

Simultaneous Occurrence of Braess' and Emission Paradoxes

Wai Yuen Szeto¹, Xiaoqing Li² and Margaret O'Mahony³

Abstract

Braess' paradox is well-known and examined. However, the emission paradox is not. Without considering the emission paradox, the road network improvement that mitigates congestion may increase harmful vehicular emissions. In this paper, we analytically examine the occurrence of the emission paradox and the simultaneous occurrence of Braess' and emission paradoxes in the classical Braess' network. We ascertain that the occurrence of the emission paradox depends on the demand for travel, the parameters of link performance functions as well as link emission factors. We also find that Braess' and emission paradoxes do not always occur at the same time, and that the emission paradox is more likely to occur than the Braess paradox in some networks. More importantly, we discover that under some conditions of parameters in link performance functions, the emission paradox does occur but Braess' paradox does not. This implies that road network design for mitigating congestion alone may not be able to avoid the increase in vehicular emissions. A more comprehensive view of road network design considering both congestion and emissions simultaneously is necessary to avoid the occurrence of the emission paradox.

Introduction

Braess (1968) presented a remarkable example and demonstrated a counterintuitive phenomenon that adding a new link can increase total system travel cost (TSTC) for all travelers. However, its impact in relation to vehicular emissions caused by network expansion is less often discussed. The impact of ignoring the emission paradox can be significant. The road network improvement may be able to mitigate congestion but may increase vehicular emissions such as nitrogen oxide, carbon monoxide, nitrogen dioxide, sulphur dioxide, ozone and particulates. These emissions are harmful to human health. Up till now, only Nagurney (2000a, 2000b) has raised this great interest recently to discuss the emission paradox. Through

¹ Associate Professor, Department of Civil Engineering, National University of Singapore, 1 Engineering Drive 2, E1A 07-03, Singapore 117576; phone: +65-65162279; cveswy@nus.edu.sg

² PhD student, Centre for Transport Research and Innovation for People (TRIP), Department of Civil, Structural and Environmental Engineering, Trinity College Dublin, Dublin 2, Ireland; phone: +353-1-8962537; lixq@tcd.ie

³ Professor, Centre for Transport Research and Innovation for People (TRIP), Department of Civil, Structural and Environmental Engineering, Trinity College Dublin, Dublin 2, Ireland; phone: +353-1-8962084; margaret.omahony@tcd.ie

specific numerical examples, Nagurney demonstrated that three distinctive paradoxical phenomena can occur as regards the total emissions generated. One of them revealed that, in the well-known Braess' network, the total emissions also increase after adding the new link to the network, which is analogous to the classical Braess' Paradox. Nagurney pointed out that so-called "improvements" to the transportation network may actually induce increases in the total emissions. This further calls attention to a careful network expansion. Nevertheless, as we know, road network expansion is an expensive investment. Due to the limited budget, transport road designers should design carefully and take into account some unexpected outcomes possibly arisen from road expansion, such as the increase of total system travel cost (TSTC) and overall vehicular emissions. Particularly, when the emission increase is ignored, road network expansion can worsen an existing environment. Therefore, it is pivotal to consider and study the simultaneous occurrence of both Braess' and emission paradoxes so that transport road designers can avoid unnecessary waste of the government budget and make sure that the design brings about an efficient and environmentally-sustained transport system.

This, thus, arises following questions to be discussed further: 1. Does the emission paradox always occur? 2. Do Braess' paradox and the emission paradox occur at the same time? 3. If not, under which condition(s) the emission paradox occur (In Braess' network example, some conditions under which Braess' paradox occurs have been discussed by Pas and Principio (1997)? The objective of this study is to answer above questions using the classical Braess' network. Moreover, we also examine more demand conditions of Braess' paradox occurrence which Pas and Principio (1997) did not consider. The remainder of this paper is organized as follows. The next section analyses the conditions under which the emission paradox occurs in Braess' Network. The third section examines the conditions under which Braess' paradox occurs. Finally, the last section gives concluding remarks.

When does Nagurney's emission paradox occur in Braess' network?

Nagurney (2000a) demonstrated that, in the well-known Braess' network, total emissions also increase after adding a new link to the network, which is analogous to the classical Braess' Paradox. In this section we study the conditions under which Nagurney' emission paradox occurs in a classical Braess' network that have not been studied before. The result will be useful in analyzing the simultaneous occurrence of both Braess' and emission paradoxes. The Braess' network is shown in figure 1. Before a new link is added (figure 1a), the network consists of four nodes, and four links, and one O/D pair $w_1 = (1,4)$. The four-link network has two paths between this O/D pair, which are respectively path 1(1-2-4) and path 2 (1-3-4). After a new link is added to connect nodes 2 and 3 (figure 1b), this network has one more path, path 3 (1-2-3-4). Before and after adding the new link, the total travel demand is the same and fixed. The following notations are adopted in this paper: c_{ij} is the travel cost on link ij ; η_k is the total cost on path k ; η'_k is the marginal of total cost on path k ; $\tilde{\eta}'_k$ is the generalized marginals of the total cost on path k ; α_{ij} is the free flow travel cost on link ij , and $\alpha_{ij} > 0$; β_{ij} is the delay parameter for link ij in which a lower

delay parameter means a wider capacity on this link, and $\beta_{ij} > 0$; x_{ij} is the flow on link ij ; f_k is the path flow on path k ; h_{ij} is the emission factor on link ij ; D is the total travel demand, and D_w is the travel demand between O/D pair w ; Q is the total emissions; TSTC is Total System Travel Cost.

According to the emission models developed by the National Environmental Protection Agency (NEPA), the volume of emissions can be calculated by multiplying the emissions factor and the vehicle activity (or link load), which is the key in the estimation of vehicular emissions (Allen, 1995; Anderson et al., 1996; DeCorla-Souza et al., 1994). This relationship in calculating vehicular emissions is also utilized in our paper as described follows:

$$Q = \sum_{ij \in A} h_{ij} x_{ij}. \quad (1)$$

where Q is the total emissions, h_{ij} is the emission factor on link $ij \in A$, which is assumed to be given for all links, and x_{ij} is the flow on link ij . According to equation (1), link flows are required to calculate the total emissions generated in the network. With the total emissions, the conditions when Nagurney's emission paradox occurs can be examined. In the following, flow patterns and the total emissions before and after adding the new link are obtained.

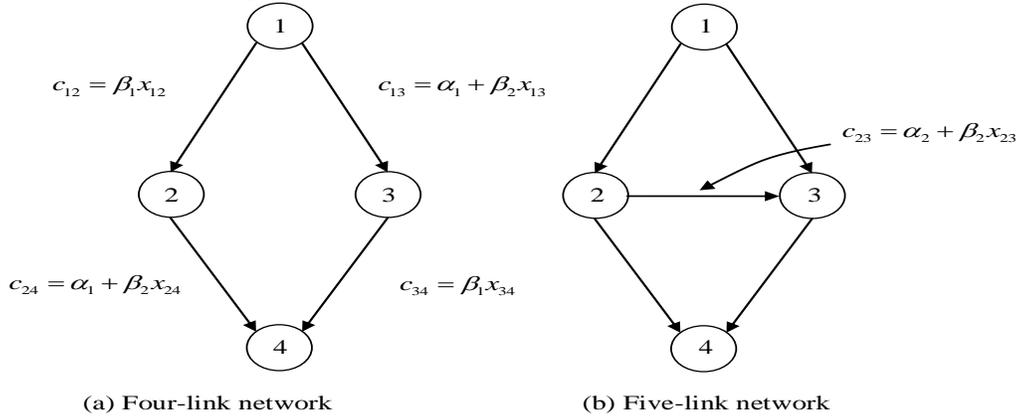


Figure 1. Braess' networks.

Before the addition of link 2-3, two paths' costs equal and flow on each path is the same when User Equilibrium condition is reached. We can obtain:

$$\eta_1 = \eta_2 = \frac{D(\beta_1 + \beta_2)}{2} + \alpha_1, \text{ and} \quad (2)$$

$$f_1 = f_2 = D/2.$$

The total emissions in the network are then calculated as follows:

$$Q^{four\ links} = h_{12}x_{12} + h_{24}x_{24} + h_{13}x_{13} + h_{34}x_{34}, \text{ or} \quad (3)$$

$$Q^{four\ links} = D(h_{12} + h_{24} + h_{13} + h_{34})/2. \quad (4)$$

After link 2-3 is added, there are three possible flow patterns, which are respectively: 1. There are flows on all three paths. (i.e., $f_1, f_2, f_3 > 0$); 2. Only the new path, path 3, is used. (i.e., $f_1 = f_2 = 0, f_3 > 0$); 3. Only the new path is not used.

(i.e., $f_1 = f_2 > 0, f_3 = 0$), which is the same as the case that new link is not added in four-link network. As we will see later, the occurrence of each flow pattern depends on the network configuration (or the parameters of each link performance function). It is noted that due to the symmetrical characteristic of Braess' network, the flows on paths 1 and 2 are always the same, i.e. $f_1 = f_2$. In the following, the three flow patterns and the corresponding total emissions are described.

All three paths are used (i.e., $f_1, f_2, f_3 > 0$). In this case, travel costs on three paths are equal, and the User Equilibrium flow pattern must satisfy the following conditions:

$$f_1 = f_2 = \frac{\alpha_2 - \alpha_1 + D(\beta_1 + \beta_2)}{\beta_1 + 3\beta_2}, \quad (5)$$

$$f_3 = D - 2f_1, \text{ and} \quad (6)$$

$$\eta_1 = \eta_2 = \eta_3 = \alpha_1 + D\beta_1 + (\beta_2 - \beta_1) \left[\frac{\alpha_2 - \alpha_1 + D(\beta_1 + \beta_2)}{\beta_1 + 3\beta_2} \right], \quad (7)$$

and the total emissions in the network are:

$$Q^{five\ links} = f_1(h_{24} + h_{13} - h_{12} - h_{34} - 2h_{23}) + D(h_{12} + h_{34} + h_{23}). \quad (8)$$

Only new path (path 3) is used (i.e., $f_1 = f_2 = 0, f_3 > 0$). In this case, paths 1 and 2 carry no flow and on

$$\begin{aligned} f_1 = f_2 &= 0, \\ f_3 &= D, \text{ and} \end{aligned} \quad (9)$$

$$\eta_1 = \eta_2 = \alpha_1 + \beta_1 D \geq \eta_3 = D(2\beta_1 + \beta_2) + \alpha_2.$$

We can then obtain:

$$D \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}. \quad (10)$$

The total emissions, in this case, are as follows:

$$Q^{five\ links} = f_3(h_{12} + h_{23} + h_{34}) = D(h_{12} + h_{23} + h_{34}).$$

Only the new path (path 3) is not used (i.e., $f_1 = f_2 > 0, f_3 = 0$). In this case, path 3 carries no flow and the total emissions are the same as described in equation (4).

The User Equilibrium flow pattern must follow:

$$\begin{aligned} f_1 = f_2 &= D/2, \\ f_3 &= 0, \text{ and} \end{aligned} \quad (11)$$

$$\eta_1 = \eta_2 = \alpha_1 + \beta_1 D/2 + \beta_2 D/2 \leq \eta_3 = \alpha_2 + \beta_1 D/2 + \beta_1 D/2.$$

From equation (11), we have:

$$(\beta_1 - \beta_2)D \geq 2(\alpha_1 - \alpha_2). \quad (12)$$

According to the above three types of flow patterns, the changes in total emissions can be expressed as follows

1. All three paths are used (i.e., $f_1, f_2, f_3 > 0$):

$$Q^{fivelinks} - Q^{fourlinks} = (f_1 - D/2)(h_{24} + h_{13} - h_{12} - h_{34} - 2h_{23}) = (f_1 - D/2)A. \quad (13)$$

where f_1 must satisfy equation (5) and $A = h_{24} + h_{13} - h_{12} - h_{34} - 2h_{23}$.

2. Only new path (path 3) is used. (i.e., $f_1 = f_2 = 0, f_3 > 0$.):

$$Q^{fivelinks} - Q^{fourlinks} = (D/2)(h_{12} + 2h_{23} + h_{34} - h_{24} - h_{13}) = (-D/2)A. \quad (14)$$

3. Only new path (path 3) is not used. (i.e., $f_1 = f_2 > 0, f_3 = 0$.):

$$Q^{fivelinks} - Q^{fourlinks} = 0. \quad (15)$$

Based on the above changes in total emissions, we can determine whether Nagurney's emission paradox occurs in Braess' network. The emission paradox occurs if and only if the change in total emission is positive (i.e., $Q^{fivelinks} - Q^{fourlinks} > 0$). According to equation (13),

$$f_1 - D/2 = f_1 - \frac{f_1 + f_2 + f_3}{2} = -f_3/2 < 0. \text{ This implies that the emission paradox}$$

occur if $A < 0$. Similarly, according to (14), $-D/2 < 0$. This implies that the emission paradox occur if $A < 0$. According to (15), the emission paradox must not occur. These implications are summarized in Table 1; the emission paradox occurs if $A < 0$ and the new path carries flow (the old paths may or may not carry flows). When $A < 0$? We assume emission factors only depend on distance traveled, or the emission factor of each link is proportional to the travel distance on that link. According to the triangle formula, $(h_{13} - h_{23} - h_{12}) < 0$ and $(h_{24} - h_{34} - h_{23}) < 0$. Then, $A = (h_{13} - h_{23} - h_{12}) + (h_{24} - h_{34} - h_{23}) < 0$. However, if emission factors depend on not only distance traveled but also some other factors, which have been pointed out by DeCorla-Souza et al. (1994), then A can be any real values. From the above expressions, we can observe that whether Nagurney's emission paradox occurs or not depends on the flow pattern on the new network as well as the emission factor of each link. The result is summarized in Table 1.

Therefore, it is essential to know the demand conditions under which the new path will carry flows and under which the new path will never carry flows when examining the occurrence of the emission paradox. According to different relationships of parameters in link performance functions, immediately in table 2 below we can summarize the demand conditions under which three flow patterns occur based on equations (10) and (12). For example, in condition 1, all three flow patterns occur but under different demand conditions. When $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} < D < \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$, there are flows on all three paths; when $D \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$, only

path 3 carries flows; however, when $D \geq \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$, path 3 carries no flow. In

condition 2, there are only two flow patterns occurred, when $D > \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$, all

three paths carry flows, when $D \leq \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$, the new path will not be used.

Parameters' conditions 3 and 4 have the same demand conditions under which flow

patterns 1 and 2 occur, and under any demand value flow pattern 3 doesn't occur. When parameters in link performance functions fall into condition 5, under any demand value, only flow pattern 1 happens, in which all three paths are utilized. It is noticeable that the same flow pattern occurs in conditions 6, 7, 8 and 9. Under any demand value only flow pattern 3 occurs and flow patterns 1 and 2 do not occur. This implies, in these cases, the new link doesn't play any role in the network after construction.

Table 1. Summary of the Occurrence of the Emission Paradox under Each Flow Pattern.

Flow patterns			
Conditions of parameters	1. $f_1, f_2, f_3 > 0$	2. $f_1, f_2 = 0, f_3 > 0$	3. $f_1, f_2 > 0, f_3 = 0$
$A < 0$	<u>The emission paradox must happen</u>		
$A \geq 0$	The emission paradox never happens		

Table 2. Summary of Demand Conditions under which Three Flow Patterns Occur According to Different relationships of Parameters in Link Performance Functions.

Flow patterns			
Conditions of Parameters	1. $f_1, f_2, f_3 > 0$	2. $f_1, f_2 = 0, f_3 > 0$	3. $f_1, f_2 > 0, f_3 = 0$
1. $\beta_1 - \beta_2 > 0;$ $\alpha_1 - \alpha_2 > 0.$	$\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} < D < \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$	$D \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$	$D \geq \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$
2. $\beta_1 - \beta_2 < 0;$ $\alpha_1 - \alpha_2 < 0.$	$D > \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$	Never happen under any demand value	$D \leq \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)}$
3. $\beta_1 - \beta_2 < 0;$ $\alpha_1 - \alpha_2 > 0.$	$D > \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$	$D \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$	Never happen under any demand value
4. $\beta_1 - \beta_2 = 0;$ $\alpha_1 - \alpha_2 > 0.$			
5. $\beta_1 - \beta_2 < 0;$ $\alpha_1 - \alpha_2 = 0.$	Must happen under any demand value	Never happen under any demand value	
6. $\beta_1 - \beta_2 > 0;$ $\alpha_1 - \alpha_2 < 0.$	Never happen under any demand value		Must happen under any demand value
7. $\beta_1 - \beta_2 = 0;$ $\alpha_1 - \alpha_2 = 0.$			
8. $\beta_1 - \beta_2 > 0;$ $\alpha_1 - \alpha_2 = 0.$			
9. $\beta_1 - \beta_2 = 0;$ $\alpha_1 - \alpha_2 < 0.$			

If the new link does not play any role in the network, the emission paradox will not occur. According to table 2 we can summarize conditions under which the emission paradox occurs or not. When $A < 0$, it is obvious that the emission paradox is more likely to occur when parameters in link performance functions satisfy conditions 1 and 2. In conditions 3, 4 and 5, the emission paradox must occur if $A < 0$, because the new link is utilized under any demand value. However, in condition 6, 7, 8 and 9, the emission paradox doesn't occur, as the new link does not play any role under any demand value after construction. After examining the demand conditions under which different flow patterns occur, we know that under conditions 6, 7, 8 and 9, there is no point to build the new link, as in any circumstance it doesn't play any role in the network. In terms of cost and benefit, the road designer should examine the network configuration carefully to avoid such redundant construction. According to parameters in Nagurney's example parameters in Braess' network, $\alpha_1=50$, $\alpha_2=10$, $\beta_1=10$, $\beta_2=1$, $D=6$, $h_{12}=h_{13}=h_{24}=h_{34}=h_{23}=0.1$, we can easily evaluate under which condition Nagurney's emission paradox occurs. As $A = h_{24} + h_{13} - h_{12} - h_{34} - 2h_{23} = -0.2 < 0$, also $\beta_1 - \beta_2 > 0, \alpha_1 - \alpha_2 > 0$ falls into condition 1, we have $D = 6 < \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)} = 8.89$. In this situation, the emission paradox occurs as the new path carries some flows.

When does Braess' Paradox Occur?

Pas and Principio (1997) have examined demand conditions under which Braess' paradox occur when $\beta_1 - \beta_2 > 0, \alpha_1 - \alpha_2 > 0$, which is condition 1 in our study. Here, we further examine all other demand conditions under which Braess' paradox occurs. From table 2 we know in conditions 6, 7, 8 and 9, the new path must not carry any flow, so Braess' paradox will not occur. Therefore, we only examine conditions 2, 3, 4, and 5 respectively. Here, derivation for the occurrence of Braess' paradox is omitted due to the length limitation of the paper. Both demand conditions for Braess' and emission paradoxes' occurrence are summarized in table 3 below. It indicates that, in condition 1, Braess' and emission paradoxes occur when the demand falls into the range described in table 3. In condition 2, Braess' paradox never happens, but the emission paradox may happen under certain demand range. In condition 3, the emission paradox must occur, however, Braess' paradox only turns out under certain demand range. In conditions 4 and 5, the emission paradox must occur, however Braess' paradox never happens. In conditions 6, 7, 8 and 9, both Braess' and emission paradoxes never take place.

Figure 2 shows the occurrence of two paradoxes in the classical Braess' network. As we can see, under Braess' network parameters with Nagurney's emission factors, there are three regions:

1. Both paradoxes do not occur (when the demand is larger than or equal to 8.89 units).
2. Both paradoxes occur at the same time (when the demand between 2.58 and 8.89 units).

3. The emission paradox occurs but Braess' paradox does not (when the demand is less than 2.58 units).

The existence of the last region implies that road network design for mitigating congestion alone may not be able to avoid the increase in vehicular emissions.

Moreover, we can observe that the range of the demand for the occurrence of the emission paradox includes the range for the occurrence of Braess' paradox, which implies that the emission paradox is more likely to occur. However, we do not know whether this is always true for any network. This is left for future studies.

Table 3. Demand Conditions under Which Braess' and Emission Paradoxes Occur.

	Braess' Paradox	Emission Paradox ($A < 0$)
1. $\beta_1 - \beta_2 > 0,$ $\alpha_1 - \alpha_2 > 0;$	When $\frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2} < D < \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)},$ Braess' paradox occurs	When $D < \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)},$ the emission paradox occurs
2. $\beta_1 - \beta_2 < 0,$ $\alpha_1 - \alpha_2 < 0;$	Braess' paradox never occurs under any demand value	When $D > \frac{2(\alpha_1 - \alpha_2)}{(\beta_1 - \beta_2)},$ the emission paradox occurs
3. $\beta_1 - \beta_2 < 0,$ $\alpha_1 - \alpha_2 > 0;$		The emission paradox must occur under any demand value
4. $\beta_1 - \beta_2 = 0,$ $\alpha_1 - \alpha_2 > 0;$		
5. $\beta_1 - \beta_2 < 0,$ $\alpha_1 - \alpha_2 = 0;$		
6, 7, 8 and 9.		Both congestion and emission paradoxes never occur under any demand value

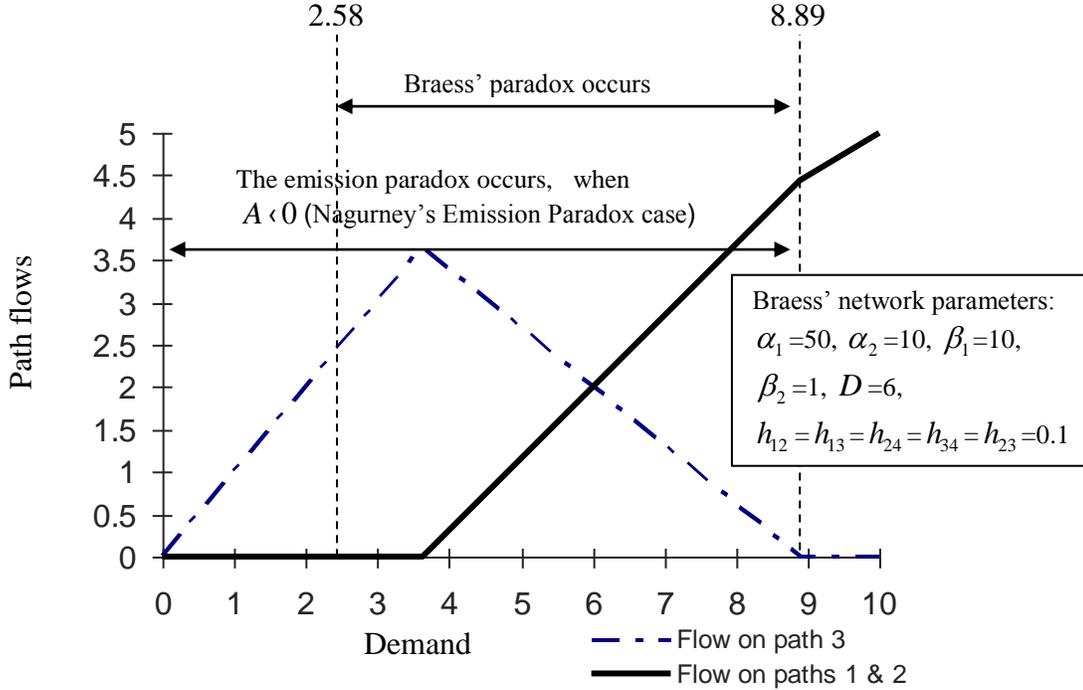


Figure 2. The occurrence of two paradoxes in Braess' network, where $\beta_1 - \beta_2 = 10 - 1 > 0$ and $\alpha_1 - \alpha_2 = 50 - 10 > 0$.

Concluding Remark

In this paper, we analytically examine the occurrence of the emission paradox and the simultaneous occurrence of Braess' and emission paradoxes in the classical Braess' network. We discover the emission paradox does not always occur. The occurrence of the emission paradox depends on the demand for travel, the parameters of link performance functions as well as link emission factors. Moreover, Braess' and emission paradoxes do not always occur at the same time. More importantly, we find that under some conditions of parameters in link performance functions, the emission paradox does occur but Braess' paradox does not. This implies that road network design for mitigating congestion alone may not be able to avoid the increase in vehicular emissions. A more comprehensive view of road network design considering both congestion and emissions simultaneously is necessary to avoid the occurrence of the emission paradox.

Acknowledgment

This research is funded under the Programme for Research in Third-Level Institutions (PRTL), administered by the Irish Higher Education Authority and the start-up grant R-264-000-229-112 from the National University of Singapore.

References

- Allen, W. G. (1995). "Model Improvements for Evaluating Pricing Strategies." In *Transportation Research Record: Journal of the Transportation Research Board*, No. 1498, TRB, National Research Council, Washington, D.C., 75–81.
- Anderson, W. P., P. S. Kanaroglou, E. J. Miller, and R. N. Buliung. (1996). "Simulating Automobile Emissions in An Integrated Urban Model." In *Transportation Research Record: Journal of the Transportation Research Board*, No. 1520, TRB, National Research Council, Washington, D.C., 71–80.
- Braess, D. (1968). "Über Ein Paradoxon Der Verkehrsplanung." *Unternehmensforschung*, 12, 258–268.
- DeCorla-Souza, P., J. Everett, J. Cosby, and P. Lim. (1994). "Trip-based Approach to Estimate Emissions with Environmental Protection Agency's MOBILE Model." In *Transportation Research Record: Journal of the Transportation Research Board*, No. 1444, TRB, National Research Council, Washington, D.C., 118–125.
- Nagurney, A. (2000a). "Congested Urban Transportation Networks and Emission Paradoxes." *Transportation Research*, 5D, 145–151.
- Nagurney, A. (2000b). *Sustainable Transportation Networks*. Edward Elgar Publishers, Cheltenham.