfel (2000)).
Cauchy-Green tensor $b$ is defined in Equation (4.6) and the Euler-Almansi strain tensor in Equation (4.7).

\[
F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{z}(\mathbf{X}, t)}{\partial \mathbf{X}} = \text{Grad} \mathbf{x}(\mathbf{X}, t) \quad \text{or} \quad F_a = \frac{\partial x_a}{\partial X_a} = \text{Grad}_{X_a} x_a \quad (4.3)
\]

\[
C = F^T F \quad \text{or} \quad C_{AB} = F_{aA} F_{aB} \quad (4.4)
\]

\[
E = \frac{1}{2} (C - I) \quad \text{or} \quad E_{AB} = \frac{1}{2} (F_{aA} F_{aB} - \delta_{AB}) \quad (4.5)
\]

\[
b = F F^T \quad \text{or} \quad b_{ab} = F_{aA} F_{bA} \quad (4.6)
\]

\[
e = \frac{1}{2} (I - F^{-T} F^{-1}) \quad \text{or} \quad e_{ab} = \frac{1}{2} (\delta_{ab} - F_{aA} F_{bA}) \quad (4.7)
\]

The above strain tensors are defined in either the reference or the current configuration and the two-point deformation gradient tensor associates with both configurations. The operation that transforms a vector or tensor-valued quantity based on the reference configuration to the current configuration is called a push-forward, Equation (4.8) presents the push-forward of the Green-Lagrange strain tensor $E$ to the Euler-Almansi strain tensor $e$. The inverse operation is called a pull-back, which transforms a vector or tensor-valued quantity based on the current configuration to the reference configuration, presented in Equation (4.9).

\[
e = \frac{1}{2} (I - F^{-T} F^{-1}) = F^{-T} \left[ \frac{1}{2} F^T (I - F^{-T} F^{-1}) F \right] F^{-1} = F^{-T} \left[ \frac{1}{2} (F^T F - I) \right] F^{-1} = F^{-T} E F^{-1} = \chi_*(E) \quad (4.8)
\]

\[
E = \frac{1}{2} (F^T F - I) = F^T \left[ \frac{1}{2} F^{-T} (F^T F - I) F^{-1} \right] F = F^T \left[ \frac{1}{2} (I - F^{-T} F^{-1}) \right] F = F^T e F = \chi^*(e) \quad (4.9)
\]

Work conjugate pairs are given by requiring the invariance of scalar quantities like the internal stress work or power with respect to configurational changes. Kirchhoff stress $\mathbf{t} = J \sigma$ defined in current configuration, $\sigma$ in Equation (4.10) (1) and the Euler-Almansi strain are work conjugates. Likewise, the deformation gradient $\mathbf{F}$ and first Piola-Kirchhoff stress $\mathbf{P}$ in Equation (4.10) (2), Green-Lagrange strain $\mathbf{E}$ and second Piola-Kirchhoff stress $\mathbf{S}$ in Equation (4.10) (3) are commonly used work conjugates. In Equation (4.10), $f$ is the force vector, $\mathbf{t}$ is the Cauchy traction vector, $\mathbf{n}$ is the unit normal vector; $A$ is the current area, $J$ is the determinant of the deformation gradient.

\[
df = \mathbf{t} dA; \quad \mathbf{t} = \sigma \mathbf{n}; \quad (1) \quad \mathbf{P} = J \sigma F^{-T}; \quad (2) \quad \mathbf{S} = J F^{-T} \sigma F^{-T} = F^{-1} \mathbf{P}; \quad (3) \quad (4.10)
\]
4.1.3 Balance principles and entropy inequality

There are two approaches to propose constitutive relationships within the thermodynamically consistent framework, i.e. to constrain the already-existing constitutive models with laws of thermodynamics as well as to derive the constitutive relationships from the laws of thermodynamics (Houlsby and Puzrin, 2007). The latter has the manifest advantage of a smaller number of constitutive equations and \textit{a priori} satisfies of the laws of thermodynamics. The fundamental balance principles, i.e. balance of mass, the momentum balance and balance of energy, are applicable to any particular material and must be satisfied at all times. The derivations of the balance principles can be found in continuum mechanics text books, such as Holzapfel (2000) and Belytschko et al. (2013). The conclusions can be summarised as follows. For a closed system with a current local mass density of $\rho(x, t)$ and a material point spatial velocity $\mathbf{v} = \dot{x}$, the dot above a variable represents the material time derivative of that quantity, the balance of mass reads:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \quad (4.11)$$

where \text{div} denotes the spatial divergence operator. The balance of linear momentum is the presence of surface tractions $\mathbf{t}(x, t, \mathbf{n})$ and body forces $\mathbf{b}(x, t)$ and reads:

$$\text{div} \mathbf{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad (4.12)$$

The balance of angular momentum leads to the symmetry of the Cauchy stress tensor:

$$\mathbf{\sigma} = \mathbf{\sigma}^T \quad (4.13)$$

The first law of thermodynamics is a fundamental axiom in mechanics describing the conservation of energy (both mechanical and thermal), which states that the time rates of kinetic energy $E_{\text{kin}}$ and internal energy $U$ of a system equal the sum of the external mechanical power $P_{\text{ext}}$ and thermal power $Q$ done on the system. In terms of the time rates, we have:

$$\frac{d}{dt} E_{\text{kin}} + \frac{d}{dt} U = P_{\text{ext}} + Q \quad (4.14)$$

In this study, only mechanical energy is considered, i.e. the balance of mechanical energy in the spatial description follows that the rate of external mechanical work $P_{\text{ext}}$ is the sum of the stress power (rate of internal mechanical work) $P_{\text{int}}$ and the change in kinetic energy $E_{\text{kin}}$ (Holzapfel, 2000):

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho \mathbf{v}^2 \, d\Omega + \int_{\partial \Omega} \mathbf{\sigma} : \mathbf{d}d\Omega = \int_{\Gamma} \mathbf{t} \cdot \mathbf{v} \, d\Gamma + \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} \, d\Omega \quad (4.15)$$

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where the rate of external mechanical work is the sum of external mechanical power per unit current surface $\Gamma$ done by surface tractions $t(x, t, n)$ and per unit current volume $\Omega$ done by body forces by unit mass $b(x, t)$; the kinetic energy $E_{\text{kin}}$ is a generalization of Newtonian mechanics to continuum mechanics; the stress power $P_{\text{int}}$ describes the response of a region $\Omega$ at time $t$ done by the stress field; $\mathbf{v} = \dot{x}$ is the velocity field; $\mathbf{d} = \text{sym} \{\text{grad} \mathbf{v}\}$ is the rate of deformation tensor. Comparing (4.15) with (4.14), the rate of internal energy reads:

$$\frac{d}{dt} U = P_{\text{int}} + Q \quad (4.16)$$

Introducing the heat flux vector $\mathbf{q}$, heat sources per unit mass and time $r$ and the specific internal energy $u$ leads to the integral equation of (4.16):

$$\frac{d}{dt} \int_{\Omega} \rho u \, d\Omega = \int_{\Omega} \mathbf{\sigma} : d \, d\Omega + \int_{\Omega} \rho \, r \, d\Omega - \int_{\partial\Omega} \mathbf{q} \cdot n \, d\Gamma \quad (4.17)$$

The second law of thermodynamics clarifies the direction of energy transfer and makes a statement on the reversibility of processes. The second law of thermodynamics is considerably more subtle than the first law (Houlsby and Puzrin, 2007). A fundamental state variable, entropy, is used in formulating the second law of thermodynamics. The entropy can be viewed as the quantitative measure of microscopic randomness and disorder. A thermodynamic process is called reversible if it is not accompanied by any entropy production, where for each cycle the material response returns to its initial state. With the specific entropy $s$, the second law of thermodynamics states that the total production of entropy per unit time, i.e. the difference between the rate of change of entropy and the rate of entropy input by means of thermal processes is never negative (with respect to the absolute temperature $\theta(x, t) \geq 0$):

$$\frac{d}{dt} \int_{\Omega} \rho s \, d\Omega - \int_{\Omega} \frac{r}{\theta} \, d\Omega + \int_{\partial\Omega} \frac{\mathbf{q} \cdot n}{\theta} \, d\Gamma \geq 0 \quad (4.18)$$

where the last two terms correspond to the entropy supply and flux through heat, respectively. Combining the balance principles and second law of thermodynamics yields the Clausius-Duhem inequality by introducing the specific Helmholtz free energy $\tilde{\psi} = u - \theta s$ (Holzapfel, 2000):

$$\mathbf{\sigma} : d - \rho s \frac{d\theta}{dt} - \rho \frac{d\tilde{\psi}}{dt} - \frac{1}{\theta} \text{grad} \theta \cdot \mathbf{q} \geq 0 \quad (4.19)$$

For isothermal processes, $d\theta/dt = 0$ and $\text{grad} \theta = 0$, (4.19) becomes:

$$\mathbf{\sigma} : d - \rho \frac{d\tilde{\psi}}{dt} \geq 0 \quad (4.20)$$
The Clausius-Duhem inequality (4.20) is the basis for the subsequent derivation of thermodynamically consistent constitutive models.

### 4.1.4 Overlay concept

The overlay concept was developed originally for describing the nonlinear behaviour of solid materials in the 1970s (Owen et al., 1974; Pande et al., 1977). The material is assumed to decompose into several property layers or overlays for mathematical convenience. By assigning the same deformation to each layer, the total stress is obtained by summing the different contributions of each overlay. Several standard material behaviours can be reproduced by combining different constitutive components, such as visco-elastoplastic responses. The overlay concept has been used in peat modelling by Teunissen and Zwannenburg (2015) to describe the matrix behaviour and the fibre behaviour of a structurally anisotropic model. In this study, the phenomenological mechanical behaviour of peat in undrained conditions was decomposed into three layers, namely the hyperelastic, hyperviscoelastic and hyperelastoplastic layers. In one-dimensional rheological systems, the model consists of spring, dashpot and friction elements. It can be straightforwardly extended into three-dimensional finite deformations by multiplicative decompositions of the deformation gradient. The total stress follows from the summation of the stress in each layer (Equation (4.21), where e stands for elastic; v stands for viscoelastic and p stands for elastoplastic).

\[ S = S_e + S_v + S_p \]  

(4.21)

The rate-independent strain recovery indicates the rate-dependent components and the rate-independent component should be connected in parallel. The nonlinear large strain model is realized by adopting a modified Neo-Hookean hyperelastic formulation (Görke et al., 2010; Nagel, 2012). Multiple viscoelastic components are used to consider different time scales discovered in rate-dependent tests. Rate-independent hysteresis in the equilibrium test is realized with a plastic flow rule proportional to the absolute value of the total strain without defining a yield condition (Haupt, 1991).

### 4.2 Hyperelastic model

The hyperelastic constitutive model is derived from the Clausius-Duhem inequality in material configuration (4.20):

\[ - \rho_0 \dot{\psi}(\mathbf{C}) + \frac{1}{2} \mathbf{S} : \dot{\mathbf{C}} \geq 0 \]  

(4.22)
The equality sign refers to the reversible case of the isotropic elastic material. The 2nd Piola-Kirchhoff stress can be derived from the Helmholtz free energy function with respect to the Right Cauchy Green tensor $C$ as

$$S = 2\rho_0 \frac{\partial \tilde{\psi}}{\partial C} = 2 \frac{\partial \psi}{\partial C}$$  \hspace{1cm} (4.23)$$

where $\tilde{\psi}$ and $\psi$ are the specific Helmholtz free energy (per unit mass) and the Helmholtz free energy density (per unit volume), respectively. A modified Neo-Hooke hyperelastic model (Görke et al., 2010; Nagel, 2012) expressed in terms of the principal invariants was adopted as

$$\psi = \frac{C_1}{\alpha} \left[ e^{\alpha(I_1 - \ln I_3 - 3)} \right] + D_2 (\ln I_3)^2$$  \hspace{1cm} (4.24)$$

where $C_1$, $D_2$ and $\alpha$ are three material parameters, of which $C_1$ is a modulus quantifying a hybrid behaviour of shear and bulk, $D_2$ quantifies bulk behaviour and $\alpha$ controls the non-linearity of the constitutive relationship; the principal invariants of the right Cauchy Green tensor $C$ are

$$I_1(C) = C : I$$
$$I_2(C) = \frac{1}{2} \left[ (C : I)^2 - C^2 : I \right]$$
$$I_3(C) = \det C = J^2$$  \hspace{1cm} (4.25)$$

The form of the hyperelastic energy function (4.24) is expressed with $I_1$ and $I_3$, therefore we get the 2nd Piola-Kirchhoff stress:

$$S = 2 \left( \frac{\partial \psi}{\partial I_1} \frac{\partial I_1}{\partial C} + \frac{\partial \psi}{\partial I_3} \frac{\partial I_3}{\partial C} \right)$$  \hspace{1cm} (4.26)$$

where the partial derivatives of $I_1$ and $I_3$ with respect to $C$ are:

$$\frac{\partial I_1}{\partial C} = I$$
$$\frac{\partial I_3}{\partial C} = (\det C) C^{-1}$$  \hspace{1cm} (4.27)$$

Thus (4.26) reads

$$S = 2 \frac{\partial \psi}{\partial I_1} I + 2 \frac{\partial \psi}{\partial I_3} \det(C) C^{-1}$$  \hspace{1cm} (4.28)$$

with

$$\frac{\partial \psi}{\partial I_1} = C_1 e^{\alpha(I_1 - \ln I_3 - 3)}$$
$$\frac{\partial \psi}{\partial I_3} = \frac{2D_2 \ln I_3 - C_1 e^{\alpha(I_1 - \ln I_3 - 3)}}{I_3}$$  \hspace{1cm} (4.29)$$

For small strains, the material parameters $C_1$ and $D_2$ can be derived from Young's modulus $E$ and Piosson's ratio $\nu$ as
\[ C_1 = \frac{E}{4(1 + \nu)} \quad \text{and} \quad D_2 = \frac{C_1 \nu}{2(1 - 2\nu)} \] (4.30)

### 4.3 Hyperviscoelastic model

#### 4.3.1 Standard Solid Model 1D Formulation

The Standard Solid model, which combines the Maxwell model and a Hookean spring in parallel, is taken as the starting viscoelastic model for further complex models.

According to the structural scheme, the total stress decomposes into a rate-independent equilibrium elastic stress \( \sigma_{eq0} \) and a rate-dependent overstress \( \sigma_{ov} \),

\[ \sigma = \sigma_{eq0} + \sigma_{ov} \] (4.31)

For small strain conditions, the total strain is additively decomposed into the elastic strain \( \epsilon_{ev} \) and the viscous strain \( \epsilon_v \):

\[ \epsilon = \epsilon_{ev} + \epsilon_v \] (4.32)

For isothermal processes, the specific Helmholtz free energy of a rheological model is simply the potential energy of the springs. The free energy function is hypothetically decomposed into the elastic and viscoelastic parts:

\[ \dot{\psi} = \dot{\psi}_{eq}(\epsilon) + \dot{\psi}_{ov}(\epsilon - \epsilon_v) \] (4.33)

\[ \sigma_{eq0} = \rho_0 \frac{\partial \dot{\psi}_{eq}(\epsilon)}{\partial \epsilon} \] (4.34)

\[ \sigma_{ov} = \rho_0 \frac{\partial \dot{\psi}_{ov}(\epsilon - \epsilon_v)}{\partial \epsilon_{ev}} \] (4.35)
4.3.2 3D Formulation

4.3.2.1 Stress and strain measures

For finite strain problems, the multiplicative decomposition of the deformation gradient of the viscoelastic material in Figure 4.2 is given as:

\[ F = F_e F_v \]  
(4.36)

The finite strain measure Green-Lagrangian strain tensor is defined as:

\[ E = \frac{1}{2} (F^T F - I) \]  
(4.37)

By the following manipulation:

\[ E = \frac{1}{2} (F^T F - I) = \frac{1}{2} (F_e^T F_e F_v - I) = \frac{1}{2} (F_v^T F_e F_v - F_e^T F_v F_v - I) \]
\[ = \frac{1}{2} F_v^T (F_e^T F_e - I) F_v + \frac{1}{2} (F_v^T F_v - I) = \frac{1}{2} (C - C_v) + \frac{1}{2} (C_v - I) = E_e + E_v \]

(4.38)

it can be seen that the additive decomposition of small strain in Equation (4.32) is equivalent to the multiplicative decomposition of deformation gradient in Equation (4.36).

Only on the intermediate configuration, the total strain decomposes additively into purely elastic strain of Green-Lagrangian type and purely inelastic strain of Almansi type, i.e.

\[ \varepsilon = F_v^{-T} E F_v^{-1} = \frac{1}{2} (F_e^T F_e - F_v^{-T} F_v^{-1}) = \varepsilon_{ev} + \varepsilon_v \]  
(4.39)

with

\[ \varepsilon_{ev} = \frac{1}{2} (F_e^T F_e - I) \]  
and  
\[ \varepsilon_v = \frac{1}{2} (I - F_v^{-T} F_v^{-1}) \]  
(4.40)

From Figure 4.2, an analogous decomposition of the 2nd Piola-Kirchhoff stress is introduced:

\[ S = S_{eq0} + S_{ov} \]  
(4.41)

where \( S_{eq0} \) is the equilibrium stress and \( S_{ov} \) is the rate-dependent overstress. As this stress decomposition is associated with the reference configuration \( \mathcal{R} \), the overstress has to be pushed to the intermediate configuration in order to be related by constitutive equations to the associated strain tensor \( \varepsilon_v \):

\[ \tau_{ov} = F_v S_{ov} F_v^T \]  
(4.42)

The contravariant Oldroyd stress rate and covariant Oldroyd strain rate are given within the concept of dual variables (Haupt, 2000):
\[
\begin{align*}
\tau_{ov} & := F_v \dot{S}_{ov} F_v^T = F_v \dot{F}_v^{-1} \tau_{ov} F_v^{-1} F_v^T = F_v \left( \dot{F}_v^{-1} \tau_{ov} F_v^{-1} + F_v^{-1} \tau_{ov} F_v^{-1} \right) F_v^T \\
& = \tau_{ov} - l_v \tau_{ov} - \tau_{ov} l_v^T \\
\tau \varepsilon & := F_v^{-T} \dot{E} F_v^{-1} = F_v^{-T} \dot{F}_v^T \varepsilon F_v^{-1} = F_v^{-1} \left( \dot{F}_v^{-1} F_v + F_v^{-1} \dot{F}_v + F_v^{-1} \dot{F}_v \right) F_v^{-1} \\
& = \dot{\varepsilon} + \varepsilon l_v + l_v^T \varepsilon 
\end{align*}
\] (4.43)

\[
\tau_{ov} = \dot{\varepsilon} + \varepsilon l_v + l_v^T \varepsilon 
\] (4.45)

The reason for the use of the concept of dual variables is the fact that the stress power as a scalar quantity is invariant to configurational transformations:

\[
S_{ov} : \dot{E} = \tau_{ov} : \dot{\varepsilon} = \tau_{ov} : \left( \varepsilon_{ev} + \dot{\varepsilon} \right) 
\] (4.46)

### 4.3.2.2 Constitutive relations

The specific Helmholtz free energy per unit mass is assumed to decompose like the stresses:

\[
\bar{\psi}(E, \varepsilon_{ev}) = \bar{\psi}_0(E) + \bar{\psi}_v(\varepsilon_{ev}) 
\] (4.47)

The constitutive equations for the equilibrium part have been defined in Section 4.2. The constitutive equations for the overstress as well as the evolution equation for the inelastic strain are derived using the isothermal Clausius-Duhem inequality:

\[
S : \dot{E} - \rho_0 \dot{\bar{\psi}} \geq 0 
\] (4.48)

The stress power is at least as large as the rate of change of the free energy. Using (4.41), (4.47) and the chain rule, (4.48) becomes

\[
\left( S_{eq0} - \rho_0 \frac{\partial \bar{\psi}_0}{\partial E} \right) : \dot{E} + S_{ov} : \dot{E} - \rho_0 \frac{\partial \bar{\psi}_v}{\partial \varepsilon_{ev}} : \varepsilon_{ev} \geq 0 
\] (4.49)

The part in the brackets equals 0 due to the reversibility of hyperelasticity. Inserting Equations (4.44) and (4.46) to (4.49) gives

\[
\tau_{ov} - \rho_0 \frac{\partial \bar{\psi}_v}{\partial \varepsilon_{ev}} : \dot{\varepsilon}_{ev} + \tau_{ov} : \dot{\varepsilon}_{ev} + \rho_0 \frac{\partial \bar{\psi}_v}{\partial \varepsilon_{ev}} : (\varepsilon_{ev} l_v + l_v^T \varepsilon_{ev}) \geq 0 
\] (4.50)

Again due to the reversibility of the hyperelastic component, the constitutive relations for the intermediate configuration can be defined as:

\[
\tau_{ov}(\varepsilon_{ev}) = \rho_0 \frac{\partial \bar{\psi}_v}{\partial \varepsilon_{ev}} 
\] (4.51)
pulling back into the reference configuration:

\[
S_{ov} = F_v^{-1} \tau_{ov} F_v^{-T} = \rho_0 F_v^{-1} \frac{\partial \tilde{\psi}_v}{\partial \epsilon_{ev}} F_v^{-T}
\]  

(4.52)

Assuming \( \tilde{\psi}_v \) is an isotropic function (rotationally invariant) of \( \epsilon_{ev} \):

\[
\frac{\partial \tilde{\psi}_v}{\partial \epsilon_{ev}} : \left( I_{ev}^T \epsilon_{ev} + \epsilon_{ev} I_v \right) = 2 \epsilon_{ev} \frac{\partial \tilde{\psi}_v}{\partial \epsilon_{ev}} : \epsilon_v
\]  

(4.53)

so that the dissipation inequality reads

\[
\rho_0 (I + 2 \epsilon_{ev}) \frac{\partial \tilde{\psi}_v}{\partial \epsilon_{ev}} : \Delta \epsilon_v \geq 0
\]  

(4.54)

Specifying the viscous strain rate on the intermediate configuration

\[
\Delta \epsilon_v := \frac{\rho_0}{\eta_v} \left( I + 2 \epsilon_{ev} \right) \frac{\partial \tilde{\psi}_v}{\partial \epsilon_{ev}}
\]  

(4.55)

with \( \eta_v \) is a non-negative viscosity, \( \hat{C}_{ev} = I + 2 \epsilon_{ev} \). Thus the non-negativity of (4.54) is satisfied for arbitrary deformation processes for \( \eta_v \geq 0 \).

As \( \tilde{\psi}_v \) is an isotropic function of \( \epsilon_{ev} \), it can be expressed in terms of the principal invariants of its argument:

\[
\tilde{\psi}_v(\epsilon_{ev}) = \tilde{\psi}_v(\hat{C}_{ev}) = \tilde{\psi}_v(I_{ev}^1, I_{ev}^2, I_{ev}^3)
\]  

(4.56)

where \( I_{ev}^1, I_{ev}^2, I_{ev}^3 \) are the principal invariants of the elastic Right Cauchy Green type tensor \( \hat{C}_{ev} = F_{ev}^T F_{ev} \). Expressing the invariants by quantities defined on the reference configuration \( \mathcal{R} \) yields

\[
I_{ev}^1 = \text{tr} \left( C C_{ev}^{-1} \right)
\]  

(4.57)

\[
I_{ev}^2 = \frac{1}{2} \left[ \text{tr}^2 \left( C C_{ev}^{-1} \right) - \text{tr} \left( C C_{ev}^{-1} \right)^2 \right]
\]  

(4.58)

\[
I_{ev}^3 = \text{det} \left( C C_{ev}^{-1} \right)
\]  

(4.59)

A constitutive relation to the similar elastic Neo-Hooke model was chosen in (4.60). Using the chain rule, Equation (4.52) derives into Equation (4.61).

\[
\psi_v = \frac{C_{1v}}{a_v} \left[ e^{a_v(I_{ev}^3 - \ln I_{ev}^3) - 1} \right] + D_{2v} \left( \ln I_{ev}^3 \right)^2
\]  

(4.60)

\[
S_{ov} = 2 F_v^{-1} \frac{\partial \psi_v}{\partial \hat{C}_{ev}} F_v^{-T} = 2 \left( \frac{\partial \psi_v}{\partial I_{ev}^1} \frac{\partial I_{ev}^1}{\partial C} + \frac{\partial \psi_v}{\partial I_{ev}^3} \frac{\partial I_{ev}^3}{\partial C} \right)
\]  

(4.61)

Using (4.57), (4.58) and (4.59)
\[
S_{ov} = 2 \left( \frac{\partial \psi_v}{\partial I_{1}^{ev}} C_v^{-1} + \frac{\partial \psi_v}{\partial I_{3}^{ev}} I_{3}^{ev} C_v^{-1} \right) \tag{4.62}
\]
with
\[
\frac{\partial I_{1}^{ev}}{\partial C_v} = C_v^{-1} \quad \text{and} \quad \frac{\partial I_{3}^{ev}}{\partial C_v} = I_{3}^{ev} C_v^{-1} \tag{4.63}
\]

The evolution equation for the inelastic (viscous) Right Cauchy Green tensor is specified by pulling back the flow rule (4.55) for the inelastic strain using \( \dot{C}_v = 2F_v^T \epsilon_v F_v \):

\[
\dot{C}_v = 2F_v^T \left[ \frac{1}{\eta_v} (I + 2\epsilon_{ev}) \frac{\partial \psi_v}{\partial \epsilon_{ev}} \right] F_v = 4 \frac{1}{\eta_v} F_v^T \left( \dot{C}_v \frac{\partial \psi_v}{\partial C_v} \right) F_v \tag{4.64}
\]

\[
= \frac{4}{\eta_v} \left( \frac{\partial \psi_v}{\partial I_{1}^{ev}} C_v + I_{3}^{ev} \frac{\partial \psi_v}{\partial I_{3}^{ev}} C_v \right) \tag{4.65}
\]

with
\[
\frac{\partial I_{1}^{ev}}{\partial C_v} = -C_v : C_v^{-1} \otimes C_v^{-1} \quad \text{and} \quad \frac{\partial I_{3}^{ev}}{\partial C_v} = -I_{3}^{ev} C_v : C_v^{-1} \otimes C_v^{-1} \tag{4.66}
\]

\[
\frac{\partial \psi_v}{\partial I_{1}^{ev}} = C_{1v} e^{\alpha (I_{1}^{ev} - \ln I_{3}^{ev} - 3)} \tag{4.67}
\]

\[
\frac{\partial \psi_v}{\partial I_{3}^{ev}} = \frac{C_{1v}}{I_{3}^{ev}} C_{1v} e^{\alpha (I_{1}^{ev} - \ln I_{3}^{ev} - 3)} + \frac{2D_{2v}^{ev}}{I_{3}^{ev}} \ln I_{3}^{ev} \tag{4.68}
\]

From Equations (4.62) and (4.65), the final definition of the viscoelatic evolution equation reads

\[
\dot{C}_v = \frac{2}{\eta_v} C_v S_{ov} C \tag{4.69}
\]

### 4.3.2.3 Model with two Maxwell elements

In order to incorporate the distinct relaxation behaviours on different time scales observed in the laboratory experiments, the above proposed model with a single Maxwell element (presented in Figure 4.2) can be extended in a series like-fashion with \( n \) Maxwell elements with independent material parameters in parallel connection with the equilibrium Maxwell spring. By introducing \( n \) viscous intermediate configurations \( F_i^v \), the total overstress is obtained as:

\[
S_{ov} = \sum_{i=1}^{n} S_{ov}^i \tag{4.70}
\]

In this study two Maxwell elements were considered for the further validations of the extended model (Figure 4.3). From Equation (4.28), (4.41) and (4.62), the total 2nd Piola-Kirchhoff stress is
Figure 4.3: Series of parallel Maxwell elements and the equilibrium hyperelastic spring element.

\[
S = 2 \left( \frac{\partial \psi_0}{\partial I_1^{\text{eq0}}} I + \frac{\partial \psi_0}{\partial I_3^{\text{eq0}}} I_3^{\text{eq0}} \mathbf{C}^{-1} \right) + \\
+ 2 \left( \frac{\partial \psi_{v1}}{\partial I_1^{\text{ev1}}} \mathbf{C}^{-1} v_1 + \frac{\partial \psi_{v1}}{\partial I_3^{\text{ev1}}} I_3^{\text{ev1}} \mathbf{C}^{-1} \right) + 2 \left( \frac{\partial \psi_{v2}}{\partial I_1^{\text{ev2}}} \mathbf{C}^{-1} v_2 + \frac{\partial \psi_{v2}}{\partial I_3^{\text{ev2}}} I_3^{\text{ev2}} \mathbf{C}^{-1} \right) \tag{4.71}
\]

with \( \psi_{v2} \) is the Helmholtz free energy density function for the second Maxwell element with independent material parameters. The viscoelastic evolution equation holds the same format as Equation (4.69).

### 4.4 Hyperviscoplastic model

To account for the irrecoverable strain of the peat stress-strain behaviour recorded in the undrained triaxial tests, plasticity is added to the proposed hyperviscoelastic model. From the compression curves of the vertical peat specimens in the undrained triaxial compression tests, no apparent yield points on the compression curves could be observed. Additionally constrained by the limited experimental data on the plastic behaviour of peat in undrained triaxial compression, a plastic flow rule without the need to define a yield surface is adopted to simulate the observed behaviour.

Figure 4.4: Hyperviscoplastic model.
Analogous to the uniaxial viscoelastic model, an elastoplastic component is added in parallel as shown in Figure 4.4. Thus the total strain in 1D small strain condition can be additively decomposed as

\[ \epsilon = \epsilon_{ep} + \epsilon_p \]  \hspace{1cm} (4.72)

The Helmholtz free energy density function becomes

\[ \psi = \psi_0(\epsilon) + \sum_{i=1}^{n} \psi^i_{v}(\epsilon - \epsilon^i_{v}) + \psi_{p}(\epsilon - \epsilon_p) \] \hspace{1cm} (4.73)

with

\[ \sigma_p = \frac{\partial \psi_p}{\partial \epsilon_e} \] \hspace{1cm} (4.74)

### 4.4.1 Constitutive Relations

The additive decomposition (4.72) corresponds to the multiplicative decomposition of the deformation gradient into elastic and plastic components:

\[ F = F_{ep} F_p \] \hspace{1cm} (4.75)

On the corresponding intermediate configuration of this multiplicative decomposition, the total strain can be decomposed additively into purely elastic Green-Lagrangian type strain and purely plastic Almansi type strain, i.e.

\[ \epsilon = F^{-\top}_p EF^{-1}_p \epsilon = \epsilon_{ep} + \epsilon_p \] \hspace{1cm} (4.76)

with

\[ \epsilon_{ep} = \frac{1}{2} \left( F^\top_{ep} F_{ep} - I \right) \] \hspace{1cm} and \hspace{1cm} \[ \epsilon_p = \frac{1}{2} \left( I - F^{-\top}_p F^{-1}_p \right) \] \hspace{1cm} (4.77)

The total 2nd Piola-Kirchhoff stress is

\[ S = S_{eq0} + \sum_{i=1}^{n} S_{ov}^i + S_p \] \hspace{1cm} (4.78)

The isothermal entropy inequality:

\[ S : \dot{E} - \rho_0 \dot{\psi} \geq 0 \] \hspace{1cm} (4.79)

i.e.

\[ \left( S_{eq0} - \rho_0 \frac{\partial \psi_0}{\partial E} \right) : \dot{E} + \sum_{i=1}^{n} S_{ov}^i : \dot{E} - \rho_0 \sum_{i=1}^{n} \frac{\partial \psi^i_v}{\partial \epsilon^i_{ev}} : \dot{\epsilon}^i_{ev} + S_p : \dot{E} - \rho_0 \frac{\partial \psi_p}{\partial \epsilon_{ep}} : \dot{\epsilon}_{ep} \geq 0 \] \hspace{1cm} (4.80)
The first item in the brackets on the LHS equals to 0 due to the reversibility of the hyperelastic model. In the hyperviscoelastic model, the viscoelastic part of the entropy inequality has been set to be non-negative. The contravariant Oldroyd strain rate with respect to the plasticity intermediate configuration (Haupt, 2000) is:

\[
\Delta \dot{\varepsilon} = F^{-T}_p \dot{E} F^{-1}_p = \dot{\varepsilon} + \varepsilon l_p + l^T_p \varepsilon
\]  

(4.81)

with the plastic velocity gradient

\[
l_p = F_p F^{-1}_p
\]

(4.82)

Since the stress power as a scalar quantity is invariant to configurational transformations:

\[
S_p : \dot{E} = \tau_p : \Delta \dot{\varepsilon} = \tau_p : \Delta (\varepsilon_{ep} + \dot{\varepsilon}_p)
\]

(4.83)

where \( \tau_p \) is the push-forward of the plastic 2nd Piola Kirchhoff stress into the intermediate configuration \( \tau_p = F_p S_p F^T_p \). Inserting Equations (4.81) and (4.83) into (4.80), we get

\[
\left( \tau_p - \rho_0 \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}} \right) : \varepsilon_{ep} + \tau_p : \varepsilon_{ep} + \rho_0 \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}} : (\varepsilon_{ep} l_p + l^T_p \varepsilon_{ep}) \geq 0
\]

(4.84)

The standard argument leads to the stress relation:

\[
\tau_p = \rho_0 \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}}
\]

(4.85)

Assuming \( \tilde{\psi}_p \) is an isotropic function of \( \varepsilon_{ep} \), thus

\[
\frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}} : (\varepsilon_{ep} l_p + l^T_p \varepsilon_{ep}) = 2 \varepsilon_{ep} \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}} : \varepsilon_{ep}
\]

(4.86)

Then (4.84) becomes

\[
\rho_0 (I + 2 \varepsilon_{ep}) \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}} : \varepsilon_{ep} \geq 0
\]

(4.87)

Specifying plastic flow rule on the intermediate configuration with a non-negative parameter \( c_p \):

\[
\Delta \dot{\varepsilon}_p := c_p \rho_0 || \dot{\varepsilon} || (I + 2 \varepsilon_{ep}) \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}}
\]

(4.88)

where the elastic Green-Lagrangian strain tensor on the intermediate configuration is \( \varepsilon_{ep} = \frac{1}{2} (\dot{\mathbf{C}}_{ep} - I) = \frac{1}{2} (F^T_{ep} \dot{\mathbf{F}}_{ep} - I) \). By pulling back Equation (4.88) into the reference configuration using \( \dot{\mathbf{C}}_p = 2 F^T_{ep} \Delta \dot{\varepsilon}_p \dot{F}_p \) and pulling back of the scalar valued \( || \dot{\varepsilon} || \) equal to \( || \dot{E} || \), we get

\[
\dot{\mathbf{C}}_p = 2 F^T_{ep} \left[ c_p \rho_0 || \dot{\varepsilon} || (I + 2 \varepsilon_{ep}) \frac{\partial \tilde{\psi}_p}{\partial \varepsilon_{ep}} \right] \dot{F}_p = 4 c_p \rho_0 || \dot{E} || F^T_{ep} \left( \dot{\mathbf{C}}_{ep} \frac{\partial \tilde{\psi}_p}{\partial \dot{\varepsilon}_{ep}} \right) \dot{F}_p
\]

(4.89)
Since $\psi_p$ is an isotropic function of $\varepsilon_{ep}$, it can be expressed in terms of the principal invariants of $\varepsilon_{ep}$, i.e. the principal invariants of $\hat{C}_{ep}$:

$$\tilde{\psi}_p(\varepsilon_{ep}) = \tilde{\psi}_p(\hat{C}_{ep}) = \tilde{\psi}_p(I_{1e}^{ep}, I_{2e}^{ep}, I_{3e}^{ep})$$  \hspace{1cm} (4.90)

$$I_{1e}^{ep} = \text{tr} \left( C C_{p}^{-1} \right)$$  \hspace{1cm} (4.91)

$$I_{2e}^{ep} = \frac{1}{2} \left[ \text{tr}^2 \left( C C_{p}^{-1} \right) - \text{tr} \left( C C_{p}^{-1} \right)^2 \right]$$  \hspace{1cm} (4.92)

$$I_{3e}^{ep} = \text{det} \left( C C_{p}^{-1} \right)$$  \hspace{1cm} (4.93)

where $I_{1e}^{ep}$, $I_{2e}^{ep}$, $I_{3e}^{ep}$ are the principal invariants of the elastic Right Cauchy Green type tensor

$$\hat{C}_{ep} = F_{ep}^T F_{ep}$$

which can be obtained by (4.91), (4.92) and (4.93). A similar strain energy density function $\psi_p$ to that of the modified Neo-Hooke model is chosen for the elastoplastic component of the rheological model:

$$\psi_p = \rho_0 \tilde{\psi}_p = \frac{C_{ep}}{\alpha_p} \left[ e^{\alpha_p (I_{1e}^{ep} - \ln I_{3e}^{ep} - 3)} - 1 \right] + D_2 (\ln I_{3e}^{ep})^2$$  \hspace{1cm} (4.94)

Thus the plastic flow rule (4.89) reads

$$\dot{C}_p = 4 c_p \| \dot{E} \| \left( \frac{\partial \psi_p}{\partial I_{1e}^{ep}} C + I_{3e}^{ep} \frac{\partial \psi_p}{\partial I_{3e}^{ep}} C_p \right) = 2 c_p \| \dot{C} \| \left( \frac{\partial \psi_p}{\partial I_{1e}^{ep}} C + I_{3e}^{ep} \frac{\partial \psi_p}{\partial I_{3e}^{ep}} C_p \right)$$  \hspace{1cm} (4.95)

with the relations

$$\frac{\partial I_{1e}^{ep}}{\partial \hat{C}_{ep}} = \mathbf{I} \quad \text{and} \quad \frac{\partial I_{3e}^{ep}}{\partial \hat{C}_{ep}} = I_{3e}^{ep} \hat{C}_{ep}^{-1}$$  \hspace{1cm} (4.96)

The plastic 2\textsuperscript{nd} Piola-Kirchhoff stress reads

$$S_p = 2 \left( \frac{\partial \psi_p}{\partial I_{1e}^{ep}} \frac{\partial I_{1e}^{ep}}{\partial \hat{C}} + \frac{\partial \psi_p}{\partial I_{3e}^{ep}} \frac{\partial I_{3e}^{ep}}{\partial \hat{C}} \right) = 2 \left( \frac{\partial \psi_p}{\partial I_{1e}^{ep}} C_{p}^{-1} + I_{3e}^{ep} \frac{\partial \psi_p}{\partial I_{3e}^{ep}} C_{p}^{-1} \right)$$  \hspace{1cm} (4.97)

with the relations:

$$\frac{\partial I_{1e}^{ep}}{\partial \hat{C}} = C_{p}^{-1} \quad \text{and} \quad \frac{\partial I_{3e}^{ep}}{\partial \hat{C}} = I_{3e}^{ep} C_{p}^{-1}$$  \hspace{1cm} (4.98)
\begin{align*}
\frac{\partial \psi_p}{\partial I_1^{ep}} &= C_{1p} e^{\alpha_{v}}(I_1^{ep} - \ln I_3^{ep} - 3) & (4.99) \\
\frac{\partial \psi_p}{\partial I_3^{ep}} &= -\frac{C_{1p}}{I_3^{ep}} e^{\alpha_{v}}(I_1^{ep} - \ln I_3^{ep} - 3) + \frac{2D_{2p}}{I_3^{ep}} \ln I_3^{ep} & (4.100) \\
\frac{\partial^2 \psi_p}{\partial (I_1^{ep})^2} &= C_{1p} \alpha_{v} e^{\alpha_{v}}(I_1^{ep} - \ln I_3^{ep} - 3) & (4.101) \\
\frac{\partial^2 \psi_p}{\partial I_1^{ep} \partial I_3^{ep}} &= \frac{\partial^2 \psi_p}{\partial I_3^{ep} \partial I_1^{ep}} = -\frac{C_{1p}}{I_3^{ep}} e^{\alpha_{v}}(I_1^{ep} - \ln I_3^{ep} - 3) & (4.102) \\
\frac{\partial^2 \psi_p}{\partial (I_3^{ep})^2} &= \frac{C_{1p}(\alpha_{p} + 1)}{(I_3^{ep})^2} e^{\alpha_{v}(I_1^{ep} - \ln I_3^{ep} - 3)} + \frac{2D_{2p}}{(I_3^{ep})^2} (1 - \ln I_3^{ep}) & (4.103)
\end{align*}

The common practice for modelling soil plasticity is to adopt the modified Cam Clay model. This research did not follow the conventional routine as 1) there were not enough experimental data to identify which yield surface and plastic potential should be chosen for peat; 2) the proposed hyperviscoplastic model can serve as a base model for further complex models; 3) the number of material parameters to be defined is drastically reduced.

### 4.5 Summary

This chapter elaborated the derivation of the large strain model adopted for modelling peat. The hyperviscoplastic model was derived accounting for the experimental behaviour found in Chapter 3, where the undisturbed peat was classified as a rate-dependent material with equilibrium hysteresis. The comprehensive model started from a modified Neo-Hooke model and was extended with two Maxwell elements to account for the rate-dependent behaviour as well as an elastoplastic component to account for the plastic behaviour, respectively, by employing the overlay concept. The constitutive relationships were derived from the Clausius-Duhem inequality, which guaranteed the derived constitutive relationships are thermodynamically consistent. The numerical implementation, verification and validation of the derived models are detailed in Chapter 5.
Chapter 5

Numerical Implementation, Verification and Validation

In this chapter, the proposed constitutive model in Chapter 4 is implemented numerically, verified against the analytical solutions and validated against the experimental data.

5.1 Methods for numerical implementation

The phenomenological models consist of partial differential equations which cannot be analytically solved. The proposed constitutive relations in Chapter 4 only contain rate formulations and are classified as initial value problems. The rate-formulations are discretised in time using a finite-difference method with an implicit backward Euler scheme, and then are solved by Newton-Raphson iterations.

5.1.1 Kelvin mapping

The constitutive theories proposed in 4 operate on three-dimensional Euclidean space using scalar, vector, and tensor-valued quantities usually up to the order of four, e.g. the fourth-order stiffness tensor $C_{ijkl}$ in the stress-strain relationship of generalized Hooke’s law in Equation (5.1). For numerical convenience, a standard matrix-vector notation is employed by replacing three-dimensional symmetric second order tensors by six-dimensional vectors and three-dimensional fourth order tensors by 6×6 matrices. In the finite element literature, the common method of mapping tensor notations to vector-matrix notations is Voigt mapping (Voigt, 1966), where different transformations are performed for stress and strain tensors.

\[
\sigma_{ij} = C_{ijkl} \epsilon_{kl} \tag{5.1}
\]

The minor symmetries of Equation (5.2)(1) resulted from the symmetry of the stress and strain tensors reduce the independent components of the stiffness tensor $C_{ijkl}$ from $81 \ (3 \times 3 \times 3 \times 3)$ to $36 \ (6 \times 6)$, and the major symmetry of $C_{ijkl}$ as in Equation (5.2)(2) reduces
the independent components to 21. The Voigt mapping of stress and strain tensors is de-
noted in (5.3) (Simo and Hughes, 2000). The use of engineering shear strains \( \gamma_{ij} = 2\epsilon_{ij} \) maintains the identity of the strain energy density as in (5.4). Correspondingly, the constitutive matrix \( C \) is formatted as a 6×6 matrix from the fourth order tensor in (5.5). The advantages of the Voigt mapping are that the elastic energy density and the elastic stiffness are preserved, while the stress and strain are treated differently and the norms of tensors are not preserved. The merits of tensor algebra are lost in the Voigt mapping, such that there is no "invariant representation", i.e. representation of change of coordinate system by left/right multiplication of transformation tensors is not possible. For complex models, the Voigt mapping brings increasing and unnecessary difficulties in keeping track of the distinctions of different quantities implemented in finite element programs (Nagel et al., 2016).

\[
C_{ijkl} = C_{jikl} = C_{ijkl}, \quad C_{ijkl} = C_{klji} \tag{5.2}
\]

\[
\sigma_{ij} \rightarrow \sigma_{1} = \sigma = [\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12} \sigma_{13} \sigma_{23}]^T
\]

\[
\epsilon_{ij} \rightarrow \epsilon_{1} = \epsilon = [\epsilon_{11} \epsilon_{22} \epsilon_{33} 2\epsilon_{12} 2\epsilon_{13} 2\epsilon_{23}]^T
\tag{5.3}
\]

\[
\sigma : \epsilon = \sigma_{ij}\epsilon_{ij} = \sigma_{1}\epsilon_{1} = \sigma\epsilon \tag{5.4}
\]

\[
C = \begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\
C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\
C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\
C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\
C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\
C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323}
\end{pmatrix} \tag{5.5}
\]

In this study, an alternative vector-matrix notation of tensors, namely the Kelvin mapping, is adopted for the numerical implementation of the constitutive theories proposed in Chapter 4. A comprehensive review of the development of Kelvin mapping can be found in Helbig (2013). The idea was firstly proposed by Lord Kelvin (Thomson, 1856) and put into the context of modern tensor algebra by several authors (Mehrabadi and Cowin, 1990; Moakher, 2008; Kowalczyk-Gajewska and Ostrowska-Maciejewska, 2009) in the context of anisotropic elasticity. Nagel et al. (2016) elaborated on the advantages of adopting the Kelvin mapping in generalized finite element implementations of material constitutive theories. The convenience of using Kelvin mapping originates in its tensor character by introducing the six-dimensional basis in (5.6).

\[
M_1 = e_1 \otimes e_1, \quad M_2 = e_2 \otimes e_2, \quad M_3 = e_3 \otimes e_3
\]

\[
M_4 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1), \quad M_5 = \frac{1}{\sqrt{2}}(e_1 \otimes e_3 + e_3 \otimes e_1), \quad M_6 = \frac{1}{\sqrt{2}}(e_2 \otimes e_3 + e_3 \otimes e_2) \tag{5.6}
\]
To transform a second order symmetric tensor $A$, and a fourth order symmetric tensor $\mathcal{A}$ with base vectors, we write

$$A = A_{ij} e_i \otimes e_j = A_I M_I \quad \text{with} \quad A_I = A : M_I \quad \text{(5.7)}$$

$$\mathcal{A} = A_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l = A_{IJ} M_I \otimes M_J \quad \text{with} \quad A_{IJ} = A : \mathcal{A} : M_I \otimes M_J \quad \text{(5.8)}$$

It can be seen that the tensor character is preserved as being presented by the Kelvin Mapping. The double contraction between second/fourth order tensors is replaced by a dot product. The mathematical bold italic font is used for denoting tensors and the mathematical sans-serif italic font for their Kelvin mappings. With (5.7) and (5.8), we get the following Kelvin mapping of the stress and strain tensor coordinates similarly to the Voigt mapping in (5.3):

$$\sigma_{ij} \rightarrow \sigma = [\sigma_{11} \sigma_{22} \sigma_{33} \sqrt{2} \sigma_{12} \sqrt{2} \sigma_{13} \sqrt{2} \sigma_{23}]^T$$

$$\varepsilon_{ij} \rightarrow \varepsilon = [\varepsilon_{11} \varepsilon_{22} \varepsilon_{33} \sqrt{2} \varepsilon_{12} \sqrt{2} \varepsilon_{13} \sqrt{2} \varepsilon_{23}]^T \quad \text{(5.9)}$$

From (5.8) the stiffness tensor $C$ (tensor notation of $C_{ijkl}$ in (5.1)) in Kelvin mapping stands

$$C = \begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1112} & \sqrt{2}C_{1113} & \sqrt{2}C_{1123} \\
C_{2211} & C_{2222} & C_{2233} & \sqrt{2}C_{2212} & \sqrt{2}C_{2213} & \sqrt{2}C_{2223} \\
C_{3311} & C_{3322} & C_{3333} & \sqrt{2}C_{3312} & \sqrt{2}C_{3313} & \sqrt{2}C_{3323} \\
\sqrt{2}C_{1211} & \sqrt{2}C_{1221} & \sqrt{2}C_{1233} & 2C_{1212} & 2C_{1213} & 2C_{1223} \\
\sqrt{2}C_{1311} & \sqrt{2}C_{1322} & \sqrt{2}C_{1333} & 2C_{1312} & 2C_{1313} & 2C_{1323} \\
\sqrt{2}C_{2311} & \sqrt{2}C_{2322} & \sqrt{2}C_{2333} & 2C_{2312} & 2C_{2313} & 2C_{2323}
\end{pmatrix} \quad \text{(5.10)}$$

When compared with the Voigt mapping in (5.5), it can be seen that the components of the Kelvin mapping stiffness tensor $C$ contain the $\sqrt{2}$ factors. The manual manipulation can be avoided by the local stress update which is consistently performed with Kelvin mapping quantities (Nagel et al., 2016).

### 5.1.2 Numerical implementation

#### 5.1.2.1 Local stress iteration

The stress integration algorithm adopted in this study is elaborated in Bucher et al. (2001) and Nagel et al. (2016). The evolution equations in the hyperviscoelastic (4.65) and hyperviscoplastic (4.95) models have to be integrated at each load step to calculate the increments of the internal tensor variables $C_v$ and $C_p$. Thus the differential equations are discretised in time using the implicit Euler backward scheme formulated in (5.11) considering a general ordinary differential equation $\dot{y} = f(y)$ for a time step in the interval $[t, t + \Delta t]$.

$$y^{t+\Delta t} = y^t + \Delta t f^{t+\Delta t} \quad \text{(5.11)}$$
The time discretization of the differential and algebraic equations necessary to integrate the stress increment can be written in the form of (5.12) by defining a residual vector $\mathbf{r}$ and the state vector $\mathbf{z}$.

$$\mathbf{r}(\mathbf{z}, \epsilon^t) = \mathbf{0}$$ (5.12)

A Taylor series expansion of (5.12) considering only linear terms in the neighbourhood of a given iterative solution of $\mathbf{r}_j$ and $\frac{\partial \mathbf{r}}{\partial \mathbf{z}}_j$:

$$-\mathbf{r}^l = \left. \frac{\partial \mathbf{r}}{\partial \mathbf{z}} \right|_j \Delta \mathbf{z}^{i+1}$$ (5.13)

Once the iteration converges, the use of the total differential of $\mathbf{r}$ directly yields the consistent tangent matrix for the global iteration:

$$\frac{d\mathbf{r}}{d\epsilon^{t+\Delta t}} = \frac{\partial \mathbf{r}}{\partial \epsilon^{t+\Delta t}} + \left( \frac{\partial \mathbf{r}}{\partial \mathbf{z}} \right)_{t+\Delta t} \frac{d\mathbf{z}^{t+\Delta t}}{d\epsilon^{t+\Delta t}} = \mathbf{0}$$ (5.14)

In the resulting linear system, the first entry of the solution $d\mathbf{z}/d\epsilon^{t+\Delta t}$:

$$\left( \frac{\partial \mathbf{r}}{\partial \mathbf{z}} \right)_{t+\Delta t} \frac{d\mathbf{z}^{t+\Delta t}}{d\epsilon^{t+\Delta t}} = -\left. \frac{\partial \mathbf{r}}{\partial \epsilon} \right|_{t+\Delta t}$$ (5.15)

is the tangent matrix $\mathbf{C} = d\mathbf{\sigma}/d\epsilon$. This approach provides quadratic convergence of the global problem due to an algorithmically consistent linearisation (Simo and Hughes, 2000).

### 5.1.2.2 Global momentum balance

The tangent matrix derived from the local stress iteration is consistent with the global momentum balance. For the quasistatic equilibrium conditions, the balance of linear momentum in the current configuration (4.12) becomes:

$$\text{div}\mathbf{\sigma} + \rho \mathbf{b} = \mathbf{0}$$ (5.16)

To prepare a finite element implementation, a weak form is obtained by the integration, over a domain $\Omega$, of the multiplication between the momentum equation (5.16) and a kinematically compatible test function $\mathbf{v}$, where $\mathbf{v} = \mathbf{0} \forall \mathbf{x} \in \partial \Omega_u$, i.e. $\mathbf{v}$ vanishes on the Dirichlet boundary $\partial \Omega_u$:

$$\int_{\Omega} \mathbf{\sigma} : \text{grad} \mathbf{v} d\Omega = \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} d\Omega + \int_{\partial \Omega_i} \mathbf{t} \cdot \mathbf{v} d\Gamma$$ (5.17)

where $\partial \Omega_i$ is the Neumann boundary with $\partial \Omega_u \cup \partial \Omega_i = \partial \Omega$ and $\partial \Omega_u \cap \partial \Omega_i = \emptyset$. Pulling back the weak form from the spatial configuration (5.17) to the reference configuration (Bucher et al., 2001; Nagel et al., 2016), we get:
\begin{align*}
\int_{\Gamma^0} S : E(U; V) d\Omega^0 &= \int_{\Gamma^0} \rho_0 B \cdot V d\Omega^0 + \int_{\partial \Omega^0} T \cdot V d\Gamma^0 \\
(5.18)
\end{align*}

Due to the nonlinearities, a Newton-Raphson iteration procedure is used to get the solution for a time increment $\Delta t$ at time $t + \Delta t$ based on the known solution at time $t$. The displacement solution $U^{i+1}$ for the $i+1$th iteration is obtained via

$$U^{i+1} = U^i + \sum_{k=1}^{i+1} \Delta U^k$$  \hspace{1cm} (5.19)

External loads are assumed to be conservative loads, where work done by conservative loads is path-independent. Linearising (5.18) in a total Lagrangian formulation then yields

\begin{align*}
\int_{\Gamma^0} \left[ S^l : \tilde{E}(\Delta U^{i+1}, V) + E(U^i, V) : \frac{dS}{dE} \bigg|_{l} : E(U^i, \Delta U^{i+1}) \right] d\Omega^0 &= \\
= \int_{\Gamma^0} \rho_0 B^{i+\Delta t} \cdot V d\Omega^0 + \int_{\partial \Omega^0} T^{i+\Delta t} \cdot V d\Gamma^0 - \int_{\Gamma^0} S^l : E(U^i, V) d\Omega^0
\end{align*}

with

\begin{align*}
E(U^i, V) &= \text{sym} \left[ F^{iT} \text{Grad} V \right] \hspace{1cm} (5.20) \\
E(U^i, \Delta U) &= \text{sym} \left[ F^{iT} \text{Grad} V \right] \hspace{1cm} (5.21) \\
\tilde{E}(\Delta U^{i+1}, V) &= \text{sym} \left[ (\text{Grad} \Delta U^{i+1})^T \text{Grad} V \right] \hspace{1cm} (5.22)
\end{align*}

More details of the above linearisation can be found in Zienkiewicz and Taylor (2000), Bucher et al. (2001), Belytschko et al. (2013) and Nagel et al. (2016). The procedure for discretising the linearised weak formulation in Kelvin mapping is detailed in Nagel et al. (2016). The vector of unknown displacement increments $\Delta \hat{U}$ is solved through the linearised system defined by the residual vector $\psi$ and the stiffness matrix $K$:

$$K^i \Delta \hat{U}^{i+1} = \psi^i$$  \hspace{1cm} (5.23)

where the stiffness matrix $K$ is consistent with the tangent matrix $C$ obtained in the converged local stress iteration (5.15).
5.2 Numerical implementation and verification of the constitutive models

5.2.1 Hyperelastic model

5.2.1.1 Numerical implementation

For the modified Neo-Hooke model, the 2nd Piola-Kirchhoff stress derived from the Helmholtz free energy function is

\[
S = 2C_1 e^{α(I_1 - \ln I_3)} I + 2\left(2D_2 \ln I_3 - C_1 e^{α(I_1 - \ln I_3)}\right) C^{-1}
\]  

(5.24)

let

\[
S = C_1 \bar{S}
\]

(5.25)

so that \(\bar{S}\) is a dimensionless stress quantity.

Kelvin mapping is used for the transition of three-dimensional symmetric second order tensors into six-dimensional vectors and three-dimensional fourth order tensors into \(6 \times 6\) matrices. According to section 5.1.2.1, the overview numerical implementation of the modified Neo-Hooke model is summarised in Box: Modified Neo-Hooke Model.

---

**Box: Modified Neo-Hooke Model**

**Residual vector:**

\[
r = \bar{S} - 2\left[e^{α(I_1 - \ln I_3)} I + \left(2 \frac{D_2}{C_1} \ln I_3 - e^{α(I_1 - \ln I_3)}\right) C^{-1}\right]
\]

(5.26)

**State vector:**

\[
z = \bar{S}
\]

(5.27)

**Jacobian:**

\[
J = \frac{∂r}{∂\bar{S}} = I
\]

(5.28)

**RHS matrix for global tangent solution:**

\[
\frac{∂r}{∂E} = 2 \frac{∂r}{∂\bar{S}} = -4 \left[αe^{α(I_1 - \ln I_3)} I \otimes I + \left(2 \frac{D_2}{C_1} - αe^{α(I_1 - \ln I_3)}\right) C^{-1} \otimes C^{-1} - αe^{α(I_1 - \ln I_3)} \left(C^{-1} \otimes I + I \otimes C^{-1}\right) - \left(2 \frac{D_2}{C_1} \ln I_3 - e^{α(I_1 - \ln I_3)}\right) C^{-1} \otimes C^{-1}\right]
\]

(5.29)
5.2.1.2 Verification of Modified Neo-Hooke model

To model the undrained laboratory experiments on peat, an analytical solution for the undrained testing conditions is derived. For the undrained triaxial compression test with a compression of $\lambda(t) = \Delta h(t)/h_0$, taking vertical as the 11 direction, the deformation gradient (5.30), the Cauchy-Green tensor (5.31) and the Green-Lagrange strain tensor (5.32) only contain deviatoric deformations as

$$ F = \begin{pmatrix} \lambda(t) & 0 & 0 \\ 0 & \frac{1}{\sqrt[3]{\lambda(t)}} & 0 \\ 0 & 0 & \frac{1}{\sqrt[3]{\lambda(t)}} \end{pmatrix} $$

(5.30)

$$ C = \begin{pmatrix} \lambda(t)^2 & 0 & 0 \\ 0 & \frac{1}{\lambda(t)} & 0 \\ 0 & 0 & \frac{1}{\lambda(t)} \end{pmatrix} \quad C^{-1} = \begin{pmatrix} \lambda(t)^{-2} & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{pmatrix} $$

(5.31)

$$ E = \frac{1}{2} (C - I) = \frac{1}{2} \begin{pmatrix} \lambda(t)^2 - 1 & 0 & 0 \\ 0 & \frac{1}{\lambda(t)} - 1 & 0 \\ 0 & 0 & \frac{1}{\lambda(t)} - 1 \end{pmatrix} $$

(5.32)

Rearranging Equation (5.24) and introducing the indeterminate pressure term $p_s$, we get

$$ S = 2 \left[ C_1 e^{\alpha(I_1 - \ln I_3 - 3)} (I - C^{-1}) + p_s C^{-1} \right] $$

(5.33)

This solution strategy is motivated by the Flory split (Flory, 1961) of the deformation gradient into the volumetric and deviatoric parts. Substituting (5.31) into (5.33) and considering the zero confining pressure applied in the triaxial cell, i.e. $S_{22} = S_{33} = 0$, the pressure $p_s$ is obtained as

$$ p_s = C_1 e^{\alpha(I_1 - \ln I_3 - 3)} (1 - \lambda(t)^{-1}) $$

(5.34)

Thus, the vertical stress $S_{11}$ becomes

$$ S_{11} = 2 \left[ C_1 e^{\alpha(I_1 - \ln I_3 - 3)} (1 - \lambda(t)^{-3}) \right] $$

(5.35)

with the 11 component of the Green-Lagrange strain tensor $E_{11} = \frac{1}{2} (\lambda(t)^2 - 1)$.

The analytical solution was used for the validation of the numerical model with the undrained triaxial loading case where vertical displacement and the confining pressure were passed as the boundary conditions to the global Newton iterations. The material parameters for the validation are $C_1 = 43$ kPa, $D_2 = 1000$ kPa and $\alpha = 0.0, 0.5, 1.0$. Figure 5.1 presents the results and it shows good verification of the numerical implementation to the analytical results of the vertical and horizontal strains and that $S_{22}(= S_{33})$ are 0 kPa within tolerance.
5.2.2 Hyperviscoelastic model

5.2.2.1 Numerical implementation with one Maxwell element

Using the relations derived in Section 4.3.2, Equation (4.41) becomes

\[
S = 2 \left( \frac{\partial \psi_0}{\partial I_{\text{eq}0}^1} I + \frac{\partial \psi_0}{\partial I_{\text{eq}0}^3} I_{\text{eq}0}^3 C^{-1} \right) + 2 \left( \frac{\partial \psi_v}{\partial I_{\text{ev}}^1} C^{-1} + \frac{\partial \psi_v}{\partial I_{\text{ev}}^3} I_{\text{ev}}^3 C^{-1} \right)
\]  \hspace{1cm} (5.36)

The viscoelastic evolution equation (4.65):

\[
C_v = 4 \eta_v \left( \frac{\partial \psi_v}{\partial I_{\text{ev}}^1} C + I_{\text{ev}}^3 \frac{\partial \psi_v}{\partial I_{\text{ev}}^3} C_v \right)
\]  \hspace{1cm} (5.37)

The numerical implementation for the hyperviscoelastic model is summarized in Box: Hyperviscoelastic Model with One Maxwell Element.

Box: Hyperviscoelastic Model with One Maxwell Element

Residual vector:

\[
r_1 = \bar{S} - \frac{2}{C_1} \left( \frac{\partial \psi_0}{\partial I_{\text{eq}0}^1} I + \frac{\partial \psi_0}{\partial I_{\text{eq}0}^3} I_{\text{eq}0}^3 C^{-1} \right) - \frac{2}{C_1} \left( \frac{\partial \psi_v}{\partial I_{\text{ev}}^1} C^{-1} + \frac{\partial \psi_v}{\partial I_{\text{ev}}^3} I_{\text{ev}}^3 C^{-1} \right)
\]  \hspace{1cm} (5.38)

\[
r_2 = C_v - \frac{C_v}{\Delta t} - \frac{4}{\eta_v} \left( \frac{\partial \psi_v}{\partial I_{\text{ev}}^1} C + I_{\text{ev}}^3 \frac{\partial \psi_v}{\partial I_{\text{ev}}^3} C_v \right)
\]  \hspace{1cm} (5.39)

where \( \bar{S} = S / C_1 \) is a dimensionless stress measure, and

continued on next page...
\[ \frac{\partial I_{e}^{v}}{\partial C} = C_{v}^{-1} \quad \text{and} \quad \frac{\partial I_{e}^{v}}{\partial C} = I_{3}^{e} C_{v}^{-1} \]  
(5.40)

\[ \frac{\partial I_{e}^{v}}{\partial \mathbf{C}} = -C : C_{v}^{-1} \otimes C_{v}^{-1} \quad \text{and} \quad \frac{\partial I_{e}^{v}}{\partial \mathbf{C}} = -I_{3}^{e} C_{v}^{-1} \otimes C_{v}^{-1} \]  
(5.41)

State vector:
\[ \mathbf{z} = [\mathbf{S}^T, C_{v}]^T \]  
(5.42)

Jacobian:
\[ J_{11} = \frac{\partial \mathbf{r}_{1}}{\partial \mathbf{S}} = I \]  
(5.43)

\[ J_{12} = \frac{\partial \mathbf{r}_{1}}{\partial \mathbf{C}} \]
\[ = \frac{2}{C_{i}} \left[ \frac{\partial^2 \psi_{v}}{\partial (I_{1}^{e})^2} C_{v}^{-1} \otimes C + \frac{\partial^2 \psi_{v}}{\partial I_{1}^{e} \partial I_{3}^{e}} I_{3}^{e} (C_{v}^{-1} \otimes C_{v} + C^{-1} \otimes C) + \right. \]
\[ + \left( \frac{\partial^2 \psi_{v}}{\partial (I_{3}^{e})^2} I_{3}^{e} + \frac{\partial \psi_{v}}{\partial I_{3}^{e}} I_{3}^{e} C^{-1} \otimes C_{v} \right) \left( C_{v}^{-1} \otimes C_{v}^{-1} \right) \]  
(5.44)

\[ J_{21} = \frac{\partial \mathbf{r}_{2}}{\partial \mathbf{S}} = 0 \]  
(5.45)

\[ J_{22} = \frac{\partial \mathbf{r}_{2}}{\partial \mathbf{C}} \]
\[ = \left( \frac{1}{\Delta t} - \frac{4}{\eta_{v}} I_{3}^{e} \frac{\partial \psi_{v}}{\partial I_{3}^{e}} \right) I + \]
\[ + \frac{4}{\eta_{v}} \left[ \frac{\partial^2 \psi_{v}}{\partial (I_{3}^{e})^2} C \otimes C + I_{3}^{e} \frac{\partial^2 \psi_{v}}{\partial I_{3}^{e} \partial I_{3}^{e}} (C \otimes C_{v} + C_{v} \otimes C) + \right. \]
\[ + \left( \frac{\partial^2 \psi_{v}}{\partial (I_{3}^{e})^2} I_{3}^{e} + \frac{\partial \psi_{v}}{\partial I_{3}^{e}} I_{3}^{e} C_{v} \otimes C_{v} \right) \left( C_{v}^{-1} \otimes C_{v}^{-1} \right) \]  
(5.46)

where \( I \) in the above equations is the Kelvin mapping (6 × 6 matrix) of the fourth order identity tensor.

continued on next page...
Box: Hyperviscoelastic Model with One Maxwell Element

RHS matrix for global tangent solution:

\[
\begin{align*}
\frac{\partial r_1}{\partial E} &= 2 \frac{\partial r_1}{\partial C} \\
&= -\frac{4}{C_1} \left( \frac{\partial^2 \psi_0}{\partial (I_0^{eq})^2} I \otimes I + \frac{\partial^2 \psi_0}{\partial I_0^{eq} \partial I_3^{eq}} I_3^{eq} (C^{-1} \otimes I + I \otimes C^{-1}) + \frac{\partial^2 \psi_v}{\partial (I_0^{eq})^2} I_3^{eq} C^{-1} \otimes C^{-1} \right) \\
&\quad + \frac{\partial^2 \psi_v}{\partial (I_0^{eq})^2} I_3 I_3^{eq} (C^{1-} \otimes C^{1-} + C^{1-} \otimes C^{1-}) + \frac{\partial \psi_v}{\partial I_0^{ev}} C_v \otimes C^{-1} \\
&\quad + \frac{\partial \psi_v}{\partial I_0^{ev}} I_3^{ev} C^{-1} \otimes C^{-1} \frac{\partial \psi_v}{\partial I_0^{ev}} I_3^{ev} C^{-1} \otimes C^{-1}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial r_2}{\partial E} &= 2 \frac{\partial r_2}{\partial C} \\
&= -\frac{8}{\eta_v} \left[ \frac{\partial^2 \psi_v}{\partial (I_0^{ev})^2} C_v \otimes C_v^{-1} + I_3^{ev} \frac{\partial^2 \psi_v}{\partial I_0^{ev} \partial I_3^{ev}} (C \otimes C^{-1} + C_v \otimes C_v^{-1}) + \frac{\partial \psi_v}{\partial I_0^{ev}} I + \frac{\partial^2 \psi_v}{\partial (I_0^{ev})^2} I_3^{ev} \left( \frac{\partial \psi_v}{\partial I_0^{ev}} C_v \otimes C^{-1} \right) \right]
\end{align*}
\]

(5.47)

where \( \frac{\partial r}{\partial E} \) is Kelvin mapping (6×1 matrix) of the second order identity tensor; \( \frac{\partial r}{\partial E} \) is Kelvin mapping (6×6 matrix) of the fourth order identity tensor.

### 5.2.2.2 Verification with one Maxwell element

The numerical implementation of the proposed hyperviscoelastic model is verified against the analytical solution provided in Nagel (2012). Taking \( D_2 = D_{2v} = 0 \) kPa and \( \alpha = \alpha_v = 0 \), the 2\textsuperscript{nd} Piola-Kirchhoff equilibrium stress and overstress follow as

\[
\begin{align*}
S_{eq} &= 2 C_1 (I - C^{-1}) \\
S_{ov} &= 2 C_{1v} (C_v^{-1} - C^{-1})
\end{align*}
\]

(5.49) \hspace{1cm} (5.50)

The Right Cauchy Green tensors and their inverse for uniaxial loading with an axial stretch/compression of \( \lambda \) are:

\[
\begin{align*}
C_{ij} &= \text{diag}(\lambda^2, 1, 1) \\
(C^{-1})_{ij} &= \text{diag}(\lambda^{-2}, 1, 1) \\
(C_v)_{ij} &= \text{diag}(\lambda_v^2, 1, 1) \\
(C_v^{-1})_{ij} &= \text{diag}(\lambda_v^{-2}, 1, 1)
\end{align*}
\]

(5.51) \hspace{1cm} (5.52)
Their axial stress components are

\[ S_{eq}^{11} = 2C_1(1 - \lambda^{-2}) \]  
\[ S_{ov}^{11} = 2C_{1v}(\lambda_{eq}^{-2} - \lambda^{-2}) \]  

The evolution equation (4.69) can be simplified to

\[ \dot{C}_v = \frac{4C_{1v}}{\eta_v}(C - C_v) \]  

with the one component of relevance

\[ \dot{\lambda}_v^2 = \frac{4C_{1v}}{\eta_v} \left( \lambda^2 - \lambda_{eq}^2 \right) \]  

At constant viscosity \( \eta_v \), constant \( C_{1v} \) and a given stretch \( \lambda \), it can be solved as

\[ \lambda_v^2 = \lambda^2 - (\lambda^2 - \lambda_{eq}^2)e^{-\frac{4C_{1v}}{\eta_v}(t-t_0)} \]  

Including one Maxwell element parallel to the equilibrium spring (Figure. 4.2), the total stress

\[ S^{11} = 2C_1(1 - \lambda^{-2}) + 2C_{1v}(\lambda_{eq}^{-2} - \lambda^{-2}) \]  

The proposed model with the numerical implementation is validated against the above analytical solution. The uniaxial test is loaded to an axial compression of \( \lambda = 0.95 \) for 2.5 min and relaxed for 7.5 min. 101 steps in total were implemented in the 10 min ramping (increase in compressive load) and relaxation. The model parameters are listed in Table 5.1. The validation results are presented in Figure 5.2 with the Green-Lagrange strain in Equation (5.32), which shows good agreement between the numerical and analytical solutions.

<table>
<thead>
<tr>
<th>( C_1 ) (kPa)</th>
<th>( D_2 ) (kPa)</th>
<th>( \alpha )</th>
<th>( C_{1v} ) (kPa)</th>
<th>( D_{2v} ) (kPa)</th>
<th>( \alpha_v )</th>
<th>( \eta_v ) (kPa-min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>43.0</td>
<td>0.0</td>
<td>0.0</td>
<td>50.0</td>
<td>0.0</td>
<td>0.0</td>
<td>200.0</td>
</tr>
</tbody>
</table>

Table 5.1: Model parameters for validation.

### 5.2.2.3 Numerical implementation with two Maxwell elements

The hyperviscoelastic model with two Maxwell elements is also validated, similarly to the model with one Maxwell element. From Equation (5.36), the extended total 2nd Piola-Kirchhoff stress is:

\[ S = 2\rho_0 \left( \frac{\partial \tilde{\psi}_0}{\partial I_{eq}^{3\xi}} I + \frac{\partial \tilde{\psi}_0}{\partial I_{eq}^{3\xi}} I_{eq}^{3\xi} C^{-1} \right) + \]
\[ + 2\rho_0 \left( \frac{\partial \tilde{\psi}_{v1}}{\partial I_{ev1}^{v1}} C_{v1}^{-1} + \frac{\partial \tilde{\psi}_{v1}}{\partial I_{ev1}^{v1}} I_{ev1}^{v1} C^{-1} \right) + 2\rho_0 \left( \frac{\partial \tilde{\psi}_{v2}}{\partial I_{ev2}^{v2}} C_{v2}^{-1} + \frac{\partial \tilde{\psi}_{v2}}{\partial I_{ev2}^{v2}} I_{ev2}^{v2} C^{-1} \right) \]  

(5.59)
Figure 5.2: Model validation with analytical solution of uniaxial test.

For the second Maxwell element, an independent Helmholtz free energy function was adopted. Thus an extra evolution equation was added in the same form as Equation (4.64). The numerical implementation of the hyperviscoelastic model with two Maxwell elements is summarized in Box: Hyperviscoelastic Model with Two Maxwell Elements.

Box: Hyperviscoelastic Model with Two Maxwell Element

Residual vector:

\[ \mathbf{r}_1 = \mathbf{S} - \frac{2}{C_1} \left( \frac{\partial \psi_0}{\partial I_1^\text{eq0}} I_1 + \frac{\partial \psi_0}{\partial I_3^\text{eq0}} I_3^\text{eq0} \mathbf{C}^{-1} \right) - \frac{2}{C_1} \left( \frac{\partial \psi_{v1}}{\partial I_1^{ev1}} \mathbf{C}_{v1}^{-1} + \frac{\partial \psi_{v1}}{\partial I_3^{ev1}} I_3^{ev1} \mathbf{C}^{-1} \right) - \frac{2}{C_1} \left( \frac{\partial \psi_{v2}}{\partial I_1^{ev2}} \mathbf{C}_{v2}^{-1} + \frac{\partial \psi_{v2}}{\partial I_3^{ev2}} I_3^{ev2} \mathbf{C}^{-1} \right) \]

\[ (5.60) \]

\[ \mathbf{r}_2 = \frac{\mathbf{c}_{v1} - \mathbf{c}_{v1}}{\Delta t} = \frac{4}{\eta_{v1}} \left( \frac{\partial \psi_{v1}}{\partial I_1^{ev1}} \mathbf{C} + I_3^{ev1} \frac{\partial \psi_{v1}}{\partial I_3^{ev1}} \mathbf{C}_{v1} \right) \]

\[ (5.61) \]

\[ \mathbf{r}_3 = \frac{\mathbf{c}_{v2} - \mathbf{c}_{v2}}{\Delta t} = \frac{4}{\eta_{v2}} \left( \frac{\partial \psi_{v2}}{\partial I_1^{ev2}} \mathbf{C} + I_3^{ev2} \frac{\partial \psi_{v2}}{\partial I_3^{ev2}} \mathbf{C}_{v2} \right) \]

\[ (5.62) \]

continued on next page...
State vector:
\[ z = [S^T, C^T_{v1}, C^T_{v2}]^T \]  

(5.63)

Jacobian:

\[ J_{11} = \frac{\partial r_1}{\partial S} = I \]  

(5.64)

\[ J_{12} = \frac{\partial r_1}{\partial C_{v1}} = \begin{bmatrix} \frac{\partial^2 \psi_{v1}}{\partial (I_{v1}^1)^2} C_{v1}^{-1} \otimes C + & \frac{\partial^2 \psi_{v1}}{\partial I_{v1}^1 \partial I_{v1}^2} I_{3}^{ev1} C_{v1}^{-1} \otimes C + C^{-1} \otimes C \end{bmatrix} + \left( \frac{\partial^2 \psi_{v1}}{\partial (I_{v1}^2)^2} I_{3}^{ev1} + \frac{\partial \psi_{v1}}{\partial I_{v1}^2} \right) I_{3}^{ev1} C_{v1}^{-1} \otimes C_{v1} \]  

\[ + \left( \frac{\partial^2 \psi_{v1}}{\partial I_{v1}^1 \partial I_{v1}^3} I_{3}^{ev1} + \frac{\partial \psi_{v1}}{\partial I_{v1}^3} \right) I_{3}^{ev1} C_{v1}^{-1} \otimes C_{v1} \]  

(5.65)

\[ J_{13} = \frac{\partial r_1}{\partial C_{v2}} = \begin{bmatrix} \frac{\partial^2 \psi_{v2}}{\partial (I_{v2}^2)^2} C_{v2}^{-1} \otimes C + & \frac{\partial^2 \psi_{v2}}{\partial I_{v2}^1 \partial I_{v2}^2} I_{3}^{ev2} C_{v2}^{-1} \otimes C + C^{-1} \otimes C \end{bmatrix} + \left( \frac{\partial^2 \psi_{v2}}{\partial (I_{v2}^3)^2} I_{3}^{ev2} + \frac{\partial \psi_{v2}}{\partial I_{v2}^3} \right) I_{3}^{ev2} C_{v2}^{-1} \otimes C_{v2} \]  

\[ + \left( \frac{\partial^2 \psi_{v2}}{\partial I_{v2}^1 \partial I_{v2}^3} I_{3}^{ev2} + \frac{\partial \psi_{v2}}{\partial I_{v2}^3} \right) I_{3}^{ev2} C_{v2}^{-1} \otimes C_{v2} \]  

(5.66)

\[ J_{21} = \frac{\partial r_2}{\partial S} = 0 \]  

(5.67)

\[ J_{22} = \frac{\partial r_2}{\partial C_{v1}} = \begin{bmatrix} \frac{1}{\Delta t} & -\frac{4}{\eta_{v1} I_{3}^{ev1}} \frac{\partial \psi_{v1}}{\partial I_{3}^{ev1}} C \otimes C + I_{3}^{ev1} \frac{\partial^2 \psi_{v1}}{\partial I_{v1}^1 \partial I_{v1}^2} I_{3}^{ev1} C_{v1}^{-1} \otimes C + C_{v1}^{-1} \otimes C \end{bmatrix} + \left( \frac{\partial^2 \psi_{v1}}{\partial (I_{v1}^2)^2} I_{3}^{ev1} + \frac{\partial \psi_{v1}}{\partial I_{v1}^2} \right) I_{3}^{ev1} C_{v1}^{-1} \otimes C_{v1} \]  

(5.68)

\[ J_{23} = \frac{\partial r_2}{\partial C_{v2}} = 0 \]  

(5.69)

\[ J_{31} = \frac{\partial r_3}{\partial S} = 0 \]  

(5.70)

\[ J_{32} = \frac{\partial r_3}{\partial C_{v1}} = 0 \]  

(5.71)

continued on next page...
Box: Hyperviscoelastic Model with Two Maxwell Element

...continued following former page

\[ J_{33} = \frac{\partial r_3}{\partial C_{v2}} \]
\[ = \left( \frac{1}{\Delta t} - \frac{4}{\eta_{v2}} I_{3} I_{3} \frac{\partial \psi_{v2}}{\partial I_{3}} \right) I_{3} + \frac{4}{\eta_{v2}} \left( \frac{\partial^2 \psi_{v2}}{\partial (I_{r} I_{v2})^2} C \otimes C + I_{3} I_{3} \frac{\partial^2 \psi_{v2}}{\partial I_{3} \partial I_{3}} (C \otimes C_{v2} + C_{v2} \otimes C) + \left( \frac{\partial^2 \psi_{v2}}{\partial (I_{r} I_{v2})^2} I_{3} I_{3} C_{v2} \otimes C_{v2} \right) (C_{v2} \otimes C_{v2}^{-1}) \right) \]  

(5.72)

Additional components of the RHS matrix for global tangent solution:

\[ \frac{\partial r_1}{\partial E} = 2 \frac{\partial r_1}{\partial C} \]
\[ = -\frac{4}{C_{1}} \left[ \frac{\partial^2 \psi_{v1}}{\partial (I_{r} I_{v1})^2} I_{3} I_{3} \frac{\partial \psi_{v1}}{\partial I_{3}} I_{3} \left( C^{-1} \otimes C + I \otimes C^{-1} \right) + \left( \frac{\partial^2 \psi_{v1}}{\partial (I_{r} I_{v1})^2} I_{3} I_{3} \frac{\partial \psi_{v1}}{\partial I_{3}} I_{3} \left( C^{-1} \otimes C^{-1} + C_{v1} \otimes C_{v1} \right) \right) \right] \]  

(5.73)

\[ \frac{\partial r_3}{\partial E} = 2 \frac{\partial r_3}{\partial C} \]
\[ = -\frac{8}{\eta_{v2}} \left[ \frac{\partial^2 \psi_{v2}}{\partial (I_{r} I_{v2})^2} C \otimes C_{v2}^{-1} + I_{3} I_{3} \frac{\partial^2 \psi_{v2}}{\partial I_{3} \partial I_{3}} (C \otimes C^{-1} + C_{v2} \otimes C_{v2}^{-1}) + \frac{\partial \psi_{v2}}{\partial I_{r} I_{v2}} + \left( \frac{\partial^2 \psi_{v2}}{\partial (I_{r} I_{v2})^2} I_{3} I_{3} \right) C_{v2} \otimes C^{-1} \right] \]  

(5.74)

(5.75)

where \( I \) in Jacobian components and in \( \frac{\partial r_1}{\partial E} \) is the Kelvin mapping (6×6 matrix) of the fourth order identity tensor, \( I \) in \( \frac{\partial r_3}{\partial E} \) is the Kelvin mapping (6×1 matrix) of the second order identity tensor.
The global tangent, in addition to Equations (5.47) and (5.48) for \( \psi_{v1} \) with material parameters \( C_{1v1}, D_{2v1}, \eta_{v1}, \alpha_{v1} \) and the viscous Right Cauchy-Green tensor \( C_{v1}, \frac{\partial \varepsilon_{r1}}{\partial \varepsilon} \) is updated as in Equation (5.74) and the third component can be determined in Equation (5.75).

### 5.2.2.4 Verification with two Maxwell elements

Including two Maxwell elements parallel to the equilibrium spring, the total stress (5.58) becomes

\[
S^{11} = 2C_1 \left( 1 - \lambda^{-2} \right) + 2C_{1v} \left( \lambda_v^{-2} - \lambda_v^{-2} \right) + 2C_{1v2} \left( \lambda_{v2}^{-2} - \lambda_{v2}^{-2} \right)
\]

(5.76)

with an additional viscous Right Cauchy Green tensor for uniaxial loading:

\[
(C_{v2})_{ij} = \text{diag}(\lambda_{v2}^{-2}, 1, 1)
\]

(5.77)

The result of the validation is presented in Figure 5.3 with the parameters in Table 5.2 and \( E_{11} \) is the 11 component of the Green Lagrange strain tensor in Equation (5.32), where the values for \( \lambda_v \) and \( \lambda_{v2} \) are the initial input values of viscous stretch for the analytical solution.

![Figure 5.3: Validation of the proposed numerical implementation with its analytical solution of the hyperviscoelastic model with two Maxwell elements.](image)

Table 5.2: Validation parameters for the hyperviscoelastic model with two Maxwell elements, where the unit of \( C_1, C_{1v1}, C_{1v2}, D_2, D_{2v1}, D_{2v2} \) is kPa; the viscosity \( (\eta_{v1}, \eta_{v2}) \) unit is kPa·min.

| \( C_1 \) | \( D_2 \) | \( \alpha \) | \( \lambda \) | \( C_{1v1} \) | \( D_{2v1} \) | \( \alpha_{v1} \) | \( \lambda_{v1}^0 \) | \( \eta_{v1} \) | \( C_{1v2} \) | \( D_{2v2} \) | \( \alpha_{v2} \) | \( \lambda_{v2}^0 \) | \( \eta_{v2} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 43.0 | 0.0 | 0.0 | 0.95 | 50.0 | 0.0 | 0.0 | 1.0 | 100.0 | 70.0 | 0.0 | 0.0 | 1.0 | 200.0 |
5.2.3 hyperviscoplastic model

5.2.3.1 Numerical implementation of the hyperviscoplastic model

Total 2\textsuperscript{nd} Piola-Kirchhoff stress of the hyperviscoplastic model is

\[ S = 2 \left( \frac{\partial \psi_0}{\partial I_{eq0}} I + \frac{\partial \psi_0}{\partial I_{eq0}^T I_3} I_{eq0} C^{-1} \right) + 2 \left( \frac{\partial \psi_v}{\partial I_{ev1}} C_{v1}^{-1} + \frac{\partial \psi_v}{\partial I_{ev1}^T I_3} I_{ev1} C^{-1} \right) + 2 \left( \frac{\partial \psi_v}{\partial I_{ev2}} C_{v2}^{-1} + \frac{\partial \psi_v}{\partial I_{ev2}^T I_3} I_{ev2} C^{-1} \right) + 2 \left( \frac{\partial \psi_p}{\partial I_{ep1}} C_{p1}^{-1} + \frac{\partial \psi_p}{\partial I_{ep1}^T I_3} I_{ep1} C^{-1} \right) \]\n
(5.78)

Combining the three (hyperelastic, hyperviscoelastic, hyperelastoplastic) components of the proposed model, the numerical implementation is summarized in Box: hyperviscoplastic Model.

Box: hyperviscoplastic Model

Residual vector:

\[ r_1 = \tilde{S} - \frac{2}{C_1} \left( \frac{\partial \psi_0}{\partial I_{eq0}} I + \frac{\partial \psi_0}{\partial I_{eq0}^T I_3} I_{eq0} C^{-1} \right) - \frac{2}{C_1} \left( \frac{\partial \psi_v}{\partial I_{ev1}} C_{v1}^{-1} + \frac{\partial \psi_v}{\partial I_{ev1}^T I_3} I_{ev1} C^{-1} \right) \]

(5.79)

\[ r_2 = \frac{C_{v1} - C_{v1}^t}{\Delta t} = \frac{4}{\eta_{v1}} \left( \frac{\partial \psi_v}{\partial I_{ev1}} C + I_{ev1} ^T \frac{\partial \psi_v}{\partial I_{ev1}^T} C_{v1} \right) \]

(5.80)

\[ r_3 = \frac{C_{v2} - C_{v2}^t}{\Delta t} = \frac{4}{\eta_{v2}} \left( \frac{\partial \psi_v}{\partial I_{ev2}} C + I_{ev2} ^T \frac{\partial \psi_v}{\partial I_{ev2}^T} C_{v2} \right) \]

(5.81)

\[ r_4 = \frac{C_{p} - C_{p}^t}{\Delta t} - 2c_p \frac{||C-C||}{\Delta t} \left( \frac{\partial \psi_p}{\partial I_{ep1}} C + I_{ep1} ^T \frac{\partial \psi_p}{\partial I_{ep1}^T} C_{p} \right) \]

(5.82)

State vector:

\[ z = \left[ \tilde{S}^T, C_{v1}^T, C_{v2}^T, C_{p}^T \right]^T \]

(5.83)

To solve the state vector, the Jacobian is of the size 4 \times 4. Comparing to the hyperviscoelastic model, the additional components of the hyperviscoplastic Jacobian are: 

continued on next page...
Jacobian:

\[
J_{14} = \frac{\partial r_1}{\partial C_p} = \frac{2}{C_1} \left[ \frac{\partial^2 \psi_p}{\partial (I_1^{ep})^2} C_p^{-1} \otimes C + \frac{\partial^2 \psi_p}{\partial I_1^{ep} \partial I_3^{ep}} I_3^{ep} \left( C_p^{-1} \otimes C_p + C^{-1} \otimes C \right) + \left( \frac{\partial^2 \psi_p}{\partial (I_3^{ep})^2} + \frac{\partial \psi_p}{\partial I_3^{ep}} \right) I_3^{ep} C^{-1} \otimes C_p \right] (C_p^{-1} \otimes C_p^{-1}) + \frac{\partial \psi_p}{\partial I_3^{ep}} C_p^{-1} \otimes C_p^{-1} \\
\]

\[
J_{24} = J_{34} = J_{41} = J_{42} = J_{43} = 0 \\
J_{44} = \frac{\partial r_1}{\partial C_p} = \frac{I}{\Delta t} \left( 1 - 2 c_p ||C - C||^2 I_3^{ep} \frac{\partial \psi_p}{\partial I_3^{ep}} \right) + \frac{2 c_p ||C - C|| I_1^{ep}}{\Delta t} \left[ \frac{\partial^2 \psi_p}{\partial (I_1^{ep})^2} C \otimes C + \left( \frac{\partial \psi_p}{\partial I_1^{ep}} I_3^{ep} \otimes C_p \right) (C_p \otimes C_p^{-1}) + \left( \frac{\partial^2 \psi_p}{\partial (I_3^{ep})^2} + \frac{\partial \psi_p}{\partial I_3^{ep}} \right) I_3^{ep} C_p \otimes C_p \right] (C_p^{-1} \otimes C_p^{-1}) \\
\]

(5.84)

(5.85)

(5.86)

continued on next page...
5.2.3.2 Verification of the hyperelastoplastic model

Similarly to the model verifications of the hyperviscoelastic models, the hyperelastoplastic model is verified against an analytical solution in uniaxial loading with a compression of...
To eliminate the viscoelastic components in the hyperviscoplastic model, the material parameters for the viscoelastic models were chosen as in Table 5.3 where the springs in the viscoelastic model were set significantly weaker than the spring resistance in the hyperelastoplastic model and trivial viscosities. Taking $D_2 = D_{2p} = 0$ kPa and $\alpha = \alpha_p = 0$, the Helmholtz free energy for the hyperelastoplastic model becomes

$$\psi = C_1 \left( I_1^{ep} - \ln I_3^{ep} - 3 \right) + C_{1p} \left( I_1^{ep} - \ln I_3^{ep} - 3 \right)$$

(5.90)

Therefore the 2nd Piola Kirchhoff equilibrium stress and plastic stress follow as

$$S_{eq0} = 2C_1 \left( I - C^{-1} \right)$$

(5.91)

$$S_p = 2C_{1p} \left( C_p^{-1} - C^{-1} \right)$$

(5.92)

The Right Cauchy Green tensors for uniaxial loading are:

$$C_{ij} = \text{diag} \left( \lambda^2, 1, 1 \right) \quad \text{and} \quad C_{ij}^{-1} = \text{diag} \left( \lambda^{-2}, 1, 1 \right)$$

(5.93)

$$\left( C_p \right)_{ij} = \text{diag} \left( \lambda_p^2, 1, 1 \right) \quad \text{and} \quad \left( C_p \right)^{-1} = \text{diag} \left( \lambda_p^{-2}, 1, 1 \right)$$

(5.94)

The plastic flow rule becomes

$$\dot{C}_p = 2c_p C_{1p} \| \dot{\lambda} \| \left( C - C_p \right)$$

(5.95)

The main component of the strains gets

$$\dot{\lambda}_p = 2c_p C_{1p} \| \dot{\lambda}^2 \| \left( \lambda^2 - \lambda_p^2 \right)$$

(5.96)

The Right Cauchy Green tensors for uniaxial loading are:

$$C_{ij} = \text{diag} \left( \lambda^2, 1, 1 \right) \quad \text{and} \quad C_{ij}^{-1} = \text{diag} \left( \lambda^{-2}, 1, 1 \right)$$

(5.93)

$$\left( C_p \right)_{ij} = \text{diag} \left( \lambda_p^2, 1, 1 \right) \quad \text{and} \quad \left( C_p \right)^{-1} = \text{diag} \left( \lambda_p^{-2}, 1, 1 \right)$$

(5.94)

The plastic flow rule becomes

$$\dot{C}_p = 2c_p C_{1p} \| \dot{\lambda} \| \left( C - C_p \right)$$

(5.95)

The main component of the strains gets

$$\dot{\lambda}_p = 2c_p C_{1p} \| \dot{\lambda}^2 \| \left( \lambda^2 - \lambda_p^2 \right)$$

(5.96)

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(5.93)

$$\left( C_p \right)_{ij} = \text{diag} \left( \lambda_p^2, 1, 1 \right) \quad \text{and} \quad \left( C_p \right)^{-1} = \text{diag} \left( \lambda_p^{-2}, 1, 1 \right)$$

(5.94)

The plastic flow rule becomes

$$\dot{C}_p = 2c_p C_{1p} \| \dot{\lambda} \| \left( C - C_p \right)$$

(5.95)

The main component of the strains gets

$$\dot{\lambda}_p = 2c_p C_{1p} \| \dot{\lambda}^2 \| \left( \lambda^2 - \lambda_p^2 \right)$$

(5.96)

Taking constant model parameters $c_p$ and $C_{1p}$, the above equation (details see Appendix B.2) is solved in the time range of $[0, t_f]$ (equivalent to the plastic compression range of $[\lambda_p0, \lambda_pf]$ as well as to the total compression range of $[\lambda_0, \lambda_f]$), giving the absolute value of total compression rate $\| \dot{\lambda}^2 \| = -\dot{\lambda}^2$ during loading and $\| \dot{\lambda}^2 \| = \dot{\lambda}^2$ during unloading, so that for the loading stage

Table 5.3: Validation parameters for the rate-independent hyperelastoplastic model.
\[ \lambda^2_p = \frac{1}{2c_p C_{1p}} (1 + 2c_p C_{1p} \lambda^2) + \left[ \lambda^2_{p0} - \lambda^2_0 - \frac{1}{2c_p C_{1p}} \right] e^{2c_p C_{1p} (\lambda^2 - \lambda_0^2)} \]  
(5.97)

and for the unloading stage

\[ \lambda^2_p = \frac{1}{2c_p C_{1p}} (2c_p C_{1p} \lambda^2 - 1) + \left[ \lambda^2_{p0} - \lambda^2_0 + \frac{1}{2c_p C_{1p}} \right] e^{2c_p C_{1p} (\lambda^2 - \lambda_0^2)} \]  
(5.98)

The total stress in the 11 direction is:

\[ S_{11} = 2C_1 (1 - \lambda^{-2}) + 2C_{1p} (\lambda^{-2} - \lambda^{-2}) \]  
(5.99)

with the component of the Green-Lagrange strain in the 11 direction as

\[ E_{11} = -\frac{1}{2} (\lambda^2 - 1) \]  
(5.100)

where the minus sign converts the compressive stress into positive in soil mechanics practice. Figure 5.4 shows that the analytical solution of the hyperelastoplastic model is congruent with its numerical implementation in both the loading and unloading cases. The analytical solutions of the plastic stretch in (5.97) and (5.98) indicate that the plastic strain is dependent on the current strain as well as on the strain history, i.e. the values of the total and plastic strains at the start of the tests.

Figure 5.4: Analytical verification of the hyperelastoplastic model.

### 5.3 Jacobian and global tangent check

The numerical Jacobian calculated with (5.101) is checked against the analytical Jacobian, where \( r_i \) stands for the \( i^{th} \) vector component of the residual vector \( \mathbf{r} \); \( z_j \) stands for the \( j^{th} \) component of the internal state vector \( \mathbf{z} \); \( \epsilon \) is a small perturbation with a value of \( 1.0 \times 10^{-8} \). The absolute value of the maximum deviation of the \( 24 \times 24 \) components between
the analytical and numerical Jacobians is $8.73 \times 10^{-8}$ with the non-zero material parameters presented in Table 5.4, which is at the same order of magnitude of tolerance.

$$J_{ij} = \frac{\Delta r_i}{\Delta z_j} = \frac{r_i(z_j + \varepsilon) - r_i(z_j - \varepsilon)}{2\varepsilon}$$  \hspace{1cm} (5.101)

The global tangent, obtained by adding the elastoplasticity, has the component $\frac{\partial r_i}{\partial \varepsilon}$ updated (Equation (5.87)) and includes the new component $\frac{\partial r_i}{\partial \varepsilon}$ as in Equation (5.88). The analytical global tangent is checked against numerical global tangent calculated using (5.102), where $\varepsilon_j$ is the $j$th component of strain vector $\varepsilon$. The maximum absolute value of the difference between the $24 \times 6$ components in the analytical and numerical global tangents having the non-zero model parameters given in Table 5.4 is $2.41 \times 10^{-7}$.

$$\frac{\Delta r_i}{\Delta E_j} = \frac{r_i(\varepsilon_j + \varepsilon) - r_i(\varepsilon_j - \varepsilon)}{2\varepsilon}$$  \hspace{1cm} (5.102)

<table>
<thead>
<tr>
<th>Modified Neo-Hookean Model</th>
<th>$C_1$</th>
<th>$D_2$</th>
<th>$\alpha$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^{st}$ Maxwell Element</td>
<td>$C_{v1}$</td>
<td>$D_{2v1}$</td>
<td>$\alpha_{v1}$</td>
<td>$\lambda_{v1}$</td>
</tr>
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<td>0.5</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$2^{nd}$ Maxwell Element</td>
<td>$C_{v2}$</td>
<td>$D_{2v2}$</td>
<td>$\alpha_{v2}$</td>
<td>$\lambda_{v2}$</td>
</tr>
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<td>0.6</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>Hyperelastic Model</td>
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<td>$D_{2p}$</td>
<td>$\alpha_p$</td>
<td>$\lambda_p$</td>
</tr>
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<td></td>
<td>1.0</td>
<td>1.5</td>
<td>0.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 5.4: Material parameters for testing the global tangent of the hyperviscoplastic model.

5.4 Load cases and parameter sensitivity tests

5.4.1 Load cases for the rate-dependent model

As the rate-dependent behaviour is solely governed by the viscoelastic components of the proposed model, different load cases were tested for the hyperviscoelastic material model to investigate its capability of simulating different testing conditions, such as loading-unloading, loading-relaxation-unloading, loading-relaxation-unloading-relaxation, creep tests, etc. Once approved with the hyperviscoelastic model, the same load cases would apply in the extended hyperviscoplastic model.

According to the loading conditions of the experiments carried out in a triaxial cell, the following load cases have been implemented numerically with the hyperviscoelastic model parameters in Table 5.2.

- single stage compression followed by relaxation (Figure 5.3);
• single stage compression followed by unloading with relaxation (Figure 5.5);
• single stage compression with relaxation followed by unloading with relaxation (Figure 5.6);
• multi-stage compression with relaxations followed by single stage unloading (Figure 5.7);
• multi-stage compression with relaxations followed by multi-stage unloading with relaxations (Figure 5.8).

5.4.2 Material parameter sensitivity tests for hyperviscoelastic model

The influence of the various viscoelastic parameters on the performance of the models is tested. Material parameter sensitivity tests on the model performance for the 6 test simulations are listed in Table 5.5. The sets have the following motivations:

• $C_{1v}$: the spring stiffness of the first Maxwell element is varied.
Figure 5.7: Multi-stage compression with relaxations followed by unloading with relaxation.

Figure 5.8: Multi-stage compression with relaxations followed by unloading with relaxation.

<table>
<thead>
<tr>
<th>Parameter (unit)</th>
<th>$C_1$ (kPa)</th>
<th>$C_{1v}$ (kPa)</th>
<th>$D_{2v}$ (kPa)</th>
<th>$\eta_{v1}$ (kPa-min)</th>
<th>$\eta_{v2}$ (kPa-min)</th>
<th>$\lambda$</th>
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<td>0.5/1.0/2.0</td>
<td>0.0</td>
<td>10.0</td>
<td>-</td>
<td>0.95</td>
</tr>
<tr>
<td>Set $\eta_v$</td>
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<td>1.0</td>
<td>0.0</td>
<td>2.5/5.0/10.0</td>
<td>-</td>
<td>0.95</td>
</tr>
<tr>
<td>Set $\lambda$</td>
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<td>1.0</td>
<td>0.0</td>
<td>10.0</td>
<td>-</td>
<td>0.8/0.9/0.95</td>
</tr>
<tr>
<td>Set 2 Maxwell</td>
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<td>1.0</td>
<td>1.0</td>
<td>10.0/25.0</td>
<td>2.5/1.0</td>
<td>0.95</td>
</tr>
<tr>
<td>Set $D_{2v}$</td>
<td>1.0</td>
<td>0.5</td>
<td>0.5/1.0/2.0</td>
<td>25.0</td>
<td>-</td>
<td>0.95</td>
</tr>
<tr>
<td>Set rate</td>
<td>1.0</td>
<td>1.0</td>
<td>0.0</td>
<td>25.0</td>
<td>5.0</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Table 5.5: Material parameter test sets.

- $\eta_{v1}$: the viscosity of the dashpot of the first Maxwell element is varied.
- $\eta_{v2}$: the viscosity of the dashpot of the second Maxwell element is varied.
- $\lambda$: the applied compression level is varied.
- $D_{2v}$: The bulk modulus of the first Maxwell element is varied. No analytical solution.
is available for this case.

- Strain rate: the loading rates are varied to observe how the peaks behave in relation to the instantaneous-load relaxation curve, also with numerical results only.

The influence of the parameters is presented in Figure 5.9. From Figure 5.9(a), it can be seen that the parameter $C_{1v}$ effects the stress peak value as well as the relaxation behaviour as shown in Equation (5.57), thus $\eta_v$ has a similar effect as $C_{1v}$ as can be seen in Figure 5.9(b). The effect of the magnitude of compression was tested with a compression rate, $\dot{\lambda}$, of 0.06/min to the stretches of 0.80, 0.90 and 0.95, respectively (5.9(c)).
(c) Set $\lambda$

(d) Set 2 Maxwell elements

(e) Set $D_{2v1}$
5.4.3 Material parameter sensitivity tests for the hyperelastoplastic model

In order to investigate the roles of the total spring component and the elastoplastic component in the hyperelastoplastic model, isochoric (undrained) loading-unloading-hold tests were carried out, by setting $D_2 = D_2p = 500$ kPa and $\alpha = \alpha_p = 0$, with the material parameters in Table 5.6. The material was loaded with a 2\textsuperscript{nd} Piola-Kirchhoff stress of 40 kPa for 10 min followed by unloading to 0 kPa in 10 min and the zero load held for 1 min. The reason for the sets of parameter chosen for testing is to investigate their influence on the plastic strain as well as on the loading-unloading curves. This was obtained by choosing:

- Set $C_1$: the total spring stiffness is varied;
- Set $C_{1p}$: the elastoplastic spring stiffness is varied;
- Set $c_p$: the plastic flow rule parameter is varied.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$C_1$ (kPa)</th>
<th>$C_{1p}$ (kPa)</th>
<th>$c_p$ (kPa$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set $C_1$</td>
<td>0.001/5.0/10.0/20.0</td>
<td>50.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Set $C_{1p}$</td>
<td>20.0</td>
<td>5.0/10.0/50.0/70.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Set $c_p$</td>
<td>0.001</td>
<td>50.0</td>
<td>0.0/0.001/0.01/0.08</td>
</tr>
</tbody>
</table>

Table 5.6: Material parameter test sets.

The Green-Lagrange strain and the plastic strain on the reference configuration with time are plotted in Figure 5.10. Note that the plastic strain, obtained via Equation (5.103) from (4.38) on the reference configuration, is not the entire plastic strain, which can be seen by (5.104). However, the analytical plastic flow rule (5.95) reflects the evolution of the plastic strain. In Figure 5.10, loading occurs with a compound slope of the two springs dependent on the amount of plasticity. The amount of plastic strain is determined by (1) the
stress in the spring of the elastoplastic component, and (2) the elastoplastic stress, which gets translated into plastic strain by the $c_p$ value. The parameter $c_p$ determines the proportion of the elastoplastic stress to generate the plastic strain and thus determines the loading slope. On unloading, the elastoplastic spring recovers first, then is reversely loaded by the total spring (upper spring in Figure 4.4). Therefore, to have a small strain recovery, the total spring must be weak in comparison to the elastoplastic spring. And $c_p$ controls the amount of the elastoplastic stress translated into plastic strain during unloading. The plastic strain at the end of unloading is the irrecoverable strain.

\[ E_p = \frac{1}{2} (C_p - I) \]  
(5.103)

\[ E_e = \frac{1}{2} (C - C_p) \]  
(5.104)
Figure 5.10: Hyperelastoplastic model material parameter tests subjected to stress-controlled loading-unloading test condition.

To obtain a reasonable initial estimate for the material parameters to be used with the hyperelastoplastic model for the equilibrium state tests on the peat materials, the material parameter sensitivities on the constitutive relationships were studied. Thus the hyperviscoelastic components ($1^\text{st}$ and $2^\text{nd}$ Maxwell elements) of the proposed model are eliminated by setting the viscoelastic parameters to values in Table 5.3. To test the hyperelastoplastic model, the experimental equilibrium test load case (axial strain rate = 0.16 %/h and 0 kPa cell confining pressure to an axial strain of 20 %) was assigned as the load and boundary condition to the simulation. When the stress reaches zero value at the unloading, the strain-controlled load case switches to stress control at zero value. The sets of testing material parameter values are listed in Table 5.7, with the results presented in Figure 5.11 plotted together with the experimental result from an undisturbed vertical peat specimen tested at an axial strain rate of 0.16 %/h to an axial strain of 20 %.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$C_1$ (kPa)</th>
<th>$D_2/D_{2p}$ (kPa)</th>
<th>$\alpha$</th>
<th>$C_{1p}$ (kPa)</th>
<th>$\alpha_p$</th>
<th>$c_p$ (kPa$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set $C_1$</td>
<td>9.0±5.0</td>
<td>500.0</td>
<td>0.0</td>
<td>50.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Set $D_2/D_{2p}$</td>
<td>9.0</td>
<td>1000.0± 500.0</td>
<td>0.0</td>
<td>50.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Set $\alpha$</td>
<td>9.0</td>
<td>500.0</td>
<td>0.5±0.5</td>
<td>50.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Set $C_{1p}$</td>
<td>9.0</td>
<td>500.0</td>
<td>0.0</td>
<td>50.0±20</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Set $\alpha_p$</td>
<td>9.0</td>
<td>500.0</td>
<td>0.0</td>
<td>50.0</td>
<td>100.0±100</td>
<td>0.1</td>
</tr>
<tr>
<td>set $c_p$</td>
<td>9.0</td>
<td>500.0</td>
<td>0.0</td>
<td>50.0</td>
<td>0.0</td>
<td>0.1 ± 0.02</td>
</tr>
</tbody>
</table>

Table 5.7: Material parameter test sets of the hyperelastoplastic model.
(a) Set $C_i$

(b) Set $D_2$ and $D_2p$

(c) Set $\alpha$
Figure 5.11: Hyperelastoplastic model material parameter sensitivity tests subject to strain controlled tests.

For the undrained test condition, the volumetric strain is taken as zero, i.e. $I_3 = 1.0$. Similar to the incompressible analytical solution in Section 5.2.1.2, the 2nd Piola-Kirchhoff stress of the modified Neo-Hooke model is rearranged and an indeterminate pressure $p_i$ is introduced as
The strain-controlled load and cell pressure of 0 kPa are passed through the load case. Thus to get the zero volumetric strain ($I_3 = 1.0$), i.e. incompressible material, the material parameter $D_2$ has to be relatively large. As presented in Figure 5.11(b), $D_2$ and $D_{2p}$ with a value of greater than 500 kPa have an unnoticeable influence on the constitutive relationship of the isochoric tests. The third invariant of the right Cauchy-Green deformation tensor, $I_3$, in the simulation was $1.0 \pm 0.005$. Similar parameter effects in the stress-controlled tests are found in the strain-controlled tests, i.e. the loading curve is a hybrid effect of the parameters $C_1$, $C_{1p}$ and $c_p$. A higher $C_1$ value indicates a larger loading stiffness and a smaller irrecoverable plastic strain. $C_{1p}$ determines the elastoplastic stress, and thus impacts on the plastic strain by the plastic flow rule parameter $c_p$. A larger $c_p$ causes a greater amount of the elastoplastic stress to generate a higher plastic strain. The shape parameters $\alpha$ and $\alpha_p$ have the same effect on the total spring and the elastoplastic spring elements, where $\alpha$ has a more sensitive impact on the constitutive relationship with the selected material parameters. The aim of the material parameter fitting process is to find the proper combination of the parameters that provides a balance between the loading stress-strain relationship and the irrecoverable plastic strain during unloading.

5.5 Material parameter fittings and model validations

The proposed isotropic constitutive model is used for parameter fitting for the vertical peat specimens in undrained test conditions. Trials of using LMFIT (Non-Linear Least-Square Minimization and Curve-Fitting for Python) (Newville et al., 2014) were carried out to obtain the best-fit model parameters against experimental data. A large tolerance of at least 5 kPa was required to auto-fit the isotropic material model to the test data of vertical peat specimens. Also on account of the specimen variations as well as the temperature influence, parameter fitting precision was not pursued. The model validation was for assessing the capability of the model to capture the large-strain, rate-dependent, relaxation behaviours with expected deviations between the numerical results and the experimental data.

5.5.1 Experimental data

To compare the numerical simulation with the experimental data of the undrained triaxial tests, the stress and strain measures should be consistent for both. For the experimental data, the stress measure is the Cauchy stress $\sigma$ and the strain measure is the engineering
strain $\varepsilon_{\text{eng}} = \frac{\Delta l}{l_0}$; whereas for the numerical model proposed in Chapter 4, the stress measure is the second Piola-Kirchhoff stress $S$ and the strain measure is Green-Lagrange strain $E$. From the compression in the vertical direction $\lambda = 1 - \varepsilon_{\text{eng}}$, the deformation gradient for the isochoric compression is obtained as in (5.30). Then the Cauchy stress and the second Piola-Kirchhoff stress are related via $S = JF^{-1}\sigma F^{-1}$ (where $J = \det(F)$) and the Green-Lagrange strain tensor $E = \frac{1}{2}(F^T F - I)$. The computational costs of pulling-back the experimental measures to the numerical model measures and the pushing-forward of the numerical measures to the experimental measures are the same. In this study, the former transformation was adopted.

5.5.2 Hyperelastoplastic model-equilibrium state tests

5.5.2.1 Equilibrium test fitting

The constitutive model takes the equilibrium elastoplastic behaviour and the rate-dependent behaviour of the tested peat into consideration. Structural anisotropy can be added into the model for future development. This model shows the potential of the adopted framework to simulate complex material with different load cases. Based on the parameter sensitivity tests in Figure 5.11, the material model with the parameters in Table 5.8 provides a good fit of the model loading stage to the undrained triaxial test data in Figure 5.12, but deviates from the unloading curve. Apart from the stress negative values reached due to the testing conditions explained in Chapter 3, the possible reasons for this difference can be (1) material model parameters, i.e. a better parameter set needs to be found to balance the loading-unloading curves accounting for the inter-specimen variations. (2) deficiencies in the plastic flow rule, where the plastic flow rule (4.95) is dependent on the current total strain, thus during active unloading, the plastic friction element is also stretched. Eliminating the plastic strain increase during unloading could reduce the irrecoverable plastic strain. (3) structural anisotropy. In the experimental tests, the horizontal fibres in the vertical specimen eliminate the horizontal bulging during compression and thus induce an increasing vertical stress as well as a larger elasticity in the vertical direction. Whereas in the isotropic model, the fibre effect is not included. (4) The strain energy density function not being specifically derived for peat material. The aforementioned four reasons provide the research outlook for the further development of the constitutive modelling for peat.

<table>
<thead>
<tr>
<th>$C_1$ (kPa)</th>
<th>$D_2$ (kPa)</th>
<th>$\alpha$</th>
<th>$C_{1p}$ (kPa)</th>
<th>$D_{2p}$ (kPa)</th>
<th>$\alpha_p$</th>
<th>$c_p$ (kPa$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.0</td>
<td>500.0</td>
<td>0.0</td>
<td>70.0</td>
<td>500.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 5.8: Material parameter for the hyperelastoplastic model.
Figure 5.12: Material parameter fitting to the hyperelastoplastic model (equilibrium test).

5.5.2.2 Hyperelastoplastic model validation

The hyperelastoplastic model with the material parameters in Table 5.8 is validated against a loading-unloading-reloading triaxial test of an undisturbed vertical peat specimen with 0 kPa cell pressure at the axial strain rate of 0.16 %/h to axial strains of 5 % and 6 %. Due to the test conditions, a relatively large negative stress was obtained in the experiment test. Correspondingly, the simulation was carried out with a continuous unloading of a strain-controlled manner. The stress-strain relationships of the experimental test and the numerical simulation are presented in Figure 5.13. The hyperelastoplastic model with the equilibrium test fitted material parameters provide good stress-strain simulation at the initial stage of the loading test as well as when the strain recovery at the minimum stress is reached. The peak 2nd Piola-Kirchhoff stress difference is within 2 kPa, which is within the expected tolerance considering the temperature influence and the specimen variations. The strain history effect on the constitutive relationships can be modelled by removing the loading from the material when tensile stress occurs. As expected, the simulated reloading curve was shifted from the experimental reloading curve by a smaller irrecoverable plastic strain. The model captures the stress history effect on the reloading test.

The hyperelastoplastic model is validated against the equilibrium test with relaxations (Figure 5.14). In the undrained triaxial test at a loading rate of 0.16 %/h, stress relaxation occurred at a relative small rate and was sensitive to the temperature change. Comparatively, the hyperelastoplastic model takes the low strain rate test as a hypothetical equilibrium state test, which can be refined by further slower tests or relaxation tests. However, the magnitude of the stress relaxation of the 0.16 %/h test was at the same order of magnitude as the temperature influence. The slower tests are only of necessity for a precise result with strict temperature control. The hyperelastoplastic model with parameters in Table 5.8 is fairly well validated against the undrained equilibrium test with relaxations.
The stress relaxation is improved by incorporating the viscoelastic components.

Compared with the conventional 1D compression models for peat at equilibrium tests, which are for drained compression test conditions, the 3D hyperelastoplastic model is derived in a finite-strain formulation and is able to capture the loading history effect on the loading-unloading-reloading tests.

### 5.5.3 Hyperviscoplastic model

#### 5.5.3.1 Hyperviscoplastic model parameter fitting

Accepting the fitted material parameters for the hyperelastoplastic model, tests at the same conditions of the equilibrium testing but at different strain rates are needed for fitting the hyperviscoplastic model. Undrained triaxial compression tests at two strain rates with relaxations were used for the parameter fitting of the hyperviscoplastic model with the two

Figure 5.13: Hyperelastoplastic model validation with loading-unloading test.

Figure 5.14: Hyperelastoplastic model validation against equilibrium test with relaxations.
rate-dependent components. Taking the undrained test as an isochoric process, the parameters $D_{2v1}$ and $D_{2v2}$ have no influence on the simulated constitutive relationship. The parameters $D_{2v1}$ and $D_{2v2}$ were assigned the value 500 kPa, which is the same as the value chosen for $D_2$ and $D_{2p}$. Analogous to the hyperelastoplastic model, $\alpha_{v1}$ and $\alpha_{v2}$ were assigned to be zero where the free energy function is simplified to Neo-Hooke model. The model parameters $C_{1v}$ and $\eta_v$ of each viscoelastic component were obtained by fitting the undrained triaxial compression tests with relaxations at the strain rates of 1.60 %/h and 16.0 %/h in Table 5.9 with the results in Figure 5.15 and Figure 5.16.

<table>
<thead>
<tr>
<th>Model</th>
<th>$C_1$ (kPa)</th>
<th>$D_2$ (kPa)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified Neo-Hookean</td>
<td>9.0</td>
<td>500.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1st Maxwell Element</td>
<td>8.0</td>
<td>500.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2nd Maxwell Element</td>
<td>40.0</td>
<td>500.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Hyperelastoplastic Model</td>
<td>70.0</td>
<td>500.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 5.9: hyperviscoplastic model parameters.

Figure 5.15: hyperviscoplastic model parameter fitting with undrained triaxial compression-relaxation test at $\dot{\varepsilon}_a$ = 1.60 %/h.

5.5.3.2 hyperviscoplastic model validation

The hyperviscoplastic model with the material parameters in Table 5.9 is validated against undrained triaxial tests on the undisturbed vertical peat specimens at axial strain rates of 160, 16.0, 1.6, 4.81 and 0.16 %/h, respectively. To compare the proposed model performance with the experimental data, the load case for the following validations is in a purely strain-controlled manner, thus negative stress during the tensile unloading is generated.
Figure 5.16: hyperviscoplastic model parameter fitting with undrained triaxial compression-relaxation test at $\dot{\varepsilon}_a = 16.0 \% / h$.

Firstly, the model with the fitted parameters is validated against an undrained triaxial lu test at an axial strain rate of $1.60 \% / h$. The model generates large tensile stress at the end of unloading as the load is continuously applied albeit the stress reaches negative values, whereas in the laboratory experiment, the load cell detached from the specimen. This can be fixed by switching the load case from strain-control to stress-control as the boundary condition when the stress reaches zero. From the analytical solution of the viscoelastic model flow rule (5.57), the constitutive relationship is dependent on the total compression reached for a period of time. Therefore, the tensile part of the unloading by the viscoelastic components in the simulation has no influence on the reloading of the proposed model. Figure 5.17 presents the validation results, of which the model provides fairly good simulation of the undrained compression.

Figure 5.17: hyperviscoplastic model validation against undrained triaxial lu test at $\dot{\varepsilon}_a = 1.60 \% / h$. 
Secondly, the validation of the slower test at 4.81 %/h in Figure 3.20 is presented in Figure 5.18. The model is capable of simulating the undrained triaxial test at the axial strain rate of 4.81 %/h.

Thirdly, the 16.0 %/h strain rate test in Figure 3.4 is used for the hyperviscoplastic model validation (Figure 5.19). Compared to the above two slower tests, the deviation between the numerical and experimental results at the peak stresses is larger for the 16.0 %/h test. The compression-relaxation test in the model parameter fitting (Figure 5.16) is carried out at an axial strain rate of 16.0 %/h, which is the same as in the test in Figure 5.19. Thus the relatively larger deviation at the peak stress may be due to the peat specimen variation as well as temperature influence. The early loading stages provide good validation of the hyperviscoplastic model.

Figure 5.18: hyperviscoplastic model validation against undrained triaxial test at $\dot{\varepsilon}_a = 4.81 \%$/h.

Figure 5.19: hyperviscoplastic model validation against undrained triaxial test at $\dot{\varepsilon}_a = 16.0 \%$/h.
To test the performance of the hyperviscoplastic model in stress relaxations, the model is validated against the undrained triaxial test at axial strain rate of 160%/h with loading and unloading step relaxations. Figure 5.20 shows that the hyperviscoplastic model is well capable of fitting the loading stage with relaxations and capturing the unloading stress relaxations. However, as stated in Section 5.5.2.1, the model can be improved for its unloading stage by fitting a better set of parameters, adapting the flow rule as well as accounting for structural anisotropy.

![Figure 5.20: hyperviscoplastic model validation against undrained triaxial test at $\dot{\epsilon}_a = 160 \text{%/h}$.](image)

With the fitted hyperviscoplastic model parameters, the undrained triaxial loading-relaxation-unloading test in Figure 5.14 is replotted in Figure 5.21. The model captures the stress relaxation in the 0.16%/h test.

![Figure 5.21: hyperviscoplastic model validation against equilibrium test with relaxations.](image)

From the above validations against undrained triaxial compression tests, the hyperviscoplastic model with the fitted parameters provides good simulations of the undrained
compression on the tested peat at various testing strain rates, but over-predicts the irrecoverable plastic strains after complete unloadings.

The undrained triaxial creep test under the calculated compressive stress of 10 kPa is used for the hyperviscoplastic model validation. The actual stress level is undetermined since the friction between the piston and the triaxial cell was unknown and unmeasurable. Trial tests were carried out. Figure 5.22 demonstrates creep simulation with a compressive 2nd Piola Kirchhoff stress of 7.2 kPa. The model over-predicts the initial creep strain and shows the ability to simulate the secondary and tertiary compression with the two time-scales. However, the secondary compression is longer than the model simulation. Due to the limitations of the current triaxial cell, additional 1D undrained creep tests on the same type of peat are needed for further calibration of time-dependent behaviour of the proposed model.

![Figure 5.22: hyperviscoplastic model validation against undrained triaxial creep test.](image)

**5.6 Discussion**

The validations of the hyperviscoplastic model demonstrated good agreement during the loading stage in the undrained triaxial tests on the tested peat material. The relatively larger discrepancy during unloading between the numerical simulation with the fitted parameters and the experimental results can be improved by correcting the load case and modifying the plastic flow rule.

1. The load cases are different between the experiments and the numerical simulations at the end of unloading. The recorded strain increment in the experiment was the relative vertical displacement between the triaxial cell and the load measurement (platen). At a certain stage of unloading, the load measurement detached from the specimen, resulting in a continuous axial strain with a zero deviator stress. Whereas
in the simulation, the load was applied by the strain increment from the experimental data, i.e. when the specimen reached zero stress, the platen was not separated from the material but continued stretching the material by the strain increment applied via the load case. This introduces a strain history effect on the starting positions of reloading tests. It can be fixed in the boundary condition by switching the load case from strain-control to stress-control when the platen lifted off the specimen.

2. The proposed plastic flow rule in the hyperviscoplastic model resulted in a plastic strain increase while the elastoplastic stress was compressive even in the unloading stage. This can be directly seen from the plastic flow rule analytical solution in Equation (5.95) and from the plastic strain plots in Figure 5.10. The increasing plastic strain during unloading could be eliminated to reduce the plastic strain during the unloading stage.

Also a better set of the material model parameters could be chosen to improve the balance between the loading stress-strain curve and the plastic strain during unloading based on a statistical collection of experimental data. The two modifications to the load case and the plastic flow rule, respectively, were studied to investigate how they improve the model validation.

5.6.1 Load case modification

The load case modification was realized by detaching the applied strain increments from the material. During the separation period between the loading platen and the material, the material was allowed to creep under a deviator stress of zero value. When the loading platen reached the position of the creeping material during reloading, the strain-controlled load was reactivated. The hyperviscoplastic model with the modified load case was re-validated against the equilibrium test in Figure 5.21. Figure 5.23 presents the modified load case which reflects the test loading conditions.

As the proposed model is capable of simulating tests carried out at different strain rates, the triaxial loading-unloading-reloading test in Figure 5.17 was used as a rate-dependent representative test with strain-cycles for the validation with the modified load case. The strain history effect is presented in Figure 5.24 and the unloading tensile stress is eliminated by the modified load case. The reloading curve was shifted to a larger irrecoverable plastic strain from the former unloading.

5.6.2 Plastic flow rule modification

The plastic flow rule was modified to reduce the plastic strain during unloading by eliminating the plastic strain increase in the unloading compressive domain. When the applied
Figure 5.23: hyperviscoplastic model validation with modified load case against undrained triaxial test at $\dot{\varepsilon} = 0.16$ %/h.

Figure 5.24: hyperviscoplastic model validation with modified load case against undrained triaxial test at $\dot{\varepsilon} = 1.60$ %/h.

load switched from loading to unloading, the material was still within the compressive domain. The current plastic flow rule in (4.95) determines that the plastic strain increment direction is the same as the current stress in the plastic element. Therefore a modification of the flow rule is obtained by setting the plastic strain increment equal to zero during unloading compressive domain. Plastic flow appears again during active tensile unloading. The modification is realized by multiplying a Heaviside step function to the current plastic flow rule. In the intermediate configuration, the current plastic flow rule (4.88) is modified to

$$
\delta \varepsilon^p = c_p \rho_0 ||\delta \varepsilon^p|| \left( \mathbf{1} + 2 \frac{\partial \Psi^p}{\partial \varepsilon} \right) \frac{\partial \Psi^p}{\partial \varepsilon} H \left[ \text{sign} \left( \rho_0 \frac{\partial \Psi^p}{\partial \varepsilon} \right) \right] \tag{5.106}
$$

where $H(\bullet)$ is the Heaviside step function given by
\[ H(x) = \begin{cases} 
0, & x < 0 \\
\frac{1}{2}, & x = 0 \\
1, & x > 0 
\end{cases} \quad (5.107) \]

and \( \text{sign}(\bullet) \) is a sign function given by
\[
\text{sign}(x) = \begin{cases} 
-1, & x < 0 \\
0, & x = 0 \\
1, & x > 0 
\end{cases} \quad (5.108) 
\]

Pulling back the plastic flow rule to the material description and removing the multiplicative positive items in the sign function, (5.106) reads
\[
\dot{C}_p = 2c_p \left| |C| \right| \left( \frac{\partial \psi_p}{\partial I_1} C + I_3^{ep} \frac{\partial \psi_p}{\partial I_3} C_p \right) H \left[ \text{sign} \left( \frac{\partial \psi_p}{\partial I_1} C + I_3^{ep} \frac{\partial \psi_p}{\partial I_3} C_p \right) \right] (\dot{C}) 
\]

(5.109)

For the numerical implementation, three terms in the Box: hyperviscoplastic Model need to be updated with the modified plastic flow rule. They are presented in Box: Modified Flow Rule.

Box: Modified Plastic Flow Rule

Residual vector:
\[
r_4 = \frac{C_p - C_t}{\Delta t} - 2c_p \left| |C - C^t| \right| \left( \frac{\partial \psi_p}{\partial I_1} C + I_3^{ep} \frac{\partial \psi_p}{\partial I_3} C_p \right) H \left[ \text{sign} \left( \frac{\partial \psi_p}{\partial I_1} C + I_3^{ep} \frac{\partial \psi_p}{\partial I_3} C_p \right) \right] (C - C^t) 
\]

(5.110)

Jacobian:
\[
J_{44} = \frac{\partial r_4}{\partial C_p} = \frac{I}{\Delta t} \left( 1 - 2c_p \left| |C - C^t| \right| I_3^{ep} \frac{\partial \psi_p}{\partial I_3} C_p \right) + \frac{2c_p \left| |C - C^t| \right|}{\Delta t} \left[ \frac{\partial^2 \psi_p}{\partial I_1^{ep} \partial I_1^{ep}} C \otimes C + I_3^{ep} \frac{\partial^2 \psi_p}{\partial I_1^{ep} \partial I_3^{ep}} (C \otimes C_p + C_p \otimes C) + I_3^{ep} \frac{\partial^2 \psi_p}{\partial I_1^{ep} \partial I_3^{ep}} \left( \frac{\partial^2 \psi_p}{\partial I_3^{ep} \partial I_3^{ep}} I_3^{ep} C \otimes C_p \right) (C_p^{-1} \otimes C_p^{-1}) \right] H \left[ \text{sign} \left( \frac{\partial \psi_p}{\partial I_1^{ep}} C + I_3^{ep} \frac{\partial \psi_p}{\partial I_3^{ep}} C_p \right) \right] (C^t) 
\]

(5.111)

continued on next page...
RHS matrix for global tangent solution:
\[
\frac{\partial r_1}{\partial E} = 2 \frac{\partial r_4}{\partial C}
\]
\[
= -\frac{4c_p}{\Delta t} \frac{||C-C'||}{||C-C'||} \left[ \frac{\partial^2 \psi_p}{\partial I_1^{ep}} C \otimes C_p^{-1} + I_3^{ep} \frac{\partial^2 \psi_p}{\partial I_3^{ep}} \left( C \otimes C^{-1} + C_p \otimes C_p^{-1} \right) + I_3^{ep} \frac{\partial^2 \psi_p}{\partial I_3^{ep}} C_p \left( C - C' \right) \right]
\]
\[
\frac{\partial^2 \psi_p}{\partial I_1^{ep}} \left( \frac{\partial^2 \psi_p}{\partial I_1^{ep}} I_1^{ep} + \frac{\partial^2 \psi_p}{\partial I_3^{ep}} I_3^{ep} \right) C_p \otimes C_p^{-1} \right] H \left[ \left( \frac{\partial \psi_p}{\partial I_1^{ep}} C + I_3^{ep} \frac{\partial \psi_p}{\partial I_3^{ep}} C_p \left( C - C' \right) \right) \right]
\]
(5.112)

The hyperelastoplastic model with the modified plastic flow rule was verified with its analytical solution with the same parameters in Table 5.3 and the same loading case. The modified plastic flow rule during unloading changed the value of \( \lambda^2_p \) when the signs of the plastic stress and the strain increment are different, resulting in \( \lambda^2_p = \lambda^2_{pf} \). When the signs are the same, the unloading plastic flow rule resumes as Equation (5.96). The verification result is shown in Figure 5.25. Comparing with Figure 5.4, the reduction in plastic strain in unloading is not apparent due to the material parameters used.

Figure 5.25: Hyperelastoplastic model verification with the modified plastic flow rule.

As the modified plastic flow rule for the loading stage remains the same, the fitted material parameters are used for the validation of the hyperviscoplastic model with the modified plastic flow rule. The undrained tests at 0.16 %/h and 1.6 %/h were used for the validation and comparison with former validations of the hyperviscoplastic model with the modified plastic flow rule. Figure 5.26 and Figure 5.27 present the simulation results of
the modified plastic flow rule with the original flow rule for the equilibrium test. The plastic strain increase during the unloading compressive domain is eliminated, by remaining constant instead. The irrecoverable plastic strain of the constitutive relationship during unloading is improved by modifying the plastic flow rule. This provides insight into improving the model performance when simulating peat by modifying the plastic flow rule when future experimental data on peat plasticity is available.

Figure 5.26: Hyperelastoplastic model validation with the modified plastic flow rule against the undrained triaxial test at $\dot{\varepsilon}_a = 0.16 \% / h$.

Figure 5.27: Hyperelastoplastic model validation with the modified plastic flow rule against the undrained triaxial test at $\dot{\varepsilon}_a = 1.6 \% / h$.

5.7 Summary

The numerical implementation of the constitutive models in Chapter 4 has been detailed, including the Kelvin mapping, local stress iteration and the global iteration of the momentum balance. To verify the numerical implementation of each component of the entire
model, analytical solutions of the modified Neo-Hooke model, hyperviscoelastic models and hyperviscoplastic model were derived and compared with the numerical solutions. The simulation results indicated a good agreement between the numerical and analytical solutions. Then the capabilities of modelling different load cases were investigated for the rate-dependent model. Model parameter sensitivities were tested for both the hyperviscoelastic model and the hyperelastoplastic model for parameter fitting purposes. The model parameters for the rate-independent hyperelastoplastic model were obtained by fitting the model to an equilibrium test defined as undrained triaxial \( \mu \) test carried out at an axial strain of 0.16 \%/h. Concerning the shape of the constitutive relationship, the shape parameter \( \alpha \) was set to 0.0. Assuming no volumetric strain occurred during the undrained test, the parameters \( D_2 \) and \( D_{2v} \) with large values had no impact on the constitutive relationship. The three remaining hyperelastoplastic model parameters, \( C_1 \), \( C_{1p} \) and \( c_p \), were obtained based on the parameter sensitivity tests as well as the phenomenological analysis of laboratory experiments. Two compression-relaxation tests carried out at two axial strain rates were used for parameter fitting of the rate-dependent hyperviscoplastic model. The comprehensive model was validated against undrained triaxial tests on the undisturbed vertical peat specimens. The model was capable of simulating the loading-unloading-reloading, loading-relaxation-unloading and loading-relaxation-unloading-relaxation load cases. A creep test using the hyperviscoplastic model showed that the model was able to capture the two-time scale creep behaviour, i.e. secondary compression and tertiary compression in undrained conditions. However, due to the inferior undrained creep test data, the triaxial undrained creep experimental results should not be used for the model calibration. Considering the specimen variations, the proposed model provided good simulation of the undrained triaxial tests carried out at five different axial strain rates for the loading stage. Two modifications, on loading boundary conditions and on plastic flow rule, were studied to improve the model validations. By changing the loading boundary condition, the tensile stress during unloading was banished, but the reloading curve was shifted to the irrecoverable plastic strain from the former stage unloading. To reduce the plastic strain during unloading, the plastic flow rule was modified by setting the plastic strain increase to zero in the unloading compressive stress domain. The modification of the plastic flow rule reduced the irrecoverable plastic strain and provided better results.
Chapter 6

Conclusions

A series of undrained triaxial tests on undisturbed peat has been carried out to characterise the peat material. Based on the characterisation, a corresponding constitutive model has been proposed. This is the first systematic study in the literature of a finite strain constitutive model in modern mechanics context within a thermodynamically consistent framework based on modelling-oriented laboratory experiments in peat materials. The laboratory experiments revealed the complex peat geomechanical behaviour and designated undrained peat as a rate-dependent elastoplastic material. The testing results indicated the possible structures of the proposed constitutive models. A constitutive model with elastic, viscoelastic, elastoplastic components was adopted for modelling peat. The model provided qualitatively good simulation results of the undrained constitutive relationship of the tested peat and demonstrated its versatility in simulating different load cases as well as its potential for further developments. The detailed summaries of the laboratory work and the numerical modelling on peat in this study are as follows.

6.1 Summary of the experimental tests

1. The undrained constitutive relationship of the peat in triaxial loading-unloading tests was found to be rate-dependent, i.e. a higher strain rate resulting in a higher initial stiffness and higher stress. Both loading and unloading stress relaxations were observed in the undrained triaxial tests.

2. Equilibrium test for the undisturbed vertical peat specimen in undrained triaxial testing was defined based on the insignificant stress relaxation rate as the test carried out at an axial strain rate of 0.16 %/hr. This equilibrium stress-strain relationship was also validated by the relaxation test terminations up to an axial strain of 6 %. The loading-unloading test results indicated that peat is a rate-dependent material with equilibrium hysteresis.
3. Triaxial loading-unloading tests were used for determining the strain recoveries of the tested peat in undrained testing conditions. Large strain recoveries were experienced for the undisturbed vertical peat specimens, where on average 15% out of 20% total strain was reversible. The elastic strain estimated from the shape of the unloading curves provided smaller but fair estimations for the actual recovered strains calculated from the starting points of reloading strains.

4. Testing strain rates and stress relaxations had insignificant impact on the elastic strain recoveries. However, the undisturbed peat specimens which experienced large strain history demonstrated a smaller elasticity, i.e. the same vertical peat specimen with a smaller initial void ratio had a higher stiffness and a smaller elastic recovery in the loading-unloading tests.

5. The stress relaxation was found to be sensitive to temperature fluctuations, with a sensitivity of about 2.1 kPa/K at the end of a one-week relaxation.

6. A linear relationship can be found between the deviator stress and logarithm of time after an initial time period for the vertical undisturbed peat specimen relaxation tests. The pore water pressure variations were very small during the stress relaxation.

7. Cell pressures had negligible effect on the elastic strain recovery and the deviator stress-axial strain behaviour of the undrained triaxial loading-unloading tests on vertical undisturbed peat specimens within an axial strain range of 5%.

8. Compared to the undrained triaxial relaxation tests, the undrained triaxial creep tests took a lot longer to arrive at the equilibrium state, even with a smaller applied stress.

9. The permeability anisotropy of undisturbed peat obtained in falling head tests was not significant, i.e. the coefficient of permeability of the horizontal specimen was marginally lower than that of the vertical specimen. The coefficient of permeability tested by the falling head method with filtered bog water and distilled water both gave acceptable values, nevertheless the coefficient of permeability tested using distilled water attained a smaller value than that using filtered bog water.

10. In the undrained triaxial loading-unloading tests, the horizontal specimens reached higher deviator stresses at axial strains of 5% and 10%, and showed a pronounced strain softening after about 10% axial strain compared to vertical specimens which did not reach failure even at an axial strain of 20%. The proportion of elastic strain to total strain of the horizontal specimens was smaller than that of the vertical specimens. The vertical specimens kept a nearly constant cross-section after the undrained triaxial loading-unloading tests while through-cracks took place in the horizontal specimens.
The rate- and relaxation-independent strain recoveries in the loading-unloading tests implied that the viscoelastic component of the proposed model could be in a parallel connection with a rate-independent component. The equilibrium hysteresis indicated a plastic element in the rate-independent component of the model. As the elastic strain recoveries were proportional to the maximum strains reached, the plastic component could be parallelly connected with a spring element with a proper plastic flow rule.

6.2 Summary of the numerical models

A phenomenological modelling framework consisting of elastic, viscoelastic and elastoplastic components connected in parallel was adopted to capture the rate-dependent peat behaviour with equilibrium hysteresis. Initial validation was performed as a prelude for further refined models, particularly for peat materials. The following conclusions can be made:

1. The constitutive relationships of the elastic, viscoelastic and elastoplastic components were derived from the isothermal Clausius-Duhem inequalities by adopting a Helmholtz free energy function for a modified Neo-Hooke model. By using the overlay concept, the process of developing the model showed the straightforwardness to extend the model to higher levels of complexity.

2. Kelvin mapping was adopted for the numerical implementation of the constitutive models. Tensor characters can be preserved by using Kelvin mapping, and the stress and strain tensors use the same mapping rule from the matrix notations to vector notations. The numerical implementation framework provided a convenient approach for the numerical solutions of various material models which were verified with analytical solutions. The described algorithm is the basis for a future finite element implementation. In such a scheme, the algorithm will be executed at each Gauss point during each global iteration and will yield algorithmically consistent tangent operators for the global Newton scheme. All subsequent tests remained at the integration point level and could thus only treat homogeneous deformation states.

3. The hyperviscoplastic model, which was formulated based on finite strain theory, showed its capability in providing phenomenologically correct simulation results in various load cases, including the relaxations in loading and unloading, strain history effect and the nonlinear constitutive relationships. The hyperelastoplastic model parameters were obtained by fitting the model to the equilibrium test on undisturbed vertical peat specimens. The process of fitting model parameters is to balance between the loading curve and the plastic strain in unloading. The hyperviscoelastic model parameters were obtained from two strain-rate relaxation tests. Considering
the specimen variations, the model provided qualitatively good simulations of the undrained triaxial compression tests carried out at five different axial strain rates. The model is able to simulate the two time scales (two "S"-shaped on the creep strain – log time curve) in undrained creep test.

4. The strain-controlled numerical validation tests against the undrained triaxial loading-unloading-reloading tests generated large tensile stress in unloading. By a modification of the load case according to the experiment loading procedure, the tensile stress during unloading was eliminated. To reduce the irrecoverable plastic strain, a modification to the plastic flow rule by eliminating the plastic strain increase in the unloading compressive stress domain was studied, which improved the model performance and provided insight into the future improvement of the proposed constitutive model.

6.3 Limitations

The limitations of the experimental works in this study lies in the peat sampling, testing apparatus, testing environment as well as the nature of the peat sample variations:

1. The tested peat material is not fully saturated, therefore the compressible air bubbles as well as the likely compressible organic solids challenge the incompressible assumption of the undrained tests in the model simulations;

2. The testing apparatus used in the laboratory conditions were designed for mineral soils. The applied load and pore water pressure of the tested peat lies in the lower band of the measurement ranges of the load cell and the pore water pressure transducer;

3. The mechanical behaviour of peat materials was highly sensitive to the temperature fluctuations, which results in constitutive data with great variations in the normal laboratory environment;

4. The size of the tested peat specimen in the standard triaxial tests was small relative to the specimen variations and structural anisotropy, which also introduced loading stability problems, i.e. specimens failed due to asymmetric loading.

5. The experiments carried out in this study categorised the tested peat material, however the amount of experimental work was insufficient to determine a strain energy function, viscoelastic and plastic flow rules, and yield conditions, particularly for the tested peat.

The limitations of the proposed model are:
1. The proposed constitutive model simulates peat material in undrained condition by ignoring the peat microstructure as well as pore water and air phases. The isotropic constitutive model was validated against undisturbed vertical peat specimens. This can be improved by extending the model to a multiphase material model with two-level porous structures and incorporating structural anisotropy.

2. The model parameters can only be obtained by fitting experimental data for the nonlinear large strain materials. A more efficient fitting scheme should be proposed and correlations between the nonlinear material parameters with the common soil mechanics parameters should be studied. Material parameter value bounds should be studied to consider the material variations.

### 6.4 Research outlook

Further tests of strain energy functions and anisotropy on peat should be carried out to gain a better understanding and provide further data for modelling purposes in peat materials. A proper testing environment with strict temperature control is needed, both for an isothermal testing condition and to obtain data for the development of a coupled thermo-mechanical model.

Structural anisotropy with a separate fibre layer should be added to the isotropic model. Accounting for the two-level microstructure of peat materials, the proposed model should be extended into a dual porosity model with inter-continuum mass exchange and water and/or gas flows. Coupled thermo-mechanical analyses are needed to take into account climate change and seasonal influence on the mechanical behaviour of peat.

The proposed model has been established in the context of the modern continuum mechanics which might not be directly accessible to the geotechnical engineers. Thus, correlations between the nonlinear material parameters and the conventional soil mechanics properties, especially the physical properties, such as moisture content, organic content, fibre content, etc., should be investigated. The complete model will need to be validated under complex in-situ conditions.
References


Alexandre, G. and I. Martins (2012). Stress relaxation under various stress and drainage conditions. *hal-00879838*.


Appendix A

Summary of constitutive models for peat

A.1 Rheological models for peat
A.2 Evolution of the application of time-line theory in peat
### Appendix A.1: Summary of rheological models for peat soils

<table>
<thead>
<tr>
<th>Research</th>
<th>Peat description</th>
<th>Model used</th>
<th>Model diagram</th>
<th>Constraint</th>
<th>Mathematical description</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edil (1982); Edil and Mochtar (1984); Edil et al. (1994)</td>
<td>Fibrous peat from Middleton, Wisconsin. ( \bar{w} = 550% ) (average water content); ( e_o = 10.5 ) initial void ratio; Organic content = 93%; Fiber content = 50%; Preconsolidation stress in the range 25 – 40 kPa.</td>
<td>Gibson and Lo (1961); Unique ( \sigma''_0 = e - e' ) relationship after Suklje (1957), Battelino (1973) and Leroueil et al. (1985).</td>
<td>constant effective stress ( (t &gt; t_a) ); ( t_a ) end of primary consolidation time determined by observational procedure proposed by Asaoka (1978)</td>
<td>( \epsilon(t) = \Delta \sigma \left[ a + b \left( 1 - e^{-\lambda \beta t} \right) \right] )</td>
<td>Model verified using data from 43 multi-stage oedometer tests performed on 63 and 285 mm diameter specimens.</td>
<td></td>
</tr>
<tr>
<td>Berry and Poskitt (1972)</td>
<td>Amorphous peat:</td>
<td>Gibson and Lo (1961) with a nonlinear spring at the top to simulate nonlinear soil compressibility</td>
<td>The principle of continuity of mass and consideration of Darcy’s law with a permeability coefficient ( k ) derived from the linear relationship between void ratio and log ( k ).</td>
<td>Continuity of mass:</td>
<td>Consolidation equations for amorphous peat are highly nonlinear and require at least 14 parameters.</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Consolidation equation:</td>
<td>Postulation of linear ( e - \log p ) relationship.</td>
<td>Validated using Rowe consolidation tests on remoulded amorphous peat.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fibrous peat</td>
<td>Double Terzaghi pot: two nonlinear springs simulate the nonlinear compressibility of fibrous peat using the same compressibility law adopted for amorphous peat (linear ( e - \log p ) relationship)</td>
<td>The creep rate is a nonlinear function of the stress increment.</td>
<td>Continuity of mass:</td>
<td>Postulation of linear ( e - \log p ) relationship; Darcy’s law for micropore water flow; The postulation of physical laws, such as the law governing the average flow rate of pore water from micropores to macropores, based purely on intuition since no experimental data was accessible.</td>
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</table>
The creep rate is a linear function of stress increment. Barden (1968) proposed an amorphous peat and clay Kelvin model in a Terzaghi pot to simulate secondary consolidation. The nonlinear dashpot gives the characteristic linear secondary compression against log $t$ and takes account of the dominant effect of load-increment ratio. This model is basically an improvement by adding time-dependent secondary compression to the Terzaghi 1D consolidation theory. The creep rate is described by the equation:

$$T_\text{c} = \frac{\Delta \rho^n - \Delta \tau}{a \beta^n}$$
Appendix A.2: Evolution of the application of time-line theory into peat

<table>
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<tbody>
<tr>
<td>theory</td>
<td>Isotache principle</td>
<td>Time-line theory (nc)</td>
<td>abc model (nc+oc)</td>
<td>EVP model</td>
<td>abc model</td>
</tr>
</tbody>
</table>

**Model**

- **Improvement feature**: Combined analytical-graphical solution; Each creep isotache corresponds to a constant void ratio rate.
- **Instant compression lines**: Over-consolidation is not included. Each time-line corresponds to time, rather than strain rate, so that the \( \dot{\varepsilon} = 0 \) condition is considered.
- **Overconsolidation time-line**: is added based on the Bjerrum time-line theory.
- **Combined isochrone-line theory** (visco-) and critical state model (elasto-plastic); Introduced the concept of "equivalent time".
- **Intrinsic time concept**: developed for clay and peat; Creep rate calculated from intrinsic time.

**Formulation**

- **Total strain** = instant strain + delayed strain,
  
  \[ \varepsilon = \varepsilon_1 + \frac{(a_1 + a_2 \log \tau_0)}{t} \]

- **Direct strain**:
  
  \[ \dot{\varepsilon} = \frac{a_0}{\dot{\varepsilon}_0} - \frac{a_1}{\dot{\varepsilon}_0} \log \frac{t}{t_1} \]

**Assumptions**

1. The speed of consolidation depends on the mean values of void ratio and of total intergranular pressure;
2. Full saturation and the speed of consolidation must correspond to the speed of pore water flow in both the primary and

**Assumption:** parallel \( e - \log p \) curves for difference times.

**Constant compression index.** Overconsolidation is not considered.

**Hypothesis B.**

**The preconsolidation pressure** (equivalent critical pressure) determined is smaller than the Casagrande method.

**Intrinsic time to calculate the creep strain rate.**

**Hencky strain.**
<table>
<thead>
<tr>
<th>Secondary phase of consolidation;</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Darcy’s law with variability of the coefficient of permeability with the development of consolidation;</td>
</tr>
<tr>
<td>4. Uniform consolidation process in all parts of the layer with one-dimensional water flow.</td>
</tr>
</tbody>
</table>

Hypothesis B.
Appendix B

Derivations and definitions

B.1 ⊙ product of Right Cauchy Green tensor

Define the notation of the circle product of the inverse of an invertible smooth tensor \( C^{-1} \) as

\[
\frac{\partial C^{-1}}{\partial C} = -C^{-1} \odot C^{-1}
\]  

(B.1)

with the \( \odot \) product as

\[
A \odot B = \frac{1}{2} \left[ (A \otimes B)^T + (A \otimes B^T) \right]
\]  

(B.2)

Then the components of \( C^{-1} \odot C^{-1} \) are

\[
(C^{-1} \odot C^{-1})_{ijkl} = \frac{1}{2} \left( C^{-1}_{ik} C^{-1}_{lj} + C^{-1}_{il} C^{-1}_{kj} \right)
\]  

(B.3)

For the detailed proof, refer to Holzapfel (2000).

B.2 Analytical solution for the hyperelastoplastic model

The analytical solution for (5.96) is obtained by considering the uniaxial compression condition, where \( \| \lambda^2 \| = -\lambda^2 \). Equation (5.96) becomes

\[
\dot{\lambda}_p^2 = -2c_p C_{1p} \dot{\lambda}_p^2 \left( \lambda^2 - \lambda_p^2 \right)
\]  

(B.4)

Set \( \lambda_p^2 = y \), \( \lambda^2 = x \) and \( 2c_p C_{1p} = C_0 \), (B.4) gets

\[
\frac{dy}{dt} = -C_0 \frac{dx}{dt} (x - y)
\]  

(B.5)

Since \( dt > 0 \), getting rid of \( dt \) and rearranging

\[
\frac{dy}{dx} = -C_0 (x - y)
\]  

(B.6)
Define \( y' = \frac{dy}{dx} \). Multiply \( e^{-C_0 x} \) to both sides of the above equation and rearrange
\[
e^{-C_0 x} y' - C_0 e^{-C_0 x} y = -C_0 x e^{-C_0 x}
\] (B.7)

Integrate both sides of the above equation
\[
\int (e^{-C_0 x} y') = \int (-C_0^2 x e^{-C_0 x})
\] (B.8)

thus
\[
e^{-C_0 x} y = \frac{1}{C_0} [e^{-C_0 x} (1 + C_0 x)] + B
\] (B.9)

where B is a constant respect to x and y. To get the value of B, using the initial boundary condition: \( \lambda_p|_{\lambda_0} = \lambda_{p0} \). Divide the both sides of the above equation by \( e^{-C_0 x} \) and replacing \( x, y \) and \( C_0 \), get the final analytical solution of (B.4):
\[
\lambda_p^2 = \frac{1}{2c_p C_{1p}} \left( 1 + 2c_p C_{1p} \lambda^2 \right) \left[ \lambda_{p0}^2 - \lambda_0^2 - \frac{1}{2c_p C_{1p}} \right] e^{2c_p C_{1p} (\lambda^2 - \lambda_0^2)}
\] (B.10)

Similarly, for unloading stage, \( |\lambda^2| = \frac{1}{\lambda_f^2} \) and the boundary condition \( \lambda_p|_{\lambda_f} = \lambda_{pf} \) (\( \lambda_f \) and \( \lambda_{pf} \) are the final stretches during the loading stage), we get the analytical solution of plastic stretch as
\[
\lambda_p^2 = \frac{1}{2c_p C_{1p}} \left( 2c_p C_{1p} \lambda^2 - 1 \right) \left[ \lambda_{pf}^2 - \lambda_f^2 + \frac{1}{2c_p C_{1p}} \right] e^{2c_p C_{1p} (\lambda_f^2 - \lambda_p^2)}
\] (B.11)

B.3 The value range of parameter \( \alpha \)

The material parameter \( \alpha \) controls the shape of the constitutive relation derived from the free Helmholtz energy of the modified Neo-Hooke model. The value range of \( \alpha \) to be fitted is constrained by the positive definite condition of the material stiffness tensor. As in this study only undrained testing condition was carried out, the simulation was correspondingly taken without volumetric change. The material stiffness (elasticity) tensor is obtained by deriving the second Piola-Kirchhoff stress from the Green-Lagrange strain tensor. Substituting the isochoric constraint, i.e. \( I_3 = 1.0 \), into (5.24):
\[
S = 2C_1 e^{\alpha(I_3-3)} (I - C^{-1})
\] (B.12)

For the undrained triaxial compression test condition with vertical compression of \( \lambda \), the right Cauchy Green tensor and its inverse are taken as (5.31). Using the chain rule, the stiffness of the modified Neo-Hookean model is
\[
\frac{\partial S}{\partial E} = 2 \frac{\partial S}{\partial C} = 4C_1 e^{\alpha(I_3-3)} \left[ \alpha (I - C^{-1}) \otimes \frac{\partial I_1}{\partial C} + C^{-1} \otimes C^{-1} \right]
\] (B.13)
The tangent stiffness tensor should be positive definite, i.e. the eigenvalues of the tensor are all positive. By substituting the deformation tensors into (B.13), the three different eigenvalues are \( \alpha(1 - \lambda^2) + \lambda^{-4} \), \( \alpha(1 - \lambda) + \lambda^2 \) and \( \alpha + \lambda^{-1} \), respectively. Thus to satisfy the positive definite constraint,

\[
\alpha \geq \max \left( \frac{-\lambda^{-4}}{1 - \lambda^2}, \frac{-\lambda^2}{1 - \lambda}, \frac{-1}{\lambda} \right) \quad \text{for} \quad 0.0 < \lambda \leq 1.0 \tag{B.14}
\]

The first and third terms are always negative, but the second term can be asymptotic to 0. Therefore the constraint for the shape parameter of the proposed constitutive relation by the free Helmholtz energy function is \( \alpha \geq 0.0 \) for \( 0.0 < \lambda \leq 1.0 \) which indicates the material can be compressed from 0 to nearly the full height in the vertical direction. However, in this study, the maximum axial strain reached in the undrained triaxial compression test was 30\%, i.e. a vertical compression of 0.7. The second term is monotonically decreasing (negative differentiation with respect to \( \lambda \)) and third terms are monotonically increasing in \( 0.7 \leq \lambda \leq 1.0 \). Differentiate the first term with respect to \( \lambda \) and we get

\[
\left( \frac{-\lambda^{-4}}{1 - \lambda^2} \right)' = \frac{2(3\lambda^2 - 2)}{\lambda^5(\lambda^2 - 1)^2} \tag{B.15}
\]

The critical points of the differentiation are 0.0, 1.0 and \( \sqrt{\frac{2}{3}} \). The maximum value of the first term is at \( \lambda = \sqrt{\frac{2}{3}} \). Therefore for the compression range of \( 0.7 \leq \lambda \leq 1.0 \), the shape parameter is in the range of \( \lambda \geq -1.0 \).