Free Field Representation and Form Factors of the Chiral Gross-Neveu Model

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Declaration

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Stephen Britton

25th July 2013

For my family.

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Abstract

The process of using the free field representation to construct form factors of two dimensional integrable models is very promising. In this thesis, this procedure is analysed and adapted for application to the chiral Gross-Neveu model. The vertex operators and Zamolodchikov-Faddeev algebra for the particles are presented, with a similar structure producing a representation of the local operators of the theory. Using these techniques, the form factors of the model are then constructed as traces over the space of Zamolodchikov-Faddeev operators, and given in terms of an integral representation. In particular, the two-particle form factors of the current operator are found, and shown to agree with previous results in the literature.

Summary

The study of form factors is interesting in itself, and because it can lead to calculation of correlation functions. Form factors can be derived by studying the axioms that were developed in the late 1980's. However, the approach advocated here has the advantage that the analytic properties do not need to be known and can be derived from the results. The method is to develop a free field representation of bosonic fields for the particles. More precisely, the particles are represented as vertex operators which are written in terms of the bosonic fields, together with lowering operators which act via an integral representation. This gives rise to a representation of the Zamolodchikov-Faddeev algebra of the particles. In addition, for the chiral Gross-Neveu model, which is the model we are primarily interested in here, we can use a fusion procedure to find the free field representation of the bound states of the theory. These bound states are also antiparticles and therefore the full particle content of the theory is given in terms of the free field representation.

Furthermore, a related bosonic field allows the construction of representations of local operators. These are constructed using similar methods to the bound states. With the particles, bound states and local operators represented by these bosonic fields, the process of calculating form factors reduces to calculating traces of the operators subject to selection rules, which can also be derived from these traces. The calculation of traces of several particles or operators can be reduced to calculating all traces between two objects, which can be any combination of particles, bound states or operators. Strictly speaking, what emerges from these calculations aren't form factors, but are generating functions for form factors. It

requires an expansion to find the form factors of physical operators. In this way, the form factors of the current operators are isolated. This is the key result as it shows the validity of the free field representation presented here.

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Contents

Chapter 1

Introduction

Modern experimental particle physics consists heavily of measuring cross-sections of particles in collisions and decay events. From these we can find information about the scattering amplitudes. Such quantities must be predicted from theories which allow either exact calculations or numerical simulation of the interactions. However, if we wish to calculate objects which are not directly related to scattering amplitudes, we need to begin with more general quantities, which we must determine from new techniques. Generally speaking, when looking at physical observables, we want to consider correlation functions, which are vital objects for any field theory. Therefore, an idealistic goal would be to develop methods to determine the correlation functions for any quantum field theory. Of course, our goal is not as ambitious as that, but the motivation is the same: namely, to understand how to isolate observable information from two-dimensional integrable models. To this end, we will explore the use of the free field representation to calculate form factors, which are building blocks for correlation functions. Form factors themselves provide us with information about the scattering between *in*-states and *out*-states. In fact, correlation functions can in theory be constructed from a weighted sum of form factors. From this perspective, an understanding of the form factors gives us an understanding of all the observables in the model.

To be more precise, in a quantum field theory, the matrix elements of local operators O between *in*- and *out*-states, $\langle out|O(x)|in\rangle$, determine the field theory correlation functions. In a two-dimensional relativistic theory, the crossing invariance allows one to express a generic matrix element in terms of analytically continued form factors which are matrix elements between the vacuum, $|vac\rangle$, and *in*-states

$$F_{a_1\ldots a_n}(\alpha_1,\ldots,\alpha_n)=\langle vac|O(0)|\alpha_1,\ldots,\alpha_n\rangle^{in}_{a_1\ldots a_n}\,. \tag{1.1}$$

Here a_i is a flavour index of the i-th particle, and α_i is its usual rapidity variable related to its energy and momentum by $E_i = m_i\cosh\alpha_i\,,\ p_i = m_i\sinh\alpha_i$. As a result, because of the link between correlation functions and form factors, and because of crossing symmetry, knowing how to find (1.1) is sufficient to find all correlation functions of the theory. Recall that correlation functions are given by

$$C_n(x_1,\cdots,x_n)=\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle\,, \tag{1.2}$$

and so if we know the form factors, we can reconstruct the correlation functions by inserting a complete set of states. This should look like

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle=\frac{\sum_\lambda\langle vac|\mathcal{O}(x_1)|\phi_\lambda\rangle\langle\phi_\lambda|\mathcal{O}(x_2)|vac\rangle}{||\phi_\lambda||^2}\,, \tag{1.3}$$

where the objects $\langle vac|\mathcal{O}(x_1)|\phi_\lambda\rangle$ and $\langle\phi_\lambda|\mathcal{O}(x_2)|vac\rangle$ are form factors defined by (1.1). It is this that motivates our study of form factors.

In an integrable two-dimensional relativistic theory the form factors satisfy a set of axioms [1]-[3], collected in section 1.3 below, whose solutions were found and studied for some models, see e.g. [3]-[10] and references therein. Finding a solution to the axioms is a complicated problem which requires understanding and employing the form factors' analytic properties. It was observed by Lukyanov [11] (by generalising the ideas in [12]) that the problem of computing the form factors can be reduced to the problem of constructing a free field representation of the Zamolodchikov-Faddeev (ZF) algebra [13, 14] for the model under consideration. The free field representation approach has been successfully applied to several models [11], [15]-[24] including the SU(2) Thirring and sine-Gordon models [11]. An advantage of this approach is that, in principle, the construction of form factors does not require a complete understanding of their analytic properties. This might be important for understanding analytic properties of form factors of nonrelativistic models. In particular, an important model to keep in mind is the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring sigma model in the light-cone gauge [25]. Even though it is relatively straightforward to generalise most of the form factor axioms to this case [26], finding a solution appears to be highly nontrivial; in particular, the analytic properties of $\mathrm{AdS}_5 \times \mathrm{S}^5$ form factors are not known. It is quite possible that Lukyanov's approach will appear to be more efficient in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case. Moreover, the analytic properties follow from the free field representation.

The goal of this thesis is to extend Lukyanov's results for the SU(2) Thirring model to a more general case of the SU(N) chiral Gross-Neveu (GN) model [27]. Previously, in [9] and [10] Bethe ansatz methods were used to find the minimal solutions for the form factor axioms. The calculation of form factors for a given operator then reduced to postulating a so-called "p-function" unique to the operator. In this manner the form factors of the current operator and stress-energy tensor were found. The results presented in this thesis use the free field representation method for the Gross-Neveu model, and were previously published in [28]. The discussion here includes some extra details and explanations. The model has a very rich spectrum of particles. Its "elementary" particles transform in the rank-1 fundamental representation of SU(N), and they can form bound states transforming in all the other fundamental representations of SU(N) [29]. Anti-particles of rank-r are rank-(N-r) particles. In particular, anti-particles of elementary particles are at the same time their bound states [29]. The exact Gross-Neveu S-matrix was found by combining the SU(N) invariance with the $1/N$ expansion [30]-[33]. The chiral Gross-Neveu model was extensively studied in the axiomatic approach in [3, 8, 10] where form factors of several local operators were constructed. It is therefore useful for understanding how the free field approach works in the case of models containing bound states and invariant under higher-rank symmetry algebras. In this thesis we construct a free field representation of the Gross-Neveu model Zamolodchikov-Faddeev algebra for elementary particles and their bound states, and find a large class of operators generating form factors of local operators through Lukyanov's trace formula [11]. A free field representation for elementary particles of the chiral Gross-Neveu model was also constructed in [34], and it agrees with our findings up to some Klein factors necessary to satisfy the Zamolodchikov-Faddeev algebra relations. We also reproduce the two particle form factors of the current operator, which validates our main goal, which is to find the free field representation of particles and local operators of the Gross-Neveu model.

The thesis is laid out as follows. In this chapter, we begin by recalling some facts about two-dimensional integrable models in section 1.1. Following this, we give an introduction to the Gross-Neveu model, particularly the properties of its scattering matrix, in section 1.2. The chapter concludes with an overview of form factors in section 1.3.

Chapter 2 begins with a discussion of the free field representation in section 2.1, before applying it to the Gross-Neveu model in section 2.2. The general process of the free field representation is explained in detail, and the process is made explicit for the Gross-Neveu model by finding the relevant constants and functions. The action of the lowering operators is discussed in detail in relation to constructing the Zamolodchikov-Faddeev algebra, with attention paid to the integrals that appear. The angular Hamiltonian is also found.

Chapter 3 is where we first consider the bound states of the Gross-Neveu model in detail. Section 3.1 gives the details of the construction of the bound state operators, although some calculations are relegated to the appendices for clarity. Similarly, section 3.2 is where the local operators are constructed as fused operators analogous to the bound states.

It is in chapter 4 that all the strands come together to calculate form factors. Firstly, in section 4.1, the general approach for using the free field representation to construct form factors is outlined, before a more detailed demonstration is given in section 4.2 of how to perform the trace calculations. With this done, the form factors for the current operator are found. This is the main result of the thesis, which demonstrates the effectiveness of the methods.

After this, we turn to future developments in chapter 5. Firstly, we discuss application of the free field representation to the Principal Chiral Field model in section 5.1.1, before turning to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring in section 5.1.2, before closing with some final remarks.

In addition there are several appendices. The first, appendix A, gives proofs that the form factors found in the text satisfy the form factor axioms. Next, appendix B provides a complete listing of functions needed for the free field representation of the Gross-Neveu model. Appendix C incorporates some calculations deemed too lengthy for section 3.1, and likewise appendix D has similar calculations for section 3.2. In appendix E some examples of the use of the principal value prescription as required in section 2.2 are given. Next, appendix F gives a derivation of the formula for computing traces from the free field representation, and appendix G describes the regularisation of the free field representation and the derivation of the selection rules. The final appendix, H, gives some background on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring that is necessary for the discussion in section 5.1.2.

1.1 Two-dimensional integrable models

Before discussing the Gross-Neveu model and defining form factors, let us begin by giving an overview of two dimensional integrable models.

1.1.1 Two dimensional field theories

We will be considering $1+1$ dimensional quantum field theories, so we recall some of the important properties of these models. In particular, we are interested in a special class of such models called integrable models. Examples of such models include the Thirring model, the Sinh-Gordon model, the Gross-Neveu model, and - at least as a conjecture - the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string world-sheet sigma model. In two dimensions, integrability is often the result of having an infinite number of conserved charges. However, Parke [35] showed that if in addition to the momentum P_μ and the internal symmetry charges I_i, (i.e. $SU(N)$ symmetry for the chiral Gross-Neveu model) we have two extra conserved charges, that this is sufficient to ensure the absence of particle production and the factorisabilty of the S-matrix. In addition, previous work by Shankar and Witten [36] showed that the existence of a higher rank conserved charge gives rise to factorisation of the three-particle S-matrix, and to the Yang-Baxter equation. As a result, integrability can be deduced from the existence of two extra higher rank conserved charges. The outline of Parke's argument [35] is as follows. Assume that we have a theory with two extra charges Q^+ and Q^-, which transform under the Lorentz group as

$$Q^+ \to \Lambda^{+m} Q^+ \,, \quad Q^- \to \Lambda^{-n} Q^- \,, \tag{1.4}$$

where $\Lambda^\pm$ are the Lorentz transformations written in light-cone coordinates, so that if $x^\pm = x_0 \pm x_1$, we have

$$x^\pm \to \Lambda^\pm x^\pm \,. \tag{1.5}$$

It is necessary that m and n are odd and $m \geq n > 1$ to ensure that the charges transform differently to each other. Parke then introduces the linear combination

$$Q_\theta \equiv (\cos\theta) Q^+/m - (\sin\theta) Q^-/n \,, \tag{1.6}$$

which he uses to explore the amplitude of scattering two *in*-states into N *out*-states. This amplitude looks like

$$\langle \phi_3, \cdots, \phi_{N+2} | S | \phi_1, \phi_2 \rangle \,, \tag{1.7}$$

where no assumption is made about the S-matrix, S, at this stage, other than expecting the usual analyticity and continuity properties apply. Since Q_θ is a conserved charge, this must be equivalent to

$$\langle \phi_3, \cdots, \phi_{N+2} | e^{-i\alpha Q_\theta} \, S \, e^{i\alpha Q_\theta} | \phi_1, \phi_2 \rangle \,, \tag{1.8}$$

for real α. By studying the overlap of the wave functions of the particle states, Parke concluded that extra particles can only appear at isolated points in the rapidity difference of the incoming particles. Such an S-matrix would not satisfy the correct analyticity and continuity properties required of the S-matrix, and hence particle production cannot exist. Moreover, the scattering amplitude must show energy-momentum conservation, which requires there to be two particles in the *out*-state, which must have the same momentum and mass as the particles in the *in*-state. Parke wrote this in the somewhat symbolic notation

$$\langle p_1, \cdots, p_N | S | q_1, q_2 \rangle \propto \delta_{N2} \delta^{(2)}(p_1 - q_1) \delta^{(2)}(p_2 - q_2) \,. \tag{1.9}$$

While a similar argument to the above concludes that the three-particle S-matrix does not allow particle production, the factorisation of the multi-particle S-matrix is perhaps better understood by the argument of Shankar and Witten [36]. They argued that a higher rank conserved charge Q^n of rank n acting on a state in the manner

$$e^{i\alpha Q^n} |p\rangle = e^{i\alpha p^n} |p\rangle \,, \tag{1.10}$$

where p is the momentum, results in shifting particles in a collision relative to each other by an amount dependent on the momentum. As a result, even if we have an apparently simultaneous three-body collision, acting by this conserved charge on all the states (which obviously must leave the scattering amplitude unchanged) changes the impact parameters and ensures that we can always view the three-particle S-matrix as a product of three two-particle S-matrices. In fact, the same logic gives rise to the Yang-Baxter equation, which will be discussed later in the context of the Zamolodchikov-Faddeev algebra.

Therefore, integrability ensures that particle number is conserved and that scattering is elastic. Furthermore, as a result of these conservation laws, scattering of many particles is reduced to a series of two particle scattering events, since at each stage in the process all the charges must be conserved. This is known as Factorised Scattering. Clearly this greatly simplifies problems involving scattering, since we only need to consider a two particle scattering matrix. Essentially, this is a statement that since the model only has one space dimension, a faster moving particle will catch a slower moving particle, and after scattering, the order of rapidities will reverse.

In addition to these properties, we also discover that the creation and annihilation operators of the asymptotic states have extra relations. Consider creation operators $A_i^\dagger(\theta)$ which create particles by acting on the vacuum, and annihilation operators $A^i(\theta)$, for which

$$A^i(\theta)|vac\rangle = 0\,. \tag{1.11}$$

With these operators, we can define two types of states, *in*-states and *out*-states as follows

$$\begin{aligned}
|\theta_1,\theta_2,\cdots,\theta_n\rangle^{(in)}_{k_1,\ldots,k_n} &= A^\dagger_{k_n}(\theta_n)\cdots A^\dagger_{k_1}(\theta_1)|vac\rangle\,, \qquad \theta_1<\theta_2<\cdots<\theta_n\,,\\
|\theta_1,\theta_2,\cdots,\theta_n\rangle^{(out)}_{k_1,\ldots,k_n} &= A^\dagger_{k_1}(\theta_1)\cdots A^\dagger_{k_n}(\theta_n)|vac\rangle\,, \qquad \theta_1<\theta_2<\cdots<\theta_n\,.
\end{aligned} \tag{1.12}$$

These are the Zamolodchikov-Faddeev (ZF) creation and annihilation operators and describe particles in the theory. These operators are intrinsically linked with the two particle scattering matrix, and have to satisfy the quadratic Zamolodchikov-

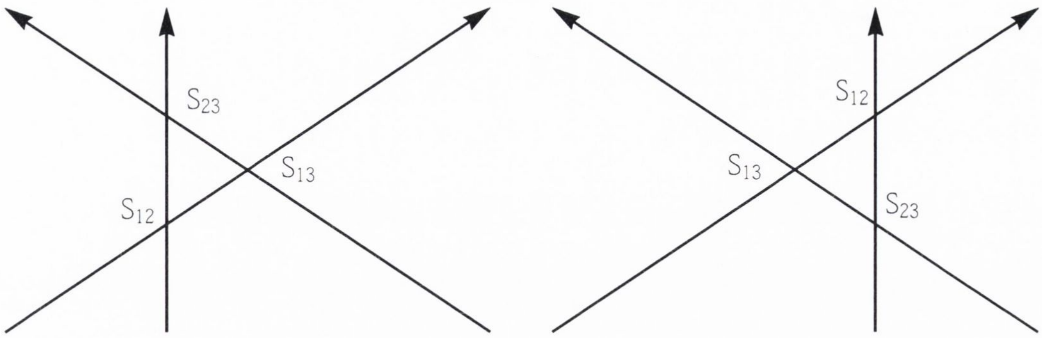

Figure 1.1: Two orders of three particle scattering.

Faddeev algebra given by the relations

$$A^{\dagger}_{k_1}(\theta_1)A^{\dagger}_{k_2}(\theta_2) = A^{\dagger}_{n_2}(\theta_2)A^{\dagger}_{n_1}(\theta_1)S^{n_1n_2}_{k_1k_2}(\theta_{12})\,, \tag{1.13}$$

$$A^{k_1}(\theta_1)A^{k_2}(\theta_2) = S^{k_1k_2}_{n_1n_2}(\theta_{12})A^{n_2}(\theta_2)A^{n_1}(\theta_1)\,, \tag{1.14}$$

$$A^{k_1}(\theta_1)A^{\dagger}_{k_2}(\theta_2) = A^{\dagger}_{n_2}(\theta_2)S^{k_1n_2}_{n_1k_2}(\theta_{21})A^{n_1}(\theta_1) + 2\pi\delta^{k_1}_{k_2}\delta(\theta_{12})\,, \tag{1.15}$$

in which we use the notation $\theta_{ij} = \theta_i - \theta_j$. From these, we can now infer the Yang-Baxter equation. We may write it as

$$\begin{aligned} &S^{b_2b_3}_{c_2c_3}(\theta_2-\theta_3)S^{b_1c_3}_{c_1a_3}(\theta_1-\theta_3)S^{c_1c_2}_{a_1a_2}(\theta_1-\theta_2) \\ &= S^{b_1b_2}_{c_1c_2}(\theta_1-\theta_2)S^{c_1b_3}_{a_1c_3}(\theta_1-\theta_3)S^{c_2c_3}_{a_2a_3}(\theta_2-\theta_3)\,. \end{aligned} \tag{1.16}$$

This is shown pictorially in Figure 1.1.1. In words, this equation tells us that the intermediate steps in a series of scattering events are irrelevent to the final state we observe. The product of the S-matrices of different orderings of scattering are the same for a model with Factorised Scattering. As a result, the *in-* and *out-*states contain all the necessary information to determine the overall S-matrix. In addition to the Yang-Baxter equation, the S-matrix for relativistic two-dimensional integrable models should have the following properties:

1. Unitarity

$$S^{b_1b_2}_{a_1a_2}(\theta)S^{c_1c_2}_{b_1b_2}(-\theta) = \delta^{c_1}_{a_1}\delta^{c_2}_{a_2}\,. \tag{1.17}$$

2. Crossing symmetry

$$S^{b_1 b_2}_{a_1 a_2}(i\pi - \theta) = C_{a_1 c} S^{c b_2}_{d a_2}(\theta) C^{d b_1} , \tag{1.18}$$

where C_{ab} is the charge conjugation matrix such that $C_{ab}C^{bc} = \delta^c_a$.

It should be mentioned that we always consider the rapidities to reside in the physical strip, $0 \leq Im\ \theta \leq \pi$, and that any values outside this strip are understood by analytic continuation.

1.2 The Gross-Neveu model

The Gross-Neveu (GN) model is a theory containing N fermions with quartic interactions in two dimensions, the study of which was led by [27]. A good review is contained in [37], and this overview summarises much of what was said there. In particular, we consider the chiral Gross-Neveu model, which has an $SU(N)$ invariant Lagrangian density given by

$$\mathcal{L} = \bar{\psi}_i i\gamma^\mu \partial_\mu \psi_i + \frac{g^2}{2}\left((\bar{\psi}_i \psi_i)^2 - (\bar{\psi}_i \gamma^5 \psi_i)^2 \right) . \tag{1.19}$$

The entities ψ_i are the N complex Dirac fields. There is a global $U(1)$ symmetry given by $\psi \to e^{i\alpha}\psi$, a global continuous chiral symmetry $\psi \to e^{i\theta\gamma_5}\psi$, and an $SU(N)$ symmetry with Noether current given by $j^\mu_{ij} = i\bar{\psi}_j \gamma^\mu \psi_i$. This model is asymptotically free, displaying a negative β function, as was established in [27]. The model also displays an apparent contradiction: according to Coleman [38], there can be no spontaneous symmetry breaking in two dimensions and hence no Goldstone bosons; however, there exists a massless boson that generates dynamical mass. This apparent contradiction was clarified by Witten [39]. There are two parts to this. Firstly, from Coleman's theorem it is clear that there can be no massive chiral particles since the chiral symmetry would be broken, but Witten showed that although the elementary fermion field, ψ, that appears in (1.19) has a non-zero chirality, the physical fermion has zero chirality, and therefore can be massive. The second part to the theorem is that there should be no Goldstone boson in two dimensions to produce this mass. Although there is indeed a massless scalar, Witten argued that it is not in fact a Goldstone boson since it does not have the required low-energy behavior that a Goldstone boson should have. Since there is no chirality breaking Green's functions (since, after all, the physical fermion has zero chirality) a Goldstone boson is not necessary and cannot exist. That fact does not preclude other massless scalars from existing however. It is such a massless scalar that generates mass in the chiral Gross-Neveu model.

In order to find the $1/N$ expansion, it is necessary to rewrite the Lagrangian (1.19) in a new form. The method followed here was originally constructed in [33] and called the operator formulation in the treatment in [37]. The idea is to write the original field in terms of new boson fields,

$$\begin{aligned}\psi_i(x) = K_i \left(\frac{\mu}{2\pi}\right)^{\frac{1}{2}} e^{-i\frac{\pi}{4}\gamma^5} \\ \times :\exp\left(i\sqrt{\frac{\pi}{N}}\left[\gamma^5\phi(x) + \int_{x^1}^{\infty} dy^1 \dot{\phi}(x^0, y^1)\right]\right): \\ \times :\exp\left(-i\sqrt{\pi}\left[\gamma^5\phi_i(x) + \int_{x^1}^{\infty} dy^1 \dot{\phi}_i(x^0, y^1)\right]\right):,\end{aligned} \tag{1.20}$$

where ϕ_i are $SU(N)$ valued fields that are pseudo-potentials of the $SU(N)$ currents that satisfy the constraint

$$\sum_{i=1}^{N} \phi_i(x) = 0\,, \tag{1.21}$$

and the field ϕ is the pseudo-potential of the conserved $U(1)$ current. Finally, K_i are Klein factors to ensure the correct anti-commutation relations for the fermions. However, these are not the physical particles, which are instead given by

$$\hat{\psi}_i(x) = K_i\sqrt{\frac{\mu}{2\pi}}\exp\left(i\sqrt{\pi}\left[\gamma^5\phi_i(x) + \int_{x^1}^{\infty} dy^1 \dot{\phi}_i(x^0, y^1)\right]\right)\,, \tag{1.22}$$

and satisfy

$$\hat{\psi}_j^\dagger(x) \sim \frac{1}{(N-1)!}\epsilon_{ji_1\cdots i_{N-1}}\hat{\psi}_{i_1}(x)\cdots\hat{\psi}_{i_{N-1}}(x)\,, \tag{1.23}$$

as a result of (1.21). These fields have unusual commutation relations. They carry spin $s = \frac{1}{2}(1 - 1/N)$ and have commutation relations

$$\hat{\psi}_i(x,t)\hat{\psi}_i(y,t) = e^{2\pi i s \epsilon(x-y)}\hat{\psi}_i(y,t)\hat{\psi}_i(x,t)\,, \tag{1.24}$$

and for the creation and annihilation operators

$$\hat{a}^\dagger(p)\hat{a}^\dagger(p') = e^{2\pi i s \epsilon(p-p')}\hat{a}^\dagger(p')\hat{a}^\dagger(p)\,. \tag{1.25}$$

There is no known scattering theory for particles with these statistics, and therefore new fields must be defined with Fermi statistics,

$$\psi'_i(x) = \exp\left(i\sqrt{\frac{\pi}{N}}\left[\gamma^5 A(x) + B(x)\right]\right)\psi_i(x)\,, \tag{1.26}$$

where $A(x)$ and $B(x)$ are independent free massless fields. In terms of these new fields, the Lagrangian (1.19) becomes

$$\begin{aligned}\mathcal{L} = {} & \bar{\psi}'_i\gamma^\mu\partial_\mu\psi'_i + \frac{g^2}{2}\left((\bar{\psi}'_i\psi'_i)^2 - (\bar{\psi}'_i\gamma^5\psi'_i)^2\right) - \frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 \\ & + \frac{\alpha}{\sqrt{N}}\bar{\psi}'_i\gamma^5\gamma^\mu\psi'_i\partial_\mu A - \frac{\beta}{\sqrt{N}}\bar{\psi}'_i\gamma^\mu\psi'_i\partial_\mu B\,,\end{aligned} \tag{1.27}$$

where α and β are unspecified coupling constants. The final step is to introduce auxiliary fields $\sigma = \bar{\psi}_i\psi_i$ and $\pi = i\bar{\psi}_i\gamma^5\psi_i$, which will change the potential term for ψ'_i, via the relation (1.26), to

$$\frac{g^2}{2}\left((\bar{\psi}'_i\psi'_i)^2 - (\bar{\psi}'_i\gamma^5\psi'_i)^2\right) \to -\frac{1}{2g}(\sigma^2+\pi^2) + \bar{\psi}'_i(\sigma + i\pi\gamma^5)\psi'_i\,. \tag{1.28}$$

It is then possible to integrate over the fields ψ'_i, leading to the effective action

$$\begin{aligned}S_{eff} = {} & -iN\mathrm{Tr}\log\left(i\gamma^\mu\partial_\mu + \sigma + i\pi\gamma^5 + \frac{\alpha}{\sqrt{N}}\gamma^5\gamma^\mu\partial_\mu A - \frac{\beta}{\sqrt{N}}\gamma^\mu\partial_\mu B\right) \\ & - \frac{1}{2g}\int d^2x(\sigma^2+\pi^2) - \frac{1}{2}\int d^2x\left((\partial_\mu A)^2 + (\partial_\mu B)^2\right)\,.\end{aligned} \tag{1.29}$$

It is possible to find the propagators for the fields from this action [33],

$$\begin{aligned}
\Delta_\pi(p) &= -\frac{2\pi i}{N}\frac{\coth\frac{\theta}{2}}{\theta}\frac{1-\frac{\alpha^2}{\pi}\left(1-\frac{\theta}{\sinh\theta}\right)}{1-\frac{\alpha^2}{\pi}}, \\
\Delta_\sigma(p) &= -\frac{2\pi i}{N}\frac{\tanh\frac{\theta}{2}}{\theta}, \\
\Delta_A(p) &= -\frac{i}{N}\frac{1}{p^2}\frac{1}{1-\frac{\alpha^2}{\pi}}, \\
\Delta_B(p) &= -\frac{i}{N}\frac{1}{p^2},
\end{aligned} \tag{1.30}$$

where θ is the rapidity, and $p^2 = -4m^2\sinh^2\frac{\theta}{2}$. From these, the $1/N$ expansion results for amplitudes can be compared to the S-matrix given below in section 1.2.1 to prove that it is indeed the correct S-matrix of the model.

The fact that this model exhibits factorised scattering was first established by Zamolodchikov and Zamolodchikov [13], and the S-matrix itself was developed in [29]-[33]. In [30] the different classes of $U(N)$ invariant factorised S-matrices are given and the minimal solutions established. The chiral GN model is in class II of this classification structure. Following from this, [31] compares the $1/N$ expansion to the perturbation result and finds that the results match. In [29] and [33] the scattering matrix involving antiparticles was established. In addition, it was noted in [29] that bound states composed of fundamental particles transform in other representations of $SU(N)$. In fact, bound states are the same as antiparticles in this model. More explicitly, rank-r antiparticles are rank-$(N-r)$ particles (or bound states). As a result, the antiparticle of a fundamental rank-1 particle is a rank-$(N-1)$ particle (bound state). For $N = 3$, this means that the bound states of two particles are the antiparticles corresponding to the fundamental particles. It was shown in [29] that the scattering between a bound state and particle produces the same amplitudes as the scattering between the particle and its antiparticle. This means that such an identification between the bound states and antiparticles is natural. The scattering amplitudes for fundamental particles may be summarised as follows

$$\begin{aligned}\langle A_d(\theta_2)A_c(\theta_1)|A_a(\theta_1)A_b(\theta_2)\rangle &= u_1(\theta_1-\theta_2)\delta_{ac}\delta_{bd}+u_2(\theta_1-\theta_2)\delta_{ad}\delta_{bc}\,,\\ \langle \bar{A}_d(\theta_2)A_c(\theta_1)|A_a(\theta_1)\bar{A}_b(\theta_2)\rangle &= t_1(\theta_1-\theta_2)\delta_{ac}\delta_{bd}+t_2(\theta_1-\theta_2)\delta_{ab}\delta_{cd}\,,\end{aligned} \tag{1.31}$$

with the functions given by

$$\begin{aligned} t_1(\theta) &= \frac{\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{1}{N}-\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2}-\frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{1}{N}+\frac{\theta}{2\pi i}\right)}\,,\\ t_2(\theta) &= -\frac{\frac{2\pi i}{N}}{i\pi-\theta}t_1(\theta)\,,\\ u_1(\theta) &= \frac{\Gamma\left(1-\frac{\theta}{2\pi i}\right)}{\Gamma\left(1-\frac{1}{N}-\frac{\theta}{2\pi i}\right)}\frac{\Gamma\left(\frac{\theta}{2\pi i}-\frac{1}{N}\right)}{\Gamma\left(\frac{\theta}{2\pi i}\right)}\,,\\ u_2(\theta) &= -\frac{\frac{2\pi i}{N}}{\theta}u_1(\theta)\,. \end{aligned} \tag{1.32}$$

An example of the application of this is to $N = 3$ with a three particle state exhibiting a pole at $\theta_1 = \theta_2 + \frac{2\pi i}{3}$ which leads to a bound state. The S-matrix is considered in more detail in the following section.

1.2.1 The S-matrix of the chiral Gross-Neveu model

The spectrum of particles of the chiral SU(N) GN model consists of N elementary particles of mass m transforming in the rank-1 fundamental representation of SU(N), and their r-particle bound states of mass $m_r = m \sin\frac{\pi r}{N}/\sin\frac{\pi}{N}$ transforming in the rank $r = 2, \ldots, N-1$ fundamental representation of SU(N). A rank-r particle with rapidity θ is created by a ZF operator $\mathcal{A}^\dagger_K(\theta)$, and annihilated by $\mathcal{A}^K(\theta)$ where $K = (k_1, \ldots, k_r)$ has integer-valued components ordered as $1 \le k_1 < k_2 < \cdots < k_r \le N$. The creation and annihilation operators satisfy the ZF algebra

$$\mathcal{A}^\dagger_{K_1}(\theta_1)\mathcal{A}^\dagger_{K_2}(\theta_2) = \mathcal{A}^\dagger_{N_2}(\theta_2)\mathcal{A}^\dagger_{N_1}(\theta_1)S^{N_1N_2}_{K_1K_2}(\theta_{12})\,, \tag{1.33}$$

$$\mathcal{A}^{K_1}(\theta_1)\mathcal{A}^{K_2}(\theta_2) = S^{K_1K_2}_{N_1N_2}(\theta_{12})\mathcal{A}^{N_2}(\theta_2)\mathcal{A}^{N_1}(\theta_1)\,, \tag{1.34}$$

$$\mathcal{A}^{K_1}(\theta_1)\mathcal{A}^\dagger_{K_2}(\theta_2) = \mathcal{A}^\dagger_{N_2}(\theta_2)S^{K_1N_2}_{N_1K_2}(\theta_{21})\mathcal{A}^{N_1}(\theta_1)+2\pi\delta^{K_1}_{K_2}\delta(\theta_{12})\,, \tag{1.35}$$

where $\theta_{ij} \equiv \theta_i - \theta_j$ and $S^{N_1N_2}_{K_1K_2}(\theta_{12})$ is the scattering matrix of particles of ranks r_1 and r_2 with rapidities θ_1 and θ_2. Since higher rank particles are bound states of elementary particles, their S-matrices are obtained from the GN S-matrix for elementary particles by the fusion procedure. It is often convenient to use the matrix form of the GN S-matrix. We introduce N-dimensional rows E^i and columns E_i with all vanishing entries except the one in the i-th position which is equal to the identity, and the matrix unities $E_i{}^j = E_i \otimes E^j$ with the only non-vanishing element on the intersection of the i-th column with the j-th row. Then the entries of the GN S-matrix for elementary particles can be combined in the following $N^2 \times N^2$ matrix

$$\mathbb{S}^{GN}(\theta) = S^{kl}_{ij}(\theta)\, E_k{}^i \otimes E_l{}^j\,. \tag{1.36}$$

Explicitly the S-matrix of the chiral GN model for elementary particles is given by

$$\mathbb{S}^{GN}(\theta) = S(\theta)\, \mathbb{R}(\theta)\,, \qquad S(\theta) = \frac{\Gamma\left(\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{N-1}{N} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{N-1}{N} + \frac{i\theta}{2\pi}\right)}\,, \tag{1.37}$$

where the scalar factor $S(\theta)$ does not have any poles in the physical strip $0 \le Im(\theta) \le \pi$ and for real θ has the nice integral form

$$S(\theta) = \exp\left(-2\,i\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{(N-1)\pi t}{N}}{\sinh \pi t} e^{\frac{\pi t}{N}} \sin\theta t\right)\,. \tag{1.38}$$

It satisfies the crossing symmetry condition

$$\prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} S(\theta + \frac{2\pi i}{N}\,k) = (-1)^{N-1}\frac{\theta - i\pi\frac{N-1}{N}}{\theta + i\pi\frac{N-1}{N}}\,, \tag{1.39}$$

and has the large θ asymptotics $S(\pm\infty) = e^{\mp i\pi\frac{N-1}{N}}$. The matrix and pole structure of the GN S-matrix is given by the standard SU(N)-invariant R-matrix

$$\mathbb{R}(\theta) = \frac{\theta\,\mathbb{I} - \frac{2\pi i}{N}\,\mathbb{P}}{\theta - \frac{2\pi i}{N}}\,, \tag{1.40}$$

where $\mathbb{I}$ is the identity operator and $\mathbb{P} = E_k{}^i \otimes E_i{}^k$ is the permutation operator which exchanges the flavour indices of the scattering particles. Introducing the projection operators

$$\mathbb{P}_s = \frac{1}{2}(\mathbb{I} + \mathbb{P}), \quad \mathbb{P}_a = \frac{1}{2}(\mathbb{I} - \mathbb{P}),$$

onto the symmetric and antisymmetric parts of the tensor product of two fundamental representations one gets

$$\mathbb{R}(\theta) = \mathbb{P}_s + \frac{\theta + \frac{2\pi i}{N}}{\theta - \frac{2\pi i}{N}} \mathbb{P}_a, \tag{1.41}$$

which exhibits the pole at $\theta = \frac{2\pi i}{N}$ in the antisymmetric part. This leads to the existence of bound states composed of two, three, and up to $N-1$ elementary particles. The (N-1)-particle bound states are identified with anti-particles of the elementary particles. In general a rank-r particle and a rank-(N-r) particle created by $\mathcal{A}^\dagger_K$ and $\mathcal{A}^\dagger_{\overline{K}}$ form a particle-antiparticle pair if $\overline{K}$ is such that $K \cup \overline{K} = \mathcal{P}(1, 2, \dots, N)$ where $\mathcal{P}$ is some permutation of $1, 2, \dots N$. In what follows, in such a pair we refer to a bound state of smaller rank (that is $r < N/2$) as a particle. If N is even, $N = 2p$, then a bound state with the label $K = (k_1, k_2, \dots, k_p)$ is considered as a particle. The ZF operators can be normalised in such a way that for a particle $\mathcal{A}^\dagger_K$ and antiparticle $\mathcal{A}^\dagger_L$ the charge conjugation matrix $C_{KL} = \epsilon_{KL}$ where $\epsilon_{KL} \equiv \epsilon_{i_1 \dots i_N}$ is skew-symmetric, and $\epsilon_{1\dots N} = 1$.

1.3 Form factors

Recalling from section 1.1, the definition of the *in*- and *out*-bases of asymptotic states (1.12), we again write them, this time explicitly expressed in terms of the ZF creation operators as follows

$$|\theta_1, \theta_2, \cdots, \theta_n\rangle^{(in)}_{K_1,\dots,K_n} = \mathcal{A}^\dagger_{K_n}(\theta_n) \cdots \mathcal{A}^\dagger_{K_1}(\theta_1)|vac\rangle, \qquad \theta_1 < \theta_2 < \cdots < \theta_n,$$
$$|\theta_1, \theta_2, \cdots, \theta_n\rangle^{(out)}_{K_1,\dots,K_n} = \mathcal{A}^\dagger_{K_1}(\theta_1) \cdots \mathcal{A}^\dagger_{K_n}(\theta_n)|vac\rangle, \qquad \theta_1 < \theta_2 < \cdots < \theta_n.$$

They are related to each other by the scattering matrix. The vacuum state $|vac\rangle$ is annihilated by $\mathcal{A}^K(\theta)$, and has the unit norm, $\langle vac|vac\rangle = 1$.

Form factors of a local operator $O(x)$ are the matrix elements of $O(0)$ between n-particle *in*-states and the vacuum state

$$F_{K_1\ldots K_n}(\theta_1,\ldots,\theta_n) = \langle vac|O(0)\mathcal{A}^\dagger_{K_n}(\theta_n)\cdots\mathcal{A}^\dagger_{K_1}(\theta_1)|vac\rangle\,. \tag{1.42}$$

Being analytically continued to complex θ_i the form factors satisfy a set of axioms. We give here the axioms for form factors as appeared in [3], the first four of which we present in a slightly generalised form similar to [26] to cover nonrelativistic models possessing the crossing symmetry invariance. Firstly, from form factors of type (1.42), we construct all matrix elements through the crossing symmetry[1]

$$\begin{aligned} &{}^{b_1\ldots b_m}_{\quad out}\langle\beta_1,\ldots,\beta_m|O(0)|\alpha_1,\ldots,\alpha_n\rangle^{in}_{a_1\ldots a_n} \\ &= C^{b_1c_1}\cdots C^{b_mc_m}F_{a_1\ldots a_nc_1\ldots c_m}(\alpha_1,\ldots,\alpha_n,\beta_1+i\pi,\ldots,\beta_m+i\pi)\,, \end{aligned} \tag{1.43}$$

where C^{ab} is the charge conjugation matrix.

Then, these form factors must satisfy the following axioms:

1. Permutation symmetry (Watson's theorem):

$$\begin{aligned} &F_{a_1\ldots a_{j+1}a_j\ldots a_n}(\alpha_1,\ldots\alpha_{j+1},\alpha_j,\ldots\alpha_n) = \\ &= S^{c_jc_{j+1}}_{a_ja_{j+1}}(\alpha_j,\alpha_{j+1})F_{a_1\ldots c_jc_{j+1},\ldots a_n}(\alpha_1,\ldots,\alpha_j,\alpha_{j+1},\ldots\alpha_n)\,. \end{aligned} \tag{1.44}$$

   Here $S^{c_jc_{j+1}}_{a_ja_{j+1}}(\alpha_j,\alpha_{j+1})$ is the S-matrix, which for relativistic models depends only on the difference $\alpha_j-\alpha_{j+1}$.

2. Double-crossing or quasi-periodicity condition:

$$F_{a_1\ldots a_n}(\alpha_1,\ldots\alpha_{n-1},\alpha_n+2\pi i) = e^{2\pi i\Omega(O,a_n)}F_{a_na_1\ldots a_{n-1}}(\alpha_n,\alpha_1,\ldots\alpha_{n-1})\,. \tag{1.45}$$

   The quantity $\Omega(O,a_n)$ appears if the n-th particle $\mathcal{A}^\dagger_{a_n}$ has nontrivial statistics with respect to operator $O(x)$.

3. Simple poles: The form factors have simple poles at the points $\alpha_j=\alpha_i+i\pi$.

[1] In a nonrelativistic model with the crossing symmetry invariance, e.g. the $AdS_5\times S^5$ superstring, the rapidity variable α should be chosen so that the energy and momentum of the corresponding particle are meromorphic functions on the rapidity plane, and the crossing symmetry transformation is realised as in any relativistic theory as the shift of α by $i\pi$: $\alpha\to\alpha+i\pi$.

Due to the property (1.44) it is sufficient to consider only $j = n$, and $i = n-1$. Then the form factors must have the expansion

$$
\begin{aligned}
i\, F_{a_1 \dots a_n}(\alpha_1, \dots, \alpha_{n-1}, \alpha_n) &= C_{a_n a'_{n-1}} \frac{F_{a'_1 \dots a'_{n-2}}(\alpha_1, \dots, \alpha_{n-2})}{\alpha_n - \alpha_{n-1} - \pi i} \\
&\times \Big(\delta^{a'_1}_{a_1} \cdots \delta^{a'_{n-1}}_{a_{n-1}} - e^{2\pi i \Omega(O, a_{n-1})} S^{a_{n-1} a_1}_{c_1 a_1}(\alpha_{n-1}, \alpha_1) \cdots \\
&\cdots S^{c_{n-4} a'_{n-3}}_{c_{n-3} a_{n-3}}(\alpha_{n-1}, \alpha_{n-3}) S^{c_{n-3} a'_{n-2}}_{a_{n-1} a_{n-2}}(\alpha_{n-1}, \alpha_{n-2}) \Big) + \mathcal{O}(1)
\end{aligned} \tag{1.46}
$$

at $\alpha_n \to \alpha_{n-1} + i\pi$.

4. Bound state poles: Let particles $\mathcal{A}^\dagger_K$ with $K \in \mathcal{K}$ be bound states of particles $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$. The rapidities α_I and α_J of $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$ are known functions f^K_{IJ} and f^K_{JI} of the rapidity α_K of the bound states $\mathcal{A}^\dagger_K$, and the scattering matrix $S^{ab}_{IJ}(\alpha_I, \alpha^\epsilon_J)$ of $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$ with $\alpha_I = f^K_{IJ}(\alpha_K)$ and $\alpha^\epsilon_J = f^K_{JI}(\alpha_K + \epsilon)$ has a pole at $\epsilon = 0$. Then the form factors with $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$ as external particles are related to those with $\mathcal{A}^\dagger_K$ as external particles through the small ϵ expansion

$$
\begin{aligned}
&F_{JIa_3 \dots a_n}(\alpha^\epsilon_J, \alpha_I, \alpha_3 \dots, \alpha_n) \\
&\quad = \frac{i}{\epsilon} \sum_{K \in \mathcal{K}} \Gamma^K_{IJ} F_{K a_3 \dots a_n}(\alpha_K, \alpha_3 \dots, \alpha_n) + \mathcal{O}(1)\,,
\end{aligned} \tag{1.47}
$$

where Γ^K_{IJ} are some constants determined by the consistency of (1.47) with itself and the previous form factor axioms. The relations (1.47) can be inverted and used to express the form factors of bound states through the form factors of the elementary particles.

For a relativistic theory $\alpha_I = \alpha_K + i\mathfrak{u}_+$, $\alpha_J = \alpha_K - i\mathfrak{u}_-$ ($\mathfrak{u}_\pm$ depend on the indices I, J, K), the scattering matrix $S^{ab}_{IJ}(\alpha)$ of $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$ has a pole at $\alpha = i\mathfrak{u}^K_{IJ}$, and $\mathfrak{u}_\pm$ are found from the equations

$$
\mathfrak{u}_+ + \mathfrak{u}_- = \mathfrak{u}^K_{IJ}\,, \quad m_I \sin \mathfrak{u}_+ = m_J \sin \mathfrak{u}_-\,, \tag{1.48}
$$

where m_I and m_J are masses of $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$, and the mass of the bound state $\mathcal{A}^\dagger_K$ is equal to $m_K = m_I \cos \mathfrak{u}_+ + m_J \cos \mathfrak{u}_-$.

The last two axioms are valid only for relativistic models and to stress this we use the letter θ for the rapidity variable.

5. Due to relativistic invariance, form factors should satisfy the equation

$$F_{a_1 \dots a_n}(\theta_1 + \zeta, \theta_2 + \zeta, \dots, \theta_n + \zeta) = \exp(\zeta s(O)) F_{a_1 \dots a_n}(\theta_1, \dots, \theta_n)\,, \qquad (1.49)$$

where $s(O)$ is the spin of the local operator $O(x)$.

6. Form factors $F_{a_1 \dots a_n}(\theta_1, \dots, \theta_n)$ must be analytic in each variable $\theta_i - \theta_j$ in the strip $0 \leq \operatorname{Im}\theta \leq 2\pi$ except for simple poles.

In appendix A, proofs that these axioms are satisfied by the trace formula used to find form factors for the GN model are given. An important observation by Lukyanov [11] reduces the problem of computing form factors to the problem of finding a representation of a so-called *extended* ZF algebra. It is generated by vertex operators $A_I(\theta)$,[2] the angular Hamiltonian $\mathbb{K}$, and the central elements Ω_I obeying the defining relations

$$\begin{aligned} A_I(\theta_1) A_J(\theta_2) &= A_L(\theta_2) A_K(\theta_1) S_{IJ}^{KL}(\theta_{12})\,, & (1.50)\\ A_I(\theta_1) A_J(\theta_2) &= -\frac{i\, C_{IJ}}{\theta_{12} - i\pi} + \mathcal{O}(1)\,, \quad \theta_{12} \to i\pi\,, & (1.51)\\ \frac{d}{d\theta} A_I(\theta) &= -[\mathbb{K}\,, A_I(\theta)] - i\Omega_I A_I(\theta)\,. & (1.52) \end{aligned}$$

The relations (1.50) and (1.51) show that one can think of $A_I(\theta)$ and $C^{IJ} A_J(\theta + i\pi)$, $C^{IJ} C_{JK} = \delta^I_K$ as representing the ZF creation and annihilation operators, respectively. For some models the relations (1.51) have to be modified by replacing C_{IJ} with $C_{IJ}\Gamma$ where Γ is an auxiliary element satisfying $\Gamma^2 = id$ which either commutes or anticommutes with A_I. In particular this is the case for the SU(2p) chiral GN model.

In addition to the relations above if the particles $\mathcal{A}^\dagger_K$ of the same mass with $K \in \mathcal{K}$ are bound states of particles $\mathcal{A}^\dagger_I$ and $\mathcal{A}^\dagger_J$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$ then the

[2]They should not be confused with the ZF creation and annihilation operators $\mathcal{A}^\dagger_I(\theta)$ and $\mathcal{A}^I(\theta)$.

vertex operators A_I and A_J must satisfy the following bootstrap conditions

$$A_I(\theta' + i\mathrm{u}_+)A_J(\theta - i\mathrm{u}_-) = \frac{i}{\theta' - \theta} \sum_{K \in \mathcal{K}} \Gamma_{IJ}^{K} A_K(\theta) + \mathcal{O}(1)\,, \quad \theta' \to \theta\,. \tag{1.53}$$

Let us now assume that a representation of the extended ZF algebra is constructed and the vertex operators act in some space π_A. According to Lukyanov a local operator O of the model under consideration corresponds to a linear operator $\Lambda(O)$ acting in π_A which satisfies the following two conditions

$$e^{\theta\mathbb{K}}\Lambda(O)e^{-\theta\mathbb{K}} = e^{\theta s(O)}\Lambda(O)\,, \quad \Lambda(O)A_I(\theta) = e^{2\pi i\,\Omega(O,I)}A_I(\theta)\Lambda(O)\,, \tag{1.54}$$

where $s(O)$ is the spin of the local operator $O(x)$, and $\Omega(O,I)$ appears if the particle $\mathcal{A}_I^\dagger$ has nontrivial statistics with respect to $O(x)$. Then, the form factor (1.42) is given by the formula

$$F_{K_1 \dots K_n}(\theta_1, \dots, \theta_n) = \mathcal{N}_O \, \frac{\mathrm{Tr}_{\pi_A}\left[e^{2\pi i\mathbb{K}}\Lambda(O)A_{K_n}(\theta_n)\cdots A_{K_1}(\theta_1)\right]}{\mathrm{Tr}_{\pi_A}\left[e^{2\pi i\mathbb{K}}\right]}\,, \tag{1.55}$$

where the normalisation constant $\mathcal{N}_O$ depends only on the local operator O and has to be fixed by other means. Assuming that (1.55) satisfies the necessary analyticity properties, and that the trace exists, the form factor axioms then follow from the cyclicity of the trace and the defining relations of the extended ZF algebra. The existence of the trace can be substantiated on a case by case basis by direct calculation.

Chapter 2

Free Field Realisation

2.1 Free field realisation of the extended ZF algebra

Another important observation by [11] is that for many models the extended ZF algebra can be realised in terms of free bosons. Let us sketch the idea of the construction. One considers particles of the same mass belonging to a highest weight irreducible representation of the symmetry algebra of the model under study. Then the highest weight vertex operator A_1 satisfies the following simple relation

$$A_1(\theta_1)A_1(\theta_2) = S(\theta_{12})A_1(\theta_2)A_1(\theta_1)\,. \tag{2.1}$$

Here the scattering matrix $S(\theta)$ of the two highest weight particles obeys $S(0) = -1$, and admits the representation

$$S(\theta) = \frac{g(-\theta)}{g(\theta)}\,, \tag{2.2}$$

where $g(\theta)$ is an analytic function without zeroes and poles in the lower half plane $Im(\theta) \leq 0$ except a simple zero at $\theta = 0$, and

$$\partial_\theta \ln g(\theta) = \mathcal{O}(\frac{1}{\theta})\,, \quad \theta \to \infty\,,\ Im(\theta) \leq 0\,. \tag{2.3}$$

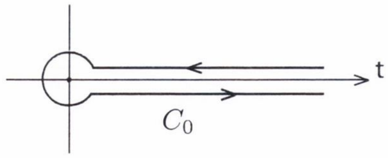

Figure 2.1: The integration contour C_0 in the integral $\int_{C_0} \frac{dt}{2\pi i} F(t) \ln(-t)$.

These properties of $g(\theta)$ imply that for $Im(\theta) < 0$ it admits the following integral representation

$$g(\theta) = \exp\left(-\int_0^\infty \frac{dt}{t} f(t) e^{-i\theta t}\right), \tag{2.4}$$

where $f(t)$ asymptotes to 1 at large t. The function $f(t)$ does not have to vanish at $t = 0$, and the integrals of the form

$$\int_0^\infty dt\, F(t) \tag{2.5}$$

will be always understood as [40]

$$\int_0^\infty dt\, F(t) \equiv \int_{C_0} \frac{dt}{2\pi i} F(t) \ln(-t), \tag{2.6}$$

where the integration contour C_0 goes from $+\infty + i0$ above the real axis, then around zero, and finally below the real axis to $+\infty - i0$, see Figure 2.1. Let us also mention that for real values of θ the scattering matrix $S(\theta)$ has the integral representation

$$S(\theta) = \exp\left(-2i \int_0^\infty \frac{dt}{t} f(t) \sin(\theta t)\right). \tag{2.7}$$

Let us now introduce the bosonic operators satisfying the commutation relations

$$[a(t), a(t')] = t f(t) \delta(t + t'), \tag{2.8}$$

where $a(t)$ and $a(-t)$ for $t > 0$ are the annihilation and creation operators, respectively

$$a(t)|0\rangle = 0 \quad \text{for } t > 0. \tag{2.9}$$

We use the operators to define the following free bosonic field

$$\phi(\theta) = Q + \int_{-\infty}^{\infty} \frac{dt}{it}\, a(t) e^{i\theta t}\,, \tag{2.10}$$

which satisfies the following relations

$$[\phi(\theta_1), \phi(\theta_2)] = \ln S(\theta_{21})\,, \quad \langle \phi(\theta_1)\phi(\theta_2)\rangle = -\ln g(\theta_{21})\,. \tag{2.11}$$

The operator Q is a zero mode coordinate operator which commutes with $a(t)$. It appears in an explicit ultraviolet regularisation of the free field [11]. In addition a regularised free field also contains the zero mode momentum operator P which annihilates the vacuum $|0\rangle$ and also commutes with $a(t)$. In fact, as will be explained, P is an element of the Cartan subalgebra of the symmetry algebra of the model. The regularised free fields for the GN model are discussed in appendix G.

The field ϕ is used to construct the basic vertex operator

$$V(\theta) =: e^{i\phi(\theta)} :\,, \tag{2.12}$$

which satisfies the ZF algebra relation (2.1) and

$$V(\theta_1)V(\theta_2) = g(\theta_{21}) : V(\theta_1)V(\theta_2) : \,. \tag{2.13}$$

The highest weight vertex operator A_1 is then realised as

$$A_1(\theta) = \omega_1 V(\theta)\,, \tag{2.14}$$

where ω_1 is a "Klein" factor which commutes with $V(\theta)$ and might be necessary to satisfy all the ZF algebra relations.

The remaining vertex operators are then obtained by acting on the highest weight vertex operator A_1 by the lowering symmetry operators ("screening charges") $\mathcal{J}_k^-$. The action of $\mathcal{J}_k^-$ on the vertex operators depends on the coproduct of the symmetry algebra. In particular in the case of a quantum group with the deformation parameter q the vertex operators are constructed as

$$A_1(\theta) = \omega_1 V(\theta)\,, \quad A_{k+1}(\theta) = \mathcal{J}_k^- A_k(\theta) - q\, A_k(\theta)\, \mathcal{J}_k^-\,, \quad k = 1, 2, \ldots\,. \tag{2.15}$$

A free field representation of $\mathcal{J}_k^-$ is found by assuming the following ansatz[1]

$$\mathcal{J}_k^- \sim \int_C d\alpha : e^{i\phi_k(\alpha)} : , \tag{2.16}$$

where the commutation relations of the free fields ϕ_k with ϕ and themselves, and the integration contours C are determined by requiring that (2.15) satisfies the extended ZF algebra. In the next section we discuss how this works for the chiral GN model.

2.2 Free field representation of the GN model ZF algebra

In this section, the work of [11] is extended to the GN model, as appeared in [28]. It appears that if one omits the Klein factors mentioned in the previous section then Lukyanov's procedure gives a free field representation for the ZF algebra of a model with a twisted S-matrix. It is invariant under the $\mathfrak{sl}(N)$ algebra with a rather unusual coproduct, which however implies the action (2.15) with $q = -1$ of the $\mathfrak{sl}(N)$ lowering generators on the vertex operators.

2.2.1 Twisted GN S-matrix

There is a simple generalisation of the standard SU(N)-invariant R-matrix $\mathbb{R}(\theta)$. One can check that the R-matrix of the form

$$\mathbb{R}^{\Sigma}(\theta) = \frac{\theta\,\Sigma - \frac{2\pi i}{N}\,\mathbb{P}}{\theta - \frac{2\pi i}{N}} , \tag{2.17}$$

where Σ is a diagonal matrix satisfies the Yang-Baxter Equation (YBE) and the unitarity condition if and only if Σ is given by

$$\Sigma = \sum_{i,j=1}^{N} s_{ij} E_{ii} \otimes E_{jj} , \quad s_{ij} s_{ji} = 1 \quad \forall\, i, j . \tag{2.18}$$

[1]This ansatz does not work if the symmetry algebra is the sum of two algebras.

In particular the coefficients s_{ij} satisfy the conditions $s_{ii} = \pm 1$ for any i. The physical unitary condition requires Σ to be unitary and therefore $s_{ij} = e^{i\phi_{ij}}$ where ϕ_{ij} are real and obey $\phi_{ij} + \phi_{ji} = 0 \; mod \; 2\pi$.

One can use $\mathbb{R}^{\Sigma}$ to define the twisted GN S-matrix as $\mathbb{S}^{\Sigma} = S(\theta)\,\mathbb{R}^{\Sigma}$. It is unclear if such a twisted S-matrix corresponds to any local field theory which would be a multi-parameter deformation of the GN model. The ZF algebra with the twisted GN S-matrix $\mathbb{S}^{\Sigma}$ has the form

$$\mathbb{A}^{\Sigma}_{1}(\theta_1)\mathbb{A}^{\Sigma}_{2}(\theta_2) = \mathbb{A}^{\Sigma}_{2}(\theta_2)\mathbb{A}^{\Sigma}_{1}(\theta_1)S^{\Sigma}_{12}(\theta_{12})\,, \tag{2.19}$$

where $\mathbb{A}^{\Sigma}$ is a row $\mathbb{A}^{\Sigma} = A^{\Sigma}_{i}\,E^{i}$ with A^{Σ}_{i} being the ZF vertex operators. The relations can be written explicitly in components

$$A^{\Sigma}_{i}(\theta_1)A^{\Sigma}_{i}(\theta_2) = s_{ii}S(\theta_{12})A^{\Sigma}_{i}(\theta_2)A^{\Sigma}_{i}(\theta_1)\,, \tag{2.20}$$

$$A^{\Sigma}_{i}(\theta_1)A^{\Sigma}_{j}(\theta_2) = S(\theta_{12})\left[\frac{s_{ij}\theta_{12}}{\theta_{12}-\frac{2\pi i}{N}}A^{\Sigma}_{j}(\theta_2)A^{\Sigma}_{i}(\theta_1) - \frac{\frac{2\pi i}{N}}{\theta_{12}-\frac{2\pi i}{N}}A^{\Sigma}_{i}(\theta_2)A^{\Sigma}_{j}(\theta_1)\right]. \tag{2.21}$$

Notice that only the transition amplitudes depend on the twist parameters s_{ij}. From eqs.(2.20, 2.21), one can see that if $s_{ii} = 1$ and $s_{ij} = -1$ for $i \neq j$, then for each pair of indices i, j the ZF relations are the same as for the SU(2) Thirring model discussed by Lukyanov. It is therefore not surprising that a free field representation for the ZF algebra of the GN model is related to the twisted S-matrix with Σ of the form

$$\Sigma^{(-1)} = \sum_{i=1}^{N} E_{ii}\otimes E_{ii} - \sum_{i\neq j}^{N} E_{ii}\otimes E_{jj} = 2\sum_{i=1}^{N} E_{ii}\otimes E_{ii} - I\,. \tag{2.22}$$

To simplify the notations in what follows, we denote the twisted S-matrix $\mathbb{S}^{\Sigma^{(-1)}}$ as $\mathbb{S}^{(-1)}$. It is easy to check that $\mathbb{S}^{(-1)}$ satisfies the invariance conditions

$$\mathbb{S}^{(-1)}\,\Delta_{(-1)}(\mathbb{J}^{-}_{k}) = \Delta^{op}_{(-1)}(\mathbb{J}^{-}_{k})\,\mathbb{S}^{(-1)}\,, \tag{2.23}$$

where $\mathbb{J}^{-}_{k} = {E_{k+1}}^{k}$ are the $\mathfrak{sl}(N)$ lowering generators in the rank-1 fundamental representation, $\Delta^{op}_{(-1)}(\mathbb{J}) \equiv \mathbb{P}\,\Delta_{(-1)}(\mathbb{J})\,\mathbb{P}$ and

$$\Delta_{(-1)}(\mathbb{J}_k^-) = \mathbb{J}_k^- \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{J}_k^- - 2({E_k}^k + {E_{k+1}}^{k+1}) \otimes \mathbb{J}_k^- \,, \tag{2.24}$$

is the coproduct. It is defined in the same way on the raising generators, and it is extended to the whole $\mathfrak{sl}(N)$ algebra via the commutation relations.

Let us introduce the ZF vertex operators Z_i satisfying the relations (2.20, 2.21) with $s_{ii} = 1$ and $s_{ij} = -1$ for $i \neq j$

$$Z_i(\theta_1)Z_i(\theta_2) = S(\theta_{12})Z_i(\theta_2)Z_i(\theta_1)\,, \tag{2.25}$$

$$Z_i(\theta_1)Z_j(\theta_2) = S(\theta_{12})\left[-\frac{\theta_{12}}{\theta_{12}-\frac{2\pi i}{N}}Z_j(\theta_2)Z_i(\theta_1) - \frac{\frac{2\pi i}{N}}{\theta_{12}-\frac{2\pi i}{N}}Z_i(\theta_2)Z_j(\theta_1)\right]\,, \tag{2.26}$$

and assume that a free field representation for Z_1 and the lowering operators χ_i^- is found. Then the coproduct (2.24) implies that all the other vertex operators are obtained through the formulae

$$Z_{i+1}(\theta) = \chi_i^- Z_i(\theta) + Z_i(\theta)\chi_i^- \,. \tag{2.27}$$

Then, one can construct operators A_i^Σ satisfying the ZF algebra with $\mathbb{S}^\Sigma$ and all $s_{ii} = 1$ through the formula

$$A_i^\Sigma(\theta) = \omega_i\, Z_i(\theta)\,, \tag{2.28}$$

where the "Klein" factors ω_i commute with Z_j and satisfy the following algebra

$$\omega_i\omega_j + s_{ij}\omega_j\omega_i = 0\,, \quad i \neq j\,. \tag{2.29}$$

In particular for the canonical (untwisted) S-matrix $s_{ij} = 1$ and if one also imposes extra conditions $\omega_i^2 = \eta_{ii}$ where η_{ii} are equal to either 1 or -1 then it is just the Clifford algebra. In the general case the relations (2.29) can be solved by representing ω_i as zero mode "vertex" operators

$$\omega_i = \Gamma_i\, e^{i\varphi_i}\,, \quad [\varphi_i, \varphi_j] = i\phi_{ij}\,, \tag{2.30}$$

where Γ_i satisfy the Clifford algebra $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\eta_{ij}$ with $\eta_{ij} = \eta_{ii}\delta_{ij}$. In what follows we find it convenient to choose η_{ii} in such a way that $\Gamma \equiv \Gamma_1\Gamma_2\cdots\Gamma_N$ satisfies the condition $\Gamma = 1$ for odd N which can be achieved by choosing $\eta_{ii} = 1$ for $i = 1, \cdots N-1$ and $\Gamma_N = \Gamma_{N-1}\cdots\Gamma_1$, while for even N the element Γ satisfies the condition $\Gamma^2 = 1$.

Thus it is sufficient to find a free field representation for Z_i only. In what follows, we will be interested only in the untwisted case and we will denote the corresponding vertex operators as A_i without any superscript.

To conclude this discussion let us also mention that introducing the twist

$$\mathbb{F}_{12} = \sum_{i,j=1}^{N} e^{-i\tau_{ij}} E_{ii} \otimes E_{jj}\,, \quad \mathbb{F}_{21} \equiv \mathbb{P}\,\mathbb{F}_{12}\,\mathbb{P}\,, \tag{2.31}$$

where the parameters τ_{ij} satisfy the conditions $\tau_{ij} - \tau_{ji} = \phi_{ij}$ mod 2π, and $\mathbb{P}$ is the permutation operator which appears in the definition of the R-matrix, (1.40). One can easily check that $\mathbb{R}^{\Sigma}$ is a twisted R-matrix

$$\mathbb{R}^{\Sigma} = \mathbb{F}_{21}\,\mathbb{R}\,\mathbb{F}_{12}^{-1}\,. \tag{2.32}$$

Since the untwisted R-matrix satisfies the invariance conditions $\mathbb{J}_{12}\ \mathbb{R} = \mathbb{R}\ \mathbb{J}_{21}$, therefore the twisted R-matrix should satisfy the invariance conditions also

$$\Delta_{\mathbb{F}}^{op}(\mathbb{J})\,\mathbb{R}^{\Sigma} = \mathbb{R}^{\Sigma}\,\Delta_{\mathbb{F}}(\mathbb{J})\,, \tag{2.33}$$

where $\mathbb{J}$ are $\mathfrak{sl}(N)$ generators, $\Delta_{\mathbb{F}}^{op}(\mathbb{J}) = \mathbb{P}\,\Delta_{\mathbb{F}}(\mathbb{J})\,\mathbb{P}$ and

$$\Delta_{\mathbb{F}}(\mathbb{J}) = \mathbb{F}_{12}\,(\mathbb{J}_1 + \mathbb{J}_2)\,\mathbb{F}_{12}^{-1}\,, \tag{2.34}$$

is the twisted coproduct.[2] The entities $\mathbb{J}_1$ and $\mathbb{J}_2$ are given by

$$\mathbb{J}_1 = \mathbb{J}_i\ E^i \otimes \mathbb{I}\,, \quad \mathbb{J}_2 = \mathbb{J}_i\ \mathbb{I} \otimes E^i\,. \tag{2.35}$$

Examples of such generators $\mathbb{J}$ are $E_{i+1}{}^{i}$ which are the $\mathfrak{sl}(N)$ lowering operators. There is however no twist which would lead to the coproduct (2.24).

[2]Strictly speaking one should consider $gl(N)$ (or $u(N)$ if $\mathbb{F}$ is unitary) because for generic $\mathbb{F}$ the coproduct of $\mathbb{J}$ which is from $\mathfrak{sl}(N)$ is not in the tensor product of two universal enveloping $\mathfrak{sl}(N)$ algebras.

2.2.2 Free fields

According to the discussion in subsection 2.1, to construct a free field representation for the vertex operators Z_k of the elementary particles of the twisted ZF algebra (2.25, 2.26) one needs a bosonic operator $a_0(t)$ for the highest weight vertex operator Z_1, and N-1 bosonic operators $a_k(t)$ for the lowering operators χ_k^-. For any μ the operators $a_\mu(t)$ and $a_\mu(-t)$ for $t > 0$ are the annihilation and creation operators, respectively: $a_\mu(t)|0\rangle = 0$ for $t > 0$. Since (N-1)-particle bound states are antiparticles of the elementary particles, only N-1 bosonic operators may be independent.

The commutation relations of the operators a_μ can be written in the uniform form

$$[a_\mu(t), a_\nu(t')] = t f_{\mu\nu}(t)\delta(t+t')\,, \quad \mu = 0, 1, \ldots, N-1\,, \tag{2.36}$$

where $f_{\mu\nu}$ must satisfy the relations $f_{\mu\nu}(-t) = f_{\nu\mu}(t)$. In addition we also impose the conditions $f_{\nu\mu}(t) = f_{\mu\nu}(t)$ which were satisfied in the $N = 2$ case, and appear to hold for general N too. We also introduce the zero mode operators Q_μ and P_μ such that $P_\mu|0\rangle = 0$. Their commutation relations are listed in appendix G but will not be important in this section.

We then define the free fields

$$\phi_\mu(\theta) = Q_\mu + \int_{-\infty}^{\infty} \frac{dt}{it}\, a_\mu(t) e^{i\theta t}\,, \tag{2.37}$$

which satisfy the following relations

$$\begin{aligned} [\phi_\mu(\theta_1), \phi_\nu(\theta_2)] &= \ln S_{\mu\nu}(\theta_2 - \theta_1)\,, \\ \langle \phi_\mu(\theta_1)\phi_\nu(\theta_2)\rangle &= -\ln g_{\mu\nu}(\theta_2 - \theta_1)\,. \end{aligned} \tag{2.38}$$

Here the S-matrices $S_{\mu\nu}$ and Green's functions $g_{\mu\nu}$ are related to $f_{\mu\nu}$ as follows

$$\begin{aligned} S_{\mu\nu}(\theta) &= \exp\left(-2i \int_0^\infty \frac{dt}{t}\, f_{\mu\nu}(t)\sin(\theta t)\right)\,, \\ g_{\mu\nu}(\theta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\, f_{\mu\nu}(t) e^{-i\theta t}\right)\,, \end{aligned} \tag{2.39}$$

and they are related to each other as

$$S_{\mu\nu}(\theta) = \frac{g_{\mu\nu}(-\theta)}{g_{\mu\nu}(\theta)}\,. \tag{2.40}$$

The fields ϕ_μ are used to construct the basic vertex operators

$$V_\mu(\theta) =: e^{i\phi_\mu(\theta)} :, \tag{2.41}$$

which obey the following relations

$$\begin{aligned} V_\mu(\theta_1)V_\nu(\theta_2) &= g_{\mu\nu}(\theta_{21}) : V_\mu(\theta_1)V_\nu(\theta_2) :, \\ V_\mu(\theta_1)V_\nu(\theta_2) &= S_{\mu\nu}(\theta_{12})V_\nu(\theta_2)V_\mu(\theta_1)\,. \end{aligned} \tag{2.42}$$

The free field realisation of the ZF algebra with the twisted S-matrix $\mathbb{S}^{(-1)}$ is constructed as follows

$$\begin{aligned} Z_1(\theta) &= \rho\, V_0(\theta)\,, \quad \rho = e^{\frac{i\pi}{N}} e^{\gamma\frac{N-1}{2N}} N^{-\frac{1}{2N}}\,, \\ \chi_k^- &= \rho_\chi \int_C d\alpha V_k(\alpha)\,, \quad \rho_\chi = \frac{e^\gamma}{2\pi}\,, \\ Z_{k+1}(\theta) &= \chi_k^-\, Z_k(\theta) + Z_k(\theta)\,\chi_k^-\,, \quad k = 1, \ldots, N-1\,. \end{aligned} \tag{2.43}$$

Here γ is Euler's constant and the normalisation constants for Z_1 and χ_k^- have been chosen for future convenience. Then the integration contour C in χ_k^- depends on operators located to the right or to the left of χ_k^- and is specified for any operator χ which involves integration as follows [11]. One first brings the product of all vertex operators in a monomial containing χ to the normal form which is considered as a regular operator. This produces a product of various Green's functions which may have poles. Then the contour C runs from $\mathrm{Re}\,\alpha = -\infty$ to $\mathrm{Re}\,\alpha = +\infty$ and it lies above all poles due to operators to the right of χ but below all poles due to operators to the left of χ. Note that if one then acts by the resulting monomial operator on other operators the contour C should be additionally deformed according to the procedure described. As an example, let us consider the monomial $V_0(\theta_L)\,\chi_1^-\,V_0(\theta_{R,1})V_2(\theta_{R,2})$ and assume for definiteness

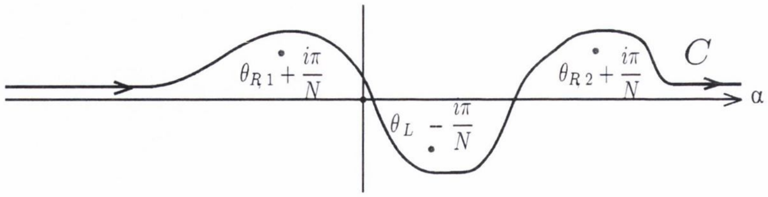

Figure 2.2: The integration contour C in the monomial $V_0(\theta_L)\,\chi_1^-\,V_0(\theta_{R,1})V_2(\theta_{R,2})$.

that $\theta_{R,1} < \theta_L < \theta_{R,2}$. One gets

$$\begin{aligned} V_0(\theta_L)\,\chi_1^-\,V_0(\theta_{R,1})V_2(\theta_{R,2}) &= \int_C d\alpha : V_0(\theta_L)V_1(\alpha)V_0(\theta_{R,1})V_2(\theta_{R,2}) : \\ &\times g_{01}(\alpha-\theta_L)g_{10}(\theta_{R,1}-\alpha)g_{12}(\theta_{R,2}-\alpha) \\ &\times g_{00}(\theta_{R,1}-\theta_L)g_{02}(\theta_{R,2}-\theta_L)g_{02}(\theta_{R,2}-\theta_{R,1})\,. \end{aligned} \tag{2.44}$$

As will be shown below the Green's functions which depend on α are equal to

$$g_{01}(\alpha) = g_{10}(\alpha) = g_{12}(\alpha) = -\frac{ie^{-\gamma}}{\alpha + \frac{i\pi}{N}}\,,$$

and therefore the poles are at

$$\alpha = \theta_L - \frac{i\pi}{N}\,, \quad \alpha = \theta_{R,1} + \frac{i\pi}{N}\,, \quad \alpha = \theta_{R,2} + \frac{i\pi}{N}\,.$$

Thus the integration contour C runs above $\theta_{R,1}+\frac{i\pi}{N}$ and $\theta_{R,2}+\frac{i\pi}{N}$ but below $\theta_L - \frac{i\pi}{N}$, see Figure 2.2.

Matrix elements of the ZF operators are therefore given by multiple integrals. One sometimes needs to compute integrals of functions which behave as $1/\alpha$ for large α. We use the principal value prescription for the integral

$$\int_{-\infty}^{\infty} dx \frac{1}{x-a} = i\pi\,, \quad \mathrm{Im}(a) > 0\,. \tag{2.45}$$

In particular, we require for $\beta > 0$, $\beta \in \mathbb{R}$ the integrals below:

$$\begin{aligned} \int_{-\infty}^{\infty} d\alpha \frac{1}{\alpha - \beta - \frac{i\pi}{N}} &= i\pi\,, \\ \int_{-\infty}^{\infty} d\alpha \frac{1}{\alpha - \beta + \frac{i\pi}{N}} &= -i\pi\,. \end{aligned} \tag{2.46}$$

We give examples of the use of these integrals and further explanation in appendix E.

To find the ZF operators A_k of the GN model, we must now incorporate the "Klein" factors. We modify (2.43) to get

$$\begin{aligned} A_k(\theta) &= \Gamma_k Z_k(\theta)\,, \quad \mathcal{J}_k^- = \Gamma_{k+1}\Gamma_k^{-1}\chi_k^-\,, \\ A_{k+1}(\theta) &= \mathcal{J}_k^-\, A_k(\theta) - A_k(\theta)\,\mathcal{J}_k^-\,, \quad k = 1, \ldots, N-1\,, \end{aligned} \tag{2.47}$$

where $\mathcal{J}_k^-$ are the $\mathfrak{sl}(N)$ algebra lowering generators.

Let us now sketch how the commutation relations between a_μ can be found. From the discussion in subsection 2.1 and eq.(1.38) we conclude that

$$f_{00}(t) = \frac{\sinh\frac{(N-1)\pi t}{N}}{\sinh \pi t} e^{\frac{\pi|t|}{N}}\,, \quad g_{00}(\theta) = e^{\frac{(N-1)(\gamma+\log(2\pi))}{N}} \frac{\Gamma\left(\frac{i\theta}{2\pi} - \frac{1}{N} + 1\right)}{\Gamma\left(\frac{i\theta}{2\pi}\right)}\,, \tag{2.48}$$

where γ is Euler's constant. Then the commutation relations between a_0 and a_1 can be easily guessed by generalising the $N = 2$ relations from [11]

$$\begin{aligned} f_{01}(t) &= -e^{\frac{\pi|t|}{N}}\,, \qquad g_{01}(\theta) = -\frac{ie^{-\gamma}}{\theta + \frac{i\pi}{N}}\,, \\ f_{11}(t) &= 1 + e^{\frac{2\pi|t|}{N}}\,, \quad g_{11}(\theta) = -e^{2\gamma}\theta\left(\theta + \frac{2i\pi}{N}\right)\,. \end{aligned} \tag{2.49}$$

It is straightforward to check that Z_1 and Z_2 indeed obey the ZF algebra relations (2.25, 2.26). To find the remaining commutation relations one notices that since χ_k^- basically are $\mathfrak{sl}(N)$ algebra generators they should have the following properties

$$\chi_k^-\chi_m^- = \chi_m^-\chi_k^- \quad \text{if} \quad |k-m| \neq 1\,, \tag{2.50}$$

and

$$\chi_k^- Z_m(\theta) = Z_m(\theta)\chi_k^- \quad \text{unless} \quad m = k \quad \text{or} \quad m = k+1\,. \tag{2.51}$$

In addition, for $m = k + 1$ one should find

$$\chi_k^- Z_{k+1}(\theta) = -Z_{k+1}(\theta)\chi_k^-\,. \tag{2.52}$$

This indeed can be checked for $k = 1$ by using the g_{01} and g_{11} functions. To this end one should use (2.43) for Z_2 and take the integration contours in χ_1^- to the real line. Then the integrand in the double integral can be symmetrised in the integration variables and gives 0, and the integrands in the single integrals appearing due to the poles in g_{01} sum up to 0 as well. Eq.(2.52) can in fact be used to find g_{11}.

It is then easy to check that the commutativity of Z_1 with χ_2^- guarantees that Z_1 and Z_3 satisfy the ZF algebra relation (2.26). The relations (2.50, 2.51) show that only adjacent operators can have nontrivial commutation relations, and therefore only the functions $g_{\mu\mu}$ and $g_{\mu,\mu\pm 1} = g_{\mu\pm 1,\mu}$ are nontrivial. Thus, using the integration contour prescription and the relations (2.42) one can represent Z_{k+1} in the following (slightly symbolic) form

$$Z_{k+1}(\theta) = \rho \int_{C_k} d\alpha_k \cdots \int_{C_1} d\alpha_1 \prod_{j=1}^{k} g^s_{j-1,j}(\alpha_j - \alpha_{j-1}) \; : V_0(\theta) V_1(\alpha_1) \cdots V_k(\alpha_k) : , \tag{2.53}$$

where $\alpha_0 \equiv \theta$ and

$$g^s_{j-1,j}(\alpha) \equiv \rho_\chi \big(g_{j-1,j}(\alpha) + g_{j-1,j}(-\alpha) \big) \, . \tag{2.54}$$

The integration contour C_j runs below the poles of $g_{j-1,j}(\alpha_j - \alpha_{j-1})$ and above the poles of $g_{j-1,j}(\alpha_{j-1} - \alpha_j)$. Strictly speaking (2.53) is a sum of 2^k integrals with contours specified by the product of $g_{j-1,j}$ functions in each integrand. Only Z_2 is given exactly by (2.53) with $g^s_{01}(\alpha) = -\frac{1/N}{\alpha^2 + \pi^2/N^2}$ and the poles of g^s_{01} at $\alpha_1 = \theta - \frac{i\pi}{N}$ and $\alpha_1 = \theta + \frac{i\pi}{N}$ lying above and below C_1, respectively.

To find the function g_{12} (and therefore f_{12}) one can use the requirement of the commutativity of Z_3 with χ_1^- which also implies that Z_2 and Z_3 satisfy the ZF algebra relation (2.26). Then one gets

$$\begin{aligned} \chi_1^- Z_3(\theta) = \rho \int_{C_1^\alpha} d\alpha_1 \int_{C_2^\beta} d\beta_2 \int_{C_1^\beta} d\beta_1 \, g^s_{01}(\beta_1 - \theta) g^s_{12}(\beta_2 - \beta_1) \\ \times \, g_{01}(\theta - \alpha_1) g_{12}(\beta_2 - \alpha_1) g_{11}(\beta_1 - \alpha_1) \; : V_0(\theta) V_1(\alpha_1) V_1(\beta_1) V_2(\beta_2) : , \end{aligned} \tag{2.55}$$

$$
\begin{aligned}
Z_3(\theta)\chi_1^- = \rho \int_{C_1^\alpha} d\alpha_1 \int_{C_2^\beta} d\beta_2 \int_{C_1^\beta} d\beta_1\, g_{01}^s(\beta_1-\theta) g_{12}^s(\beta_2-\beta_1) \\
\times\, g_{01}(\alpha_1-\theta) g_{12}(\alpha_1-\beta_2) g_{11}(\alpha_1-\beta_1)\ : V_0(\theta)V_1(\alpha_1)V_1(\beta_1)V_2(\beta_2) : \ .
\end{aligned} \tag{2.56}
$$

If the integration contours C_1^α and C_1^β, and C_2^β were the same in all the integrals (recall that both (2.55) and (2.56) are sums of 4 integrals), e.g. they all would coincide with the real line, then one could symmetrise the integrands with respect to α_1 and β_1. Then assuming that g_{12} has only one pole (as it is for g_{01}) and imposing the requirement that the symmetrised integrands in (2.55) and (2.56) are equal to each other, one immediately finds that $g_{12} = g_{01}$. Making the integration contours coincide with the real line produces extra terms due to the poles of the g_{01}'s, and one has to check that these extra terms cancel out as well. One can do this, and a lengthy computation indeed shows that if $g_{12} = g_{01}$, then $\chi_1^- Z_3(\theta) = Z_3(\theta)\chi_1^-$. The function g_{22} is found by imposing the ZF algebra relation (2.25) on Z_3 and appears to be equal to g_{11}. The same considerations are used to determine the remaining functions, and one finally reaches the natural conclusion

$$
f_{j,j+1}(t) = f_{01}(t)\,, \quad g_{j,j+1}(\theta) = g_{01}(\theta)\,, \quad f_{jj}(t) = f_{11}(t)\,, \quad g_{jj}(\theta) = g_{11}(\theta)\,. \tag{2.57}
$$

The functions $f_{\mu\nu}$, $g_{\mu\nu}$ and $S_{\mu\nu}$ are listed explicitly in appendix B. Thus all the functions $g_{j-1,j}^s$ are given by

$$
g_{j-1,j}^s(\alpha) = -\frac{1/N}{\alpha^2+\pi^2/N^2}\,, \quad j = 1,\ldots,N-1\,, \tag{2.58}
$$

and the integration contour C_j in (2.53) runs below the pole of $g_{j-1,j}(\alpha_j-\alpha_{j-1})$ at $\alpha_j = \alpha_{j-1} - \frac{i\pi}{N}$ and above the pole of $g_{j-1,j}(\alpha_{j-1}-\alpha_j)$ at $\alpha_j = \alpha_{j-1} + \frac{i\pi}{N}$.

The free field representation for the ZF algebra with the twisted S-matrix $\mathbb{S}^{(-1)}$ first appeared in [34] where it was obtained by taking a proper limit of the free boson realisation of the type-I vertex operators of the $A_{N-1}^{(1)}$ spin chain constructed in [41]. However it is claimed in [34] that it is a representation of the ZF algebra with the canonical S-matrix which disagrees with our consideration. It is noticed in [34] that the commutation relations for the operators a_k can be written in the nice form

$$[a_i(t), a_j(t')] = t\frac{\sinh\frac{a_{ij}\pi t}{N}}{\sinh\frac{\pi t}{N}}e^{\frac{\pi|t|}{N}}\delta(t+t')\,, \quad i,j = 1,2,\ldots,N-1\,, \tag{2.59}$$

where $a_{ij} = 2\delta_{ij} - \delta_{i-1,j} - \delta_{i+1,j}$ is the Cartan matrix of type A_{N-1}. Then, the operator a_0 is expressed as the following linear combination of a_k

$$a_0(t) = -\sum_{k=1}^{N-1}\frac{\sinh\frac{(N-k)\pi t}{N}}{\sinh \pi t}a_k(t)\,. \tag{2.60}$$

Finally, another linear combination of a_k

$$a_N(t) = -\sum_{k=1}^{N-1}\frac{\sinh\frac{k\pi t}{N}}{\sinh \pi t}a_k(t)\,, \tag{2.61}$$

is used in [34] to construct the vertex operator $V_N(\theta) =: e^{i\phi_N(\theta)}$: for the bound state $Z_{12\ldots N-1}$ which is the antiparticle of Z_N. Vertex operators for bound states will be discussed in more detail in section 3.1.

2.2.3 The angular Hamiltonian

The next step in constructing a free field representation is to find the angular Hamiltonian. The most general Hamiltonian we might expect would be of the form

$$\mathbb{K} = i\int_0^\infty dt\sum_{i,j=1}^{N-1}h_{ij}(t)a_i(-t)a_j(t)\,, \tag{2.62}$$

where the (anti-)hermiticity condition for $\mathbb{K}$ requires the functions h_{ij} to obey $h_{ij} = h_{ji}$, and we assume that h_{ij} are even fuctions of t: $h_{ij}(-t) = h_{ij}(t)$. We wish to satisfy the relations (1.52)

$$\frac{d}{d\theta}Z_j(\theta) = -\left[\mathbb{K}, Z_j(\theta)\right]\,, \tag{2.63}$$

where we set $\Omega_I = 0$ because as we will see in a moment they do vanish for the representation we consider. Computing the derivative of (2.53) with respect to θ

it is straightforward to show that

$$\frac{d}{d\theta}Z_{k+1}(\theta) = \rho\int_{C_k} d\alpha_k \cdots \int_{C_1} d\alpha_1 \prod_{j=1}^{k} g^s_{j-1,j}(\alpha_j - \alpha_{j-1}) \\ \times \sum_{j=0}^{k}\left(\frac{\partial}{\partial\alpha_j}\right) : V_0(\alpha_0)V_1(\alpha_1)\cdots V_k(\alpha_k) : . \qquad (2.64)$$

We can see this by looking at some examples. Firstly, we consider the simplest case, $Z_1(\theta)$. We first note that to take derivatives of the vertex operators, we have for example

$$\begin{aligned} \frac{d}{d\theta}V_0(\theta) &= \frac{d}{d\theta}\exp i\int_{-\infty}^{\infty}\frac{dt}{it}\,a_0(t)e^{i\theta t} \\ &= i\int_{-\infty}^{\infty} dt\, a_0(t)e^{i\theta t}\left(\exp\int_{-\infty}^{\infty} dt\, a_0(t)e^{i\theta t}\right) \\ &= i\int_{-\infty}^{\infty} dt\, a_0(t)e^{i\theta t}V_0(\theta)\,. \end{aligned} \qquad (2.65)$$

Applying this to $Z_1(\theta)$, we find

$$\begin{aligned} \frac{d}{d\theta}Z_1(\theta) =& \frac{d}{d\theta}\rho : V_0(\theta) : \\ =& i\,\rho\int_{-\infty}^{\infty} dt\, a_0(t)e^{i\theta t}Z_1(\theta) \\ =& -i\,\rho\int_{-\infty}^{\infty} dt\sum_{j=1}^{N-1} a_j(t)\frac{\sinh\frac{N-j}{N}\pi t}{\sinh\pi t}e^{i\theta t}Z_1(\theta)\,, \end{aligned} \qquad (2.66)$$

Next we must consider the operator $Z_2(\beta)$. First, we take the derivative

$$\begin{aligned} &\frac{d}{d\theta}Z_2(\theta) = \\ &i\,\rho\int_{-\infty}^{\infty} dt\, a_0(t)e^{i\theta t}Z_2(\theta) \\ &+ i\,\rho\int_C d\alpha\left(\frac{d}{d\theta}g^s_{01}(\alpha-\theta)\right) : V_0(\theta)V_1(\alpha) : . \end{aligned} \qquad (2.67)$$

The first term acts in the same way as Z_1, but now there is an additional term that arises from differentiating $g^s_{01}(\alpha-\theta)$. It may be observed that

$$\frac{d}{d\theta}g^s_{01}(\alpha-\theta) = -\frac{d}{d\alpha}g^s_{01}(\alpha-\theta)\,, \qquad (2.68)$$

and therefore we may rearrange the second term as follows:

$$
\begin{aligned}
i\rho \int_C & d\alpha \left(\frac{d}{d\theta} g^s_{01}(\alpha - \theta) \right) : V_0(\theta) V_1(\alpha) : \\
&= -i\rho \int_C d\alpha \left(\frac{d}{d\alpha} g^s_{01}(\alpha - \theta) \right) : V_0(\theta) V_1(\alpha) : \\
&= -i\rho \int_C d\alpha \frac{d}{d\alpha} \left(g^s_{01}(\alpha - \theta) : V_0(\theta) V_1(\alpha) : \right) \\
&+ i\rho \int_C d\alpha\, g^s_{01}(\alpha - \theta) \frac{d}{d\alpha} \left(: V_0(\theta) V_1(\alpha) : \right) \\
&= i\rho \int_{-\infty}^{\infty} dt\, a_1(t) Z_2(\theta) e^{i\alpha t} ,
\end{aligned} \tag{2.69}
$$

where we note that we expect the total derivative term to vanish. Hence we may write

$$
\frac{d}{d\theta} Z_2(\theta) = i\rho \int_{-\infty}^{\infty} dt\, a_0(t) Z_2(\theta) e^{i\theta t} + i\rho \int_{-\infty}^{\infty} dt\, a_1(t) Z_2(\theta) e^{i\alpha t} , \tag{2.70}
$$

which more correctly, should be written

$$
\begin{aligned}
\frac{d}{d\theta} Z_2(\theta) = & i\rho \int_C d\alpha g^s_{01}(\alpha - \theta) \\
& \times \left(\frac{d}{d\theta} + \frac{d}{d\alpha} \right) : \exp \int_{-\infty}^{\infty} \frac{dt}{it} a_0(t) e^{i\theta t} + a_1(t) e^{i\alpha t} : .
\end{aligned} \tag{2.71}
$$

Since in general we have

$$
\frac{d}{d\alpha_k} g^s_{kk+1}(\alpha_{k+1} - \alpha_k) = -\frac{d}{d\alpha_{k+1}} g^s_{kk+1}(\alpha_{k+1} - \alpha_k) , \tag{2.72}
$$

it is clear that this derivation can be extended to Z_{k+1}, giving (2.64).

Thus it is sufficient to find $\mathbb{K}$ such that

$$
[\mathbb{K}, V_\mu(\theta)] = -\frac{d}{d\theta} V_\mu(\theta) , \quad \mu = 0, 1, \ldots, N . \tag{2.73}
$$

Using the same reasoning as for Z_{k+1}, we get

$$
-\frac{d}{d\theta} V_\mu(\theta) = -i \int_{-\infty}^{\infty} dt\, e^{i\theta t} : a_\mu(t) e^{i\phi_\mu(\theta)} : , \tag{2.74}
$$

and since

$$
\begin{aligned}
[\mathbb{K}, i\phi_\mu(\theta)] &= \left[i \int_0^\infty dt \sum_{i,j=1}^{N-1} h_{ij}(t) a_i(-t) a_j(t), i \int_{-\infty}^\infty \frac{dt'}{it'} a_\mu(t') e^{i\theta t'} \right] \\
&= i \int_0^\infty dt \int_{-\infty}^\infty \frac{dt'}{t'} \sum_{i,j=1}^{N-1} h_{ij}(t) \left[a_i(-t) a_j(t), a_\mu(t') \right] e^{i\theta t'} \\
&= i \int_0^\infty dt \int_{-\infty}^\infty \frac{dt'}{t'} \sum_{i,j=1}^{N-1} h_{ij}(t) \Big(- t f_{i\mu}(-t) \delta(-t + t') a_j(t) \\
&\quad + t f_{j\mu}(t) \delta(t + t') a_i(-t) \Big) e^{i\theta t'} \qquad (2.75) \\
&= -i \int_0^\infty dt' \sum_{i,j=1}^{N-1} h_{ij}(t') f_{i\mu}(t') a_j(t') e^{i\theta t'} \\
&\quad - i \int_{-\infty}^0 dt' \sum_{i,j=1}^{N-1} h_{ij}(-t') f_{j\mu}(-t') a_i(t') e^{i\theta t'} \\
&= -i \int_{-\infty}^\infty dt' \sum_{i,j=1}^{N-1} h_{ij}(t') f_{i\mu}(t') a_j(t') e^{i\theta t'}
\end{aligned}
$$

we find

$$
[\mathbb{K}, V_\mu(\theta)] =: [\mathbb{K}, i\phi_\mu(\theta)] e^{i\phi_\mu(\theta)} := -i \int_{-\infty}^\infty dt\, e^{i\theta t} : f_{\mu k}(t) h_{kj}(t) a_j(t) e^{i\phi_\mu(\theta)} : . \qquad (2.76)
$$

Thus h_{ij} can be found from the equations

$$
\sum_{k=1}^{N-1} f_{0k}(t) h_{kj}(t) = -\frac{\sinh \frac{N-j}{N}\pi t}{\sinh \pi t}, \qquad \sum_{k=1}^{N-1} f_{ik}(t) h_{kj}(t) = \delta_{ij} . \qquad (2.77)
$$

Solving these equations one gets

$$
h_{ij}(t) = \frac{\sinh \frac{i}{N}\pi t \sinh \frac{N-j}{N}\pi t}{\sinh \frac{\pi t}{N} \quad \sinh \pi t} e^{-\frac{\pi}{N}t}, \quad h_{ji}(t) = h_{ij}(t), \quad i \leq j . \qquad (2.78)
$$

It is worth mentioning that the matrix h, with the entries h_{ij}, is inverse to the matrix f, with the entries f_{ij}, which will be important in computing the form factors.

Chapter 3

Composite Operators

3.1 Bound states

Let us recall that a rank-r particle created by a ZF operator $\mathcal{A}_K^\dagger(\theta)$, $K = (k_1, \ldots, k_r)$, $1 \leq k_1 < k_2 < \cdots < k_r \leq N$, is a bound state of elementary particles $\mathcal{A}_{k_j}^\dagger$. Thus the vertex operators A_K can be obtained from the vertex operators A_{k_j} by using the fusion procedure. We first construct the vertex operators Z_K for the bound states of the twisted S-matrix $\mathbb{S}^{(-1)}$ which will be normalised so that they satisfy the relations

$$Z_K(\theta' + i\pi)Z_L(\theta) = -\frac{i\delta_{KL}}{\theta' - \theta} + \mathcal{O}(1)\,, \quad \theta' \to \theta\,. \tag{3.1}$$

The vertex operator A_K of a rank-r particle is then given by

$$A_K = \Gamma_K Z_K\,, \quad \Gamma_K \equiv \Gamma_{k_1}\Gamma_{k_2}\cdots\Gamma_{k_r}\,, \tag{3.2}$$

and the formula (3.1) for A_K takes the form

$$A_K(\theta' + i\pi)A_L(\theta) = -\frac{i\,\Gamma\,\epsilon_{KL}}{\theta' - \theta} + \mathcal{O}(1)\,, \quad \theta' \to \theta\,, \tag{3.3}$$

where $\Gamma \equiv \Gamma_1\Gamma_2\cdots\Gamma_N$. Since $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\eta_{ij}$ with $\eta_{ij} = \eta_{ii}\delta_{ij}$ then for odd N one can choose the first $N-1$ η_i's to be 1, and $\Gamma_N = \Gamma_{N-1}\Gamma_{N-2}\cdots\Gamma_1$ so that $\Gamma_N^2 = \eta_N = (-1)^{\frac{N-1}{2}}$. Then $\Gamma = 1$ and the relation (3.3) takes the usual form. On the other hand if N is even then Γ is not proportional to the identity matrix but one can choose η_N so that it obeys $\Gamma^2 = 1$. To satisfy the form factor axioms one then should insert under the trace in the form factor formula (1.55) the projection operator $\frac{1}{2}(1+\Gamma)$.

3.1.1 Fused vertex operators

It is clear from the ZF algebra (or the S-matrix) that if $i \neq j$ then Z_i and Z_j can form a two-particle bound state because

$$Z_i(\theta_1)Z_j(\theta_2) = -\frac{\frac{2\pi i}{N}S(\frac{2\pi i}{N})}{\theta_{12} - \frac{2\pi i}{N}} \left(Z_i(\theta_2)Z_j(\theta_1) + Z_j(\theta_2)Z_i(\theta_1)\right) + \dots, \qquad \theta_{12} \to \frac{2\pi i}{N}. \tag{3.4}$$

Let us introduce the following fused vertex operators of rank-r

$$\mathcal{Z}_{k_1\dots k_r}(\theta) \equiv \lim_{\epsilon_{i+1,i}\to 0} \prod_{j=2}^{r} (i\epsilon_{j+1,j})\, Z_{k_1}(\theta_1^\epsilon)Z_{k_2}(\theta_2^\epsilon)\cdots Z_{k_r}(\theta_r^\epsilon)\,,$$
$$\theta_j^\epsilon \equiv \theta + i\mathfrak{u}_{r-2j+1} + \epsilon_j\,, \tag{3.5}$$

where all indices k_a are different (if two indices coincide the fused operator vanishes), $\mathfrak{u}_k \equiv \frac{\pi}{N}k$, $\epsilon_1 = 0$ and all $\epsilon_{jk} \equiv \epsilon_j - \epsilon_k$ do not vanish until one takes the limits. The fused operators satisfy the following relation

$$\mathcal{Z}_{k_1\dots k_r}(\theta) = \lim_{\epsilon\to 0} i\epsilon\, \mathcal{Z}_{k_1\dots k_p}(\theta + i\mathfrak{u}_{r-p})\mathcal{Z}_{k_{p+1}\dots k_r}(\theta - i\mathfrak{u}_p + \epsilon)\,, \tag{3.6}$$

where p is any integer between 1 and r.

By using the fused vertex operators we can write

$$Z_i(\theta_1)Z_j(\theta_2) = \frac{i}{\theta_{12} - \frac{2\pi i}{N}} \mathcal{Z}_{ij}(\theta) + \dots, \qquad \theta_j \to \theta + i\mathfrak{u}_{3-2j}\,. \tag{3.7}$$

Note that $\mathcal{Z}_{ij} = \mathcal{Z}_{ji}$ because the twisted S-matrix has a pole in the symmetric channel, and moreover the associativity of the ZF algebra implies that a general rank-r fused operator (3.5) is also symmetric under the exchange of its indices.

It is clear that a two-particle bound state ZF vertex operator is given by

$$Z_{ij}(\theta) = \mathcal{N}_2\, \mathcal{Z}_{ij}(\theta)\,, \tag{3.8}$$

where $\mathcal{N}_2$ is a normalisation constant. The mass of the two-particle bound state is equal to $m_2 = m \sin \mathfrak{u}_2 / \sin \mathfrak{u}_1$ where m is the mass of elementary particles. It is not difficult to see that Z_i and Z_{jk} with all indices different can also form a bound state which is a three-particle bound state of the mass $m_3 = m \sin \mathfrak{u}_3 / \sin \mathfrak{u}_1$, and the corresponding ZF vertex operator can be defined as

$$\begin{aligned} Z_{ijk}(\theta) &= \mathcal{N}_3\, \mathcal{Z}_{ijk}(\theta) \\ &= \mathcal{N}_3 \lim_{\epsilon \to 0} i\epsilon\, Z_i(\theta + i\mathfrak{u}_2) \mathcal{Z}_{jk}(\theta - i\mathfrak{u}_1 + \epsilon) \\ &= \mathcal{N}_3 \lim_{\epsilon \to 0} i\epsilon\, \mathcal{Z}_{ij}(\theta + i\mathfrak{u}_1) Z_k(\theta - i\mathfrak{u}_2 + \epsilon)\,. \end{aligned} \tag{3.9}$$

This procedure can be repeated and one introduces the ZF vertex operator for a r-particle bound state of mass $m_r = m \sin \mathfrak{u}_r / \sin \mathfrak{u}_1$ by the formula

$$Z_{k_1 \ldots k_r}(\theta) = \mathcal{N}_r\, \mathcal{Z}_{k_1 \ldots k_r}(\theta)\,. \tag{3.10}$$

Since $\mathcal{Z}_{k_1 \ldots k_r}$ is symmetric under the exchange of the indices we can use the canonical ordering $k_1 < k_2 < \cdots < k_r$ which also shows that the bound states of the twisted model are indeed in one-to-one correspondence with the bound states of the chiral GN model. The normalisation constants $\mathcal{N}_r$ have to be chosen so that the bound state ZF vertex operators satisfy the relations (3.1)

$$Z_K(\theta' + i\pi) Z_{\overline{K}}(\theta) = \mathcal{N}_r \mathcal{N}_{N-r}\, \mathcal{Z}_K(\theta' + i\pi) \mathcal{Z}_{\overline{K}}(\theta) = -\frac{i}{\theta' - \theta} + \mathcal{O}(1)\,, \quad \theta' \to \theta\,, \tag{3.11}$$

where $K = (k_1, \ldots, k_r)$ is a bound state index, and $K \cup \overline{K} = (1, 2, \ldots, N)$ (after reordering the indices). It is not difficult to see that $\mathcal{N}_r \mathcal{N}_{N-r}$ is independent of K because according to (3.6),

$$\lim_{\epsilon \to 0} i\epsilon\, \mathcal{Z}_K(\theta + i\pi) \mathcal{Z}_{\overline{K}}(\theta + \epsilon) = \mathcal{Z}_{12 \ldots N}(\theta + i\mathfrak{u}_k) \equiv \mathcal{V}_N \quad \Longrightarrow \quad \mathcal{N}_r \mathcal{N}_{N-r} = -\frac{1}{\mathcal{V}_N}\,, \tag{3.12}$$

where one has to show that $\mathcal{V}_N$ is indeed a constant. The computation of $\mathcal{V}_N$ is outlined in appendix C where it is shown that with our choice of ρ and ρ_χ it is equal to -1, and therefore in what follows we choose $\mathcal{N}_r = 1$ for any r.

3.1.2 Highest weight bound state vertex operators

As was discussed in section 2.2, all vertex operators Z_k for elementary particles can be obtained from the highest weight vertex operator Z_1 by acting on it with the lowering symmetry operators χ_k^-. It appears that the same is true for the bound state vertex operators. Any rank-r vertex operator is generated from the highest weight vertex operator $Z_{12\ldots r}$. Indeed, one has (dropping $i\epsilon$ and θ for clarity)

$$\begin{aligned} Z_{1\ldots r-1,r+1} &= Z_{1\ldots r-1}Z_{r+1} = Z_{1\ldots r-1}\big(\chi_r^- Z_r + Z_r\chi_r^-\big) \\ &= \chi_r^- Z_{1\ldots r-1}Z_r + Z_{1\ldots r-1}Z_r\chi_r^- = \chi_r^- Z_{1\ldots r} + Z_{1\ldots r}\chi_r^- \, . \end{aligned} \tag{3.13}$$

It is clear then that $Z_{1\ldots r-1,j+1}$ for $j \geq r$ is given by

$$Z_{1\ldots r-1,j+1}(\theta) = \chi_j^- Z_{1\ldots r-1,j}(\theta) + Z_{1\ldots r-1,j}(\theta)\chi_j^- \, , \tag{3.14}$$

as desired. To obtain $Z_{1\ldots r-2,r,r+1}$ we act on $Z_{1\ldots r-1,r+1}$ by χ_{r-1}^-. Having found $Z_{1\ldots r-2,r,r+1}$ we then construct all $Z_{1\ldots r-2,r,j+1}$, and then $Z_{1\ldots r-2,r+1,j+2}$, and so on.

The simplest case is provided by rank-(N-1) vertex operators. They are obtained from $\overline{Z}_N \equiv Z_{12\ldots N-1}$ which is the antiparticle of Z_N. Acting on $\overline{Z}_N$ with χ_{N-1}^- one creates $\overline{Z}_{N-1} \equiv Z_{1\ldots N-2,N}$ which is the antiparticle of Z_{N-1}. Then, acting on $\overline{Z}_{N-1}$ with χ_{N-2}^- one creates $\overline{Z}_{N-2} \equiv Z_{1\ldots N-3,N-1,N}$ which is the antiparticle of Z_{N-2}, and so on

$$\overline{Z}_k(\theta) = \chi_k^- \overline{Z}_{k+1}(\theta) + \overline{Z}_{k+1}(\theta)\chi_k^- \, , \quad k = N-1, \ldots, 1 \, . \tag{3.15}$$

The resulting formula agrees with the one in [34].

The highest weight vertex operators $Z_{12\ldots r}$ can be simplified to an explicit form which contains no integrals at all. The derivation is presented in appendix C and here we just state the result

$$Z_{12\ldots r}(\theta) = C_{N,r}V_{(r)}(\theta) \, , \quad V_{(r)}(\theta) \equiv \, : \prod_{k=0}^{r-1}\prod_{j=k+1}^{r} V_k\big(\theta + i\mathrm{u}_{r+k-2j+1}\big) : \, . \tag{3.16}$$

Here the normalisation constant $C_{N,r}$ is given by

$$C_{N,r} = e^{\frac{i\pi r}{N}} e^{\gamma \frac{r(N-r)}{2N}} N^{-\frac{r}{2N}} (2\pi)^{\frac{(r-1)(2N-r)}{2N}} \prod_{j=1}^{r-1} \frac{1}{\Gamma\left(\frac{j}{N}\right)}, \tag{3.17}$$

and the fused vertex operator $V_{(r)}$ can be written in the usual form

$$V_{(r)}(\theta) = :e^{i\phi_{(r)}(\theta)}:, \tag{3.18}$$

where we define new "fused" fields

$$\phi_{(r)}(\theta) = :\sum_{k=0}^{r-1} \sum_{j=k+1}^{r} \phi_k\big(\theta + i\mathrm{u}_{r+k-2j+1}\big): = \int_{-\infty}^{\infty} \frac{dt}{it} a_{(r)}(t) e^{i\theta t}, \tag{3.19}$$

with "fused" creation operators given by

$$a_{(r)}(t) = \sum_{k=0}^{r-1} a_k \frac{\sinh \frac{r-k}{N}\pi t}{\sinh \frac{\pi t}{N}}. \tag{3.20}$$

Notice that $V_{(1)} = V_0$ and $V_{(N-1)} = V_N$ as follows from (2.60, 2.61). The requirement that the fused vertex operator $Z_{12\ldots N}$ is the constant $C_{N,N} = \mathcal{V}_N = -1$, or equivalently $V_{(N)} = 1$, leads to the relation (2.60) between a_0 and a_k which can be imposed because it is consistent with the commutation relations (2.36) between a_μ. For the highest weight vertex operator $\overline{Z}_N$ the formula also simplifies

$$\overline{Z}_N(\theta) = C_{N,N-1} V_N(\theta), \tag{3.21}$$

where

$$C_{N,N-1} = -e^{\gamma \frac{(N-1)}{2N}} N^{\frac{1}{2N}} (-2\pi)^{-1/N} \Gamma\left(\frac{N-1}{N}\right). \tag{3.22}$$

This concludes the construction of the free field representation of the extended ZF algebra, and now we turn to the determination of operators Λ representing local operators of the chiral GN model.

3.2 Local operators

In this section we construct a large set of operators which commute with the ZF vertex operators, and can be used to generate form factors of local operators of the chiral GN model. The consideration generalises the one in [11] where the $N = 2$ case was discussed in detail.

3.2.1 Primed ZF operators

We follow [11] and introduce auxiliary operators a'_μ, Q'_μ related to a_μ, Q_μ by

$$a'_\mu(t) = -e^{-\frac{\pi|t|}{N}} a_\mu(t)\,, \quad Q'_\mu = -Q_\mu\,. \tag{3.23}$$

The commutation relations of the operators a'_μ have the form

$$[a'_\mu(t), a'_\nu(t')] = t f''_{\mu\nu}(t)\delta(t+t')\,, \quad [a'_\mu(t), a_\nu(t')] = t f'_{\mu\nu}(t)\delta(t+t')\,, \tag{3.24}$$

where $f''_{\mu\nu}$ and $f'_{\mu\nu}$ satisfy the same symmetry relations as $f_{\mu\nu}(t)$, and are listed explicitly in appendix B. We also define the free fields

$$\phi'_\mu(\theta) = Q'_\mu + \int_{-\infty}^{\infty} \frac{dt}{it} a'_\mu(t) e^{i\theta t}\,, \quad \mu = 0, 1, \ldots, N\,, \tag{3.25}$$

which satisfy the following relations

$$\begin{aligned} \langle \phi'_\mu(\theta_1)\phi'_\nu(\theta_2)\rangle &= -\ln g''_{\mu\nu}(\theta_2-\theta_1)\,, \\ \langle \phi_\mu(\theta_1)\phi'_\nu(\theta_2)\rangle &= \langle \phi'_\mu(\theta_1)\phi_\nu(\theta_2)\rangle = -\ln g'_{\mu\nu}(\theta_2-\theta_1)\,, \end{aligned} \tag{3.26}$$

where the functions can be found in appendix B.2.

The fields ϕ'_μ are used to construct the primed vertex operators

$$V'_\mu(\theta) =: e^{i\phi'_\mu(\theta)} :\,, \tag{3.27}$$

and the primed Z'_{k+1}

$$\begin{aligned} Z'_1(\theta) &= \rho' V'_0(\theta)\,, \quad \rho' = e^{\gamma\frac{(N-1)}{2N}} N^{\frac{1}{2N}}\,, \quad \chi^+_k = \rho_\chi \int_C d\alpha V'_k(\alpha)\,, \\ Z'_{k+1}(\theta) &= Z'_k(\theta)\,\chi^+_k + \chi^+_k\, Z'_k(\theta)\,, \quad k = 1, \ldots, N-1\,, \end{aligned} \tag{3.28}$$

where the integration contour C in χ_k^+ is determined in the same way as for χ_k^-. Then one gets the following representation

$$\begin{aligned} Z'_{k+1}(\theta) = \rho' \int_{C_k} d\alpha_k \cdots \int_{C_1} d\alpha_1 \\ \times \prod_{j=1}^{k} g''^{\,s}_{j-1,j}(\alpha_j - \alpha_{j-1}) \, : V'_0(\theta) V'_1(\alpha_1) \cdots V'_k(\alpha_k) : , \end{aligned} \tag{3.29}$$

where $\alpha_0 \equiv \theta$ and

$$g''^{\,s}_{j-1,j}(\alpha) \equiv \rho_\chi \big(g''_{j-1,j}(\alpha) + g''_{j-1,j}(-\alpha) \big) = \frac{1/N}{\alpha^2 + \pi^2/N^2} \, , \quad j = 0, 1, \ldots, N-1 \, . \tag{3.30}$$

The integration contour C_j runs above the pole of $g''_{j-1,j}(\alpha_{j-1} - \alpha_j)$ at $\alpha_j = \alpha_{j-1} - \frac{i\pi}{N}$ and below the pole of $g''_{j-1,j}(\alpha_j - \alpha_{j-1})$ at $\alpha_j = \alpha_{j-1} + \frac{i\pi}{N}$ because $g''_{j,j+1}(\theta) = -\frac{ie^{-\gamma}}{\theta - \frac{i\pi}{N}}$. Thus for $\theta \in \mathbb{R}$ all the contours coincide with the real line.

The primed operators A'_k of the GN model are then constructed as

$$\begin{aligned} A'_k(\theta) &= \Gamma_k^{-1} Z'_k(\theta) \, , \quad \mathcal{J}_k^+ = \Gamma_k \Gamma_{k+1}^{-1} \chi_k^- \, , \\ A'_{k+1}(\theta) &= A'_k(\theta) \, \mathcal{J}_k^+ - \mathcal{J}_k^+ \, A'_k(\theta) \, , \quad k = 1, \ldots, N-1 \, , \end{aligned} \tag{3.31}$$

where $\mathcal{J}_k^+$ are the $\mathfrak{sl}(N)$ algebra raising generators.

Similar to the χ_k^- operators, χ_k^+ satisfy the following properties

$$\chi_k^+ \chi_m^+ = \chi_m^+ \chi_k^+ \quad \text{if} \quad |k - m| \neq 1 \, , \tag{3.32}$$

and

$$\chi_k^+ Z'_m(\theta) = Z'_m(\theta) \chi_k^+ \quad \text{unless} \quad m = k \quad \text{or} \quad m = k+1 \, . \tag{3.33}$$

Then, for $m = k + 1$ one finds

$$\chi_k^+ Z'_{k+1}(\theta) = -Z'_{k+1}(\theta) \chi_k^+ \, , \tag{3.34}$$

because in the primed case the integration contour in χ_k^+ runs between the poles, and the integrand can therefore be symmetrised in the integration variables to give the result. In addition to these properties there are a number of similar mixed relations between primed and unprimed operators. The obvious relations are

$$\chi_k^+\chi_{k+1}^- = -\chi_{k+1}^-\chi_k^+ \,, \quad \chi_k^-\chi_{k+1}^+ = -\chi_{k+1}^+\chi_k^- \,,$$

$$\chi_k^+\chi_m^- = \chi_m^-\chi_k^+ \,, \quad \text{if} \quad |k-m| > 1 \,,$$

$$\chi_k^+ Z_m(\theta) = Z_m(\theta)\chi_k^+ \,, \quad \chi_k^- Z'_m(\theta) = Z'_m(\theta)\chi_k^- \quad \text{if} \quad m < k \,,$$

$$\chi_k^+ Z_k(\theta) = -Z_k(\theta)\chi_k^+ \,, \quad \chi_k^- Z'_k(\theta) = -Z'_k(\theta)\chi_k^- \,. \tag{3.35}$$

Then one can show that

$$\left[\chi_k^+ \,, \chi_k^-\right] \simeq P_k \,. \tag{3.36}$$

This relation only holds for arbitrary matrix elements of Z_k's and Z'_k's but not in the operator form which is reflected in $\simeq$. This is the most important relation, and together with

$$\left[P_k, Z_k\right] = Z_k \,, \quad \left[P_k, Z'_k\right] = -Z'_k \,, \tag{3.37}$$

it can be used to prove that

$$\chi_k^+ Z_{k+1}(\theta) + Z_{k+1}(\theta)\chi_k^+ \simeq Z_k(\theta) \,, \quad \chi_k^- Z'_{k+1}(\theta) + Z'_{k+1}(\theta)\chi_k^- \simeq Z'_k(\theta) \,, \tag{3.38}$$

$$\chi_k^+ Z_m(\theta) \simeq Z_m(\theta)\chi_k^+ \,, \quad \chi_k^- Z'_m(\theta) \simeq Z'_m(\theta)\chi_k^- \quad \text{if} \quad m > k+1 \,.$$

In order to avoid interrupting the discussion here, the proofs of these relations are postponed until section 3.2.2. A straightforward computation then shows that the primed ZF operators satisfy the following relations

$$Z'_i(\theta_1) Z'_i(\theta_2) = S''(\theta_{12}) Z'_i(\theta_2) Z'_i(\theta_1) \,, \tag{3.39}$$

$$Z'_i(\theta_1) Z'_j(\theta_2) = S''(\theta_{12}) \left[-\frac{\theta_{12}}{\theta_{12} + \frac{2\pi i}{N}} Z'_j(\theta_2) Z'_i(\theta_1) + \frac{\frac{2\pi i}{N}}{\theta_{12} + \frac{2\pi i}{N}} Z'_i(\theta_2) Z'_j(\theta_1) \right] \,, \tag{3.40}$$

where

$$S''(\theta) = g''(-\theta)/g''(\theta) = \frac{\Gamma\left(1 - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{N}\right)}{\Gamma\left(\frac{i\theta}{2\pi} + 1\right)\Gamma\left(\frac{1}{N} - \frac{i\theta}{2\pi}\right)}\,. \tag{3.41}$$

$S''(\theta)$ has a pole at $\theta = 2\pi i/N$, and a zero at $\theta = -2\pi i/N$ which cancels the pole in the brackets of (3.40). Let us introduce the following notation

$$\theta^{[\pm]} \equiv \theta \pm \frac{i\pi}{N}\,. \tag{3.42}$$

Since the ZF relations are regular at $\theta_{12} = -\frac{2\pi i}{N}$, the operator product $Z_i'(\theta^{[-]})Z_j'(\theta^{[+]})$ is regular too. On the other hand since $S''(\theta)$ has a pole at $\theta = 2\pi i/N$ the product $Z_i'(\theta'^{[+]})Z_j'(\theta^{[-]})$ would have a pole at $\theta' = \theta$ for any i, j unless

$$Z_i'(\theta'^{[-]})Z_j'(\theta^{[+]}) = Z_j'(\theta'^{[-]})Z_i'(\theta^{[+]}) \quad \text{for } \theta' \sim \theta\,, \tag{3.43}$$

because then the expression in the brackets in (3.40) would have a zero at $\theta_{12} = 2\pi i/N$ which cancels the pole. To prove this we first notice that

$$Z_i'(\theta'^{[-]})Z_i'(\theta^{[+]}) = 0 \quad \text{for } \theta' \sim \theta \tag{3.44}$$

because $S''(\theta)$ has a zero at $\theta = -2\pi i/N$. Then

$$Z_i'(\theta^{[-]})Z_{i+1}'(\theta^{[+]}) = Z_i'(\theta^{[-]})\chi_i^+ Z_i'(\theta^{[+]}) = Z_{i+1}'(\theta^{[-]})Z_i'(\theta^{[+]})\,, \tag{3.45}$$

and

$$\begin{aligned} &Z_i'(\theta^{[-]})Z_{i+2}'(\theta^{[+]}) \\ &= Z_i'(\theta^{[-]})\left[\chi_{i+1}^+ Z_{i+1}'(\theta^{[+]}) + Z_{i+1}'(\theta^{[+]})\chi_{i+1}^+\right] = Z_{i+2}'(\theta^{[-]})Z_i'(\theta^{[+]})\,, \end{aligned}$$

because Z_i' commutes with χ_{i+2}^+. The same proof works for $j - i > 2$.

Writing

$$S''(\theta) = \frac{R''}{\theta - \frac{2\pi i}{N}} + \text{regular}\,, \quad R'' = -\frac{2i\pi\Gamma\left(\frac{N+1}{N}\right)}{\Gamma\left(\frac{2}{N}\right)\Gamma\left(\frac{N-1}{N}\right)}\,, \tag{3.46}$$

we can now find

$$
\begin{aligned}
&Z_i'(\theta^{[+]})Z_i'(\theta^{[-]}) = -R''\,\partial_\theta Z_i'(\theta^{[-]})Z_i'(\theta^{[+]}) = R''\,Z_i'(\theta^{[-]})\partial_\theta Z_i'(\theta^{[+]})\,,\\
&Z_i'(\theta^{[+]})Z_j'(\theta^{[-]}) + Z_j'(\theta^{[+]})Z_i'(\theta^{[-]}) = -R''\frac{N}{2\pi i}\,Z_j'(\theta^{[-]})Z_i'(\theta^{[+]})\\
&Z_i'(\theta^{[+]})Z_j'(\theta^{[-]}) - Z_j'(\theta^{[+]})Z_i'(\theta^{[-]})\\
&\qquad = R''\big(\partial_\theta Z_i'(\theta^{[-]})Z_j'(\theta^{[+]}) - \partial_\theta Z_j'(\theta^{[-]})Z_i'(\theta^{[+]})\big)\\
&\qquad = R''\big(Z_j'(\theta^{[-]})\partial_\theta Z_i'(\theta^{[+]}) - Z_i'(\theta^{[-]})\partial_\theta Z_j'(\theta^{[+]})\big)\,.
\end{aligned}
\tag{3.47}
$$

Eqs.(3.43, 3.44) show that a natural analogue of the fused vertex operators $\mathcal{Z}_{k_1\ldots k_r}$ in the primed case is

$$
Z'_{k_1\ldots k_r}(\theta) \equiv \lim_{\epsilon_{i+1,i}\to 0} Z'_{k_1}(\theta_1^\epsilon)Z'_{k_2}(\theta_2^\epsilon)\cdots Z'_{k_r}(\theta_r^\epsilon)\,,\quad \theta_j^\epsilon \equiv \theta - i\mathfrak{u}_{r-2j+1} + \epsilon_j\,,\tag{3.48}
$$

where the regularisation parameters ϵ_j are such that $\epsilon_1 = 0$ and the differences $\epsilon_{jk} \equiv \epsilon_j - \epsilon_k$ are non-zero until one takes the limits. The result of this is that the differences of the rapidities in the normal-ordered product are shifted by $\epsilon_{i+1,i}$ and so any poles or zeroes that might appear will be shifted away from these values until the limits are taken. The evaluation of these poles and zeroes is carried out in appendix D, where equation (D.3.1) should be interpreted in the same way as equation (3.48) is here. The primed fused operators are symmetric under the exchange of their indices (if two indices coincide it vanishes) and satisfy the relation

$$
Z'_{k_1\ldots k_r}(\theta) = \lim_{\epsilon\to 0} Z'_{k_1\ldots k_p}(\theta - i\mathfrak{u}_{r-p})Z'_{k_{p+1}\ldots k_r}(\theta + i\mathfrak{u}_p + \epsilon)\,,\tag{3.49}
$$

where p is any integer between 1 and r. Just as it was for $Z_{k_1\ldots k_r}$, any primed fused operator can be obtained from the lowest weight fused operators $Z'_{12\ldots r}$ by acting on them with the symmetry generators χ_k^+.

It is shown in appendix D that the lowest weight primed operators $Z'_{12\ldots r}$ can be also reduced to the following explicit form

$$
Z'_{12\ldots r}(\theta) = D_{N,r}V'_{(r)}(\theta)\,,\quad V'_{(r)}(\theta) \equiv\, :\prod_{k=0}^{r-1}\prod_{j=k+1}^{r} V_k'(\theta - i\mathfrak{u}_{r+k-2j+1}):\,,\tag{3.50}
$$

where the normalisation constant $D_{N,r}$ is given by

$$D_{N,r} = e^{-\frac{\gamma(r)(r-N)}{2N}} N^{\frac{r}{2N}} (2\pi)^{-\frac{(r-1)r}{2N}} \prod_{m=1}^{r-1} \Gamma\left(1 - \frac{m}{N}\right), \tag{3.51}$$

and the fused primed vertex operator $V'_{(r)}(\theta) = \, : e^{i\phi'_{(r)}(\theta)} :$ is given by

$$\phi'_{(r)}(\theta) = \int_{-\infty}^{\infty} \frac{dt}{it} a'_{(r)}(t) e^{i\theta t}, \quad a'_{(r)}(t) = -e^{-\frac{\pi|t|}{N}} a_{(r)}(t). \tag{3.52}$$

Due to the relation (2.60) between a_0 and a_k, and our choice of ρ' and ρ_χ, the fused primed operator $Z'_{12\ldots N}$ is just the constant $D_{N,N}$ equal to 1

$$Z'_{12\ldots N} = D_{N,N} = 1. \tag{3.53}$$

For the rank-(N-1) lowest weight vertex operator, the formula also simplifies to

$$\overline{Z}'_N \equiv Z'_{12\ldots N-1}(\theta) = D_{N,N-1} V'_{(N-1)}(\theta) = D_{N,N-1} V'_{\bar{N}}(\theta), \tag{3.54}$$

where

$$D_{N,N-1} = \frac{e^{\frac{\gamma(N-1)}{2N}} N^{-\frac{1}{2N}} (2\pi)^{\frac{N-1}{N}}}{\Gamma\left(\frac{1}{N}\right)}. \tag{3.55}$$

3.2.2 Proof of relations in previous section

First we check the conjecture $\left[\chi_k^+, \chi_k^-\right] \simeq P_k$. We consider inserting the commutator in the matrix element,

$$\begin{aligned}
&\langle \bar{A}_k(\theta_1)| \left[\chi_k^+, \chi_k^-\right] |A_k(\theta_2)\rangle \\
&\sim \rho_\chi^2 \int_{C_\delta^-} d\delta \int_{C_\gamma^-} d\gamma g'_{k-1k}(\delta - \alpha_1) g_{k-1k}(\gamma - \alpha_1) g'_{kk}(\delta - \alpha_2) g_{kk}(\gamma - \alpha_2) \\
&\quad \times g'_{kk}(\gamma - \delta) g'_{kk+1}(\beta - \delta) g_{kk+1}(\beta - \gamma) \\
&- \rho_\chi^2 \int_{C_\delta^+} d\delta \int_{C_\gamma^+} d\gamma g'_{k-1k}(\delta - \alpha_1) g_{k-1k}(\gamma - \alpha_1) g'_{kk}(\delta - \alpha_2) g_{kk}(\gamma - \alpha_2) \\
&\quad \times g'_{kk}(\delta - \gamma) g'_{kk+1}(\beta - \delta) g_{kk+1}(\beta - \gamma),
\end{aligned} \tag{3.56}$$

noting that an insertion of P_k will leave the matrix element unchanged. Here we note that α_1 and α_2 are the integration variables in V_{k-1} and V_k respectively and that β is the integration variable in V_{k+1}. First, we take the integral in δ. In the first term we have the poles at $\delta \to \gamma + \frac{i\pi}{N}$ and at $\delta \to \alpha_2 - \frac{i\pi}{N}$, and in the second term we have poles at $\delta \to \gamma - \frac{i\pi}{N}$ and at $\delta \to \alpha_2 - \frac{i\pi}{N}$. The contributions from the poles in α_2 cancel. In the remaining terms, we shift the δ contour so that it follows the γ contour, which leaves a double integral and the two residue contributions from shifting the δ contour. Then, we integrate with respect to γ. The double integral is zero because the integrand vanishes. The contribution from the single integral is cancelled by the contribution from the double pole at $\delta \to \gamma - \frac{i\pi}{N}$, $\gamma \to \alpha_1 - \frac{i\pi}{N}$. The only remaining contribution is from the double pole at $\delta \to \gamma + \frac{i\pi}{N}$, $\gamma \to \theta_2 + \frac{i\pi}{N}$ which gives 1. Therefore we have

$$\langle \bar{A}_k(\theta_1)| \left[\chi_k^+, \chi_k^-\right] |A_k(\theta_2)\rangle = \langle \bar{A}_k(\theta_1) A_k(\theta_2)\rangle = \langle \bar{A}_k(\theta_1)|P_k|A_k(\theta_2)\rangle \, , \tag{3.57}$$

and therefore

$$\left[\chi_k^+, \chi_k^-\right] \simeq P_k \, . \tag{3.58}$$

Next, we see that

$$\left[P_k, Z_k\right] = Z_k \, , \quad \left[P_k, Z_k'\right] = -Z_k' \, , \tag{3.59}$$

follows from the regularized forms of the fields given in Appendix G. The next equations to check are

$$\chi_k^+ Z_{k+1}(\theta) + Z_{k+1}(\theta)\chi_k^+ \simeq Z_k(\theta) \, , \quad \chi_k^- Z_{k+1}'(\theta) + Z_{k+1}'(\theta)\chi_k^- \simeq Z_k'(\theta) \, . \tag{3.60}$$

For the first of these, we have

$$\begin{aligned}
\chi_k^+ Z_{k+1} + Z_{k+1}\chi_k^+ &= \chi_k^+\chi_k^- Z_k + \chi_k^+ Z_k \chi_k^- + \chi_k^- Z_k \chi_k^+ + Z_k \chi_k^- \chi_k^+ \\
&= \chi_k^+\chi_k^- Z_k - Z_k \chi_k^+ \chi_k^- - \chi_k^- \chi_k^+ Z_k + Z_k \chi_k^- \chi_k^+ \\
&= \left[\chi_k^+, \chi_k^-\right] Z_k - Z_k \left[\chi_k^+, \chi_k^-\right] \\
&\simeq P_k Z_k - Z_k P_k \\
&\simeq \left[P_k, Z_k\right] \\
&\simeq Z_k \, .
\end{aligned} \tag{3.61}$$

Similarly, by exchanging χ_k^+ and χ_k^- and replacing Z_k by Z'_k we can see that $\chi_k^- Z'_{k+1} + Z'_{k+1}\chi_k^- \simeq Z'_k$. The final relations to check are

$$\chi_k^+ Z_m(\theta) \simeq Z_m(\theta)\chi_k^+ , \quad \chi_k^- Z'_m(\theta) \simeq Z'_m(\theta)\chi_k^- \quad \text{if} \quad m > k+1 . \tag{3.62}$$

Again, we can check one and the other will follow by exchanging χ_k^+ and χ_k^- and replacing Z_k by Z'_k. We have

$$\begin{aligned}
\chi_k^+ Z_m &= \chi_k^+ \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^- \left(\chi_{k+1}^- Z_{k+1} + Z_{k+1}\chi_{k+1}^-\right) + \cdots\right) \\
&= \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^- \left(\chi_k^+\chi_{k+1}^- Z_{k+1} + \chi_k^+ Z_{k+1}\chi_{k+1}^-\right) + \cdots\right) \\
&\simeq \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^- \left(-\chi_{k+1}^-\chi_k^+ Z_{k+1} + \left(Z_k - Z_{k+1}\chi_k^+\right)\chi_{k+1}^-\right) + \cdots\right) \\
&\simeq \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^-\big(- \chi_{k+1}^- \left(Z_k - Z_{k+1}\chi_k^+\right)\right. \\
&\qquad \left. + \left(Z_k - Z_{k+1}\chi_k^+\right)\chi_{k+1}^-\big) + \cdots\right) \\
&\simeq \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^- \left(\chi_{k+1}^- Z_{k+1}\chi_k^+ - Z_{k+1}\chi_k^+\chi_{k+1}^-\right) + \cdots\right) \\
&\simeq \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^- \left(\chi_{k+1}^- Z_{k+1}\chi_k^+ + Z_{k+1}\chi_{k+1}^-\chi_k^+\right) + \cdots\right) \\
&\simeq \left(\chi_m^-\chi_{m-1}^- \cdots \chi_{k+2}^- \left(\chi_{k+1}^- Z_{k+1} + Z_{k+1}\chi_{k+1}^-\right) + \cdots\right)\chi_k^+ \\
&\simeq Z_m\chi_k^+ ,
\end{aligned} \tag{3.63}$$

as expected.

3.2.3 Λ and T operators

To discuss the operators Λ which are used to generate form factors of local operators we need the following algebra of the ZF vertex operators and the primed operators

$$Z_i(\theta_1)Z'_j(\theta_2) = -(-1)^{\delta_{ij}} S'(\theta_{12})Z'_j(\theta_2)Z_i(\theta_1) , \tag{3.64}$$

where

$$S'(\theta) = \frac{g'(-\theta)}{g'(\theta)} = -\frac{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{2N}\right)}{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{2N}\right)} . \tag{3.65}$$

Obviously, $S'(\theta)$ has a pole at $\theta = -\pi i/N$, and a zero at $\theta = \pi i/N$. In addition, for any θ, $S'(\theta)$ satisfies the following important relation

$$\prod_{k=1}^{N} S'\Big(\theta + 2k\frac{i\pi}{N}\Big) = (-1)^{N-1} . \tag{3.66}$$

The commutation relations (3.43), (3.44) and (3.64) take a simpler form if one introduces the primed operators for the GN model

$$A'_j(\theta) = \Gamma_j^{-1} Z'_j(\theta) , \quad A'_{k_1 \dots k_r}(\theta) = \Gamma_{k_1}^{-1} \cdots \Gamma_{k_r}^{-1} Z'_{k_1 \dots k_r}(\theta) . \tag{3.67}$$

Then the relations above take the form

$$A'_i(\theta^{[-]}) A'_j(\theta^{[+]}) = -A'_j(\theta^{[-]}) A'_i(\theta^{[+]}) , \tag{3.68}$$

$$A_i(\theta_1) A'_j(\theta_2) = S'(\theta_{12}) A'_j(\theta_2) A_i(\theta_1) . \tag{3.69}$$

Let us introduce the following operators

$$\Lambda^{\mathcal{P}}_{j_1 j_2 \dots j_N}(\theta) = \Gamma' A'_{j_1}\big(\theta_{\mathcal{P}(1)}\big) \cdots A'_{j_N}\big(\theta_{\mathcal{P}(N)}\big) = \Gamma' \prod_{a=1}^{N} A'_{j_a}\big(\theta_{\mathcal{P}(a)}\big) , \tag{3.70}$$

where $\mathcal{P}$ is any permutation of $1, 2, \dots N$, $\Gamma' \equiv \Gamma_N \cdots \Gamma_1$, and $\theta_k \equiv \theta - i u_{N-2k+1}$. Then taking into account (3.66), one concludes that these operators commute with A_i for both N odd and even

$$A_i(\theta_1) \Lambda^{\mathcal{P}}_{j_1 \dots j_N}(\theta_2) = \Lambda^{\mathcal{P}}_{j_1 \dots j_N}(\theta_2) A_i(\theta_1) . \tag{3.71}$$

It is worth mentioning that the indices j_a are arbitrary and some of them may coincide. However, it should be stated that these operators do not satisfy the first equation of (1.54). The physical operators will appear in the form factors in a series expansion, see section 4.3. If the permutation $\mathcal{P}$ is trivial, $\mathcal{P} = id$ then

$$\Lambda^{id}_{j_1 \dots j_N}(\theta) = \epsilon_{j_1 \dots j_N} . \tag{3.72}$$

The simplest nontrivial class of the operators $\Lambda^{\mathcal{P}}_{j_1 \dots j_N}$ is obtained for a cyclic permutation $\mathcal{P}_\lambda \equiv (N - \lambda + 1, \dots, N, 1, \dots, N - \lambda)$ and $\mathcal{P}_0 = \mathcal{P}_N = id$. Taking into account (3.48) and (3.67) one finds

$$\Lambda^{\mathcal{P}_\lambda}_{j_1\cdots j_N}(\theta) = \Gamma' A'_{j_1\cdots j_\lambda}(\theta + i\mathrm{u}_{N-\lambda}) A'_{j_{\lambda+1}\cdots j_N}(\theta - i\mathrm{u}_\lambda)\,. \tag{3.73}$$

It is clear that the operators $\Lambda^{\mathcal{P}}_{j_1\cdots j_N}$ are not linearly independent. If for a generic set of indices $j_1,\ldots,j_N$ in a permutation $\mathcal{P} = (p_1,\ldots,p_{k-1},p_k,\ldots,p_N)$ one has $|p_k - p_{k-1}| \geq 2$ for some k then by using the ZF algebra for the primed operators one can exchange the positions of the operators $A'_{j_{k-1}}$ and A'_{j_k} and express $\Lambda^{\mathcal{P}}_{j_1\cdots j_N}$ as a linear combination of $\Lambda^{\mathcal{P}'}_{j_1\cdots j_N}$ with $\mathcal{P}' = (p_1,\ldots,p_k,p_{k-1},\ldots,p_N)$. This allows one to choose a convenient basis of the operators $\Lambda^{\mathcal{P}}_{j_1\cdots j_N}$. In a given permutation $\mathcal{P}$ one first moves N to the left either to the first position or to $N-1$. Then one moves $N-1, N$ to the left either to the first position or to $N-2$. Finally one gets the permutation $\mathcal{P}' = (N-\lambda_1+1,\ldots,N,\mathcal{P}_{N-\lambda_1})$ where $\mathcal{P}_{N-\lambda_1}$ is a permutation of $1,\ldots,N-\lambda_1$. Repeating the procedure one eventually gets the following permutation

$$\begin{aligned}\mathcal{P}_{\vec{\lambda}} =& (\{N-\lambda_1+1,\ldots,N\},\{N-\lambda_1-\lambda_2+1,\ldots,N-\lambda_1\},\\ &\ldots,\{1,\ldots,N-\sum_{b=1}^{r-1}\lambda_b\})\,,\end{aligned} \tag{3.74}$$

which consists of r length λ_k sequences of consecutive integers. The corresponding operators Λ are then given by

$$\Lambda^{\mathcal{P}_{\vec{\lambda}}}_{j_1\cdots j_N}(\theta) = \Gamma' A'_{j_1\cdots j_{\lambda_1}}(\theta_1) A'_{j_{\lambda_1+1}\cdots j_{\lambda_1+\lambda_2}}(\theta_2)\cdots A'_{j_{N-\lambda_r+1}\cdots j_N}(\theta_r)\,, \tag{3.75}$$

where $\theta_k = \theta + i\mathrm{u}_{N-\lambda_k-2\sum_{i=1}^{k-1}\lambda_i}$, and $\sum_{i=1}^{r}\lambda_i = N$. These operators are obviously antisymmetric under the exchange of indices in each of the vertex operators $A'_{j_1\cdots j_k}$.

In addition if one considers a linear combination of $\Lambda^{\mathcal{P}}_{j_1\cdots j_N}$ which is anti-symmetric under the exchange of the indices $j_k, j_{k+1},\ldots,j_{k+n}$ for some k and n then in the permutation $\mathcal{P}$ one can always reorder $p_k,p_{k+1},\ldots,p_{k+n}$ so that $p_k < p_{k+1} < \ldots < p_{k+n}$. As a result if in the operator (3.75) the right boundary of the sequence λ_j and the left boundary of the sequence λ_{j+1} lie between the positions k and $k+n$ then these sequences can be united into one sequence of length $\lambda_2 + \lambda_3$. In what follows we will always consider operators (3.75).

The operators $\Lambda^{\mathcal{P}}_{j_1 \cdots j_N}$ and their products form an overcomplete basis of an algebra of operators commuting with A_i and can be used to generate form factors of many local operators of the chiral GN model. The local operators however would have the trivial index $\Omega(O, I)$ equal to 0.

Another set of interesting operators is

$$T_r(\theta) = \frac{\Gamma'}{N!} \epsilon^{a_1 a_2 \ldots a_N} A'_{a_1 \ldots a_r}(\theta - \frac{i\pi}{2}) \partial_\theta A'_{a_{r+1} \ldots a_N}(\theta + \frac{i\pi}{2}), \quad r = 1, \ldots, N-1. \tag{3.76}$$

Taking into account the identity (3.66), and the relations (3.69) and (3.72), one finds

$$\left[T_r(\theta), A_i(\beta)\right] = \partial_\theta \log s_r(\theta - \beta)\, A_i(\beta), \tag{3.77}$$

where

$$s_r(\theta) \equiv \prod_{j=1}^{r} S'\big(-\theta - i\mathrm{u}_{r-2j+1} - \frac{i\pi}{2}\big) = \prod_{j=1}^{r} \frac{1}{S'\big(\theta + i\mathrm{u}_{r-2j+1} + \frac{i\pi}{2}\big)}. \tag{3.78}$$

The functions s_r are not real unless $r = N/2$. It might be convenient to take the following linear combinations of T_r

$$T_r^+(\theta) = T_r(\theta) + T_{N-r}(\theta), \quad T_r^-(\theta) = i\big(T_r(\theta) - T_{N-r}(\theta)\big), \tag{3.79}$$

which lead to real functions $s_r^\pm$

$$s_r^+(\theta) = (-1)^{N-1} \prod_{j=1}^{r} \frac{S'\big(\theta + i\mathrm{u}_{r-2j+1} - \frac{i\pi}{2}\big)}{S'\big(\theta + i\mathrm{u}_{r-2j+1} + \frac{i\pi}{2}\big)}, \tag{3.80}$$

$$s_r^-(\theta) = i\,(-1)^{N-1} \prod_{j=1}^{r} \frac{1}{S'\big(\theta + i\mathrm{u}_{r-2j+1} + \frac{i\pi}{2}\big) S'\big(\theta + i\mathrm{u}_{r-2j+1} - \frac{i\pi}{2}\big)}. \tag{3.81}$$

The meaning of the operators T_r was discussed in [11]. They are generating functions for the integrals of motion. To describe these, let us focus on one of these functions, which we will denote $I(\alpha)$. Then, for integrals of motion with spin s,

I_s, such that

$$
\begin{aligned}
I_s|vac\rangle &= 0\,, \\
I_s|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle &= \gamma^{(s)}\sum_{k=1}^{n}\exp(s\theta_k)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle\,,
\end{aligned}
\tag{3.82}
$$

with $\gamma^{(s)}$ constants, the generating function is given by

$$
\begin{aligned}
I(\alpha) &= \sum_{s>0} I_s \exp(-s\alpha)\,, \quad \exp(\alpha)\to\infty\,, \\
I(\alpha) &= \sum_{s>0} I_{-s}\exp(s\alpha)\,, \quad \exp(\alpha)\to 0\,.
\end{aligned}
\tag{3.83}
$$

This gives rise to the requirement (3.77) for T_r and in general for any $I(\alpha)$, which we now write as

$$
\big[I(\alpha), A_i(\theta)\big] = \partial_\theta \log s(\alpha-\theta)\, A_i(\theta)\,. \tag{3.84}
$$

Futhermore, if for some local operators $\Lambda(O)$ we consider its commutator with the integrals of motions,

$$
O(x,s) = [O(x), I_s]\,, \tag{3.85}
$$

the result, $O(x,s)$, is also a local operator. Due to this, the form factors for the operator $O(x,s)$ may be found from the form factors of $O(x)$ as

$$
F^{O(x,s)}_{a_1,\cdots,a_n}(\theta_1,\cdots\theta_n) = \sum_{k=1}^{n}\partial_\alpha \log s(\alpha-\theta_k) F^{O(x)}_{a_1,\cdots,a_n}(\theta_1,\cdots\theta_n)\,. \tag{3.86}
$$

Let us finally mention that as in the $N=2$ case the operators $\Lambda^{\mathcal{P}}_{j_1\cdots j_N}$ and T_r form a quadratic algebra.

Chapter 4

Form Factors of the Gross-Neveu Model

4.1 A General Approach to Form Factors

Following [11], we expect that, up to an overall normalisation constant, form factors of a local operator should be generated by the following functions

$$\begin{aligned} &\mathcal{F}^{\mathcal{V}}_{M_1\ldots M_n}(\theta'_1,\ldots,\theta'_k|\theta_1,\ldots,\theta_n) \\ &\quad = \frac{\mathrm{Tr}_{\pi_A}\left[\frac{1}{2}(1+\Gamma)e^{2\pi i\mathbb{K}}\Lambda^{\mathcal{V}}(\theta'_k,\ldots,\theta'_1)A_{M_n}(\theta_n)\cdots A_{M_1}(\theta_1)\right]}{\mathrm{Tr}_{\pi_A}\left[\frac{1}{2}(1+\Gamma)e^{2\pi i\mathbb{K}}\right]}\,. \end{aligned} \tag{4.1}$$

Here $\Lambda^{\mathcal{V}}(\theta'_k,\ldots,\theta'_1)$ are linear combinations of the products of k operators $\Lambda^{\mathcal{P}}_{\mathbf{j}}(\theta')$ with $\mathbf{j}=\{j_1\ldots j_N\}$ transforming in an irreducible representation $\mathcal{V}$ of $\mathfrak{su}(N)$

$$\Lambda^{\mathcal{V}}(\theta'_k,\ldots,\theta'_1)=\sum c^{\mathcal{V}}_{\mathbf{j}_k\ldots\mathbf{j}_1}\,\Lambda^{\mathcal{P}_k}_{\mathbf{j}_k}(\theta'_k)\cdots\Lambda^{\mathcal{P}_1}_{\mathbf{j}_1}(\theta'_1)\,, \tag{4.2}$$

and $A_{M_i}(\theta_i)$ is a rank-r_i bound state vertex operator. These functions are therefore combinations of traces of the form

$$\frac{\mathrm{Tr}_{\pi_A}\left[\frac{1}{2}(1+\Gamma)e^{2\pi i\mathbb{K}}\Big(\prod_{j=1}^{N}\prod_{a=1}^{m'_j}A'_j(\theta'_{j,a})\Big)\Big(\prod_{j=1}^{N}\prod_{a=1}^{m_j}A_j(\theta_{j,a})\Big)\right]}{\mathrm{Tr}_{\pi_A}\left[\frac{1}{2}(1+\Gamma)e^{2\pi i\mathbb{K}}\right]}\,, \tag{4.3}$$

where $\sum_{j=1}^{N}m'_j=M'=kN$ and $\sum_{j=1}^{N}m_j=M=\sum_{j=1}^{n}r_j$ are the total numbers of A'_j and A_j operators. Such a trace does not vanish only if m_j and m'_j satisfy the following selection rules

$$m_j - m'_j = r\,, \quad M - M' = rN \text{ for some integer } r\,. \tag{4.4}$$

This formula shows that if A transforms in the fundamental representation of $\mathfrak{su}(N)$, then A' transforms in the antifundamental representation, and the form factor vanishes unless a decomposition of the product of all A and A' into irreducible representations of $\mathfrak{su}(N)$ contains a singlet, which is a natural requirement. Since $A_k = \Gamma_k Z_k$, one can also see from (4.4) that up to a sign (4.3) is equal to

$$\frac{\mathrm{Tr}_{\pi_Z}\left[e^{2\pi i\mathbb{K}}\left(\prod_{j=1}^{N}\prod_{a=1}^{m'_j} Z'_j(\theta'_{j,a})\right)\left(\prod_{j=1}^{N}\prod_{a=1}^{m_j} Z_j(\theta_{j,a})\right)\right]}{\mathrm{Tr}_{\pi_Z}\left[e^{2\pi i\mathbb{K}}\right]}\,. \tag{4.5}$$

The selection rules (4.4) just follow from the requirement that the product of all Z and Z' does not depend on the zero mode operators Q_μ. A careful derivation of (4.4) would employ an explicit ultraviolet regularisation of the free fields similar to the one used in [11]. This is done in appendix G where the rules (4.4) are derived.

It is thus clear that the functions (4.1) are combinations of multiple integrals with integrands of the form

$$R^{\nu_1\ldots\nu_p}_{\mu_1\ldots\mu_q}(\alpha'_1,\ldots,\alpha'_p|\beta_1,\ldots,\beta_q) = \langle\langle V'_{\nu_p}(\alpha'_p)\cdots V'_{\nu_1}(\alpha'_1)V_{\mu_q}(\beta_q)\cdots V_{\mu_1}(\beta_1)\rangle\rangle\,, \tag{4.6}$$

where the sets $\{\alpha'_1,\ldots,\alpha'_p\}$ and $\{\beta_1,\ldots,\beta_q\}$ contain θ'_j- and θ_j-related rapidities respectively, and for any operator W acting in π_Z we define

$$\langle\langle W\rangle\rangle \equiv \frac{\mathrm{Tr}_{\pi_Z}\left[\,e^{2\pi i\mathbb{K}}\,W\,\right]}{\mathrm{Tr}_{\pi_Z}\left[\,e^{2\pi i\mathbb{K}}\,\right]}\,. \tag{4.7}$$

It is shown in section 4.2 that for any operator W which is the product of free field exponents

$$W = U_n(\theta_n)\cdots U_1(\theta_1)\,, \quad U_j(\theta) =: e^{i\phi_j(\theta)} := e^{i\phi_j^-(\theta)}\,e^{i\phi_j^+(\theta)}\,, \tag{4.8}$$

one obtains $\langle\langle W\rangle\rangle$ by applying Wick's theorem

$$\langle\langle U_n(\theta_n)\cdots U_1(\theta_1)\rangle\rangle = \prod_{j=1}^{n} C_{U_j}\prod_{j>k} G_{U_jU_k}(\theta_k-\theta_j)\,, \tag{4.9}$$

where

$$C_{U_j} = \langle\langle U_j(\theta_j)\rangle\rangle = \exp\big(- \langle\langle \phi_j^-(0)\phi_j^+(0)\rangle\rangle\big) \,, \tag{4.10}$$

$$G_{U_jU_k}(\theta_k - \theta_j) = \exp\big(- \langle\langle \phi_j(\theta_j)\phi_k(\theta_k)\rangle\rangle\big) \,. \tag{4.11}$$

The constants C_{V_μ}, $C_{V'_\mu}$, and the functions $G_{\mu\nu} \equiv G_{V_\mu V_\nu}$, $G'_{\mu\nu} \equiv G_{V'_\mu V_\nu}$ and $G''_{\mu\nu} \equiv G_{V'_\mu V'_\nu}$ are computed in section 4.2.

The integration contours in (4.1) are chosen in the same way as for the vacuum expectation values

$$\langle 0|\Lambda^{\mathcal{V}}(\theta'_k, \ldots, \theta'_1) A_{M_n}(\theta_n) \cdots A_{M_1}(\theta_1)|0\rangle \,, \tag{4.12}$$

that is the integration contour C in $\chi_k^\pm$ runs from $\mathrm{Re}\,\alpha = -\infty$ to $\mathrm{Re}\,\alpha = +\infty$ and it lies above all poles of $g_{k\mu}$-functions due to operators to the right of $\chi_k^\pm$ but below all poles due to operators to the left of $\chi_k^\pm$. $G_{k\mu}$-functions however have more poles, and in addition to this rule one also requires that the contour C is in the simply-connected region which contains all the poles of $g_{k\mu}$ but no other poles of $G_{k\mu}$.

4.2 Traces of vertex operators

4.2.1 General formula

We want to compute traces of products of vertex operators defined as

$$V(\theta) =: \exp(i\phi(\theta)) : \,, \tag{4.13}$$

where $\phi(\theta)$ is a linear combination of the independent oscillators $a_i(t)$

$$\phi(\theta) = \int_{-\infty}^{\infty} \frac{dt}{it}\, c_i(t)\, a_i(t) e^{i\theta t} = \int_0^\infty dt\, \bar{\alpha}_i(t)\, a_i(t) + \int_0^\infty dt\; a_i^\dagger(t)\beta_i(t) \,, \tag{4.14}$$

with the coefficients given by

$$\bar{\alpha}_i(t) = \frac{1}{it}\, c_i(t) e^{i\theta t} \,, \quad \beta_i(t) = -\frac{1}{it}\, c_i(-t) e^{-i\theta t} \,, \quad a_i^\dagger(t) = a_i(-t) \,. \tag{4.15}$$

It is sufficient to understand how to compute

$$\mathrm{Tr}_F\left(\exp(2\pi i K)V(\theta)\right), \quad K = i\int_0^\infty dt \sum_{i,j=1}^{N-1} h_{ij}(t)a_i^\dagger(t)a_j(t), \tag{4.16}$$

where F is the Fock space where $a_i(t)$ act. It is not difficult to show that if one has one set of oscillators $a, a^\dagger$ such that

$$[a, a^\dagger] = 1, \quad K = i\,h\,a^\dagger a, \quad V(\theta) =: \exp(i\phi):, \quad \phi = \bar{\alpha}\,a + a^\dagger \beta, \tag{4.17}$$

then the trace of the normal-ordered exponential of ϕ is given by

$$\mathrm{Tr}_F\ e^{2\pi i K} : \exp(i\phi) := \frac{1}{1-e^{-2\pi h}} \exp\left(\frac{\bar{\alpha}\beta}{1-e^{2\pi h}}\right). \tag{4.18}$$

Proof of this is given in appendix F. We want to generalise the formula to the case of several coupled oscillators, and we can drop the t-dependence because the commutation relations are ultra-local in t. So we consider

$$[a_i, a_j^\dagger] = f_{ij}, \quad K = i\,h_{ij}\,a_i^\dagger a_j, \quad V(\theta) =: \exp(i\phi):, \quad \phi = \bar{\alpha}_i\,a_i + a_i^\dagger \beta_i, \tag{4.19}$$

where $\bar{f}_{ij} = f_{ji}$ and $\bar{h}_{ij} = h_{ji}$, that is the matrices f and h are hermitian. Since f is hermitian it can be diagonalised with a unitary matrix U

$$UfU^\dagger = D^2, \quad D_{ij} = d_i\delta_{ij}, \quad d_i > 0, \tag{4.20}$$

with the oscillators transforming into a new basis $b = Ua$, $b^\dagger = a^\dagger U^\dagger$. The new oscillators satisfy the relations

$$[b_i, b_j^\dagger] = d_i^2\delta_{ij}, \quad K = i\,b_i^\dagger(UhU^\dagger)_{ij}\,b_j, \quad \phi = (\bar{\alpha}U^\dagger)_i\,b_i + b_i^\dagger(U\beta)_i. \tag{4.21}$$

We then rescale b_i to get the canonical commutation relations for another set of oscillators

$$b_i = d_i c_i, \quad b_i^\dagger = d_i c_i^\dagger, \quad [c_i, c_j^\dagger] = \delta_{ij}, \tag{4.22}$$

which now results in an altered form of the hamiltonian and fields

$$K = i\,c_i^\dagger(DUhU^\dagger D)_{ij}\,c_j, \quad \phi = (\bar{\alpha}U^\dagger D)_i\,c_i + c_i^\dagger(DU\beta)_i. \tag{4.23}$$

The matrix $DUhU^\dagger D$ is obviously hermitian and can be diagonalised with a unitary matrix W

$$DUhU^\dagger D = W^\dagger Q W\,, \quad Q_{ij} = q_i\delta_{ij}\,, \tag{4.24}$$

and introducing the new oscillators $A = Wc\,,\ A^\dagger = c^\dagger W^\dagger$ one gets

$$[A_i, A_j^\dagger] = \delta_{ij}\,, \quad K = i\,q_i A_i^\dagger A_i\,, \quad \phi = (\bar{\alpha}U^\dagger D W^\dagger)_i\, A_i + A_i^\dagger (WDU\beta)_i\,, \tag{4.25}$$

and therefore in the same way as we found (4.18), we can now write

$$\mathrm{Tr}_F\; e^{2\pi i K} : \exp(i\phi) := \prod_i \frac{1}{1-e^{-2\pi q_i}} \exp\left(\frac{\bar{\alpha}_i' \beta_i'}{1-e^{2\pi q_i}}\right)\,, \tag{4.26}$$

where the new coefficients are

$$\bar{\alpha}_i' = (\bar{\alpha}U^\dagger D W^\dagger)_i\,, \quad \beta_i' = (WDU\beta)_i\,. \tag{4.27}$$

Formula (4.26) can be brought to the form

$$\mathrm{Tr}_F\; e^{2\pi i K} : \exp(i\phi) := \frac{1}{\det(1-e^{-2\pi h f})} \exp\left(\bar{\alpha}_i \left(f \frac{1}{1-e^{2\pi h f}}\right)_{ij} \beta_j\right)\,. \tag{4.28}$$

There is also a useful identity

$$f \frac{1}{1-e^{2\pi h f}} = \frac{1}{1-e^{2\pi f h}} f\,. \tag{4.29}$$

Fortunately, thanks to (2.77), $fh = t$, where we take into account that the actual commutation relations are $[a_i(t), a_j^\dagger(t')] = t f_{ij}\delta(t-t')$. Thus introducing the integral over t one gets

$$\frac{\mathrm{Tr}_F\,(\exp(2\pi i K) V(\theta))}{\mathrm{Tr}_F\,(\exp(2\pi i K))} = \exp\left(\int_0^\infty dt\, \frac{\bar{\alpha}_i(t)\, t f_{ij}(t)\, \beta_j(t)}{1-e^{2\pi t}}\right)\,. \tag{4.30}$$

This formula agrees with the prescription in [18]. To show this let's consider the case of two vertex operators,

$$\mathrm{Tr}_F\,(\exp(2\pi i K) V_1 V_2)\,, \tag{4.31}$$

where the vertex operators are given by

$$V_k =: \exp(i\phi_k) :, \quad \phi_k = \int_0^\infty dt\, \bar{\alpha}_i^{(k)}(t)\, a_i(t) + \int_0^\infty dt\; a_i^\dagger(t)\beta_i^{(k)}(t)\,. \tag{4.32}$$

Normal-ordering these two vertex operators will give us

$$V_1 V_2 = g_{12} : V_1 V_2 :, \quad g_{12} = \exp\left(- \int_0^\infty dt\, \bar{\alpha}_i^{(1)}(t)\, t f_{ij}(t)\, \beta_j^{(2)}(t) \right), \tag{4.33}$$

and the trace will become

$$\begin{aligned} &\frac{\mathrm{Tr}_F\left(\exp(2\pi i K) V_1 V_2\right)}{\mathrm{Tr}_F\left(\exp(2\pi i K)\right)} \\ &\quad = \exp\left(\int_0^\infty dt\, \left(-\bar{\alpha}_i^{(1)}(t)\, t f_{ij}(t)\, \beta_j^{(2)}(t) + \frac{\bar{\alpha}_i(t)\, t f_{ij}(t)\, \beta_j(t)}{1 - e^{2\pi t}} \right)\right), \end{aligned} \tag{4.34}$$

where the new coefficients are now given by

$$\bar{\alpha}_i = \bar{\alpha}_i^{(1)} + \bar{\alpha}_i^{(2)}, \quad \beta_j(t) = \beta_j^{(1)} + \beta_j^{(2)}\,. \tag{4.35}$$

We can write this as

$$\frac{\mathrm{Tr}_F\left(\exp(2\pi i K) V_1 V_2\right)}{\mathrm{Tr}_F\left(\exp(2\pi i K)\right)} = C_1 C_2 G_{12}\,, \tag{4.36}$$

with the constants given by

$$C_k = \frac{\mathrm{Tr}_F\left(\exp(2\pi i K) V_k\right)}{\mathrm{Tr}_F\left(\exp(2\pi i K)\right)}\,, \tag{4.37}$$

and the remaining function as

$$G_{12} = \exp\left(- \int_0^\infty dt\, \left(\frac{\bar{\alpha}_i^{(1)}(t)\, t f_{ij}(t)\, \beta_j^{(2)}(t)}{1 - e^{-2\pi t}} - \frac{\bar{\alpha}_i^{(2)}(t)\, t f_{ij}(t)\, \beta_j^{(1)}(t)}{1 - e^{2\pi t}} \right)\right). \tag{4.38}$$

We can look at skipping the intermediate steps of this process by introducing the traces of the oscillators

$$\langle\langle a_j(t) a_k(t') \rangle\rangle = \frac{t f_{jk}(t)}{1 - e^{-2\pi t}} \delta(t + t')\,, \tag{4.39}$$

which give us the following traces for vertex operators

$$\langle\langle \phi_1 \phi_2 \rangle\rangle = -\log G_{12}\,. \tag{4.40}$$

4.2.2 Traces of single V's

To compute the traces of V_μ and V'_μ we note that the coefficients in the expansion of the fields are

$$\bar{\alpha}_\mu(t) = \bar{\alpha}'_\mu(t) = -\frac{ie^{i\theta t}}{t}, \quad \beta_\mu(t) = \beta'_\mu(t) = \frac{ie^{-i\theta t}}{t}, \tag{4.41}$$

and therefore from (4.30) one gets

$$C_\mu \equiv \frac{\mathrm{Tr}_F\left(\exp(2\pi i K)V_\mu(\theta)\right)}{\mathrm{Tr}_F\left(\exp(2\pi i K)\right)} = \exp\left(\int_0^\infty \frac{dt}{t}\frac{f_{\mu\mu}(t)}{1-e^{2\pi t}}\right), \tag{4.42}$$

and a similar formula for V' with the obvious replacement $f \to f'$.

Computing the integrals, using here and below the integrals given in section B.2.1 of appendix B, one gets the following constants

$$\begin{aligned} \log C_j &= -\log\Gamma\left(1-\frac{1}{N}\right) + \gamma\left(\frac{1}{N}-1\right) + \frac{\log(2\pi)}{N}, \\ C_j &= \frac{e^{\gamma\left(\frac{1}{N}-1\right)}(2\pi)^{\frac{1}{N}}}{\Gamma\left(\frac{N-1}{N}\right)}, \end{aligned} \tag{4.43}$$

for the vertex operators V_j and for both V_0 and V_N we get

$$\begin{aligned} \log C_0 = \log C_N = 1 &- \frac{3}{2N} + \frac{1}{2N^2} - \frac{\gamma(N-1)^2}{2N^2} \\ &+ \frac{1-N}{N}\log\Gamma\left(2-\frac{1}{N}\right) \\ &+ \left(-\frac{1}{2N^2}+\frac{1}{N}-1\right)\log(2\pi) + \psi^{(-2)}\left(2-\frac{1}{N}\right), \end{aligned} \tag{4.44}$$

where $\psi^{(-2)}(z)$ is a polygamma function, given by

$$\psi^{(-2)}(z) = \int_0^z dt \log\Gamma(t) \,. \tag{4.45}$$

Similarly, the primed vertex operators can be inserted in (4.30). For V'_j we find

$$C'_j = \frac{e^{\gamma\left(-\frac{1}{N}-1\right)}(2\pi)^{-1/N}}{\Gamma\left(\frac{N+1}{N}\right)}, \tag{4.46}$$

and likewise for V_0' or V_N' we get the constant

$$\begin{aligned}\log C_0' &= -\frac{\gamma\,(N^2-1)}{2N^2} - \frac{1}{2N^2} + \left(\frac{1}{2N^2} + \frac{1}{2}\right)\log(2\pi) - \frac{1}{2N} \\ &\quad -\psi^{(-2)}\left(1+\frac{1}{N}\right) + \frac{\log\left(\Gamma\left(1+\frac{1}{N}\right)\right)}{N}\,. \end{aligned} \tag{4.47}$$

4.2.3 Traces of $V_\mu V_\nu$ and functions $G_{\mu\nu}$

To compute the traces of two vertex operators, we use (4.36) and (4.38) which takes the form

$$G_{\mu\nu}(\beta_2-\beta_1) = \exp\big(-\langle\langle\phi_\mu(\beta_1)\phi_\nu(\beta_2)\rangle\rangle\big)\,. \tag{4.48}$$

This can be written in a convenient integral form

$$G_{\mu\nu}(\beta) = \exp\Big(-\int_0^\infty \frac{dt}{t}\frac{f_{\mu\nu}(t)}{\sinh\pi t}\cos\left(\beta+i\pi\right)t\Big)\,. \tag{4.49}$$

Written in this form, it becomes apparent that these functions satisfy the relations

$$G_{\mu\nu}(\beta-2\pi i) = G_{\mu\nu}(-\beta)\,, \quad S_{\mu\nu}(\beta) = \frac{G_{\mu\nu}(-\beta)}{G_{\mu\nu}(\beta)}\,, \tag{4.50}$$

which are necessary to satisfy the form factor axioms. This is explained in appendix A.

Computing the functions, and in each case extracting a constant, we get the following representations

$$\begin{aligned} G_{00}(\beta) &= C_{00}\exp\Big(-2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t}{\sinh^2\pi t}e^{\frac{\pi t}{N}}\sinh^2\left(\frac{\pi t}{2}-\frac{i\beta t}{2}\right)\Big)\,, \\ G_{00}'(\beta) &= C_{00}'\exp\left(2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t}{\sinh^2\pi t}\sinh^2\left(\frac{\pi t}{2}-\frac{i\beta t}{2}\right)\right)\,, \\ G_{00}''(\beta) &= C_{00}''\exp\Big(-2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t}{\sinh^2\pi t}e^{-\frac{\pi t}{N}}\sinh^2\left(\frac{\pi t}{2}-\frac{i\beta t}{2}\right)\Big)\,, \end{aligned} \tag{4.51}$$

with the constants that appear given by

$$
\begin{aligned}
C_{00} &= \exp\Big(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t}{\sinh^2\pi t}e^{\frac{\pi t}{N}}\Big)\\
&= \frac{2^{\frac{N-1}{N^2}-\frac{5}{12}}\pi^{\frac{N-1}{N^2}-1}}{A^3}\Gamma\left(\frac{3}{2}-\frac{1}{N}\right)^{\frac{2}{N}-1}e^{\frac{(N-1)(N+\gamma-1)}{N^2}+2\psi^{(-2)}\left(\frac{3}{2}-\frac{1}{N}\right)}\,,\\
C'_{00} &= \exp\Big(\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t}{\sinh^2\pi t}\Big)\\
&= \left(\frac{\pi}{2}\right)^{1-\frac{1}{N}}\left(\frac{(N-1)\sec\left(\frac{\pi}{2N}\right)}{N}\right)^{1-\frac{1}{N}}e^{\frac{1-N}{N}-2\psi^{(-2)}\left(\frac{3}{2}-\frac{1}{2N}\right)+2\psi^{(-2)}\left(\frac{N+1}{2N}\right)}\,,\\
C''_{00} &= \exp\Big(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t}{\sinh^2\pi t}e^{-\frac{\pi t}{N}}\Big)\\
&= A^3 2^{\frac{1}{N^2}-\frac{1}{N}+\frac{17}{12}}\pi^{\frac{1}{N^2}-\frac{1}{N}+1}\Gamma\left(\frac{1}{2}+\frac{1}{N}\right)^{\frac{2}{N}-1}e^{-\frac{\gamma(N-1)+1}{N^2}-2\psi^{(-2)}\left(\frac{1}{2}+\frac{1}{N}\right)}\,.
\end{aligned}
\tag{4.52}
$$

These constants correspond to a special point of the functions, where the cos term vanishes in (4.49) at $\beta = -i\pi$. In other words, $C_{00} = G_{00}(-i\pi)$, etc. The integral representations are well-defined for

$$
\begin{aligned}
G_{00}(\beta):&\quad -2\pi < \mathrm{Im}(\beta) < 0 \quad \text{and} \quad N > 1\,,\\
G'_{00}(\beta):&\quad -\frac{\pi}{N} - 2\pi < \mathrm{Im}(\beta) < \frac{\pi}{N} \quad \text{and} \quad N > 1\,,\\
G''_{00}(\beta):&\quad -\frac{2\pi}{N} - 2\pi < \mathrm{Im}(\beta) < \frac{2\pi}{N} \quad \text{and} \quad N > 1\,.
\end{aligned}
\tag{4.53}
$$

Note that $G_{00}(\beta) = G_{NN}(\beta)$, $G'_{00}(\beta) = G'_{NN}(\beta)$, and $G''_{00}(\beta) = G''_{NN}(\beta)$. Similarly, one can get the following representations

$$
\begin{aligned}
G_{0N}(\beta) &= C_{0N}\exp\Big(-2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t}{N}}{\sinh^2\pi t}e^{\frac{\pi t}{N}}\sinh^2\Big(\frac{\pi t}{2}-\frac{i\beta t}{2}\Big)\Big)\,,\\
G'_{0N}(\beta) &= C'_{0N}\exp\Big(2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t}{N}}{\sinh^2\pi t}\sinh^2\Big(\frac{\pi t}{2}-\frac{i\beta t}{2}\Big)\Big)\,,\\
G''_{0N}(\beta) &= C''_{0N}\exp\Big(-2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t}{N}}{\sinh^2\pi t}e^{-\frac{\pi t}{N}}\sinh^2\Big(\frac{\pi t}{2}-\frac{i\beta t}{2}\Big)\Big)\,,
\end{aligned}
\tag{4.54}
$$

with the constants given by

$$
\begin{aligned}
C_{0N} &= \exp\Big(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t}{N}}{\sinh^2 \pi t} e^{\frac{\pi t}{N}}\Big) \\
&= (2\pi)^{\frac{1}{N^2}+1}\Gamma\left(\frac{N-1}{N}\right)^{-2/N} e^{\frac{N+\gamma-1}{N^2}-2\psi^{(-2)}\left(\frac{N-1}{N}\right)}, \\
C'_{0N} &= \exp\Big(\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t}{N}}{\sinh^2 \pi t}\Big) \\
&= \left(\Gamma\left(1-\frac{1}{2N}\right)\Gamma\left(1+\frac{1}{2N}\right)\right)^{1/N} e^{-\frac{1}{N}+2\psi^{(-2)}\left(1-\frac{1}{2N}\right)-2\psi^{(-2)}\left(1+\frac{1}{2N}\right)}, \\
C''_{0N} &= \exp\Big(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t}{N}}{\sinh^2 \pi t} e^{-\frac{\pi t}{N}}\Big) \\
&= (2\pi)^{-\frac{1}{N^2}-1}\Gamma\left(1+\frac{1}{N}\right)^{-2/N} e^{\frac{N-\gamma+1}{N^2}+2\psi^{(-2)}\left(1+\frac{1}{N}\right)}.
\end{aligned}
\tag{4.55}
$$

Again, we see that $C_{0N} = G_{0N}(-i\pi)$, etc. The integral representations are well-defined for

$$
\begin{aligned}
G_{0N}(\beta): &\quad \frac{2\pi}{N} - 3\pi < \mathrm{Im}(\beta) < \pi - \frac{2\pi}{N}, \\
G'_{0N}(\beta): &\quad -3\pi + \frac{\pi}{N} < \mathrm{Im}(\beta) < \pi - \frac{\pi}{N} \quad \text{and} \quad N > 1, \\
G''_{0N}(\beta): &\quad -3\pi < \mathrm{Im}(\beta) < \pi \quad \text{and} \quad N > 1.
\end{aligned}
\tag{4.56}
$$

The remaining functions can be computed exactly. The first set of these is

$$
\begin{aligned}
G_{jj}(\beta) &= \frac{i\,4^{\frac{1}{N}+1} e^{\frac{2\gamma}{N}}\pi^{-\frac{N+2}{N}}\sinh\left(\frac{\beta}{2}\right)}{\Gamma\left(-\frac{i\beta}{2\pi}-\frac{1}{N}+1\right)\Gamma\left(\frac{i\beta}{2\pi}-\frac{1}{N}\right)}, \\
G'_{jj}(\beta) &= \frac{1}{2\left(\cos\left(\frac{\pi}{N}\right)-\cosh(\beta)\right)}, \\
G''_{jj}(\beta) &= \frac{i\,2e^{-\frac{2\gamma}{N}}(2\pi)^{\frac{N-2}{N}}\sinh\left(\frac{\beta}{2}\right)}{\Gamma\left(-\frac{i\beta}{2\pi}+\frac{1}{N}+1\right)\Gamma\left(\frac{i\beta}{2\pi}+\frac{1}{N}\right)},
\end{aligned}
\tag{4.57}
$$

where the integral representations are well-defined for

$$
\begin{aligned}
G_{jj}(\beta): &\quad \frac{2\pi}{N} - 2\pi < \mathrm{Im}(\beta) < -\frac{2\pi}{N} \quad \text{and} \quad N > 2, \\
G'_{jj}(\beta): &\quad \frac{\pi}{N} - 2\pi < \mathrm{Im}(\beta) < -\frac{\pi}{N} \quad \text{and} \quad N > 1, \\
G''_{jj}(\beta): &\quad -2\pi < \mathrm{Im}(\beta) < 0 \quad \text{and} \quad N > 0.
\end{aligned}
\tag{4.58}
$$

Finally, we have

$$
\begin{aligned}
G_{j,j+1}(\beta) &= e^{-\frac{\gamma}{N}}(2\pi)^{-\frac{N+1}{N}}\Gamma\left(-\frac{i\beta}{2\pi}-\frac{1}{2N}+1\right)\Gamma\left(\frac{i\beta}{2\pi}-\frac{1}{2N}\right), \\
G'_{j,j+1}(\beta) &= 2i\sinh\left(\frac{\beta}{2}\right), \\
G''_{j,j+1}(\beta) &= e^{\gamma/N}(2\pi)^{\frac{1}{N}-1}\Gamma\left(-\frac{i\beta}{2\pi}+\frac{1}{2N}+1\right)\Gamma\left(\frac{i\beta}{2\pi}+\frac{1}{2N}\right),
\end{aligned}
\tag{4.59}
$$

where the integral representations are well-defined for

$$
\begin{aligned}
G_{j,j+1}(\beta):&\quad \frac{\pi}{N}-2\pi<\mathrm{Im}(\beta)<-\frac{\pi}{N} \quad\text{and}\quad N>1\,, \\
G'_{j,j+1}(\beta):&\quad -2\pi<\mathrm{Im}(\beta)<0 \quad\text{and}\quad N>0\,, \\
G''_{j,j+1}(\beta):&\quad -\frac{\pi}{N}-2\pi<\mathrm{Im}(\beta)<\frac{\pi}{N} \quad\text{and}\quad N>0\,.
\end{aligned}
\tag{4.60}
$$

From these explicit expressions, we can see that the functions satisfy the following identities

$$
G_{00}(\beta-i\mathfrak{u}_1)G_{0N}(\beta+i\mathfrak{u}_{N-1})=\frac{1}{G_{01}(\beta)}=G_{00}(\beta+i\mathfrak{u}_1)G_{0N}(\beta-i\mathfrak{u}_{N-1})\,, \tag{4.61}
$$

$$
G_{00}(\beta+i\mathfrak{u}_{N-1})G_{0N}(\beta-i\mathfrak{u}_1)=\frac{1}{G_{01}(\beta+i\mathfrak{u}_{N-2})}\,, \tag{4.62}
$$

$$
G_{00}(\beta-i\mathfrak{u}_{N-1})G_{0N}(\beta+i\mathfrak{u}_1)=\frac{1}{G_{01}(\beta-i\mathfrak{u}_{N-2})}\,, \tag{4.63}
$$

and G' and G'' functions obey the same identities too.

4.2.4 Traces of $V_{1\dots r}V_{1\dots s}$ and functions $G_{1\dots r;1\dots s}$

To compute the traces involving fused operators, we use (4.36) and (4.38) which takes the form

$$
G_{1\dots r;1\dots s}(\beta_2-\beta_1)=\exp\big(-\langle\langle\phi_{1\dots r}(\beta_1)\phi_{1\dots s}(\beta_2)\rangle\rangle\big)\,, \tag{4.64}
$$

and therefore we can write them in the convenient form

$$
G_{1\dots r;1\dots s}(\beta)=\exp\Big(-\int_0^\infty\frac{dt}{t}\frac{f_{1\dots r;1\dots s}(t)}{\sinh\pi t}\cos\left(\beta+i\pi\right)t\Big)\,. \tag{4.65}
$$

Unlike in the previous section, we do not extract the constant and leave the integral representation in its original form. The trace over two fused operators results in

$$\begin{aligned}
G_{1\ldots r;1\ldots s}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t r}{N}\sinh\frac{\pi t(N-s)}{N}}{\sinh^2\pi t\sinh\frac{\pi t}{N}}e^{\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),\\
G'_{1\ldots r;1\ldots s}(\beta) &= \exp\left(\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t r}{N}\sinh\frac{\pi t(N-s)}{N}}{\sinh^2\pi t\sinh\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),\\
G''_{1\ldots r;1\ldots s}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t r}{N}\sinh\frac{\pi t(N-s)}{N}}{\sinh^2\pi t\sinh\frac{\pi t}{N}}e^{-\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),
\end{aligned} \tag{4.66}$$

for $\quad s > r\,.$

The ranges for which these integral representation are well-defined is as follows ($N > 1$):

$$\begin{aligned}
G_{1\ldots r;1\ldots s}(\beta):&\quad -2\pi-\frac{(r-s)\pi}{N}<\mathrm{Im}(\beta)<\frac{(r-s)\pi}{N},\\
G'_{1\ldots r;1\ldots s}(\beta):&\quad -2\pi-\frac{\pi}{N}-\frac{(r-s)\pi}{N}<\mathrm{Im}(\beta)<\frac{\pi}{N}+\frac{(r-s)\pi}{N},\\
G''_{1\ldots r;1\ldots s}(\beta):&\quad -2\pi-\frac{2\pi}{N}-\frac{(r-s)\pi}{N}<\mathrm{Im}(\beta)<\frac{2\pi}{N}+\frac{(r-s)\pi}{N}.
\end{aligned} \tag{4.67}$$

Note that these formulae include the cases $r = 1$ and $s = 1$, which gives the highest weight particle of rank 1. Some particular cases of interest are

$$\begin{aligned}
G_{0;1\ldots s}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t(N-s)}{N}}{\sinh^2\pi t}e^{\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),\\
G'_{0;1\ldots s}(\beta) &= \exp\left(\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t(N-s)}{N}}{\sinh^2\pi t}\cos(\beta+i\pi)t\right),\\
G''_{0;1\ldots s}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t(N-s)}{N}}{\sinh^2\pi t}e^{-\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),
\end{aligned} \tag{4.68}$$

for $r = 1$ which gives the elementary particle with index 0 and

$$\begin{aligned}
G_{1\ldots r;N}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t r}{N}}{\sinh^2\pi t}e^{\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),\\
G'_{1\ldots r;N}(\beta) &= \exp\left(\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t r}{N}}{\sinh^2\pi t}\cos(\beta+i\pi)t\right),\\
G''_{1\ldots r;N}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{\pi t r}{N}}{\sinh^2\pi t}e^{-\frac{\pi t}{N}}\cos(\beta+i\pi)t\right),
\end{aligned} \tag{4.69}$$

for $s = N-1$ which gives the the fused rank $N-1$ particle, which takes index N. For the remaining traces of fused particles with V_j, we find

$$\begin{aligned} G_{r;1\ldots r}(\beta) &= \exp\left(\int_0^\infty \frac{dt}{t}\frac{1}{\sinh \pi t} e^{\frac{\pi t}{N}} \cos(\beta + i\pi)t\right), \\ G'_{r;1\ldots r}(\beta) &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{1}{\sinh \pi t} \cos(\beta + i\pi)t\right), \\ G''_{r;1\ldots r}(\beta) &= \exp\left(\int_0^\infty \frac{dt}{t}\frac{1}{\sinh \pi t} e^{-\frac{\pi t}{N}} \cos(\beta + i\pi)t\right). \end{aligned} \tag{4.70}$$

Remarkably, we find a relationship with the functions $G_{j,j+1}$:

$$\begin{aligned} G_{r;1\ldots r}(\beta) &= G_{r,r+1}(\beta), \\ G'_{r;1\ldots r}(\beta) &= G'_{r,r+1}(\beta), \\ G''_{r;1\ldots r}(\beta) &= G''_{r,r+1}(\beta). \end{aligned} \tag{4.71}$$

4.3 Form factors of the current operators

It is clear from the $SU(2)$ result [11] that a linear combination of the operators $\Lambda^{\mathcal{P}}_{j_1 j_2 \cdots j_N}(\alpha)$ should generate form factors of the current operators $J_i^{\pm k}$. The $\mathfrak{su}(N)$ symmetry obviously tells us that it should be proportional to a linear combination of $\epsilon^{j_1 \ldots j_{N-1} k} \Lambda^{\mathcal{P}}_{j_1 \ldots i \ldots j_{N-1}}(\alpha)$ where one inserts the index i in the sequence $j_1 \ldots j_{N-1}$ at some position λ. According to the discussion in section 3.2 there are three relevant types of permutations, and therefore three types of operators to be considered. The first type is

$$\Lambda^{\mathcal{P}_\lambda}{}^i_k(\alpha) = \frac{\Gamma' \epsilon_{j_1 \ldots j_{N-1} k}}{(\lambda-1)!(N-\lambda)!} A'_{i j_1 \cdots j_{\lambda-1}}(\alpha + i\mathfrak{u}_{N-\lambda}) A'_{j_\lambda \cdots j_{N-1}}(\alpha - i\mathfrak{u}_\lambda), \tag{4.72}$$

where $\mathcal{P}_\lambda \equiv (N-\lambda+1, \ldots, N, 1, \ldots, N-\lambda)$ and $\lambda = 1, \ldots, N-1$. Then

$$\Lambda^{\mathcal{P}_{\lambda-1}}{}^i_k(\alpha) = \frac{\Gamma' \epsilon_{j_1 \ldots j_{N-1} k}}{(\lambda-1)!(N-\lambda)!} A'_{j_1 \cdots j_{\lambda-1}}(\alpha + i\mathfrak{u}_{N-\lambda+1}) A'_{j_\lambda \cdots j_{N-1} i}(\alpha - i\mathfrak{u}_{\lambda-1}), \tag{4.73}$$

where $\mathcal{P}_{\lambda-1} \equiv (N-\lambda+2, \ldots, N, 1, \ldots, N-\lambda+1)$ and $\lambda = 2, \ldots, N$. Finally

$$\begin{aligned} \Lambda^{\mathcal{P}_{\lambda-1,1}}{}^i_k(\alpha) = \frac{\Gamma' \epsilon_{j_1 \ldots j_{N-1} k}}{(\lambda-1)!(N-\lambda)!} & A'_{j_1 \cdots j_{\lambda-1}}(\alpha + i\mathfrak{u}_{N-\lambda+1}) \\ & \times A'_i(\alpha + i\mathfrak{u}_{N-2\lambda-1}) A'_{j_\lambda \cdots j_{N-1}}(\alpha - i\mathfrak{u}_\lambda), \end{aligned} \tag{4.74}$$

where $\mathcal{P}_{\lambda-1,1} \equiv (N-\lambda+2,\ldots,N,N-\lambda+1,1,\ldots,N-\lambda)$ and $\lambda = 2,\ldots,N-1$. Strictly speaking one should consider only the traceless parts of the operators.

We propose that form factors of the current operators $J_k^{\pm i}$ are generated either by

$$\Lambda^{\mathcal{P}_1}{}^i_k(\alpha) = \frac{\Gamma' \, \epsilon_{j_1\ldots j_{N-1}k}}{(N-1)!} A'_i(\alpha + i\mathrm{u}_{N-1}) A'_{j_1\ldots j_{N-1}}(\alpha - i\mathrm{u}_1)\,, \tag{4.75}$$

or by

$$\Lambda^{\mathcal{P}_{N-1}}{}^i_k(\alpha) = \frac{\Gamma' \, \epsilon_{j_1\ldots j_{N-1}k}}{(N-1)!} A'_{j_1\ldots j_{N-1}}(\alpha + i\mathrm{u}_1) A'_i(\alpha - i\mathrm{u}_{N-1})\,. \tag{4.76}$$

All operators of these types can be obtained from the relevant highest weight operators

$$\begin{aligned} \Lambda^{\mathcal{P}_1}{}^1_N(\alpha) &= \Gamma' A'_1(\alpha + i\mathrm{u}_{N-1}) A'_{1\ldots N-1}(\alpha - i\mathrm{u}_1) \\ &= D_{N,N-1}\Gamma' A'_1(\alpha + i\mathrm{u}_{N-1}) \bar{A}'_N(\alpha - i\mathrm{u}_1)\,, \end{aligned} \tag{4.77}$$

and

$$\begin{aligned} \Lambda^{\mathcal{P}_{N-1}}{}^1_N(\alpha) &= \Gamma' A'_{1\ldots N-1}(\alpha + i\mathrm{u}_1) A'_1(\alpha - i\mathrm{u}_{N-1}) \\ &= D_{N,N-1}\Gamma' \bar{A}'_N(\alpha + i\mathrm{u}_1) A'_1(\alpha - i\mathrm{u}_{N-1})\,, \end{aligned} \tag{4.78}$$

by acting on them with the lowering $\mathcal{J}_k^-$ operators.

Computing the simplest nontrivial form factors generated by these operators one finds

$$\begin{aligned} \mathcal{F}^{\mathcal{P}_1}(\alpha|\theta_1,\theta_2) \;&\equiv\; \langle\langle \Lambda^{\mathcal{P}_1}{}^1_N(\alpha)\, \bar{A}_N(\theta_2) A_1(\theta_1)\rangle\rangle \\ &=\; (-1)^{N-1} D_{N,N-1} C_{N,N-1} \\ &\quad\; \times \langle\langle V'_0(\alpha + i\mathrm{u}_{N-1}) V'_N(\alpha - i\mathrm{u}_1)\, V_N(\theta_2) V_0(\theta_1)\rangle\rangle \\ &=\; (-1)^{N-1} D_{N,N-1} C_{N,N-1} C'_0 C'_N C_0 C_N \\ &\quad\; \times G''_{0N}(-i\pi) G_{0N}(\theta_1 - \theta_2) \\ &\quad\; \times G'_{00}(\theta_1 - \alpha - i\mathrm{u}_{N-1}) G'_{0N}(\theta_1 - \alpha + i\mathrm{u}_1) \\ &\quad\; \times G'_{00}(\theta_2 - \alpha + i\mathrm{u}_1) G'_{0N}(\theta_2 - \alpha - i\mathrm{u}_{N-1})\,. \end{aligned} \tag{4.79}$$

Taking into account the identities (4.61-4.63) and (4.59), one obtains

$$\mathcal{F}^{\mathcal{P}_1}(\alpha|\theta_1,\theta_2) = (-1)^N \mathcal{N}_{\mathcal{F}_1} \frac{G_{0N}(\theta_1-\theta_2)}{4\sinh\frac{1}{2}(\theta_1-\alpha-iu_{N-2})\sinh\frac{1}{2}(\theta_2-\alpha)}, \tag{4.80}$$

where

$$\mathcal{N}_{\mathcal{F}_1} \equiv D_{N,N-1}C_{N,N-1}C_0'C_N'C_0C_NG_{0N}''(-i\pi). \tag{4.81}$$

According to [11] form factors are generated by expanding $\mathcal{F}^{\mathcal{P}_1}$ in powers of $e^{\pm\alpha}$

$$\begin{aligned}\mathcal{F}^{\mathcal{P}_1}(\alpha|\theta_1,\theta_2) &= \sum_{s=1}^{\infty} e^{-s\alpha}\,\mathbb{F}_s^{\mathcal{P}_1}(\theta_1,\theta_2),\\ \mathcal{F}^{\mathcal{P}_1}(\alpha|\theta_1,\theta_2) &= \sum_{s=-\infty}^{-1} e^{-s\alpha}\,\mathbb{F}_s^{\mathcal{P}_1}(\theta_1,\theta_2).\end{aligned} \tag{4.82}$$

This expansion is necessary, because as mentioned in section 3.2.3, the operator representations that give rise to the generating functions (4.79) produce representations of operators that do not satisfy the first equation in (1.54). However, each individual term in the expansion satisfies the condition due the fact that (2.63) can be extended to primed fields and hence to the operators constructed from these fields, and also the fact that the expansion contains exponential terms $e^{-s\alpha}$ rather than the sinh functions that appear in (4.80). As a result, the operator for a given spin s, which comes from the expansion (4.82) satisfies the condition. Then the form factors of the components $J_N^{\pm 1}$ of the current operators are proportional to $\mathbb{F}_{\pm 1}^{\mathcal{P}_1}(\theta_1,\theta_2)$, respectively. Explicitly one finds

$$\mathbb{F}_s^{\mathcal{P}_1}(\theta_1,\theta_2) = (-1)^N \mathcal{N}_{\mathcal{F}_1} e^{-s\frac{i\pi(N-2)}{2N}} e^{s\frac{\theta_1+\theta_2}{2}} G_{0N}(\theta_1-\theta_2), \quad s=\pm 1, \tag{4.83}$$

which up to a constant is the same as in [3, 10].

On the other hand the operator $\Lambda^{\mathcal{P}_{N-1}}{}_N^1(\alpha)$ leads to

$$\begin{aligned}\mathcal{F}^{\mathcal{P}_{N-1}}(\alpha|\theta_1,\theta_2) &\equiv \langle\langle \Lambda^{\mathcal{P}_{N-1}}{}_N^1(\alpha)\,\bar{A}_N(\theta_2)A_1(\theta_1)\rangle\rangle \\ &= \mathcal{N}_{\mathcal{F}_1}\frac{G_{0N}(\theta_1-\theta_2)}{4\sinh\frac{1}{2}(\theta_1-\alpha+iu_{N-2})\sinh\frac{1}{2}(\theta_2-\alpha)},\end{aligned} \tag{4.84}$$

and its expansion in powers of $e^{\pm\alpha}$ produces

$$\mathbb{F}_s^{\mathcal{P}_{N-1}}(\theta_1,\theta_2) = \mathcal{N}_{\mathcal{F}_1} e^{s\frac{i\pi(N-2)}{2N}} e^{s\frac{\theta_1+\theta_2}{2}} G_{0N}(\theta_1-\theta_2)\,, \quad s=\pm 1\,, \tag{4.85}$$

which up to a constant agrees with (4.83). It is thus reasonable to expect that both $\Lambda^{\mathcal{P}_1}{}^1_N$ and $\Lambda^{\mathcal{P}_{N-1}}{}^1_N$ generate form factors of the current operators $J_N^{\pm 1}$. This is the key result for this thesis and [28], since the agreement with the findings of the authors of [3, 10] confirms that the free field representation produces the correct form factors. The constant that appears is the result of a combination of normal-ordering, traces over single vertex operators, and a contribution from taking the trace of the two vertex operators that appear in $\Lambda^{\mathcal{P}_1}{}^1_N(\alpha)$. However, this is not guaranteed to be the correct normalisation for the form factor since the form factor axioms provide no way to determine this. Other methods are necessary to calculate the normalisation.

Let us also mention that the SU(N) symmetry of the model allows one to express the traces of any operator $\Lambda^{\mathcal{P}_1}{}^i_k(\alpha)$ (or $\Lambda^{\mathcal{P}_{N-1}}{}^i_k(\alpha)$) in terms of those of the highest weight operator $\Lambda^{\mathcal{P}_1}{}^1_N(\alpha)$. In particular one gets the following formula for the functions generating the particle-antiparticle form factors

$$\langle\langle \Lambda^{\mathcal{P}_1}{}^i_k(\alpha)\, \bar{A}_l(\theta_2) A_j(\theta_1)\rangle\rangle = \delta_{ij}\delta_{kl}\, \mathcal{F}^{\mathcal{P}_1}(\alpha|\theta_1,\theta_2)\,. \tag{4.86}$$

The SU(N) symmetry of the traces of operators $\Lambda^{\mathcal{V}}(\theta'_k,\ldots,\theta'_1)$ with the ZF operators follows from the fact that the identities (3.38) also hold under the traces of products of these operators. Indeed, concentrating for definiteness on the first identity in (3.38), one can see that it is sufficient to show that under the trace

$$\big[[\chi^+_{j+1}\,,\, \chi^-_{j+1}], V_j(\theta)\big] \simeq V_j(\theta)\,. \tag{4.87}$$

The relevant part in the trace comes from

$$\langle\langle e^{2\pi i K}\Big(\prod_{\mu=j}^{j+2}\prod_{k=1}^{n'_\mu} V'_\mu(\theta'_{\mu,k})\Big)\Big(\prod_{\mu=j}^{j+2}\prod_{k=1}^{n_\mu-\delta_{\mu j}} V_\mu(\theta_{\mu,k})\Big)V_j(\theta)\rangle\rangle\,, \tag{4.88}$$

where we assume without loss of generality that $V_j(\theta)$ is located to the right of all the other V_k's. Then, replacing it with the left hand side of (4.87), one gets that the following identity should hold

$$I_+ + I_- = 1\,, \tag{4.89}$$

where

$$I_\pm = \int_{C_\pm} d\alpha\, \mathcal{I}_\pm\,, \tag{4.90}$$

with

$$\begin{aligned}
\mathcal{I}_\pm &= \pm\rho_\chi^2 C_1 C_1' R_\pm \\
&\times \Big(G'_{j,j+1}(\theta - \alpha \mp \frac{i\pi}{N}) G_{j,j+1}(\theta - \alpha) \\
&\quad - G'_{j,j+1}(\alpha \pm \frac{i\pi}{N} - \theta) G_{j,j+1}(\alpha - \theta)\Big) \\
&\times \prod_{\mu=j}^{j+2} \prod_{k=1}^{n'_\mu} G''_{\mu,j+1}(\alpha \pm \frac{i\pi}{N} - \theta'_{\mu,k}) G'_{\mu,j+1}(\alpha - \theta'_{\mu,k}) \\
&\times \prod_{\mu=j}^{j+2} \prod_{k=1}^{n_\mu - \delta_{\mu j}} G'_{\mu,j+1}(\alpha \pm \frac{i\pi}{N} - \theta_{\mu,k}) G_{\mu,j+1}(\alpha - \theta_{\mu,k})\,,
\end{aligned} \tag{4.91}$$

and

$$R_\pm \equiv \mp 2\pi i \,\mathrm{Res}\, G'_{j+1,j+1}(\alpha - \alpha')|_{\alpha' = \alpha \pm \frac{i\pi}{N}} = \frac{\pi}{\sin\frac{\pi}{N}}\,. \tag{4.92}$$

The constants C_1 and C_1' are given by (4.43) and (4.46). The integration contour C_+ runs above $\theta + \frac{i\pi}{N}$ and below $\theta_{\mu.k} - \frac{i\pi}{N}$, while C_- runs below $\theta - \frac{i\pi}{N}$ and $\theta_{\mu,k} - \frac{i\pi}{N}$. It is not difficult to check that all the poles of the integrand $\mathcal{I}_+$ lie below C_+, while all the poles in $\mathcal{I}_-$ lie above C_-. Moreover, if n'_μ and n_μ satisfy the selection rules, then one finds that at large α

$$\mathcal{I}_\pm \to \mp\frac{1}{2\pi i\, \alpha}\,, \quad \pm\mathrm{Im}(\alpha) > 0\,, \quad \alpha \to \infty\,, \tag{4.93}$$

and therefore the principal value prescription gives

$$I_\pm = \frac{1}{2}\,, \tag{4.94}$$

as required. This completes the proof of the SU(N) symmetry of the traces of operators $\Lambda^{\nu}(\theta'_k, \ldots, \theta'_1)$.

Chapter 5

Looking Ahead

One of the issues not addressed by the previous considerations is how to use the free field representation on a model that has an algebra structure that is the direct sum of two Lie algebras. The only previously known bosonisation for such a situation is for the two-parameter family of integrable models [18]. Unfortunately, this is very model specific and so would have to be redesigned in order to apply to another model. Below, we outline some of the issues that these models give rise to, with the examples of the Principal Chiral Field (PCF) model and $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring sigma model.

5.1 Models with underlying direct sum Lie algebra structure

Before discussing the problems encountered in these models, first let us give the S-matrices of the models.

5.1.1 Principal Chiral Field Model

The S-matrix of the $SU(N)$ Principal Chiral Field model is given by

$$\hat{S}_{12}(\theta) = \chi_{CDD}(\theta) \cdot S_0(\theta) \frac{\hat{R}(\theta)}{\theta - \frac{2\pi i}{N}} \otimes S_0(\theta) \frac{\hat{R}(\theta)}{\theta - \frac{2\pi i}{N}} , \tag{5.1}$$

$$S_0(\theta) = -\frac{\Gamma\left(\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{N} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{N} + \frac{i\theta}{2\pi}\right)}, \qquad \chi_{CDD}(\theta) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{N}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{N}\right)}, \tag{5.2}$$

where the standard $SU(N)$ R-matrix is $\hat{R}(\theta) = \theta - \frac{2\pi i}{N}\hat{P}$ and $\hat{P}$ is the permutation operator which exchanges the spins of the scattering particles. The details of the S-matrix properties and structure can be found in [13], [30], and [42]. Note that in the case $N = 2$, the scalar CDD factor is simply -1. As a result, the $SU(2)$ case has the S-matrix,

$$\hat{S}_{12}(\theta) = -S_0(\theta)\frac{\hat{R}(\theta)}{\theta - \frac{2\pi i}{2}} \otimes S_0(\theta)\frac{\hat{R}(\theta)}{\theta - \frac{2\pi i}{2}} = -S^L(\theta) \otimes S^R(\theta), \tag{5.3}$$

$$S_0(\theta) = \frac{\Gamma\left(\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right)}. \tag{5.4}$$

The PCF S-matrix contains two copies of the GN S-matrix in a tensor product, with the addition of a CDD factor. We also note that the CDD factor may also be written as

$$\chi_{CDD} = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{N}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{N}\right)} = \frac{\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{N}\right)\Gamma\left(1 - \frac{1}{N} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{i\theta}{2\pi} - \frac{1}{N}\right)\Gamma\left(1 + \frac{1}{N} - \frac{i\theta}{2\pi}\right)}. \tag{5.5}$$

5.1.2 $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

The $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring sigma model is conjectured to be integrable. As a result it is valid to attempt to treat it in the same manner as the GN model and the PCF model. The most striking difference is that there is no Lorentz invariance, and hence some approaches would need to be generalised. This should not, however, prove too much of a problem as there is the concept of generalised rapidities, see [25] for details, through which all formulae should be expressible. Also, [26] has some generalisations for form factors in terms of these generalised rapidities. For more details of the model, see [25]. In addition, some basic information to complement what appears here is presented in appendix H. Below, the generalised rapidity variables are represented by z_i. The most convenient sector of the theory to consider is the $\mathfrak{su}(2)$ sector, although ultimately all sectors will need to considered. The S-matrix of the $\mathfrak{su}(2)$ sector is given by

$$S(z_1, z_2) = \frac{x_1^+ \, x_2^-}{x_1^- \, x_2^+} \frac{1}{\sigma(x_1^\pm, x_2^\pm)^2} \frac{u_1 - u_2 - \frac{2i}{g}}{u_1 - u_2 + \frac{2i}{g}}, \tag{5.6}$$

where $\sigma(x_1^\pm, x_2^\pm)$ is the dressing phase and the spectral parameters u_k are expressed in terms of $x_k^\pm$ as follows

$$u_k = \frac{1}{2}\left(x_k^+ + \frac{1}{x_k^+} + x_k^- + \frac{1}{x_k^-}\right) = x_k^+ + \frac{1}{x_k^+} - \frac{i}{g} = x_k^- + \frac{1}{x_k^-} + \frac{i}{g}, \tag{5.7}$$

with the x parameters defined by

$$\frac{x^+}{x^-} = e^{ip}, \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}, \tag{5.8}$$

where g is the string tension and p is the momentum of the particle state. We also use the notation

$$x_i^+ \equiv x^+(z_i), \qquad x_i^- \equiv x^-(z_i). \tag{5.9}$$

In terms of the u-parameters the last term in (5.6) is the same as the S-matrix of the Heisenberg spin chain. It exhibits a pole at $u_1 - u_2 = -\frac{2i}{g}$ which corresponds to a bound state of two fundamental particles from the $\mathfrak{su}(2)$ sector.

5.1.3 A naive approach

When considering the PCF model, having noted that it contains two copies of the GN model, we might be tempted to try to build up the free field representation by writing two free field representations of the GN model, and then find a separate free field representation for the CDD factor to build up the algebra in the manner

$$Z_{i\otimes j}(\theta) = Z_i^L(\theta) Z_j^R(\theta) V_{CDD}(\theta). \tag{5.10}$$

While this can be done in such a way as to satisfy the Zamolodchikov-Faddeev algebra relations for the highest weight state, this approach will not ultimately be successful. As was mentioned in section 2.1 in a footnote to equation (2.16), the ansatz for the lowering operators defined in (2.16) used for the two GN copies will not work. It might similarly be tempting in the $AdS_5 \times S^5$ case to try to break up the S-matrix and apply the free field representation procedure to each part to make things easier, but this is unlikely to work. Despite this, it is relatively easy to guess a form of the function g_{00} that would at least allow the highest weight Zamolodchikov-Faddeev algebra to be satisfied. For the PCF model, we could supplement the two GN model copies of g_{00} with

$$g_{CDD}(\theta) = \frac{2\pi}{\Gamma\left(\frac{1}{2}+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right)} = 2\cosh\left(\frac{\theta}{2}\right), \tag{5.11}$$

which would seem to work until we try to find the lowering operators. Simliarly, for the $AdS_5 \times S^5$ model, we could try to find the g-function for the entire S-matrix (5.6) in one go and arrive at

$$g(z_1, z_2) = \sqrt{\frac{x_1^- \, x_2^+}{x_1^+ \, x_2^-}}\, \sigma(x_1^\pm, x_2^\pm)\, \frac{u_1 - u_2}{u_1 - u_2 - \frac{2i}{g}}, \tag{5.12}$$

which has the nice properties:

- If $z_1 = z_2 = z$ then $u_1 = u_2$ and $g(z,z) = 0$. In other words it has a simple zero at $z_2 - z_1 = 0$ or $u_2 - u_1 = 0$.
- It has no zeroes or poles for $\mathrm{Im}(u_1 - u_2) \leq 0$ except the zero mentioned above.

However, as for the PCF case, when we consider the lowering operators, we run into difficulties.

5.1.4 The problems

As mentioned in the previous section, satisfying the Zamolodchikov-Faddeev algebra is relatively easy, even in the case of a model for the direct sum of two Lie algebras. The much more difficult problem is satisfying the residue condition

$$i\, Z_A(\theta_2) Z_B(\theta_1) = \frac{C_{AB}}{\theta_2 - \theta_1 - i\pi} + \ldots. \tag{5.13}$$

This will only be satisfied with an appropriate choice of lowering operators, which is why the previous comments on the lowering operators are important. If we consider for a moment the PCF model, we realise that the two copies of the GN model can both be solved in such a way as to provide a residue condition consistent with (5.13) for each. However, attempting to remove one of these poles via the CDD factor will not work, because the conditions on the g-function would not allow a zero in the upper half plane if we wish to remain consistent with the requirements of section 2.1. Therefore, in order to make progress here, or indeed in the $AdS_5 \times S^5$ case, we would need to treat the entire S-matrix in one go, rather than try to build it up from smaller blocks. This is obviously a very challenging problem to address, and is worthy of further investigation. Hopefully, if the correct free field representation can be found, the process of finding the form factors should follow the one given here in a straightforward manner.

5.2 Conclusion

The goal was to develop the free field representation of the chiral SU(N) Gross-Neveu model. This was done by constructing vertex operators of the fundamental particles and bound states. These vertex operators contain up to $N - 1$ integrals that result from the action of the lowering operators on the highest weight operator. Despite these integrals, it can be shown that the Zamolodchikov-Faddeev algebra relations are satisfied and hence these operators are Zamolodchikov-Faddeev operators. Although applied here to the Gross-Neveu model, the general approach advocated in section 2.1 should provide an outline of how to proceed for other models. It should be applicable to any two dimensional integrable model invariant under a simple Lie algebra. The application of this general method in section 2.2 to the Gross-Neveu model shows explicitly how this method works.

Using similar methods, we also constructed a large class of operators, Λ, commuting with the Zamolodchikov-Faddeev operators. These are constructed from a related set of vertex operators. The representations of the particles, bound states and operators are then used to construct generating functions of the form factors of local operators through the trace formula. In particular, we proposed two operators Λ which generate the form factors of the current operator. This matched previous results [9] and [10], where the off-shell Bethe ansatz approach was used instead. This of course provides support for the success of the free field representation found. Finding the correct operator representation that will give rise to generating functions that contain form factors of the stress-energy tensor remains an open problem. It would also be interesting to try to establish how the free field approach is related to the off-shell Bethe ansatz approach. It is not unreasonable to suspect that there may be a deep connection between the two. Obviously, the resulting form factors are the same (as they must be), so there should be some way to understand the correlation between the methods.

In light of the free field approach that was advocated here, and as expanded upon earlier, further areas of interest might include finding the free field representation of the SU(N) Principal Chiral Field model, see section 5.1, which is closely related to the Gross-Neveu model, and also applying the approach to the $AdS_5 \times S^5$ superstring sigma model in the light-cone gauge. For the Principal Chiral Field model, the main complication is that the model is invariant under the direct sum of two Lie algebras and the ansatz (2.16) for the lowering symmetry operators used in the thesis should be modified: a problem that is not immediately obvious how to resolve. This is an area that might be very fruitful to explore, since understanding how to deal with this algebra structure would greatly increase the applicability of the free field representation. As is the case for the Gross-Neveu model, we would like to be able to identify the form factors of the current operator and the stress-energy tensor for the Principal Chiral Field model.

Finally, as also mentioned in section 5.1, similar methods should allow the free field Zamolodchikov-Faddeev algebra representation of the $AdS_5 \times S^5$ superstring sigma model in the light-cone gauge to be developed. In this case, we would want to identify the operators Λ corresponding to the target space fields. As mentioned earlier, finding form factors for this model is complicated by the fact that their analytic properties are not known. Since the free field realisation does not require a full understanding of these properties, it is hoped that this approach may be able to shed some light on these form factors and their properties.

Appendix A

Proof of the form factor axioms

Here we give outlines of proofs that the traces in the main text satisfy the form factor axioms given in section 1.3. Axiom 6 is an immediate consequence of the functions that we find when computing the traces in section 4.2. Since these functions are analytic except for simple poles, the axiom follows. The remaining axioms are discussed in more detail.

A.1 Axiom 1: Watson's theorem

This axiom, (1.44), can be proven directly from the Zamolodchikov-Faddeev algebra since the S-matrix terms are scalar and will not be affected by the action of the trace. It is also possible to check that

$$S(\theta) = S_{00}^{00}(\theta) = \frac{G_{00}(-\theta)}{G_{00}(\theta)}, \tag{A.1.1}$$

which directly checks that for the highest weight states Watson's theorem is satisfied. Numerical checks can also be carried out for other states that incorporate lowering operators. Note that unlike the case for Green's functions, we cannot perform these integrations analytically using the principal value prescription, so numerical methods are the only way to perform these checks.

A.2 Axiom 2: Double crossing

This axiom, (1.45), follows easily from the properties of the traces in section 4.2 and from knowledge of the mutual locality index, $\Omega(\mathcal{O}, \Psi)$. We can show it directly from the trace formula,

$$\begin{aligned}
&F_{a_1,\cdots,a_n}(\theta_1,\cdots,\theta_n+2\pi i) \\
&\quad= \mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K}\Lambda(O)Z_{a_n}(\theta_n+2\pi i)\cdots Z_{a_1}(\theta_1)\right] \\
&\quad= e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K}Z_{a_n}(\theta_n+2\pi i)\Lambda(O)\cdots Z_{a_1}(\theta_1)\right] \\
&\quad= e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathrm{Tr}_{\pi_Z}\left[Z_{a_n}(\theta_n)e^{2\pi i K}\Lambda(O)Z_{a_{n-1}}(\theta_{n-1})\cdots Z_{a_1}(\theta_1)\right] \\
&\quad= e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K}\Lambda(O)Z_{a_{n-1}}(\theta_{n-1})\cdots Z_{a_1}(\theta_1)Z_{a_n}(\theta_n)\right] \\
&\quad= e^{2\pi i\Omega(\mathcal{O},\Psi)}F_{a_n,a_1,\cdots,a_{n-1}}(\theta_n,\theta_1,\cdots,\theta_{n-1}),
\end{aligned} \tag{A.2.1}$$

using (1.54), the fact[1] that $Z_i(\theta+\zeta) = e^{-\zeta K}Z_i(\theta)e^{\zeta K}$, and the cyclicity of the trace. Alternatively, we can look at the properties of the functions $G_{\mu\nu}$.

For the identity operator, this is $\Omega(\mathcal{O}, \Psi) = 0$, in which case, all we need to consider is the fact that

$$G_{\mu\nu}(\alpha - 2\pi i) = G_{\mu\nu}(-\alpha), \tag{A.2.2}$$

which ensures that for the identity operator axiom 2 holds, since the change from α to $-\alpha$ changes the order of the rapidites (and hence particles) in the form factor. It should be noted that although the form factors of the identity operator should be zero (which indeed can be checked to be true), it is nonetheless useful to check the axioms in this case since it is relatively simple to do so. The interpretation of this is that if we were to insert an operator into the form factor, we would expect the axioms to be satisfied, and therefore the part of the form factor that only depends on the particles should satisfy the axioms in the absence of the operator. It should also be noted that we can always insert the identity operator in any form factor, and we therefore need to be sure that such an insertion will not break any

[1] This can be shown by integrating (1.52) over ζ around a fixed value of θ.

of the axioms.

For the current operators, we would expect that $\Omega(\mathcal{O},\Psi) = 1/2$, see [11]. Then, considering (4.83), we see that the rapidity appears in both G_{0N} and in $\exp\left(s\frac{\theta_1+\theta_2}{2}\right)$. Since G_{0N} satisfies (A.2.2), the exponential term is the only remaining term which might be affected. Looking at what we have in this case, we only have two particles, so the axiom should appear as

$$F_{a_1,a_2}(\theta_1,\theta_2-2\pi i) = e^{\pi i}F_{a_2,a_1}(\theta_2,\theta_1) = -F_{a_2,a_1}(\theta_2,\theta_1)\,. \tag{A.2.3}$$

The exponential term in the form factor for the left hand side gives

$$\exp\left(s\frac{\theta_1+\theta_2-2\pi i}{2}\right) = \exp(si\pi)\exp\left(s\frac{\theta_1+\theta_2}{2}\right)\,, \tag{A.2.4}$$

and for the right hand side is still

$$\exp\left(s\frac{\theta_1+\theta_2}{2}\right)\,. \tag{A.2.5}$$

The only difference is the factor $\exp(si\pi) = -1$ since $s = \pm 1$, which is precisely what is required. Therefore this axiom also holds for the current operator.

A.3 Axiom 3: The residue condition

Axiom 3, which is the residue condition is given in (1.46). We recall from section 1.3 and from [11] the expression of form factors in terms of the trace,

$$F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) = \mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K}\Lambda(O)Z_{a_n}(\theta_n)\cdots Z_{a_1}(\theta_1)\right]\,. \tag{A.3.1}$$

It is convenient to use matrix notations as in [25]. Multiplying (A.3.1) by the row $E^{a_1}\otimes E^{a_2}\cdots\otimes E^{a_n}$ one gets

$$\mathbb{F}_{12\ldots n}(\theta_1,\ldots,\theta_n) = \mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K}\Lambda(O)\mathbb{Z}_n(\theta_n)\cdots\mathbb{Z}_1(\theta_1)\right]\,, \tag{A.3.2}$$

where $\mathbb{Z} = Z_i E^i$ is a row, and the subscript shows the location of $\mathbb{Z}$ in the tensor product $E^{a_1} \otimes E^{a_2} \cdots \otimes E^{a_n}$. The ZF algebra in terms of $\mathbb{Z}$ takes the form

$$\mathbb{Z}_1\mathbb{Z}_2 = \mathbb{Z}_2\mathbb{Z}_1 S_{12}\,. \tag{A.3.3}$$

Since Z_i satisfy

$$Z_i(\theta')Z_j(\theta) = -\frac{iC_{ij}}{\theta' - \theta - i\pi} + O(1)\,, \quad \theta' \to \theta + i\pi\,, \tag{A.3.4}$$

which can be written as

$$\mathbb{Z}_1(\theta')\mathbb{Z}_2(\theta) = -\frac{i\mathbb{C}_{12}}{\theta' - \theta - i\pi}\,, \quad \mathbb{C}_{12} \equiv C_{ij}E^i \otimes E^j\,, \tag{A.3.5}$$

one expects to find a pole in (A.3.2) at, e.g., $\theta_n = \theta_{n-1} + i\pi$, that is

$$\begin{aligned} &\mathbb{F}_{12\ldots n}(\theta_1, \ldots, \theta_n) \\ &\quad = \frac{\mathbb{R}_{12\ldots n-1,n}(\theta_1, \ldots, \theta_{n-2}, \theta_{n-1})}{\theta_n - \theta_{n-1} - i\pi} + O(1)\,, \quad \theta_n \to \theta_{n-1} + i\pi\,, \end{aligned} \tag{A.3.6}$$

where $\mathbb{R}_{12\ldots n-1,n}(\theta_1, \ldots, \theta_{n-1})$ is the residue of $\mathbb{F}_{12\ldots n}(\theta_1, \ldots, \theta_n)$ at $\theta_n = \theta_{n-1} + i\pi$.

Taking $\mathbb{Z}_{n-1}$ and bringing it in front of $\mathbb{Z}_n$ one gets

$$\begin{aligned} &\mathbb{F}_{12\ldots n}(\theta_1, \ldots, \theta_n) \\ &\quad = e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathrm{Tr}_{\pi_Z}\left[e^{2\pi iK}\Lambda(O)\mathbb{Z}_{n-1}(\theta_{n-1} + 2\pi i)\mathbb{Z}_n(\theta_n)\mathbb{Z}_{n-2}\cdots\mathbb{Z}_1\right] \\ &\quad \times S_{n-1,1}S_{n-1,2}\cdots S_{n-1,n-2} \\ &\quad = e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathbb{F}_{12\ldots n,n-1}(\theta_1, \ldots, \theta_n, \theta_{n-1} + 2\pi i)S_{n-1,1}S_{n-1,2}\cdots S_{n-1,n-2}\,. \end{aligned} \tag{A.3.7}$$

Then for $\theta_n \sim \theta_{n-1} + i\pi$ one gets

$$\begin{aligned} &\mathbb{F}_{12\ldots n}(\theta_1, \ldots, \theta_n) \\ &\quad = -e^{2\pi i\Omega(\mathcal{O},\Psi)}\frac{\mathbb{R}_{12\ldots n,n-1}(\theta_1, \ldots, \theta_{n-2}, \theta_n)}{\theta_n - \theta_{n-1} - i\pi}S_{n-1,1}S_{n-1,2}\cdots S_{n-1,n-2}\,. \end{aligned} \tag{A.3.8}$$

Thus the residues must satisfy the equation

$$\begin{aligned} &\mathbb{R}_{12\ldots n-1,n}(\theta_1, \ldots, \theta_{n-2}, \theta_{n-1}) \\ &\quad = -e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathbb{R}_{12\ldots n,n-1}(\theta_1, \ldots, \theta_{n-2}, \theta_n)S_{n-1,1}S_{n-1,2}\cdots S_{n-1,n-2}\,. \end{aligned} \tag{A.3.9}$$

A solution to this equation can be written in the form

$$\begin{aligned}
&\mathbb{R}_{12\ldots n-1,n}(\theta_1,\ldots,\theta_{n-2},\theta_{n-1}) \\
&\quad = r\,\mathbb{F}_{12\ldots n}(\theta_1,\ldots,\theta_{n-2})\left(\mathbb{C}_{n,n-1} - e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathbb{C}_{n-1,n}S_{n-1,1}\cdots S_{n-1,n-2}\right),
\end{aligned} \tag{A.3.10}$$

if $\Omega(\mathcal{O},\Psi)$ satisfies $e^{4\pi i\Omega(\mathcal{O},\Psi)} = 1$ and r is a constant. Let's show that (A.3.10) solves (A.3.9). We have

$$\begin{aligned}
&\mathbb{R}_{12\ldots n,n-1}(\theta_1,\ldots,\theta_{n-2},\theta_n) \\
&\quad = r\,\mathbb{F}_{12\ldots n}(\theta_1,\ldots,\theta_{n-2})\left(\mathbb{C}_{n-1,n} - e^{2\pi i\Omega(\mathcal{O},\Psi)}\mathbb{C}_{n,n-1}S_{n,1}\cdots S_{n,n-2}\right).
\end{aligned} \tag{A.3.11}$$

We need to show that

$$\mathbb{C}_{n,n-1}S_{n,1}\cdots S_{n,n-2}S_{n-1,1}S_{n-1,2}\cdots S_{n-1,n-2} = \mathbb{C}_{n,n-1}\,. \tag{A.3.12}$$

This is indeed the case because the crossing equation implies

$$\begin{aligned}
&\mathbb{C}_{n,n-1}S_{n,a}(i\pi-\theta)S_{n-1,a}(-\theta) = \mathbb{C}_{n,n-1} \\
\Leftrightarrow\quad &\mathbb{C}_{n,n-1}S_{n,a}(i\pi-\theta) = \mathbb{C}_{n,n-1}S_{n-1,a}(\theta)\,,
\end{aligned} \tag{A.3.13}$$

for any a.

A.4 Axiom 4: Bound state residue

Axiom 4 ensures that the bound state form factors are constructed from the fundamental particle form factors, (1.47). The process of checking this is very similar to the process for calculating the bound states given in appendix C. Here we give an example of the calculation that shows the pole appearing. To simplify matters, consider a three particle form factor of the unit operator (we won't worry about selection rules here since the axioms should apply anyway) of the form

$$F_{121}(\theta_1,\theta_2,\theta_3)\,, \tag{A.4.1}$$

with $\theta_1 = \theta - \frac{i\pi}{N} + \epsilon$, $\theta_2 = \theta + \frac{i\pi}{N}$. The integral contains the terms

$$G_{00}(\theta_1-\theta_2)G_{01}(\theta_1-\alpha)G^s_{01}(\alpha-\theta_2)G_{01}(\alpha-\theta_3)\,, \tag{A.4.2}$$

with a pinched pole at $\alpha = \theta_2 - \frac{i\pi}{N} = \theta_1 + \frac{i\pi}{N}$. Expanding this around $\epsilon = 0$ gives the contribution

$$-\frac{ie^{-\frac{2\gamma}{N}}(2\pi)^{1-\frac{2}{N}}\Gamma\left(1-\frac{1}{N}\right)^2}{\epsilon} G_{00}\left(-\frac{2\pi i}{N}\right) G_{01}(\theta-\theta_3)\,. \tag{A.4.3}$$

In addition, we also have the contribution from

$$\begin{aligned} G_{00}(\theta_1-\theta_3)G_{00}(\theta_2-\theta_3) &= G_{00}\left(\theta-\theta_3-\frac{i\pi}{N}\right) G_{00}\left(\theta-\theta_3+\frac{i\pi}{N}\right) \\ &= \frac{G_{0;12}(\theta-\theta_3)}{G_{jj+1}(\theta-\theta_3)}\,, \end{aligned} \tag{A.4.4}$$

which follows from the integral representation

$$\begin{aligned} &G_{00}\left(\theta-\theta_3-\frac{i\pi}{N}\right) G_{00}\left(\theta-\theta_3+\frac{i\pi}{N}\right) \\ &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-1}{N}\pi t e^{\frac{\pi t}{N}}}{\sinh \pi t^2}\left(\cos\left(\theta-\theta_3+\frac{i\pi}{N}+i\pi\right)t\right.\right. \\ &\quad \left.\left. +\cos\left(\theta-\theta_3-\frac{i\pi}{N}+i\pi\right)t\right)\right) \\ &= \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{N-2}{N}\pi t e^{\frac{\pi t}{N}}}{\sinh \pi t^2}\cos\left(\theta-\theta_3+i\pi\right)t\right. \\ &\quad \left. +\frac{e^{\frac{\pi t}{N}}}{\sinh \pi t}\cos\left(\theta-\theta_3+i\pi\right)t\right) \\ &= \frac{G_{0;12}(\theta-\theta_3)}{G_{jj+1}(\theta-\theta_3)}\,, \end{aligned} \tag{A.4.5}$$

and we recall the formula (4.70) and we also note that the integral representation of $G_{jj+1}(\theta)$ is given by

$$G_{jj+1}(\theta) = \exp\left(\int_0^\infty \frac{dt}{t}\frac{e^{\frac{\pi t}{N}}}{\sinh \pi t}\cos\left(\theta+i\pi\right)t\right)\,. \tag{A.4.6}$$

As a result, we arrive at

$$F_{121}(\theta_1,\theta_2,\theta_3) = \frac{i}{\epsilon}\Gamma_{12}^{(12)} F_{(12)1}(\theta,\theta_3)\,, \tag{A.4.7}$$

where the constant is

$$\Gamma_{12}^{(12)} = -e^{-\frac{2\gamma}{N}}(2\pi)^{1-\frac{2}{N}}\Gamma\left(1-\frac{1}{N}\right)^2 G_{00}\left(-\frac{2\pi i}{N}\right)\,, \tag{A.4.8}$$

and we note that the bracketed indices (12) denote the bound state. The form factor of the bound state with the particle for the unit operator is

$$F_{(12)1}(\theta, \theta_3) = G_{0;12}(\theta - \theta_3)\,. \tag{A.4.9}$$

This example shows how the residue condition is satisfied for this simple case. In general, it should be noted that the fact the the functions $G_{\mu\nu}$ contain the same bound state creating poles as $g_{\mu\nu}$ ensures that this axiom is satisfied.

A.5 Axiom 5: Relativistic invariance

Axiom 5, (1.49), says that form factors are invariant under a relativistic shift. Since the S-matrix and the functions $G_{\mu\nu}(\theta)$ depend on rapidity differences, if the same shift, ζ, is applied to all rapidities, there is clearly no effect on these functions. For functions G_{jj+1} which will contain at most one of these rapidities, the shift has the effect of shifting the position of the poles, but does not change the relative positions of poles in different rapidity variables. Therefore, the integration contour is equivalent and the result is the same. The other contribution is from the operator. For the identity operator, the spin is zero, and no more work is needed. For the current operators, we can have spin of $s = \pm 1$, and the form factors are given in (4.83). Notice that the only rapidity dependence other than in G_{0N} appears in the exponential $\exp\left(s\frac{\theta_1+\theta_2}{2}\right)$. Under the shift $\theta_1 \to \theta_1 + \zeta$, $\theta_2 \to \theta_2 + \zeta$, this gives the extra contribution

$$\exp\left(s\frac{(\theta_1+\zeta)+(\theta_2+\zeta)}{2}\right) = \exp(s\,\zeta)\exp\left(s\frac{\theta_1+\theta_2}{2}\right)\,. \tag{A.5.1}$$

The extra term $\exp(s\,\zeta)$ is exactly the exponential term that appears in (1.49).

Appendix B

Various functions

B.1 f_{AB}-functions

Since f_{AB}, f''_{AB} and f'_{AB} satisfy the relations $\mathrm{f}_{AB}(t) = \mathrm{f}_{BA}(t) = \mathrm{f}_{BA}(-t)$ where f_{AB} is any of the three functions and the indices A, B are either $\mu, \nu = 0, \dots, N$ or $(r), (s) = (1) \dots, (N-1)$, we list only nonvanishing functions $f_{\mu\nu}$ with $\mu \leq \nu$, $f_{(r)\mu}$ and $f_{(r)(s)}$ with $r \leq s$ and $t > 0$

$$f_{00}(t) = f_{NN}(t) = \frac{\sinh \frac{(N-1)\pi t}{N}}{\sinh \pi t} e^{\frac{\pi t}{N}}, \quad f_{jj}(t) = 1 + e^{\frac{2\pi t}{N}}, \quad j = 1, \dots, N-1, \tag{B.1.1}$$

$$f_{j,j+1}(t) = -e^{\frac{\pi t}{N}}, \quad j = 0, 1, \dots, N-1, \quad f_{0N}(t) = \frac{\sinh \frac{\pi t}{N}}{\sinh \pi t} e^{\frac{\pi t}{N}}, \tag{B.1.2}$$

$$f_{(r)0}(t) = \frac{\sinh \frac{\pi t(N-r)}{N}}{\sinh \pi t} e^{\frac{\pi t}{N}}, \quad f_{(r)j}(t) = -\delta_{rj} e^{\frac{\pi t}{N}}, \quad j = 1, \dots, N-1, \tag{B.1.3}$$

$$f_{(r)(s)}(t) = \frac{\sinh \frac{\pi t r}{N} \sinh \frac{\pi t(N-s)}{N}}{\sinh \pi t \sinh \frac{\pi t}{N}} e^{\frac{\pi t}{N}}, \tag{B.1.4}$$

$$f''_{ab}(t) = e^{-\frac{2\pi t}{N}} f_{ab}(t), \quad f'_{ab}(t) = -e^{-\frac{\pi t}{N}} f_{ab}(t). \tag{B.1.5}$$

B.2 $g_{\mu\nu}$- and $S_{\mu\nu}$-functions

Since $f_{\mu\nu}(t) = f_{\mu\nu}(t)$ the functions $g_{\mu\nu}$ and $S_{\mu\nu}$ satisfy the same relations $g_{\mu\nu}(\theta) = g_{\nu\mu}(\theta)$, $S_{\mu\nu}(\theta) = S_{\nu\mu}(\theta)$, and we again list only nontrivial (not equal to 1) functions

$$g_{00}(\theta) = g_{NN}(\theta) = e^{\frac{(N-1)(\gamma+\log(2\pi))}{N}} \frac{\Gamma\left(\frac{i\theta}{2\pi} - \frac{1}{N} + 1\right)}{\Gamma\left(\frac{i\theta}{2\pi}\right)}, \tag{B.2.1}$$

$$g_{jj}(\theta) = -e^{2\gamma}\theta\left(\theta + \frac{2i\pi}{N}\right), \quad j = 1, \ldots, N-1, \tag{B.2.2}$$

$$g_{j,j+1}(\theta) = -\frac{ie^{-\gamma}}{\theta + \frac{i\pi}{N}}, \quad j = 0, 1, \ldots, N-1, \tag{B.2.3}$$

$$g_{0N}(\theta) = \frac{e^{\frac{\gamma+\log(2\pi)}{N}}\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{2}\right)}{\Gamma\left(\frac{i\theta}{2\pi} - \frac{1}{N} + \frac{1}{2}\right)}, \tag{B.2.4}$$

$$S_{00}(\theta) = S_{NN}(\theta) = S(\theta) = \frac{\Gamma\left(\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{N-1}{N} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{N-1}{N} + \frac{i\theta}{2\pi}\right)}, \tag{B.2.5}$$

$$S_{jj}(\theta) = \frac{\theta - \frac{2i\pi}{N}}{\theta + \frac{2i\pi}{N}}, \quad j = 1, \ldots, N-1, \tag{B.2.6}$$

$$S_{j,j+1}(\theta) = \frac{\theta + \frac{i\pi}{N}}{-\theta + \frac{i\pi}{N}}, \quad j = 0, 1, \ldots, N-1, \tag{B.2.7}$$

$$S_{0N}(\theta) = \frac{\Gamma\left(-\frac{i\theta}{2\pi} + \frac{1}{2}\right)\Gamma\left(\frac{i\theta}{2\pi} - \frac{1}{N} + \frac{1}{2}\right)}{\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{2}\right)\Gamma\left(-\frac{i\theta}{2\pi} - \frac{1}{N} + \frac{1}{2}\right)}. \tag{B.2.8}$$

The same is true for $g''_{\mu\nu}$

$$g''_{00}(\theta) = g''_{NN}(\theta) = \frac{e^{\frac{(N-1)(\gamma+\log(2\pi))}{N}}\Gamma\left(\frac{i\theta}{2\pi} + 1\right)}{\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{N}\right)}, \tag{B.2.9}$$

$$g''_{jj}(\theta) = -e^{2\gamma}\theta\left(\theta - \frac{2i\pi}{N}\right), \quad j = 1, \ldots, N-1, \tag{B.2.10}$$

$$g''_{j,j+1}(\theta) = -\frac{ie^{-\gamma}}{\theta - \frac{i\pi}{N}}, \quad j = 0, 1, \ldots, N-1, \tag{B.2.11}$$

$$g''_{0N}(\theta) = \frac{e^{\frac{\gamma+\log(2\pi)}{N}}\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{N} + \frac{1}{2}\right)}{\Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{2}\right)}, \tag{B.2.12}$$

and for $g'_{\mu\nu}$

$$g'_{00}(\theta) = g'_{NN}(\theta) = \frac{e^{\gamma\left(\frac{1}{N}-1\right)}(2\pi)^{\frac{1}{N}-1}\Gamma\left(\frac{1}{2}\left(\frac{i\theta}{\pi}+\frac{1}{N}\right)\right)}{\Gamma\left(\frac{i\theta}{2\pi}-\frac{1}{2N}+1\right)}, \tag{B.2.13}$$

$$g'_{jj}(\theta) = -\frac{e^{-2\gamma}}{\left(\theta-\frac{i\pi}{N}\right)\left(\theta+\frac{i\pi}{N}\right)}, \quad j = 1, \ldots, N-1, \tag{B.2.14}$$

$$g'_{j,j+1}(\theta) = ie^{\gamma}\theta, \quad j = 0, 1, \ldots, N-1, \tag{B.2.15}$$

$$g'_{0N}(\theta) = \frac{e^{-\frac{\gamma}{N}}(2\pi)^{-1/N}\Gamma\left(\frac{1}{2}\left(\frac{i\theta}{\pi}-\frac{1}{N}+1\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{i\theta}{\pi}+\frac{1}{N}+1\right)\right)}. \tag{B.2.16}$$

B.2.1 Regularised integrals

Let us introduce the following functions

$$\begin{aligned} F_2(z,a) &= \frac{z^2}{2} - \frac{3z}{2} - \gamma\left(\frac{z^2}{2} - z\right) + (z-1)\log\Gamma(2-z) \\ &+ -\psi^{(-2)}(2-z) - \left(\frac{z^2}{2} - z\right)\log a, \end{aligned} \tag{B.2.17}$$

$$\begin{aligned} F_1(z,a) &= F_2(z+1,a) - F_2(z,a) \\ &= -\gamma(z-\frac{1}{2}) + \log\Gamma(1-z) - z\log a + \frac{1}{2}\log\left(\frac{a}{2\pi}\right), \end{aligned} \tag{B.2.18}$$

where $\psi^{(-2)}(z)$ is given by

$$\psi^{(-2)}(z) = \int_0^z dt \log\Gamma(t). \tag{B.2.19}$$

The following integrals are useful in computing $G_{\mu\nu}$ functions and can be expressed in terms of $F_i(z,a)$

$$\int_0^\infty \frac{dt}{t}\frac{e^{wat}-e^{zat}}{(e^{at}-1)^2} = F_2(w,a) - F_2(z,a), \tag{B.2.20}$$

$$\int_0^\infty \frac{dt}{t}\frac{e^{zat}}{e^{at}-1} = F_1(z,a), \tag{B.2.21}$$

$$\int_0^\infty \frac{dt}{t}e^{zt} = F_1(z+1,1) - F_1(z,1) = -\gamma - \log(-z). \tag{B.2.22}$$

Appendix C

Bound state vertex operators

In this appendix we derive the expressions for the highest weight bound state vertex operators $Z_{12\ldots r}$ and the normalisation constants $\mathcal{N}_r$ and $C_{N,r}$. For convenience we will perform the computations for the fused vertex operators $\mathcal{Z}_{k_1\ldots k_r}$.

C.1 Rank-2 fused vertex operators $\mathcal{Z}_{1b}$

Consider first $\mathcal{Z}_{ab}(\theta) \equiv i\epsilon\, Z_a(\theta_1) Z_b(\theta_2)$, $a < b$, in the limit $\epsilon \to 0$ where $\theta_1 = \theta + \frac{i\pi}{N}$ and $\theta_2 = \theta - \frac{i\pi}{N} + \epsilon = \theta_1 - \frac{2i\pi}{N} + \epsilon$ and θ is arbitrary. We have

$$\begin{aligned}
\mathcal{Z}_{ab}(\theta) &= i\epsilon\, Z_a(\theta_1) Z_b(\theta_2) \\
&= i\epsilon\rho^2 \int_{C^\alpha_{a-1}} d\alpha_{a-1} \cdots \int_{C^\alpha_1} d\alpha_1 \int_{C^\beta_{b-1}} d\beta_{b-1} \cdots \int_{C^\beta_1} d\beta_1 \\
&\quad \times \prod_{m=1}^{a-1} g^s_{m-1,m}(\alpha_{m,m-1}) \prod_{n=1}^{b-1} g^s_{n-1,n}(\beta_{n,n-1}) \\
&\quad \times \prod_{j=0}^{a-1} g_{jj}(\beta_j - \alpha_j) g_{j,j-1}(\beta_{j-1} - \alpha_j) g_{j,j+1}(\beta_{j+1} - \alpha_j) \\
&\quad \times :\prod_{m=0}^{a-1} V_m(\alpha_m) \prod_{n=0}^{b-1} V_n(\beta_n):,
\end{aligned} \tag{C.1.1}$$

where $\alpha_0 = \theta_1$, $\beta_0 = \theta_2$ and $\alpha_{ij} = \alpha_i - \alpha_j$, $\beta_{ij} = \beta_i - \beta_j$. Since $g_{j,j}(-2\pi i/N) = 0$ for $j > 1$ and we have a factor of ϵ, naively (C.1.1) vanishes in the limit $\epsilon \to 0$. The only way this would not happen is if in the limit $\epsilon \to 0$ two poles of $g_{k,k\pm1}$-

functions pinch one of the integration contours in (C.1.1). Let us recall that the integration contour C_m^α runs above the pole of $g^s_{m-1,m}(\alpha_{m,m-1})$ at $\alpha_m = \alpha_{m-1} + \frac{i\pi}{N}$, and below the pole of $g^s_{m-1,m}(\alpha_{m,m-1})$ at $\alpha_m = \alpha_{m-1} - \frac{i\pi}{N}$ and above the pole of $g_{m,m-1}(\beta_{m-1} - \alpha_m)$ at $\alpha_m = \beta_{m-1} + \frac{i\pi}{N}$. Then the integration contour C_m^β runs above the pole of $g^s_{m-1,m}(\beta_{m,m-1})$ at $\beta_m = \beta_{m-1} + \frac{i\pi}{N}$, below the pole of $g^s_{m-1,m}(\beta_{m,m-1})$ at $\beta_m = \beta_{m-1} - \frac{i\pi}{N}$, and below the pole of $g_{m-1,m}(\beta_m - \alpha_{m-1})$ at $\beta_m = \alpha_{m-1} - \frac{i\pi}{N}$. Thus for the contour C_1^α one gets the poles

$$\begin{aligned}
\text{below } C_1^\alpha : \alpha_1 &= \theta_1 + \frac{i\pi}{N}, \quad \operatorname{Res} g^s_{0,1}(\alpha_{1,0}) = \frac{e^\gamma}{2\pi} \operatorname{Res} g_{0,1}(\alpha_{0,1}) = \frac{i}{2\pi}, \\
\alpha_1 &= \theta_2 + \frac{i\pi}{N} = \theta_1 - \frac{i\pi}{N} + \epsilon, \quad \operatorname{Res} g_{0,1}(\beta_0 - \alpha_1) = ie^{-\gamma}, \qquad \text{(C.1.2)} \\
\text{above } C_1^\alpha : \alpha_1 &= \theta_1 - \frac{i\pi}{N}, \quad \operatorname{Res} g^s_{0,1}(\alpha_{1,0}) = \frac{e^\gamma}{2\pi} \operatorname{Res} g_{0,1}(\alpha_{1,0}) = \frac{1}{2\pi i},
\end{aligned}$$

and one sees that the poles at $\theta_1 - \frac{i\pi}{N}$ and $\theta_1 - \frac{i\pi}{N} + \epsilon$ do pinch the integration contour C_1^α in the limit $\epsilon \to 0$. Then for the contour C_1^β one gets the poles at

$$\begin{aligned}
\text{below } C_1^\beta : \beta_1 &= \theta_2 + \frac{i\pi}{N} = \theta_1 - \frac{i\pi}{N} + \epsilon, \\
&\operatorname{Res} g^s_{0,1}(\beta_{1,0}) = \frac{e^\gamma}{2\pi} \operatorname{Res} g_{0,1}(\beta_{0,1}) = \frac{i}{2\pi}, \qquad \text{(C.1.3)} \\
\text{above } C_1^\beta : \beta_1 &= \theta_2 - \frac{i\pi}{N}, \quad \operatorname{Res} g^s_{0,1}(\beta_{1,0}) = \frac{e^\gamma}{2\pi} \operatorname{Res} g_{0,1}(\beta_1 - \theta_2) = \frac{1}{2\pi i}, \\
\beta_1 &= \theta_1 - \frac{i\pi}{N}, \quad \operatorname{Res} g_{0,1}(\beta_1 - \theta_1) = -ie^{-\gamma},
\end{aligned}$$

and the poles at $\theta_1 - \frac{i\pi}{N}$ and $\theta_1 - \frac{i\pi}{N} + \epsilon$ again pinch the integration contour C_1^β. Since one of the poles is below and the other is above the integration contour, only the contribution from one of them should be taken into account. We will always choose the one which is the pole of $g^s_{m-1,m}$ to get rid of one of these functions in (C.1.1). If $a > 1$, $b > 1$ then two integration contours are pinched at the same time and one has to sum the contributions coming from each of the contours.

Let us now consider $a = 1$ and compute the integral over β_1. If $a = 1$ then there are no integrals over α_j and we get

$$\begin{aligned}
\mathcal{Z}_{1b}(\theta) &= i\epsilon\, Z_1(\theta_1) Z_b(\theta_2) \\
&= i\epsilon\rho^2 \int_{C^\beta_{b-1}} d\beta_{b-1} \cdots \int_{C^\beta_1} d\beta_1 g_{00}(\theta_{21}) g_{01}(\beta_1 - \theta_1) g^s_{01}(\beta_{1,0}) \\
&\quad \times \prod_{n=2}^{b-1} g^s_{n-1,n}(\beta_{n,n-1}) \ : V_0(\theta_1) \prod_{n=0}^{b-1} V_n(\beta_n) : \\
&= \rho^2 C^1_{N,2} \int_{C^\beta_{b-1}} d\beta_{b-1} \cdots \int_{C^\beta_2} d\beta_2 \prod_{n=2}^{b-1} g^s_{n-1,n}(\beta_{n,n-1}) \ : V_0(\theta_1) \prod_{n=0}^{b-1} V_n(\beta_n) : ,
\end{aligned} \tag{C.1.4}$$

where we take into account that $i\epsilon\, g_{0,0}(\theta_{21}) g_{0,1}(\theta_{21} + \frac{i\pi}{N}) = (2\pi)^{\frac{N-1}{N}} e^{-\frac{\gamma}{N}} / \Gamma\left(\frac{1}{N}\right)$ and $\beta_1 = \theta_2 + \frac{i\pi}{N} = \theta_1 - \frac{i\pi}{N} = \theta$ in (C.1.4) in the limit $\epsilon \to 0$, and introduce the constant

$$C^1_{N,2} \equiv (2\pi)^{\frac{N-1}{N}} e^{-\frac{\gamma}{N}} \frac{1}{\Gamma\left(\frac{1}{N}\right)} = \lim_{\epsilon \to 0} \int_C d\beta_1\, i\epsilon\, g_{0,0}(\theta_{21}) g_{0,1}(\beta_1 - \theta_1) g^s_{0,1}(\beta_{1,0}) , \tag{C.1.5}$$

with the integration contour specified above.

In particular for the highest weight fused operator one gets

$$\mathcal{Z}_{12}(\theta) = C_{N,2} : V_0(\theta + \frac{i\pi}{N}) V_0(\theta - \frac{i\pi}{N}) V_1(\theta) := C_{N,2} V_{(2)}(\theta) , \tag{C.1.6}$$

where $C_{N,2} = \rho^2 C^1_{N,2} = (-1)^{\frac{2}{N}} \frac{e^{\gamma\frac{N-2}{N}} N^{-\frac{1}{N}} (2\pi)^{\frac{N-1}{N}}}{\Gamma(\frac{1}{N})}$.

C.2 Rank-3 fused vertex operators $\mathcal{Z}_{12c}$

Let us now consider $\mathcal{Z}_{12c}(\theta) = i\epsilon_1\, i\epsilon\, Z_1(\theta_1) Z_2(\theta_2) Z_c(\theta_3)$ for $2 < c$ with $\theta_1 = \theta + \frac{2i\pi}{N}$, $\theta_2 = \theta + \epsilon_1$, $\theta_3 = \theta - \frac{2i\pi}{N} + \epsilon$, and take the limit $\epsilon_1 \to 0$ after using (C.1.6) for $i\epsilon_1 Z_1(\theta_1) Z_b(\theta_2)$

$$\begin{aligned}
\mathcal{Z}_{12c}(\theta) &= i\epsilon\, \mathcal{Z}_{12}(\beta_1) Z_c(\theta_3) \\
&= \rho^3 C^1_{N,2} \int_{C^\gamma_{c-1}} d\gamma_{c-1} \cdots \int_{C^\gamma_1} d\gamma_1\, i\epsilon \prod_{r=1}^{c-1} g^s_{r-1,r}(\gamma_{r,r-1}) \\
&\quad \times g_{00}(\theta_{31}) g_{00}(\theta_{32}) g_{01}(\gamma_1 - \theta_1) g_{01}(\gamma_1 - \theta_2) \\
&\quad \times g_{01}(\theta_3 - \beta_1) g_{11}(\gamma_1 - \beta_1) g_{01}(\gamma_2 - \beta_1) \\
&\quad \times : V_0(\theta_1) V_0(\theta_2) V_1(\beta_1) \prod_{r=0}^{c-1} V_r(\gamma_r) : ,
\end{aligned} \tag{C.2.1}$$

where $\beta_1 \equiv \theta_2 + \frac{i\pi}{N} = \theta + \frac{i\pi}{N}$. We want to integrate over γ_1 and the relevant terms in the integrand are

$$\begin{aligned}
& i\epsilon\, g^s_{01}(\gamma_{01}) g_{00}(\theta_{31}) g_{00}(\theta_{32}) g_{01}(\gamma_1 - \theta_1) g_{01}(\gamma_1 - \theta_2) g_{01}(\theta_3 - \beta_1) g_{11}(\gamma_1 - \beta_1) \\
&= g_{00}(-\frac{4i\pi}{N}) g_{00}(-\frac{2i\pi}{N}) g_{01}(-\frac{3i\pi}{N}) g^s_{01}(\gamma_{01}) \\
&\quad \times g_{01}(\gamma_1 - \theta_1) g_{01}(\gamma_1 - \theta_2) g_{11}(\gamma_1 - \beta_1)\,.
\end{aligned} \tag{C.2.2}$$

The integral over γ_1 is taken in the same way as the one over β_1 in the previous subsection, and we get that the expression above becomes

$$C^1_{N,3}\, g_{1,1}(-\frac{2i\pi}{N} + \epsilon)\,, \tag{C.2.3}$$

where we introduce the constant

$$\begin{aligned}
C^1_{N,3} &\equiv C^1_{N,2}\, g_{0,0}(-\frac{4i\pi}{N}) g_{1,0}(-\frac{3i\pi}{N})^2 \\
&= \lim_{\epsilon \to 0} \int_C d\gamma_1\, i\epsilon\, g_{0,0}(-\frac{4i\pi}{N}) g_{1,0}(-\frac{3i\pi}{N}) g_{0,0}(-\frac{2i\pi}{N}) \\
&\quad \times g_{0,1}(\gamma_1 - \theta_1) g^s_{0,1}(\gamma_{0,1}) g_{0,1}(\gamma_1 - \theta_2)\,,
\end{aligned} \tag{C.2.4}$$

and use that the pole is located at $\gamma_1 = \theta_3 + \frac{i\pi}{N} = \theta_2 - \frac{i\pi}{N} + \epsilon$. Since $g_{1,1}(-\frac{2i\pi}{N}) = 0$ we have to take the integral over γ_2 to get a finite result in the limit $\epsilon \to 0$. The relevant terms in the integrand now are

$$g_{1,1}(-\frac{2i\pi}{N} + \epsilon) g^s_{1,2}(\gamma_{1,2}) g_{1,2}(\gamma_2 - \beta_1)\,, \tag{C.2.5}$$

and the poles are at $\gamma_2 = \gamma_1 \pm \frac{i\pi}{N} = \theta_2 - \frac{i\pi}{N} + \epsilon \pm \frac{i\pi}{N}$ and $\gamma_2 = \beta_1 - \frac{i\pi}{N} = \theta_2$. Thus the contour is pinched at $\gamma_2 = \theta_2$, and we get the extra factor

$$C^2_{N,3} \equiv \frac{2\pi e^{\gamma}}{N} = \lim_{\epsilon \to 0} \int_C d\gamma_2\, g_{1,1}(-\frac{2i\pi}{N} + \epsilon) g^s_{1,2}(\gamma_{1,2}) g_{1,2}(\gamma_2 - \beta_1)\,, \tag{C.2.6}$$

where one uses that $g_{11}(-\frac{2i\pi}{N} + \epsilon) g_{12}(\epsilon - \frac{i\pi}{N}) = \frac{2e^{\gamma}\pi}{N}$. Thus one gets

$$\begin{aligned}
\mathcal{Z}_{12c}(\theta) &= \rho^3 C^1_{N,2} C^1_{N,3} C^2_{N,3} \int_{C^{\gamma}_{c-1}} d\gamma_{c-1} \cdots \int_{C^{\gamma}_3} d\gamma_3 \prod_{r=3}^{c-1} g^s_{r-1,r}(\gamma_{r,r-1}) \\
&\quad \times\ : V_0(\theta_1) V_0(\theta_2) V_0(\theta_3) V_1(\beta_1) \prod_{r=1}^{c-1} V_r(\gamma_r) :\,,
\end{aligned} \tag{C.2.7}$$

where $\gamma_1 = \theta_3 + \frac{i\pi}{N} = \theta - \frac{i\pi}{N}$, $\gamma_2 = \theta_2 = \theta$. In particular for the highest weight fused operator one gets

$$\begin{aligned} &\mathcal{Z}_{123}(\theta) \\ &= C_{N,3} : V_0(\theta + \frac{2i\pi}{N})V_0(\theta)V_0(\theta - \frac{2i\pi}{N})V_1(\theta + \frac{i\pi}{N})V_1(\theta - \frac{i\pi}{N})V_2(\theta) : \\ &= C_{N,3}V_{(3)}(\theta)\,, \end{aligned} \tag{C.2.8}$$

where $C_{N,3} = \rho^3 C^1_{N,2}C^1_{N,3}C^2_{N,3} = (-1)^{\frac{3}{N}} e^{\frac{3\gamma(N-3)}{2N}} N^{-\frac{3}{2N}} (2\pi)^{2-\frac{3}{N}} \frac{1}{\Gamma(\frac{2}{N})\Gamma(\frac{1}{N})}$.

C.3 Rank-k fused highest weight vertex operator $\mathcal{Z}_{1\ldots k}$

This pattern continues for the rank-k fused highest weight vertex operator $\mathcal{Z}_{1\ldots k}$ which contains no integrals at all. So, we consider

$$\begin{aligned} \mathcal{Z}_{12\ldots k}(\theta) &\equiv \lim_{\epsilon_a \to 0} i\epsilon_1 \cdots i\epsilon_{k-1}\, Z_1(\theta_1) Z_2(\theta_2) \cdots Z_k(\theta_k)\,, \\ \theta_j &\equiv \theta + \frac{i\pi}{N}(k - 2j + 1) + \epsilon_j\,, \end{aligned} \tag{C.3.1}$$

and we should find that it is equal to

$$\begin{aligned} \mathcal{Z}_{12\ldots k}(\theta) &= C_{N,k} : \prod_{j=1}^{k} \prod_{a_j=0}^{j-1} V_{a_j}(\theta_j + \frac{i\pi}{N}a_j) : \\ &= C_{N,k} : \prod_{r=0}^{k-1} \prod_{j=r+1}^{k} V_r(\theta_j + \frac{i\pi}{N}r) : . \end{aligned} \tag{C.3.2}$$

Note that the V_{k-1} operator only enters as $V_{k-1}(\theta)$.

To find $C_{N,k}$ we use induction. Introducing the notation

$$\theta_j^{[\pm r]} \equiv \theta_j \pm \frac{i\pi}{N} r\,, \tag{C.3.3}$$

we have ($\alpha_0 = \theta_{k+1} = \theta_k - \frac{2i\pi}{N}$)

$$
\begin{aligned}
&\mathcal{Z}_{12\ldots k,k+1}(\theta) \\
&= i\epsilon\,\rho C_{N,k} \int_{C_k^\alpha} d\alpha_k \cdots \int_{C_1^\alpha} d\alpha_1 \prod_{m=1}^{k} g^s_{m-1,m}(\alpha_{m,m-1}) \\
&\quad \times \prod_{r=0}^{k-1} \prod_{j=r+1}^{k} g_{r,r-1}(\alpha_{r-1} - \theta_j^{[+r]}) g_{r,r}(\alpha_r - \theta_j^{[+r]}) g_{r,r+1}(\alpha_{r+1} - \theta_j^{[+r]}) \\
&\quad \times : \prod_{r=0}^{k-1} \prod_{j=r+1}^{k} V_r(\theta_j^{[+r]}) \prod_{m=0}^{k} V_m(\alpha_m) : .
\end{aligned}
\tag{C.3.4}
$$

To integrate over α_j it is better to rearrange

$$
\begin{aligned}
&\prod_{r=0}^{k-1} \prod_{j=r+1}^{k} g_{r,r-1}(\alpha_{r-1} - \theta_j^{[+r]}) g_{r,r}(\alpha_r - \theta_j^{[+r]}) g_{r,r+1}(\alpha_{r+1} - \theta_j^{[+r]}) \\
&= \prod_{j=1}^{k-1} g_{00}\big(-\frac{2i\pi}{N}(k-j+1)\big) \prod_{j=2}^{k} g_{10}\big(-\frac{2i\pi}{N}(k-j+\frac{3}{2})\big) g_{00}\big(-\frac{2i\pi}{N}\big) \\
&\times \prod_{m=1}^{k} \prod_{j=m}^{k} g_{m-1,m}(\alpha_m - \theta_j^{[+(m-1)]}) \prod_{j=m+1}^{k} g_{m,m}(\alpha_m - \theta_j^{[m]}) \\
&\prod_{j=m+2}^{k} g_{m+1,m}(\alpha_m - \theta_j^{[+(m+1)]}) .
\end{aligned}
\tag{C.3.5}
$$

Thus the integrand for α_1 is

$$
\begin{aligned}
&\prod_{j=1}^{k-1} g_{00}\big(-\frac{2i\pi}{N}(k-j+1)\big) \prod_{j=2}^{k} g_{10}\big(-\frac{2i\pi}{N}(k-j+\frac{3}{2})\big) g_{00}\big(-\frac{2i\pi}{N}\big) \\
&\times g^s_{01}(\alpha_{10}) \prod_{j=1}^{k} g_{01}(\alpha_1 - \theta_j) \prod_{j=2}^{k} g_{11}(\alpha_1 - \theta_j^{[+1]}) \prod_{j=3}^{k} g_{21}(\alpha_1 - \theta_j^{[+2]}) .
\end{aligned}
\tag{C.3.6}
$$

We see that the two poles which pinch the contour are at $\alpha_1 = \theta_{k+1} + \frac{i\pi}{N} = \theta_k - \frac{i\pi}{N} + \epsilon = \theta_k^{[-1]} + \epsilon$ and $\alpha_1 = \theta_k^{[-1]}$ giving the following contribution

$$
C^1_{N,k+1}\, g_{1,1}\big(-\frac{2i\pi}{N} + \epsilon\big) , \tag{C.3.7}
$$

where

$$C^1_{N,k+1} \equiv C^1_{N,2} \prod_{j=1}^{k-1} g_{00}\big(-\frac{2i\pi}{N}(k-j+1)\big) \prod_{j=2}^{k} g_{01}\big(-\frac{2i\pi}{N}(k-j+\frac{3}{2})\big)$$
$$\times \prod_{j=1}^{k-1} g_{01}(-\frac{2i\pi}{N}(k-j+\frac{1}{2})) \prod_{j=2}^{k-1} g_{11}(-\frac{2i\pi}{N}(k-j+1))$$
$$\times \prod_{j=3}^{k} g_{01}(-\frac{2i\pi}{N}(k-j+\frac{3}{2})) \tag{C.3.8}$$
$$= C^1_{N,2}\, g_{01}\big(-\frac{i\pi}{N}(2k-1)\big)^2 \prod_{j=2}^{k} g_{00}\big(-\frac{2i\pi}{N}j\big) \prod_{j=2}^{k-1} g_{01}\big(-\frac{i\pi}{N}(2j-1)\big)^3$$
$$\times \prod_{j=2}^{k-1} g_{11}(-\frac{2i\pi}{N}j)\,.$$

Then the integrand for α_2 is

$$g^s_{12}(\alpha_{21}) \prod_{j=2}^{k} g_{12}(\alpha_2-\theta_j^{[+1]}) \prod_{j=3}^{k} g_{22}(\alpha_2-\theta_j^{[+2]}) \prod_{j=4}^{k} g_{32}(\alpha_2-\theta_j^{[+3]})\,. \tag{C.3.9}$$

We see that the two poles which pinch the contour are at $\alpha_2=\alpha_1+\frac{i\pi}{N}=\theta^{[+2]}_{k+1}=\theta_k+\epsilon$ and $\alpha_0=\theta_k$ giving the following contribution

$$C^2_{N,k+1}\, g_{2,2}\big(-\frac{2i\pi}{N}+\epsilon\big) = C^2_{N,k+1}\, g_{11}\big(-\frac{2i\pi}{N}+\epsilon\big)\,, \tag{C.3.10}$$

where

$$C^2_{N,k+1} \equiv C^2_{N,3} \prod_{j=2}^{k-1} g_{12}(-\frac{2i\pi}{N}(k-j+\frac{1}{2})) \prod_{j=3}^{k-1} g_{22}(-\frac{2i\pi}{N}(k-j+1))$$
$$\times \prod_{j=4}^{k} g_{32}(-\frac{2i\pi}{N}(k-j+\frac{3}{2})) \tag{C.3.11}$$
$$= C^2_{N,3}\, g_{01}\big(-\frac{i\pi}{N}(2k-3)\big) \prod_{j=2}^{k-2} g_{01}\big(-\frac{i\pi}{N}(2j-1)\big)^2 \prod_{j=2}^{k-2} g_{11}(-\frac{2i\pi}{N}j)\,.$$

The integral over α_m for $m<k-1$ is computed in the same way with the pole at $\alpha_m=\alpha_{m-1}+\frac{i\pi}{N}=\theta^{[+m]}_{k+1}=\theta^{[+(m-2)]}_k+\epsilon$ and gives

$$C^m_{N,k+1}\, g_{1,1}\big(-\frac{2i\pi}{N}+\epsilon\big)\,, \tag{C.3.12}$$

where

$$
\begin{aligned}
C_{N,k+1}^{m} \equiv C_{N,3}^{2} & \prod_{j=m}^{k-1} g_{01}(-\frac{2i\pi}{N}(k-j+\frac{1}{2})) \prod_{j=m+1}^{k-1} g_{11}(-\frac{2i\pi}{N}(k-j+1)) \\
& \times \prod_{j=m+2}^{k} g_{01}(-\frac{2i\pi}{N}(k-j+\frac{3}{2})) \\
& = C_{N,3}^{2}\, g_{01}\big(-\frac{i\pi}{N}(2k-2m+1)\big) \prod_{j=2}^{k-m} g_{01}\big(-\frac{i\pi}{N}(2j-1)\big)^{2} \\
& \times \prod_{j=2}^{k-m} g_{11}(-\frac{2i\pi}{N}j) .
\end{aligned}
\tag{C.3.13}
$$

For $m = k-1$ and $m = k$ we get

$$
C_{N,k+1}^{k-1} = C_{N,3}^{2}\, g_{0,1}\big(-\frac{3i\pi}{N}\big), \quad C_{N,k+1}^{k} = C_{N,3}^{2} .
$$

Thus, the result is

$$
C_{N,k+1} = \rho C_{N,k} \prod_{j=1}^{k} C_{N,k+1}^{j} . \tag{C.3.14}
$$

Computing the product one finds

$$
\prod_{j=1}^{k} C_{N,k+1}^{j} = 2\pi\, e^{-\frac{k\gamma}{N}} (2\pi)^{-\frac{k}{N}} \frac{1}{\Gamma\left(\frac{k}{N}\right)} . \tag{C.3.15}
$$

The relation above can be easily solved giving the final result

$$
\begin{aligned}
C_{N,k+1} &= \rho^{k+1} (2\pi)^{k} \left(e^{-\frac{\gamma}{N}} (2\pi)^{-\frac{1}{N}}\right)^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \frac{1}{\Gamma\left(\frac{j}{N}\right)} \\
&= (-1)^{\frac{k+1}{N}} e^{\gamma\frac{(k+1)(N-k-1)}{2N}} N^{-\frac{k+1}{2N}} (2\pi)^{\frac{k(2N-k-1)}{2N}} \prod_{j=1}^{k} \frac{1}{\Gamma\left(\frac{j}{N}\right)} .
\end{aligned}
\tag{C.3.16}
$$

By using the identity

$$
\prod_{j=1}^{N-1} \frac{1}{\Gamma\left(\frac{j}{N}\right)} = \sqrt{N} (2\pi)^{-\frac{N-1}{2}} , \tag{C.3.17}
$$

one gets $C_{N,N} = \mathcal{V}_N = -1$, and (3.22) for $C_{N,N-1}$.

Appendix D

Primed fused operators

In this appendix we derive the expressions for the highest weight fused primed vertex operators $Z'_{12\dots r}$. The derivation almost repeats the one for the highest weight bound state vertex operators $\mathcal{Z}_{12\dots r}$ considered in appendix C. It is therefore unnecessary to repeat the details and only an outline is given.

D.1 Rank-2 fused primed vertex operator Z'_{1b}

Consider first $Z'_{ab}(\theta) \equiv Z'_a(\theta_1)Z'_b(\theta_2)$, $a < b$, in the limit $\epsilon \to 0$ where $\theta_1 = \theta - \frac{i\pi}{N}$ and $\theta_2 = \theta + \frac{i\pi}{N} + \epsilon = \theta_1 + \frac{2i\pi}{N} + \epsilon$ and θ is arbitrary. We notice that the rapidities here are shifted in the opposite direction when compared to those in appendix C. In addition, all Green's functions will be of the type $g''_{\mu\nu}$. Therefore we will see $g''_{00}\left(\frac{2\pi i}{N} + \epsilon\right)$ instead of $g_{00}\left(-\frac{2\pi i}{N} + \epsilon\right)$, etc. The other major difference is that we don't have an initial factor of $i\epsilon$, but instead we have that $g''_{00}\left(\frac{2\pi i}{N}\right) = 0$. Other than these differences, the derivation is essentially identical.

With this in mind, let us now calculate the fused primed vertex operator $Z'_{1b}(\theta)$. Since $a = 1$ there are no integrals over α_j, and computing the integral over β_1 one gets

$$
\begin{aligned}
&Z'_{1b}(\theta) \\
&= \rho'^2 D^1_{N,2} \int_{C^\beta_{b-1}} d\beta_{b-1} \cdots \int_{C^\beta_2} d\beta_2 \prod_{n=2}^{b-1} g''^{s}_{n-1,n}(\beta_{n,n-1}) : V'_0(\theta_1) \prod_{n=0}^{b-1} V'_n(\beta_n) :,
\end{aligned} \tag{D.1.1}
$$

where we take into account that $g''_{0,0}(\theta_{21}) g''_{0,1}(\theta_{21} - \frac{i\pi}{N}) = e^{-\frac{\gamma}{N}} (2\pi)^{-1/N} \Gamma\left(\frac{N-1}{N}\right)$ and $\beta_1 = \theta_2 - \frac{i\pi}{N} = \theta_1 + \frac{i\pi}{N}$ in (D.1.1) in the limit $\epsilon \to 0$, and introduce the constant

$$
D^1_{N,2} \equiv (2\pi)^{-\frac{1}{N}} e^{-\frac{\gamma}{N}} \Gamma\left(\frac{N-1}{N}\right) = \lim_{\epsilon \to 0} \int_C d\beta_1 \, g''_{0,0}(\theta_{21}) g''_{0,1}(\beta_1 - \theta_1) g''^{s}_{0,1}(\beta_{1,0}) , \tag{D.1.2}
$$

with the integration contour specified in the usual way.

In particular for the highest weight fused primed operator one gets

$$
Z'_{12}(\theta) = D_{N,2} : V'_0(\theta - \frac{i\pi}{N}) V'_0(\theta + \frac{i\pi}{N}) V'_1(\theta) := D_{N,2} V'_{(2)}(\theta) , \tag{D.1.3}
$$

where $D_{N,2} = \rho'^2 D^1_{N,2} = e^{\gamma \frac{N-2}{N}} N^{\frac{1}{N}} (2\pi)^{-\frac{1}{N}} \Gamma\left(\frac{N-1}{N}\right)$.

D.2 Rank-3 fused primed vertex operators Z'_{12c}

Let us now consider $Z'_{12c}(\theta) = Z'_1(\theta_1) Z'_2(\theta_2) Z'_c(\theta_3)$, $2 < c$, $\theta_1 = \theta - \frac{2i\pi}{N}$, $\theta_2 = \theta + \epsilon_1$, $\theta_3 = \theta + \frac{2i\pi}{N} + \epsilon$, take the limit $\epsilon_1 \to 0$ and use (D.1.3) for $Z'_1(\theta_1) Z'_2(\theta_2)$. The relevant poles are at $\beta_1 \equiv \theta_2 - \frac{i\pi}{N} = \theta - \frac{i\pi}{N}$, $\gamma_1 = \theta_3 - \frac{i\pi}{N} = \theta_2 + \frac{i\pi}{N} + \epsilon$ and $\gamma_2 = \beta_1 + \frac{i\pi}{N} = \theta_2$, which can be used to reduce the expression to

$$
\begin{aligned}
Z'_{12c}(\theta) = \rho'^3 D^1_{N,2} D^1_{N,3} D^2_{N,3} &\int_{C^\gamma_{c-1}} d\gamma_{c-1} \cdots \int_{C^\gamma_3} d\gamma_3 \prod_{r=3}^{c-1} g''^{s}_{r-1,r}(\gamma_{r,r-1}) \\
&\times : V'_0(\theta_1) V'_0(\theta_2) V'_0(\theta_3) V'_1(\beta_1) \prod_{r=1}^{c-1} V'_r(\gamma_r) :,
\end{aligned} \tag{D.2.1}
$$

with the new constants defined as

$$
\begin{aligned}
D^1_{N,3} &\equiv D^1_{N,2} g''_{00}(\frac{4i\pi}{N}) g''_{10}(\frac{3i\pi}{N})^2 \\
&= \lim_{\epsilon \to 0} \int_C d\gamma_1 \, g''_{00}(\frac{4i\pi}{N}) g''_{10}(\frac{3i\pi}{N}) g''_{00}(\frac{2i\pi}{N} + \epsilon) g''_{01}(\gamma_1 - \theta_1) \\
&\times g''^{s}_{01}(\gamma_{01}) g''_{01}(\gamma_1 - \theta_2) ,
\end{aligned} \tag{D.2.2}
$$

and

$$D^2_{N,3} \equiv -\frac{2\pi e^\gamma}{N} = \lim_{\epsilon\to 0}\int_C d\gamma_2\, g''_{1,1}(\frac{2i\pi}{N}+\epsilon) g''^{s}_{1,2}(\gamma_{1,2}) g''_{1,2}(\gamma_2-\beta_1)\,. \tag{D.2.3}$$

In particular for the highest weight fused primed operator one gets

$$\begin{aligned} Z'_{123}(\theta) &= D_{N,3} : V'_0(\theta-\frac{2i\pi}{N})V'_0(\theta)V'_0(\theta+\frac{2i\pi}{N})V'_1(\theta-\frac{i\pi}{N})V'_1(\theta+\frac{i\pi}{N})V'_2(\theta) : \\ &= D_{N,3} : V'_{(3)}(\theta)\,, \end{aligned} \tag{D.2.4}$$

where $D_{N,3} = \rho'^3 D^1_{N,2} D^1_{N,3} D^2_{N,3} = e^{\frac{3\gamma(N-3)}{2N}} N^{\frac{3}{2N}} (2\pi)^{-3/N} \Gamma\left(\frac{N-2}{N}\right)\Gamma\left(\frac{N-1}{N}\right)$.

D.3 Rank-k fused highest weight primed vertex operator $Z'_{1\ldots k}$

This pattern continues for the rank-k fused highest weight primed vertex operator $Z_{1\ldots k}$ which contains no integrals at all. So, considering

$$\begin{aligned} Z'_{12\ldots k}(\theta) &\equiv \lim_{\epsilon_a\to 0} Z'_1(\theta_1)Z'_2(\theta_2)\cdots Z'_k(\theta_k)\,, \\ \theta_j &\equiv \theta - \frac{i\pi}{N}(k-2j+1)+\epsilon_j\,, \end{aligned} \tag{D.3.1}$$

one should find that it is equal to

$$Z'_{12\ldots k}(\theta) = \rho'^k D_{N,k} : \prod_{j=1}^{k}\prod_{r_j=0}^{j-1} V'_{r_j}(\theta_j - \frac{i\pi}{N}r_j) := D_{N,k} : \prod_{r=0}^{k-1}\prod_{j=r+1}^{k} V'_r(\theta_j-\frac{i\pi}{N}r) : . \tag{D.3.2}$$

To find $D_{N,k}$ one can use induction. The results are the following:

$$\begin{aligned} D^1_{N,k+1} \equiv & D^1_{N,2}\, g''_{01}\big(\frac{i\pi}{N}(2k-1)\big)^2 \prod_{j=2}^{k} g''_{00}(\frac{2i\pi}{N}j) \\ & \times \prod_{j=2}^{k-1} g''_{01}\big(\frac{i\pi}{N}(2j-1)\big)^3 \prod_{j=2}^{k-1} g''_{11}(\frac{2i\pi}{N}j)\,, \end{aligned} \tag{D.3.3}$$

and

$$D^2_{N,k+1} \equiv D^2_{N,3}\, g''_{0,1}\big(\frac{i\pi}{N}(2k-3)\big) \prod_{j=2}^{k-2} g''_{0,1}\big(\frac{i\pi}{N}(2j-1)\big)^2 \prod_{j=2}^{k-2} g''_{1,1}(\frac{2i\pi}{N}j)\,.$$

The general case is

$$D^m_{N,k+1} \equiv D^2_{N,3}\, g''_{0,1}\big(\frac{i\pi}{N}(2k-2m+1)\big) \prod_{j=2}^{k-m} g''_{0,1}\big(\frac{i\pi}{N}(2j-1)\big)^2 \prod_{j=2}^{k-m} g''_{1,1}\big(\frac{2i\pi}{N}j\big)\,.$$

For $m = k-1$ and $m = k$ we get

$$D^{k-1}_{N,k+1} = D^2_{N,3}\, g''_{0,1}\big(\frac{3i\pi}{N}\big)\,, \quad D^k_{N,k+1} = D^2_{N,3}\,.$$

Thus, the result is

$$D_{N,k+1} = \rho' D_{N,k} \prod_{j=1}^{k} D^j_{N,k+1}\,. \tag{D.3.4}$$

Computing the product one finds

$$\prod_{j=1}^{k} D^j_{N,k+1} = e^{-\gamma\frac{k}{N}} (2\pi)^{-\frac{k}{N}} \Gamma\left(\frac{N-k}{N}\right)\,. \tag{D.3.5}$$

The relation above can be easily solved giving the final result

$$\begin{aligned} D_{N,k+1} &= \rho'^{k+1} \left(e^{-\frac{\gamma}{N}}(2\pi)^{-\frac{1}{N}}\right)^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \Gamma\left(\frac{N-j}{N}\right) \\ &= e^{-\frac{\gamma(k+1)(k-N+1)}{2N}} N^{\frac{k+1}{2N}} (2\pi)^{-\frac{k(k+1)}{2N}} \prod_{m=1}^{k} \Gamma\left(1-\frac{m}{N}\right)\,. \end{aligned} \tag{D.3.6}$$

By using the identity

$$\prod_{j=1}^{N-1} \Gamma\left(\frac{N-j}{N}\right) = \frac{1}{\sqrt{N}} (2\pi)^{\frac{N-1}{2}}\,, \tag{D.3.7}$$

one gets $D_{N,N} = 1$, and (3.55) for $D_{N,N-1}$.

Appendix E

Principal value prescription examples

We now consider examples of using the principal value prescription in calculating Operator Product Expansions of two separate particle operators, which here we denote $Z_j(\beta)$. The first of these is the type that appears in the Operator Product Expansion

$$\begin{aligned} Z_1(\beta_1)Z_2(\beta_2) = &- \rho^2\rho_\chi e^{-2\gamma} g(\beta_2-\beta_1) \\ &\times \int_{C_1} d\alpha_1 \frac{:V_0(\beta_1)V_0(\beta_2)V_1(\alpha_1):}{\left(\alpha_1-\beta_2+\frac{i\pi}{N}\right)\left(\beta_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\beta_1+\frac{i\pi}{N}\right)}, \end{aligned} \tag{E.0.1}$$

which contains the most straightforward of the integrals. A useful thing to note is that integrands of this type decompose as follows

$$\begin{aligned} \frac{1}{(x+a)(x+b)} &= \frac{1}{b-a}\frac{1}{(x+a)} + \frac{1}{a-b}\frac{1}{(x+b)}, \\ \frac{1}{(x+a)(x+b)(x+c)} &= \frac{1}{(b-a)}\frac{1}{(c-a)}\frac{1}{(x+a)} \\ &\quad + \frac{1}{(a-b)}\frac{1}{(c-b)}\frac{1}{(x+b)} + \frac{1}{(a-c)}\frac{1}{(b-c)}\frac{1}{(x+c)}, \end{aligned} \tag{E.0.2}$$

and so on. Then, after deforming the contour, we see that the various integrands in (E.0.2) decompose into a linear combination of these integrals with coefficients that mirror the residues that we produce when we deform the contours. To see this, consider

$$\begin{aligned}
\mathrm{Res}_{x\to -a}\frac{1}{(x+a)(x+b)(x+c)} &= \frac{1}{(a-b)(a-c)}\,, \\
\mathrm{Res}_{x\to -b}\frac{1}{(x+a)(x+b)(x+c)} &= -\frac{1}{(a-b)(b-c)}\,, \\
\mathrm{Res}_{x\to -c}\frac{1}{(x+a)(x+b)(x+c)} &= -\frac{1}{(a-c)(c-b)}\,,
\end{aligned} \tag{E.0.3}$$

which exactly match the coefficients in the relevant decomposition in (E.0.2). In general then, we have

$$\frac{1}{\prod_{i=1}^{m}(x+a_i)} = \sum_{i=1}^{m}\frac{1}{x+a_i}\mathrm{Res}_{x\to -a_i}\frac{1}{\prod_{i=1}^{m}(x+a_i)}\,. \tag{E.0.4}$$

Therefore, when the integrand is of this form, the residues alone will give the correct result up to a constant which must be calculated by performing the integrals. Futhermore, it can also be checked that for each integrand that appears in Operator Product Expansions of vertex operators, such a decomposition can be made. The full result comes from adding the contribution from the residues (due to shifting the contour of integration to the real line) to the contribution from the integration along the real line using principle values.

A slightly more complicated example is

$$\begin{aligned}
Z_1(\beta_1)Z_3(\beta_2) = i\rho^2\rho_\chi^2 e^{-3\gamma} g(\beta_2-\beta_1) &\int_{C_1} d\alpha_1 \int_{C_2} d\alpha_2 \\
\times\, &\frac{:V_0(\beta_1)V_0(\beta_2)V_1(\alpha_1)V_2(\alpha_2):}{\left(\alpha_1-\beta_2+\frac{i\pi}{N}\right)\left(\beta_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\beta_1+\frac{i\pi}{N}\right)} \\
\times\, &\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)}\,,
\end{aligned} \tag{E.0.5}$$

where the α_2 integrand decomposes easily via the first equality in (E.0.2) to give

$$\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)} = \frac{N}{2\pi i}\left(\frac{1}{\alpha_2-\alpha_1+\frac{i\pi}{N}}+\frac{1}{\alpha_1-\alpha_2+\frac{i\pi}{N}}\right)\,, \tag{E.0.6}$$

which may also be written as

$$
\begin{aligned}
&\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)} \\
&\quad=\frac{1}{\alpha_2-\alpha_1+\frac{i\pi}{N}}\mathrm{Res}_{\alpha_2\to\alpha_1-\frac{i\pi}{N}}\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)} \\
&\quad+\frac{1}{\alpha_1-\alpha_2+\frac{i\pi}{N}}\mathrm{Res}_{\alpha_2\to\alpha_1+\frac{i\pi}{N}}\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)}\,.
\end{aligned}
\tag{E.0.7}
$$

After this decomposition, we may use (2.46) to integrate over α_2 along the real axis to get

$$
\frac{N}{2\pi i}((-i\pi)-(i\pi))=-N\,,
\tag{E.0.8}
$$

while the contour deformation that took us to the real axis will give the extra contribution

$$
\begin{aligned}
&2\pi i\Big(\mathrm{Res}_{\alpha_2\to\alpha_1-\frac{i\pi}{N}}\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)} \\
&-\mathrm{Res}_{\alpha_2\to\alpha_1+\frac{i\pi}{N}}\frac{1}{\left(\alpha_2-\alpha_1+\frac{i\pi}{N}\right)\left(\alpha_1-\alpha_2+\frac{i\pi}{N}\right)}\Big) \\
&=2\pi i\left(\frac{N}{2\pi i}-\left(-\frac{N}{2\pi i}\right)\right)=2N\,.
\end{aligned}
\tag{E.0.9}
$$

Overall therefore, the α_2 integration simply gives a value of $-N+2N=N$, and the remaining integrand over α_1 is the same as in $Z_1(\beta_1)Z_2(\beta_2)$. In fact, it is easy to see that the integrand in the product $Z_i(\beta_1)Z_j(\beta_2)$ for $j-i>1$ can always be easily reduced in this fashion to the integrand in $Z_i(\beta_1)Z_{i+1}(\beta_2)$.

Appendix F

Derivation of trace formula

In this appendix we prove formula (4.18) which shows how the free field representation can be converted into traces over the space of Zamolodchikov-Faddeev states. We begin with the trace of an operator

$$\mathrm{Tr}_F \; e^{2\pi i K}\mathcal{O}\,, \tag{F.0.1}$$

where here the operator is a general vertex operator

$$\mathcal{O} =: e^{i\bar{\alpha}a+i\beta a^\dagger} : \,. \tag{F.0.2}$$

For simplicity, we consider a and $a^\dagger$ to be canonical creation and annihilation operators,

$$\left[a, a^\dagger\right] = \delta_{m+n,0}\,. \tag{F.0.3}$$

Generalising to many non-canonical oscillators is performed in the main text. With these oscillators, we now have the Hamiltonian

$$K = iha^\dagger a\,. \tag{F.0.4}$$

The first step is to insert in (F.0.1) a complete set of states,

$$|\psi_n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,, \qquad a_n|0\rangle = 0\,, \tag{F.0.5}$$

which will give us a new expression for the trace

$$\mathrm{Tr}_F\, e^{2\pi i K}\mathcal{O} = \sum_n \langle \psi_n | e^{2\pi i K}\mathcal{O} | \psi_n \rangle\,, \tag{F.0.6}$$

from which we can extract the Hamiltonian term to give,

$$\mathrm{Tr}_F\, e^{2\pi i K}\mathcal{O} = \sum_n e^{-2\pi h n} \langle \psi_n | \mathcal{O} | \psi_n \rangle\,. \tag{F.0.7}$$

Since the operator is normal-ordered, we can immediately apply the useful formula

$$e^{i\bar{\alpha} a}(a^\dagger)^n|0\rangle = \sum_{k=0}^{n} \frac{(i\bar{\alpha})^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|\psi_{n-k}\rangle\,, \tag{F.0.8}$$

which allows us to write (F.0.6) as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-\alpha\beta)^k}{(k!)^2}\frac{n!}{(n-k)!}e^{-2\pi h n}\,. \tag{F.0.9}$$

We can simplify these summations by noticing that we can exchange the order of the sums if they become

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \to \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\,. \tag{F.0.10}$$

Moreover, the sum over n now becomes

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}e^{-2\pi h n} &= \sum_{n=0}^{\infty}\frac{(n+k)!}{n!}e^{-2\pi h(n+k)} \\ &= \frac{1}{2}k!e^{-2\pi h(k-1)}\left(1-e^{-2\pi h}\right)^{-k}(\coth(\pi h)-1)\,.\end{aligned} \tag{F.0.11}$$

We now must replace this in (F.0.9), and perform the summation over k,

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-\bar{\alpha}\beta)^k}{(k!)^2}\frac{1}{2}k!e^{-2\pi h(k-1)}\left(1-e^{-2\pi h}\right)^{-k}(\coth(\pi h)-1) \\ &= \sum_{k=0}^{\infty}\frac{\left(-\bar{\alpha}\beta e^{-2\pi h}\frac{1}{1-e^{-2\pi h}}\right)^k}{k!}e^{2\pi h}\frac{1}{2}(\coth(\pi h)-1) \\ &= \frac{1}{1-e^{-2\pi h}}e^{-\bar{\alpha}\beta\frac{e^{-2\pi h}}{1-e^{-2\pi h}}} \\ &= \frac{1}{1-e^{-2\pi h}}e^{\bar{\alpha}\beta\frac{1}{1-e^{2\pi h}}}\,,\end{aligned} \tag{F.0.12}$$

as required. Hence, we have

$$\mathrm{Tr}_F \; e^{2\pi i K}\mathcal{O} = \frac{1}{1-e^{-2\pi h}} \exp\left(\frac{\bar{\alpha}\beta}{1-e^{2\pi h}}\right) . \tag{F.0.13}$$

Appendix G

Regularised free fields: selection rules

We assume that $\phi_\mu^{(\epsilon)}(\theta)$ are defined on the finite interval

$$-\frac{\pi}{\epsilon} \leq \theta \leq \frac{\pi}{\epsilon}, \tag{G.0.1}$$

and satisfy the commutation relations

$$\left[\phi_\mu^{(\epsilon)}(\theta_1), \phi_\nu^{(\epsilon)}(\theta_2)\right] = \ln S_{\mu\nu}^{(\epsilon)}(\theta_2 - \theta_1), \tag{G.0.2}$$

where $S_{\mu\nu}^{(\epsilon)}(\theta)$ goes to $S_{\mu\nu}(\theta)$ as $\epsilon \to 0$ for finite θ and

$$S_{\mu\nu}^{(\epsilon)}\left(-\frac{\pi}{\epsilon}\right) = S_{\mu\nu}(-\infty). \tag{G.0.3}$$

Taking into account the formulae from appendix B.2 one finds that

$$\phi_\mu^{(\epsilon)}(\theta) = Q_\mu - \epsilon\theta P_\mu + \phi_\mu^{(\epsilon,osc)}(\theta), \tag{G.0.4}$$

where $\phi_\mu^{(\epsilon,osc)}(\theta)$ is periodic on $[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]$, and the zero modes P_μ, Q_μ commute with $\phi_\mu^{(\epsilon,osc)}(\theta)$ and satisfy the algebra

$$[P_\mu, Q_\nu] = ia_{\mu\nu}, \quad [Q_\mu, Q_\nu] = [P_\mu, P_\nu] = 0, \quad \mu, \nu = 0, 1, \ldots, N, \tag{G.0.5}$$

where $a_{ij} = 2\delta_{ij} - \delta_{i-1,j} - \delta_{i+1,j}$ is the Cartan matrix of type A_{N-1} for $i, j = 1, 2, \ldots, N-1$, and Q_0, P_0, Q_N, P_N are expressed in terms of Q_j, P_j as

$$\begin{aligned} Q_0 &= -\sum_{k=1}^{N-1} \frac{N-k}{N} Q_k\,, \qquad & P_0 &= -\sum_{k=1}^{N-1} \frac{N-k}{N} P_k\,, \\ Q_N &= -\sum_{k=1}^{N-1} \frac{k}{N} Q_k\,, \qquad & P_N &= -\sum_{k=1}^{N-1} \frac{k}{N} P_k\,, \end{aligned} \tag{G.0.6}$$

and therefore $a_{00} = a_{NN} = \frac{N-1}{N} = s$, $a_{01} = a_{N-1,N} = -1$, and all the remaining $a_{\mu\nu} = 0$.

The oscillatory part can be expanded in a Fourier series

$$\phi_\mu^{(\epsilon,osc)}(\theta) = \sum_{m\neq 0} \frac{1}{im} a_\mu(\epsilon m) \exp(im\epsilon\theta)\,, \tag{G.0.7}$$

where $a_\mu(\epsilon m)$ satisfy the following commutation relations

$$[a_\mu(\epsilon m), a_\nu(\epsilon n)] = m f_{\mu\nu}(\epsilon m)\delta_{m+n,0}\,, \quad i, j = 1, 2, \ldots, N-1\,, \tag{G.0.8}$$

where $f_{\mu\nu}(\epsilon m)$ are given in appendix B. In the limit $\epsilon \to 0$ with $\epsilon m = t$ and $\epsilon n = t'$ kept fixed, $\delta_{m+n,0}/\epsilon$ goes to $\delta(t+t')$ and one recovers the previous formulae.

The primed fields are defined in the same way

$$\phi_\mu'^{(\epsilon)}(\theta) = -Q_\mu + \epsilon\theta P_\mu + \phi_\mu'^{(\epsilon,osc)}(\theta)\,. \tag{G.0.9}$$

Notice that since

$$[P_j, \chi_k^-] = -a_{jk}\chi_k^-\,, \quad [P_j, \chi_k^+] = a_{jk}\chi_k^+\,, \tag{G.0.10}$$

χ_k^- is a lowering operator and χ_k^+ is a raising operator.

Thus one can use the formulae from the main text and use the zero modes only to get $N-1$ selection rules from the requirement that no dependence of Q_j should appear in the trace formula. Assuming that we have the trace of the form

$$\mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K} \Big(\prod_{\mu=0}^{N-1} \prod_{k=1}^{n'_\mu} V'_\mu(\theta'_{\mu,k}) \Big) \Big(\prod_{\mu=0}^{N-1} \prod_{k=1}^{n_\mu} V_\mu(\theta_{\mu,k}) \Big) \right]\,, \tag{G.0.11}$$

one gets

$$\frac{N-j}{N}n_0 - n_j - \frac{N-j}{N}n'_0 + n'_j = 0\,, \quad j = 1, \ldots, N-1\,, \tag{G.0.12}$$

or equivalently

$$(N-j)(n_0 - n'_0) - N(n_j - n'_j) = 0\,, \quad j = 1, \ldots, N-1\,. \tag{G.0.13}$$

For $N = 2$ one gets

$$n_0 - n'_0 - 2(n_1 - n'_1) = 0\,, \tag{G.0.14}$$

which agrees with [11].

If one considers $Z_1 Z_2 \cdots Z_N$ one finds that for this operator

$$n_0 = N\,,\ n_1 = N-1, \ldots,\ n_j = N-j\,, \tag{G.0.15}$$

which implies that there is no Q_j in this operator because from (G.0.12) one gets $n_j = \frac{N-j}{N}n_0$. In fact from (G.0.12) one sees that for the identity operator n_0 must be an integer multiple of N. For an arbitrary operator $n_0 - n'_0$ must be an integer multiple of N.

The selection rules take a simpler form if one uses Z_j and Z'_j operators. Then assuming that we have the trace of the form

$$\mathrm{Tr}_{\pi_Z}\left[e^{2\pi i K}\Big(\prod_{j=1}^{N}\prod_{a=1}^{m'_j} Z'_j(\theta'_{j,a})\Big)\Big(\prod_{j=1}^{N}\prod_{a=1}^{m_j} Z_j(\theta_{j,a})\Big)\right]\,, \tag{G.0.16}$$

one gets

$$n_\mu = \sum_{k=\mu+1}^{N} m_k\,, \quad n'_\mu = \sum_{k=\mu+1}^{N} m'_k\,, \quad \mu = 0, \ldots, N-1\,. \tag{G.0.17}$$

Thus the selection rules (G.0.12) take the form

$$\sum_{k=1}^{j}(m_k - m'_k) = \frac{j}{N}(M - M')\,, \quad M = \sum_{k=1}^{N} m_k\,, \quad M' = \sum_{k=1}^{N} m'_k\,, \tag{G.0.18}$$

where M and M' are the total numbers of Z and Z' operators. The selection rules

(G.0.18) then immediately imply

$$m_j - m_j' = \frac{1}{N}(M - M')\,, \quad j = 1, \ldots, N-1\,, \tag{G.0.19}$$

and therefore

$$m_j - m_j' = k\,, \quad M - M' = kN \text{ for some integer } k\,. \tag{G.0.20}$$

This formula shows that if Z transforms in the fundamental irrep of $\mathfrak{su}(N)$ then Z' transforms in the antifundamental irrep (and it would be more appropriate to use upper indices for Z'), and the form factor does not vanish only if the product of all Z and Z' is a singlet of $\mathfrak{su}(N)$ which is a natural requirement.

Appendix H

$\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring Sigma Model

The material given here is not supposed to be a comprehensive overview. It is merely an outline of some of the details mentioned in section 5.1.2 that are relevant to constructing a free field representation. For a much fuller account, see [25]. The first thing to recall is that the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring has a 1+1 dimensional world-sheet, on which a two dimensional sigma model is described. There are two copies of a centrally extended $\mathfrak{psu}(2|2)$ symmetry, although for the purposes of finding the free field representation, looking at the $\mathfrak{su}(2)$ sector would be the best way to start. A result of this $\mathfrak{psu}(2|2)$ symmetry is that there is a useful parameterisation

$$\frac{x^+}{x^-} = e^{ip}, \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}, \tag{H.0.1}$$

where g is the string tension and p is the momentum of the particle state. Although there is no conclusive proof, it is generally assumed that this sigma model is integrable, and as a result we have the Yang-Baxter equation and the Zamolodchikov-Faddeev algebra, which obviously are vital for the free field representation to be successful. The elementary particles have the dispersion relation

$$\epsilon(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}, \tag{H.0.2}$$

and there are also bound states to be considered, which are constructed from a fusion procedure.

One of the main difficulties that this model poses compared to the GN model is that it does not possess Lorentz symmetry. Therefore, the idea of having an S-matrix which depends on the difference of the rapidites of the interacting particles is lost. Instead a more general idea is required. As a replacement, there is the concept of generalised rapidity, z, which resides on an elliptic curve, which is in fact a torus. In terms of these generalised rapidities, the $\mathfrak{su}(2)$ sector S-matrix is conjectured to be

$$S(z_1, z_2) = \frac{x_1^+ \, x_2^-}{x_1^- \, x_2^+} \frac{1}{\sigma(x_1^\pm, x_2^\pm)^2} \frac{u_1 - u_2 - \frac{2i}{g}}{u_1 - u_2 + \frac{2i}{g}}, \tag{H.0.3}$$

where $\sigma(x_1^\pm, x_2^\pm)$ is the dressing phase and the spectral parameters u_k are expressed in terms of $x_k^\pm$ as follows

$$u_k = \frac{1}{2}\left(x_k^+ + \frac{1}{x_k^+} + x_k^- + \frac{1}{x_k^-} \right) = x_k^+ + \frac{1}{x_k^+} - \frac{i}{g} = x_k^- + \frac{1}{x_k^-} + \frac{i}{g}. \tag{H.0.4}$$

It should also be noted that the notation

$$x_i^+ \equiv x^+(z_i)\,, \qquad x_i^- \equiv x^-(z_i)\,, \tag{H.0.5}$$

has been used in the above equations. This S-matrix transforms under both copies of the symmetry algebra. Since a torus has two periodic directions, we have two periods, $2\omega_1$ and $2\omega_2$, which together take the place of $2\pi i$ as the period for the rapidity variable. Of course, the interest in these periods is to define the crossing symmetry. In the relativistic case, we would have (1.18), but here we have the more general crossing equations

$$\begin{aligned} S(z_1, z_2)\, S(z_1 + \omega_2, z_2) &= f(x_1^\pm, x_2^\pm)^2\,, \\ S(z_1, z_2)\, S(z_1, z_2 - \omega_2) &= f(x_1^\pm, x_2^\pm)^2\,, \end{aligned} \tag{H.0.6}$$

where the function $f(x_1^\pm, x_2^\pm)$ is defined by

$$f(x_1^\pm, x_2^\pm) = \frac{(x_1^- - x_2^-)\left(1 - \frac{1}{x_1^- x_2^+}\right)}{(x_1^+ - x_2^-)\left(1 - \frac{1}{x_1^+ x_2^+}\right)}, \tag{H.0.7}$$

with the S-matrix the same as in (H.0.3).

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