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Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type II equations

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Abstract

We consider the ABF background underlying the η-deformed $AdS_5 \times S^5$ sigma model. This background fails to satisfy the standard IIB supergravity equations which indicates that the corresponding sigma model is not Weyl invariant, i.e. does not define a critical string theory in the usual sense. We argue that the ABF background should still define a UV finite theory on a flat 2d world-sheet implying that the η-deformed model is scale invariant. This property follows from the formal relation via T-duality between the η-deformed model and the one defined by an exact type IIB supergravity solution that has 6 isometries albeit broken by a linear dilaton. We find that the ABF background satisfies candidate type IIB scale invariance conditions which for the R–R field strengths are of the second order in derivatives. Surprisingly, we also find that the ABF background obeys an interesting modification of the standard IIB supergravity equations that are first order in derivatives of R–R fields. These modified equations explicitly depend on Killing vectors of the ABF background and, although not universal, they imply the universal scale invariance conditions. Moreover, we show that it is precisely the non-isometric dilaton of the T-dual solution that

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leads, after T-duality, to modification of type II equations from their standard form. We conjecture that the modified equations should follow from κ-symmetry of the η-deformed model. All our observations apply also to η-deformations of $AdS_3 \times S^3 \times T^4$and $AdS_2 \times S^2 \times T^6$models.

1. Introduction

The study of integrable deformations of the $AdS_5 \times S^5$ superstring sigma model is an important direction in the search for new solvable examples of AdS/CFT duality. An interesting one-parameter integrable generalisation of the classical $AdS_5 \times S^5$ Green–Schwarz action related to the quantum deformation of the underlying supergroup symmetry was found in [1]. Just from the construction of this "η-model" (based on a particular current–current deformation of the supercoset action [2] generalising the bosonic model of [3]) there is no a priori reason why it should define a scale invariant (UV finite) 2d theory and, moreover, why it should preserve the conformal (Weyl) invariance and hence still correspond to a consistent superstring theory as the undeformed $AdS_5 \times S^5$ model does.[3]

The only indication in this direction is that the η-model action, like the original $AdS_5 \times S^5$ action, is invariant under a version of fermionic κ-symmetry [1], which reduces the number of fermions by half. However, the usual claim that κ-symmetry implies the corresponding action can be interpreted as that of a GS superstring propagating in a background that is a consistent type II supergravity solution (and hence defines a consistent critical superstring theory) assumes that the κ-symmetry is of the standard GS "projector" form [5]. This is most probably not the case for the η-model at higher orders in fermions. Indeed, it was found in [6,7] that the target space background corresponding to the η-model action [1], interpreted as a GS action, does not represent a type IIB supergravity solution.

Starting with the GS Lagrangian written in superspace form ($Z^M = (x^m, \theta^\alpha)$)

$$L = (\sqrt{h} h^{ab} E^r_M E^s_N \eta_{rs} - \epsilon^{ab} B_{MN}) \partial_a Z^M \partial_b Z^N \ , \tag{1.1}$$

one can solve the standard type II superspace constraints and Bianchi identities for $E(Z)$, $B(Z)$ (which imply the supergravity equations) in order to express the GS action in terms of component fields. One then observes that the dilaton ϕ (which is part of the dilaton superfield $\Phi(Z)$ that is introduced in the process of solving the constraints) enters the world-sheet action (i) in the combination $\mathcal{F} = e^\phi F$ with the R–R field strengths starting at order θ^2 and (ii) via derivatives $\partial_m \phi$ starting at order θ^4 (see [8] and the references therein). This action has classical Weyl invariance and κ-symmetry, which will be broken, in general, by quantum corrections. As for the bosonic string [9], to cancel the 2d stress tensor trace anomaly requires adding the familiar 1-loop dilaton counterterm $\sim \int d^2 z \sqrt{h} R^{(2)} \Phi(Z)$ (see [10,11] and the references therein).[4]

The case relevant to our discussion below is a special isometric type II solution for which the metric G_{mn}, B-field B_{mn} and R–R fields $\mathcal{F}_{m_1 \ldots m_n}$ are invariant while ϕ is linear in the isometric

[3] This is in contrast, e.g., to the integrable deformation [4] based on TsT duality transformations, which preserve conformality. In particular, the TsT deformed background is a solution of type IIB supergravity.

[4] This additional term is certainly required to reproduce the standard 1-loop Weyl-invariance conditions for the G and B-field couplings or supergravity equations in NS–NS sector. This term should also be required to cancel the quantum anomaly of κ-symmetry.

directions. In this case the GS action will depend on the isometric coordinates only through their 2d derivatives and can thus be T-dualised. As we shall see, in this case the T-dual model will be scale invariant but may not be Weyl invariant (one may not be able to cancel the Weyl anomaly by a local counterterm), i.e. may not correspond to a type II supergravity solution.

The ABF background [6,7] includes the 10d metric G, the B-field and the R–R fields $\mathcal{F}_n$ ($n = 1, 3, 5$) that are extracted from the quadratic fermionic part of the action of [1] put into the usual GS form,

$$\begin{aligned}\mathcal{A} = -T \int d^2 z \Big[& \frac{1}{2}(\eta^{ab} G_{mn} - \epsilon^{ab} B_{mn}) \partial_a x^m \partial_b x^n \\ & + i\bar{\theta}\Gamma_m D\theta \, \partial x^m + \bar{\theta}\Gamma_m \, \mathcal{F} \cdot \Gamma \, \Gamma_n \theta \, \partial x^m \partial x^n + \ldots \Big] . \end{aligned} \tag{1.2}$$

For the standard GS action in a type IIB supergravity background $\mathcal{F}_n$ are interpreted as the products of the dilaton and the R–R field strengths $\mathcal{F}_n = e^{\phi} F_n$ but in the η-model case there is no independent information about the dilaton, and there exists no dilaton field that completes $G, B, \mathcal{F}_n$ of the ABF background to a type IIB solution [7].

While not solving the standard type IIB equations directly this ABF background still turns out to be very special: it is related by T-duality to an exact type IIB supergravity solution [12, 13]. The latter HT background involves a non-diagonal metric $\hat{G}$, an imaginary 5-form $\hat{F}_5$ and the dilaton $\hat{\phi}$, and the T-duality applied in all 6 isometric directions acts only on the fields $\hat{G}$ and $\hat{\mathcal{F}}_5 = e^{\hat{\phi}} \hat{F}_5$ entering the corresponding GS action (1.2) on a flat 2d background. The GS action for any type II solution (and thus for the HT background) should be Weyl invariant and, in particular, scale invariant. As the T-duality applied to the GS action [14] is a simple path integral transformation, the T-duality relation between the ABF and HT backgrounds implies [12] that the η-model action should define a scale invariant 2d theory at least to 1-loop order.

However, there may be a problem with Weyl invariance for the η-model action on a curved 2d background. The HT dilaton $\hat{\phi}$ has a term linearly depending on the isometric directions of $\hat{G}$ and $\hat{\mathcal{F}}_5$ and thus one cannot directly apply the standard T-duality transformation rules [15] to the full HT background to get a full T-dual supergravity solution, and thus the Weyl invariance of the T-dual sigma model requires further investigation.[5] This is of course consistent with the observation [7] that the ABF background does not satisfy the full set of type IIB supergravity equations.

The aim of the present paper is to further clarify and extend these observations. We shall demonstrate that the relation by formal T-duality between the ABF and HT backgrounds implies that the former, while not a supergravity solution, should satisfy the following two generalisations or "modifications" of the type II supergravity equations:

(i) the scale invariance conditions for the type II superstring sigma model (with equations on the R–R fields $\mathcal{F}$ being of 2nd order in derivatives)

[5] The 1-loop Weyl invariance conditions of the NSR or GS type II superstring sigma model are believed to be equivalent to the field equations of type II supergravity. While this is a well-established fact in the NS–NS sector [9,16] this was never demonstrated directly with the R–R couplings included (for some related work, mostly for the heterotic string, see [10,11,18]). Given that the linearised equations for all the supergravity fields follow from the condition of marginality of the corresponding NSR vertex operators and that the type II action is a leading term in the string effective action reconstructed from the superstring S-matrix on flat space, it is usually assumed that the superstring sigma model defining consistent critical string theories should correspond (to leading order in α') to backgrounds that solve the 10d supergravity equations.

(ii) a set of equations that are structurally similar to those of type II supergravity (with 1st-order equations for the R–R fields $\mathcal{F}$) but involving, instead of derivatives of the dilaton, a certain co-vector Z_m playing now the role of the dilaton one-form and a Killing vector I^m responsible for the "modification" of the equations from their standard form.[6]

While the scale invariance conditions are universal, the second set of equations (which we shall refer to as "I-modified" type II equations) only apply to particular backgrounds with isometric G, B and $\mathcal{F}$-fields, which are related by formal T-duality to a type II solution $(\hat{G}, \hat{B}, \hat{\mathcal{F}}, \hat{\phi})$ with the dilaton $\hat{\phi}$ containing a term linear in the isometric coordinates. Such a dilaton background, breaking isometries by a linear term only, is special. As the type II supergravity equations written in terms of the $\mathcal{F}$-fields only depend on the dilaton through its *derivatives*, they remain independent of the isometric directions. As a result, the standard type II supergravity equations for the T-dual solution $(\hat{G}, \hat{B}, \hat{\mathcal{F}}, \hat{\phi})$ can be re-interpreted as certain modified type II equations for the original fields $(G, B, \mathcal{F})$, also depending on the vectors Z and I. The Killing vector I^m dependence is fixed by the term linear in isometric coordinates in the dilaton, while the vector Z_m is determined by applying the standard T-duality rules to the part of the dilaton independent of the isometric coordinates.[7]

It is possible to express the modified equations for the NS–NS fields in terms of just one single vector $X_m = Z_m + I_m$, which is the vector that appears in the scale invariance conditions. The superstring scale invariance conditions generalise the familiar one-loop scale invariance conditions for the bosonic sigma model with couplings G_{mn} and B_{mn} (cf. (1.2))

$$\beta^G_{mn} \equiv R_{mn} - \tfrac{1}{4} H_{mkl} H_n{}^{kl} = -D_m X_n - D_n X_m \ , \tag{1.3}$$

$$\beta^B_{mn} \equiv \tfrac{1}{2} D^k H_{kmn} = X^k H_{kmn} + \partial_m Y_n - \partial_m Y_n \ . \tag{1.4}$$

Here the terms involving X_m [17] and Y_m do not contribute to on-shell UV divergences or, equivalently, reflect the freedom of renormalisation by reparametrisations and B-field gauge transformations. The X_m terms drop out of the action if the sigma model field x^m is subject to the classical equations, or, equivalently, they can be absorbed in a field renormalisation, $x^m \to x^m + X^m \log\epsilon$. The origin of the $X^k H_{kmn}$ term can be understood either by starting with a counterterm proportional to $(D_m X_n + D_n X_m)\partial_a x^n \partial_b x^n + \ldots$, integrating by parts and using the equations of motion for x^m, or by observing that B_{mn} transforms under a combination of reparametrisations and gauge transformations as $X^k \partial_k B_{mn} + \partial_m X^k B_{kn} - \partial_n X^k B_{km} + \partial_m Y'_n - \partial_n Y'_m = X^k H_{kmn} + \partial_m Y_n - \partial_n Y_m$ where Y'_m or Y_m drop out of the sigma model action upon integration by parts.

The Weyl invariance conditions are equivalent to the vanishing of the trace of the 2d stress tensor operator on a curved 2d background. For the NSR type II superstring sigma model they can be satisfied provided one adds the dilaton term $\sim R^{(2)}\phi(x)$ [9,19–21,16]: they are a stronger form of the scale invariance conditions (1.3), (1.4) with X_m and Y_m no longer arbitrary, but given by

$$X_m = \partial_m \phi \ , \qquad\qquad Y_m = 0 \ . \tag{1.5}$$

The Weyl invariance equations (1.3), (1.4), (1.5) imply the "central charge" identity [9,22]

$$\partial_m \bar{\beta}^\phi = 0 \ , \qquad \bar{\beta}^\phi \equiv R - \tfrac{1}{12} H^2_{mnk} + 4D^2\phi - 4\partial^m\phi\partial_m\phi \ , \tag{1.6}$$

[6] In what follows we shall not distinguish between co-vectors and vectors, referring to both X^m and X_m as vectors.

[7] Let us stress that $\hat{F}$ and $\hat{\phi}$ explicitly depend on the isometric coordinates. It is $\hat{G}$, $\hat{B}$, $\hat{\mathcal{F}}$, $d\hat{\phi}$, and G, B, $\mathcal{F}$, Z that are invariant under the isometries generated by the Killing vectors $\hat{I}^m$ and I^m respectively. That is, Lie derivatives of the fields along the corresponding Killing vector are zero.

i.e. that the effective dilaton "β-function" is a constant (which should be zero in critical string theory). The full set of Weyl invariance equations for G, B and ϕ follows from the effective action with the same form as the NS–NS sector of the type II supergravity action ($\mathcal{F} \equiv e^{\phi} F$)

$$S = \int d^d x \sqrt{G} \left(e^{-2\phi} \bar{\beta}^{\phi} + \sum F F + \ldots \right)$$
$$= \int d^d x \sqrt{G}\, e^{-2\phi} \left(\bar{\beta}^{\phi} + \sum \mathcal{F}\mathcal{F} + \ldots \right), \tag{1.7}$$

where we have indicated the presence of the R–R field strength terms for future reference.

The generalisation of the scale invariance conditions to the presence of R–R fields is given by (1.3), (1.4) with extra $\mathcal{F}\mathcal{F}$ terms, together with a set of *second*-derivative equations for the R–R fields $\mathcal{F}$ that directly enter the GS action (1.2), $\frac{1}{2} D^2 \mathcal{F} + \ldots = X \partial \mathcal{F} + \mathcal{F} \partial X$. Here the r.h.s. stands for reparametrisation (Lie derivative) terms with the same X-vector as in (1.3), (1.4) and dots indicate non-linear terms. In the special case when $X_m = \partial_m \phi$ these equations are the consequence of the type IIB equations or Weyl invariance conditions, which are 1st order in $F = e^{-\phi} \mathcal{F}$, i.e. $d \star F + \ldots = 0$ and $dF + \ldots = 0$.[8] These universal scale invariance conditions will be satisfied by the ABF background for a particular choice of the vectors X_m and Y_m.

To explain the origin of the second "I-modified" set of equations let us first ignore the R–R fields and assume that there exists the following metric-dilaton background that solves the Weyl invariance equations (i.e. $R_{mn} + 2 D_m D_n \phi = 0$, $\bar{\beta}^{\phi} = \text{const}$)

$$\hat{ds}^2 = e^{2\hat{a}(x)} [d\hat{y} + \hat{A}_{\mu}(x) dx^{\mu}]^2 + g_{\mu\nu}(x) dx^{\mu} dx^{\nu}, \qquad \hat{\phi} = -c\,\hat{y} + f(x). \tag{1.8}$$

Here the metric has an isometry which is broken by the linear term in the dilaton ($c = \text{const}$). Examples of such non-trivial solutions[9] can be found by taking special limits of gauged WZW backgrounds [13]. T-dualising this metric, we find a diagonal metric G and B-field, i.e.

$$ds^2 = e^{2a(x)} dy^2 + g_{\mu\nu}(x) dx^{\mu} dx^{\nu}, \qquad B = \hat{A}_{\mu}(x)\, dy \wedge dx^{\mu}, \qquad a = -\hat{a}. \tag{1.9}$$

For $c = 0$ (i.e. when $\hat{\phi}$ is isometric) these fields together with the T-duality transformed dilaton $\phi = \hat{\phi} - \hat{a}$ would solve the standard Weyl invariance equations (1.3), (1.4) with $X_m = \partial_m \phi$, $Y_m = 0$. For non-zero c the equation $\hat{R}_{mn} + 2 \hat{D}_m \hat{D}_n \hat{\phi} = 0$ (for the original background (1.8)) expressed in terms of the dual fields G, B will contain additional c-dependent terms obstructing (for non-constant $a(x)$) the introduction of a new dilaton scalar. Still, they can be put in a more general form $R_{mn} + D_m X_n + D_n X_m = 0$ with a special vector X given by[10]

$$X_m dx^m \equiv I_m dx^m + Z_m dx^m = c\, e^{-2a}\, dy + \left[\partial_{\mu} (\hat{\phi} - \hat{a}) + c\, \hat{A}_{\mu} \right] dx^{\mu}. \tag{1.10}$$

The dilaton equation $\bar{\beta}^{\phi} = 0$ for the original background (1.8) also can be rewritten as the following generalised equation (cf. (1.6))[11]

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[8] The relation between the 1st-order and 2nd-order equations on $\mathcal{F}$ has the same spirit as the relation between the Dirac and the Klein–Gordon (squared Dirac) equations for spinor fields.

[9] It is important that dilaton has a linear term in a "warped" isometric direction of the metric, i.e. $a(x)$, $A_{\mu}(x)$ are non-constant, otherwise the effect of adding the linear dilaton would be trivial.

[10] The need to introduce the vector X_m, which is *not* simply a gradient of a scalar, is therefore directly related to the feature $\partial_{\hat{y}} \hat{\phi} = -c \neq 0$.

[11] Note that this equation is not present in the list of scale invariance conditions, and Weyl invariance conditions require this relation to hold with $X_m = \partial_m \phi$ for some ϕ.

$$\bar\beta^X \equiv R - \tfrac{1}{12}H^2_{mnk} + 4D^m X_m - 4X^m X_m = 0\,, \tag{1.11}$$

that is satisfied for the T-dual background.

The T-dual background (G, B) defines a sigma model that is scale invariant on a flat 2d background (satisfying equations (1.3), (1.4) with $Y_m = X_m$) but which is not Weyl invariant. The trace of stress tensor $T = \beta^G_{mn}\partial_a x^m \partial^a x^n + \beta^B_{mn}\epsilon^{ab}\partial_a x^m \partial_b x^n$ is a total derivative $T = \nabla^a N_a$, $N_a = 2(X_m\partial_a x^m + \epsilon_a{}^b Y_m \partial_b x^m)$ (up to terms proportional to the x^m equations of motion). This cannot be cancelled by a local counterterm (the classical dilaton term) unless $X_m = \partial_m\phi$, $Y_m = 0$ [19,20], which is not the case for the ABF background. The sigma models based on (1.9) (with explicit backgrounds given below) thus represent particular examples of 2d scale invariant theories that avoid the Zamolodchikov–Polchinski theorem [23] due to their non-compactness (and/or non-unitarity related to the presence of time-like directions). It thus remains unclear if such backgrounds related by formal T-duality to Weyl invariant models (1.8) can also be associated somehow with a consistent critical string theory.

As we shall see below, a similar generalisation of the full set of the bosonic type II supergravity equations also exists in the presence of R–R fields $\mathcal{F}_n$ that have the same isometries as the metric (i.e. when (1.8) is extended to an analog of the HT solution [12]). Thus in general, given a type II solution with non-isometric linear dilaton there will be an associated ("T-dual" or ABF-like) background solving such a modified set of type II equations.

The rest of this paper is organised as follows. In section 2 we shall present the general scale invariance conditions for the couplings G, B of the sigma model (1.2) that generalise (1.3), (1.4) to the presence of the R–R fields $\mathcal{F}$ and show that there exist such vectors $X = Z + I$ and Y that these equations are satisfied by the ABF background. In section 3 we shall derive a modification of the standard 1st-order IIB supergravity equations of the R–R fluxes that is "driven" by the special isometry vector I and which are satisfied by the ABF background. In section 4 we shall show that combining these 1st-order equations one can find 2nd-order equations for $\mathcal{F}$ that have the right structure (when generalised to arbitrary vector X) to be interpreted as scale invariance conditions on the R–R couplings. In section 5 we explain how the standard type II supergravity equations for a solution with the dilaton linear along the isometric directions is mapped to the modified equations for T-dual solution.

Our notation and some useful relations are summarised in Appendix A. In Appendix B we present the explicit form of the ABF background and the T-dual type IIB HT solution. Appendix C contains the derivation of the identity $\partial_m\bar\beta^X = 0$ from the modified type II equations which is closely related to the on-shell conservation of R–R stress tensor. In Appendix D, starting with the modified type II equations, we derive the 2nd-order equations for the R–R fields that are candidates for the corresponding scale invariance conditions. In Appendix E we remark on an alternative derivation of the relation (2.13) for the vector Z, which plays the role of the dilaton one-form in the modified equations. In Appendix F we summarise the analogs of the ABF and HT backgrounds in the $AdS_2 \times S^2 \times T^6$and $AdS_3 \times S^3 \times T^4$cases and give the corresponding expressions for the vectors X, Y and I that solve the scale invariance and modified type II equations. In Appendix G we explain how the 2nd-order equations for the R–R couplings $\mathcal{F}$ emerge as the one-loop conditions of scale invariance for the GS sigma model (1.2).

2. Scale invariance conditions and modified type II equations: NS–NS sector

The scale invariance conditions for the bosonic sigma model (1.3), (1.4) have a straightforward generalisation to the GS superstring case with non-zero R–R couplings $\mathcal{F} = e^\phi F$ (see

Appendix G). The $\bar{\theta}\mathcal{F}\theta\partial x\partial x$ terms in the GS action (1.2) should lead to one-loop diagrams (with one bosonic and one fermionic line) contributing logarithmic UV divergences $\sim \mathcal{F}\mathcal{F}\partial x\partial x$. These terms will produce extra $\mathcal{F}\mathcal{F}$ terms in the β-functions in (1.3) and (1.4). In particular, the analog of the Einstein equation (1.3) should pick up the R–R stress tensor term and the B-field equation (1.4), the $\mathcal{F}\mathcal{F}$ term as in the II supergravity equations.[12] This is expected as for $X_m = \partial_m\phi$, $Y_m = 0$ the resulting equations are the Weyl invariance equations that should be equivalent to the type II supergravity equations.

The scale invariance equations for the $\mathcal{F}$-fields (to be discussed in section 4) will not, however, have the familiar supergravity form of 1st-order equations for $\mathcal{F}$ (these should follow from the Weyl invariance conditions). Instead they will be of 2nd order, $D^2\mathcal{F} + \ldots = X$-dependent terms, and for $X_m = \partial_m\phi$ will be a consequence of the 1st-order supergravity equations.

Explicitly, the scale invariance conditions (1.3) and (1.4) generalise to

$$\beta^G_{mn} \equiv R_{mn} - \tfrac{1}{4}H_{mkl}H_n{}^{kl} - \mathcal{T}_{mn} = -D_m X_n - D_n X_m \ , \tag{2.1}$$

$$\beta^B_{mn} \equiv \tfrac{1}{2}D^k H_{kmn} + \mathcal{K}_{mn} = X^k H_{kmn} + \partial_m Y_n - \partial_n Y_m \ , \tag{2.2}$$

$$\mathcal{T}_{mn} \equiv \tfrac{1}{2}\mathcal{F}_m\mathcal{F}_n + \tfrac{1}{4}\mathcal{F}_{mpq}\mathcal{F}_n{}^{pq} + \tfrac{1}{4\times4!}\mathcal{F}_{mpqrs}\mathcal{F}_n{}^{pqrs} - \tfrac{1}{2}G_{mn}(\tfrac{1}{2}\mathcal{F}_k\mathcal{F}^k + \tfrac{1}{12}\mathcal{F}_{kpq}\mathcal{F}^{kpq}) \ , \tag{2.3}$$

$$\mathcal{K}_{mn} \equiv \tfrac{1}{2}\mathcal{F}^k\mathcal{F}_{kmn} + \tfrac{1}{12}\mathcal{F}_{mnklp}\mathcal{F}^{klp} \ . \tag{2.4}$$

Here $\mathcal{F}_m$, $\mathcal{F}_{mnk}$, $\mathcal{F}_{mnklp}$ are R–R fields of type IIB supergravity (for notation see Appendix A). For $X_m = \partial_m\phi$, $Y_m = 0$ these equations follow from type IIB supergravity action (1.7). $\mathcal{T}_{mn}$ is the familiar stress tensor that follows from the type IIB action (1.7) upon variation over G_{mn}.[13]

As was noted in [12], the existence of the HT solution related to the ABF background by T-duality, suggests that the GS sigma model for the latter defined on a flat 2d background should be scale invariant (at least to leading, 1-loop, order). Our key observation is that indeed there exist vectors X_m and Y_m such that eqs. (2.1) and (2.2) are satisfied for the ABF background (B.1). The vector X_m required to satisfy (2.1) turns out to be (see Appendix B for notation)

$$\begin{aligned} X \equiv X_m dx^m = {} & c_0\,\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt + c_1\,\rho^2\sin^2\zeta\, d\psi_2 + c_2\,\frac{\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1 \\ & + c_3\,\frac{1-r^2}{1+\varkappa^2 r^2}d\varphi + c_4\, r^2\sin^2\xi\, d\phi_2 + c_5\,\frac{r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi}d\phi_1 \\ & + \frac{\varkappa^2\rho^4\sin 2\zeta}{2(1+\varkappa^2\rho^4\sin^2\zeta)}d\zeta + \frac{1}{\rho}\Big(1 - \frac{3}{1-\varkappa^2\rho^2} + \frac{2}{1+\varkappa^2\rho^4\sin^2\zeta}\Big)d\rho \\ & + \frac{\varkappa^2 r^4\sin 2\xi}{2(1+\varkappa^2 r^4\sin^2\xi)}d\xi + \frac{1}{r}\Big(1 - \frac{3}{1+\varkappa^2 r^2} + \frac{2}{1+\varkappa^2 r^4\sin^2\xi}\Big)dr \ . \end{aligned} \tag{2.5}$$

X_m can be split in the following way

$$X_m = I_m + Z_m \ , \qquad D_m I_n + D_n I_m = 0 \ , \qquad D^m I_m = 0 \ , \tag{2.6}$$

[12] For an argument supporting this in the NSR formalism see [18].

[13] Note that in the first (NS–NS) term of (1.7) one does not need to vary the $\sqrt{G}$ factor as its contribution vanishes after use of the dilaton equation $\bar{\beta}^\phi = 0$ in (1.6). This equation is not required for scale invariance.

where $I^m = \sum_{i=1}^{6} c_i (I^{(i)})^m$. The index i labels the 6 isometric directions $y^i = (t, \psi_2, \psi_1, \varphi, \phi_2, \phi_1)$ of the 10d ABF metric and c_i are arbitrary constant coefficients. $(I^{(i)})^m$ are the 6 independent commuting Killing vectors of the ABF background: the Lie derivatives of the G, B and $\mathcal{F}$-fields in [6] along I^m all vanish. If we split the coordinates as $x^m = (y^i, x^\mu)$ where $\mu = 1, 2, 3, 4$ labels the non-isometric directions $x^\mu = (\zeta, \rho, \xi, r)$, then

$$I_m = \sum_{i=1}^{6} \delta^i_m c_i G_{ii}(x^\mu) \ , \qquad I^m = \delta^i_m c_i = \text{const} \ , \qquad Z_m = \delta^\mu_m Z_\mu(x^\nu) \ . \tag{2.7}$$

The vector Y_m required to satisfy (2.2) on the ABF background is found to be[14]

$$\begin{aligned} Y \equiv Y_m dx^m = {} & 4\varkappa \frac{1+\rho^2}{1-\varkappa^2\rho^2} dt + 2\varkappa \frac{\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} d\psi_1 \\ & + 4\varkappa \frac{1-r^2}{1+\varkappa^2 r^2} d\varphi - 2\varkappa \frac{r^2\cos^2\xi}{1+\varkappa^2 r^4 \sin^2\xi} d\phi_1 \\ & + \frac{\varkappa^2\rho^4\sin 2\zeta}{2(1+\varkappa^2\rho^4\sin^2\zeta)} d\zeta + \frac{1}{\rho}\Big(1 - \frac{3}{1-\varkappa^2\rho^2} + \frac{2(\varkappa^{-1}c_2 - 1)}{1+\varkappa^2\rho^4\sin^2\zeta}\Big) d\rho \\ & + \frac{\varkappa^2 r^4 \sin 2\xi}{2(1+\varkappa^2 r^4\sin^2\xi)} d\xi + \frac{1}{r}\Big(1 - \frac{3}{1+\varkappa^2 r^2} - \frac{2(\varkappa^{-1}c_5 + 1)}{1+\varkappa^2 r^4 \sin^2\xi}\Big) dr \ . \end{aligned} \tag{2.8}$$

We observe that if we fix c_i in (2.5) to the following specific values

$$c_0 = c_3 = 4\varkappa \ , \qquad c_1 = c_4 = 0 \ , \qquad c_2 = -c_5 = 2\varkappa \ , \tag{2.9}$$

then Y_m and X_m coincide

$$Y_m = X_m \ . \tag{2.10}$$

The next surprising observation is that for these specially chosen values of c_i in (2.9) the vector X_m satisfies also a direct generalisation (1.11) of the dilaton equation (1.6) ($\partial_m\phi \to X_m$)[15]:

$$\bar\beta^X \equiv R - \tfrac{1}{12} H^2_{mnk} + 4 D_k X^k - 4 X_k X^k = 0 \ . \tag{2.11}$$

As we shall show in Appendix C this $\bar\beta^X$ satisfies the generalisation of the dilaton identity (1.6)

$$\partial_m \bar\beta^X = 0 \ . \tag{2.12}$$

The reason for this particular choice of c_i in (2.9) can be traced to the form of the linear terms in the dilaton $\hat\phi$ of the T-dual HT solution (B.3). That is the presence of the I-term in X_m in (2.6) reflects the presence of the non-isometric linear terms in $\hat\phi$. Therefore, these terms drive the modification of the equations satisfied by the ABF background from their standard type II form. In this sense the Z_m part of X_m may be interpreted as the analog of $\partial_m\phi$ in the modified equations. Indeed, one can check that for I^m in (2.7) with c_i chosen as in (2.9) the following relation is satisfied

$$\partial_m Z_n - \partial_n Z_m + I^k H_{kmn} = 0 \ . \tag{2.13}$$

[14] Y is of course defined modulo a total derivative.

[15] Since $D_n X^n = D_\mu Z^\mu$, $X^m X_m = G^{ij} c_i c_j + G^{\mu\nu} Z_\mu Z_\nu$ this equation does not depend on signs of c_i.

This may be interpreted as a modified "dilaton Bianchi identity": if I_m is formally set to zero then Z_m becomes a derivative of a scalar, $\partial_m\phi$. In general, assuming that I_m represents an isometry of the B-field, i.e. the Lie derivative $(\mathcal{L}_I B)_{mn} = I^k\partial_k B_{mn} + B_{kn}\partial_m I^k - B_{km}\partial_n I^k$ vanishes (modulo a gauge transformation term $\partial_m U_n - \partial_n U_m$), we can solve (2.13) as[16]

$$Z_m = \partial_m\phi + B_{km}I^k \,, \tag{2.14}$$

where $\partial_m\phi$ term represents the trivial "zero-mode" solution. In the particular case of the ABF background with Z_m and I_m given by (2.5), (2.6), (2.7) and c_i fixed as in (2.9) we find

$$X_m = Y_m = I_m + Z_m = \partial_m\phi + (G_{km} + B_{km})I^k \,, \tag{2.15}$$

$$\phi = \tfrac{1}{2}\log\frac{(1-\kappa^2\rho^2)^3(1+\kappa^2 r^2)^3}{(1+\kappa^2\rho^4\sin^2\zeta)(1+\kappa^2 r^4\sin^2\xi)} \,. \tag{2.16}$$

The scalar ϕ in (2.16) is precisely the one that is found [12] by applying the standard T-duality transformation rule to the isometric part of the dilaton $\hat\phi$ of the HT solution in (B.3) (cf. (1.10)).

3. Modified type II equations: first-order equations for R–R couplings

Let us now explore what modification of the type IIB equations for the R–R couplings is satisfied by the ABF background.

The standard equations of type IIB supergravity [28] in the R–R sector written in terms of the rescaled $\mathcal{F} = e^\phi F$ field strengths are pairs of dynamical equations and Bianchi identities (see Appendix A for notation)[17]

$$D^m\mathcal{F}_m - Z^m\mathcal{F}_m - \tfrac{1}{6}H^{mnp}\mathcal{F}_{mnp} = 0\,, \qquad d\mathcal{F}_1 - Z\wedge\mathcal{F}_1 = 0, \tag{3.1}$$

$$D^p\mathcal{F}_{pmn} - Z^p\mathcal{F}_{pmn} - \tfrac{1}{6}H^{pqr}\mathcal{F}_{mnpqr} = 0, \qquad d\mathcal{F}_3 - Z\wedge\mathcal{F}_3 + H_3\wedge\mathcal{F}_1 = 0\,, \tag{3.2}$$

$$D^r\mathcal{F}_{rmnpq} - Z^r\mathcal{F}_{mnpq} + \tfrac{1}{36}\varepsilon_{mnpqrstuvw}H^{rst}\mathcal{F}^{uvw} = 0, \quad d\mathcal{F}_5 - Z\wedge\mathcal{F}_5 + H_3\wedge\mathcal{F}_3 = 0. \tag{3.3}$$

Here $Z = Z_m dx^m = d\phi$ is the dilaton one-form. The five-form $\mathcal{F}_5$ is also required to satisfy the self-duality equation $\star\mathcal{F}_5 = \mathcal{F}_5$ which implies the equivalence of the first and second equation in (3.3).

An a priori surprising observation is that there exist direct generalisations of the 1st-order equations (3.1)–(3.3) involving $Z = Z_m dx^m$ and $I = I_m dx^m$ in (2.5), (2.6), with fixed values of the coefficients c_i as given in (2.9), which are solved by the ABF background (B.1). Explicitly, the equations for the one-form $\mathcal{F}_1$ in (B.1) are

$$D^m\mathcal{F}_m - Z^m\mathcal{F}_m - \tfrac{1}{6}H^{mnp}\mathcal{F}_{mnp} = 0\,, \qquad I^m\mathcal{F}_m = 0\,, \tag{3.4}$$

$$(d\mathcal{F}_1 - Z\wedge\mathcal{F}_1)_{mn} - I^p\mathcal{F}_{mnp} = 0\,. \tag{3.5}$$

[16] In general, we find $Z_m = \partial_m\phi + B_{km}I^k - U_m$. Under gauge transformations of B the vector U_m transforms so that ϕ may be assumed to be invariant. In the particular case of the ABF background (B.1) with the B-field chosen in the manifestly symmetric form we have $U_m = 0$.

[17] Note that all equations including (2.2) are invariant under the simultaneous change of sign of H_3 and F_3, or of H_3, F_1 and F_5. The choice of sign of H_3 or B can be changed by parity.

We have added the condition $I^m \mathcal{F}_m = 0$ as an independent equation on $\mathcal{F}_1$.[18]

Similarly, the equations that generalise (3.2) and are satisfied for the three-form $\mathcal{F}_3$ in (B.1) are found to be

$$D^p \mathcal{F}_{pmn} - Z^p \mathcal{F}_{pmn} - \tfrac{1}{6} H^{pqr} \mathcal{F}_{mnpqr} - (I \wedge \mathcal{F}_1)_{mn} = 0\,, \tag{3.6}$$

$$(d\mathcal{F}_3 - Z \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{mnpq} - I^r \mathcal{F}_{mnpqr} = 0\,. \tag{3.7}$$

The equations satisfied by $\mathcal{F}_5$ of the ABF background are found to be

$$D^r \mathcal{F}_{rmnpq} - Z^r \mathcal{F}_{rmnpq} + \tfrac{1}{36} \varepsilon_{mnpqrstuvw} H^{rst} \mathcal{F}^{uvw} - (I \wedge \mathcal{F}_3)_{mnpq} = 0\,, \tag{3.8}$$

$$(d\mathcal{F}_5 - Z \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{mnpqrs} + \tfrac{1}{6} \varepsilon_{mnpqrstuvw} I^t \mathcal{F}^{uvw} = 0\,. \tag{3.9}$$

These two are equivalent in view of the self-duality of $\mathcal{F}_5$.

These modified equations (3.4)–(3.9) reduce back to (3.1), (3.2), (3.3) if we drop all terms with I_m and assume that $dZ = 0$, i.e. if we set

$$Z_m \to \partial_m \phi\,, \qquad I_m \to 0\,. \tag{3.10}$$

The structure of (3.4)–(3.9) supports the interpretation of Z as a generalised "dilaton one-form", while the isometry vector I effectively drives the deformation of the standard type IIB equations.

An interesting observation is that there exist certain combinations of the equations (3.4)–(3.9) that depend on Z and I only through the combination $X = Z + I$, which entered the NS–NS equations of the previous section. These are found by adding together equations of equal form degree, for example, the equation of motion for the R–R three-form and the Bianchi identity for the R–R one-form. The resulting X-dependent equations are given by

$$D^m \mathcal{F}_m - X^m \mathcal{F}_m - \tfrac{1}{6} H^{mnp} \mathcal{F}_{mnp} = 0\,, \tag{3.11}$$

$$D^p \mathcal{F}_{pmn} - X^p \mathcal{F}_{pmn} - \tfrac{1}{6} H^{pqr} \mathcal{F}_{mnpqr} + (d\mathcal{F}_1 - X \wedge \mathcal{F}_1)_{mn} = 0\,, \tag{3.12}$$

$$\begin{aligned} &D^r \mathcal{F}_{rmnpq} - X^r \mathcal{F}_{rmnpq} + \tfrac{1}{36} \varepsilon_{mnpqrstuvw} H^{rst} \mathcal{F}^{uvw} + (d\mathcal{F}_3 - X \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{mnpq} \\ &\quad = 0\,. \end{aligned} \tag{3.13}$$

Using the self-duality of $\mathcal{F}_5$ the last equation can be also written as

$$(d\mathcal{F}_5 - X \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{pqrlmn} - \tfrac{1}{6} \varepsilon_{pqrlmnvstu} (D^v \mathcal{F}^{stu} - X^v \mathcal{F}^{stu} - \mathcal{F}^v H^{stu}) = 0\,. \tag{3.14}$$

As will be discussed below, these three equations are already sufficient for deriving candidates for the scale invariance equations for the $\mathcal{F}$-fields, which are 2nd order in derivatives.

It is useful to rewrite (3.1)–(3.3) in the notation of forms (see Appendix A for conventions). To do so we introduce the dual forms defined by

$$\mathcal{F}_1 = \star \mathcal{F}_9\,, \qquad \mathcal{F}_3 = -\star \mathcal{F}_7\,, \qquad \mathcal{F}_5 = \star \mathcal{F}_5\,, \qquad \mathcal{F}_7 = -\star \mathcal{F}_3\,, \qquad \mathcal{F}_9 = \star \mathcal{F}_1\,. \tag{3.15}$$

Then the complete set of the type II supergravity equations for R–R strengths and Bianchi identities (3.1)–(3.3) is given by[19]

[18] Alternatively, one can derive this equation from the Bianchi equation (3.5), the invariance of $\mathcal{F}_1$ under the isometry, the orthogonality of I and Z, and the condition that Z is not an exact one-form. Indeed, multiplying (3.5) by I^m one finds $\partial_n(I^m \mathcal{F}_m) - Z_n I^m \mathcal{F}_m = 0$. Thus, if $I^m \mathcal{F}_m \neq 0$ then $Z = d\ln(I^m \mathcal{F}_m)$. We find, however, it more convenient to add $I^m \mathcal{F}_m = 0$ as an independent equation, and infer from it the orthogonality of I and Z.

[19] We assume that $\mathcal{F}_n = 0$ for $n < 0$ and $n > 10$.

$$d\mathcal{F}_{2n+1} - Z \wedge \mathcal{F}_{2n+1} + H_3 \wedge \mathcal{F}_{2n-1} = 0\,, \qquad n = 0, 1, \dots\,,$$
$$d \star \mathcal{F}_{2n+1} - Z \wedge \star\mathcal{F}_{2n+1} - H_3 \wedge \star\mathcal{F}_{2n+3} = 0\,, \qquad n = 0, 1, \dots\,, \tag{3.16}$$

where $Z = d\phi$.

The "I-modified" equations (3.4)–(3.9) are given by[20]

$$d\mathcal{F}_{2n+1} - Z \wedge \mathcal{F}_{2n+1} + H_3 \wedge \mathcal{F}_{2n-1} - \star(I \wedge \star\mathcal{F}_{2n+3}) = 0\,, \qquad n = -1, 0, \dots\,,$$
$$d \star \mathcal{F}_{2n+1} - Z \wedge \star\mathcal{F}_{2n+1} - H_3 \wedge \star\mathcal{F}_{2n+3} + \star(I \wedge \mathcal{F}_{2n-1}) = 0\,, \qquad n = 0, 1, \dots\,. \tag{3.17}$$

Due to (3.15) the two equations in (3.16) are equivalent and the same is true for (3.17).

Let us note that the deformed R–R equations (3.17) together with the relation (2.13) or $dZ + \iota_I H_3 = 0$ imply the following relation

$$\mathcal{L}_I \mathcal{F}_{2n+1} = (I \cdot Z)\mathcal{F}_{2n+1}\,. \tag{3.18}$$

Thus the condition that the $\mathcal{F}$-fields are invariant under the isometry I is equivalent to the condition $I \cdot Z = 0$, which is clearly satisfied for the ABF background as is evident from (2.5), (2.7).

4. Second-order equations for R–R couplings as scale invariance conditions

Let us return to the discussion of the scale invariance conditions for the couplings of the GS sigma model (1.2) in section 2 and consider the equations for the R–R couplings $\mathcal{F}$ that should follow from the requirement of (1-loop) UV finiteness of the 2d model. One can argue that the conditions analogous to eqs. (2.1), (2.2) for the G and B-field couplings should have the form

$$\beta^{\mathcal{F}}_{k_1 \dots k_s} \equiv \frac{1}{2} D^2 \mathcal{F}_{k_1 \dots k_s} + \dots = X^m \partial_m \mathcal{F}_{k_1 \dots k_s} + \sum_i \mathcal{F}_{k_1 \dots m \dots k_s} \partial_{k_i} X^m\,, \tag{4.1}$$

where we have omitted possible non-linear terms such as $R\mathcal{F} + DH\mathcal{F} + \dots$ on the l.h.s. The X-dependent Lie derivative term on the r.h.s. reflects, as in (2.1), (2.2), the reparametrisation (or off-shell x^m-renormalisation) freedom. For example, starting with the linearised RG equation $\frac{d\mathcal{F}_n(x)}{dt} = \beta^{\mathcal{F}}_n = \frac{1}{2}\partial^2 \mathcal{F}_n(x)$, $t = \log \epsilon$ and doing the coordinate redefinition $x^m \to x^m + tX^m$, one ends up with $\frac{d\mathcal{F}_n(x)}{dt} = \frac{1}{2}\partial^2 \mathcal{F}_n(x) - X^m \partial_m \mathcal{F}_n - \mathcal{F}_m \partial_n X^m$.

We shall discuss the computation of 1-loop logarithmic UV divergences for the GS action (1.2) in Appendix G clarifying the structure of $\beta^{\mathcal{F}}$.

For $X_m = \partial_m \phi$ the equations (4.1) should be a consequence of stronger Weyl invariance conditions,[21] which should be equivalent to the type II supergravity equations (3.1)–(3.3) or (3.16) where $Z = X = d\phi$. Indeed, combining ("squaring") the familiar $dF + \dots = 0$, $d \star F + \dots = 0$

[20] Note that here we include $n = -1$ as in the deformed theory it is no longer trivial: it gives the second equation in (3.4), i.e. $\star(I \wedge \star\mathcal{F}_1) = I^m \mathcal{F}_m = 0$.

[21] For example, using the NSR approach on a flat background we may consider the R–R vertex operators built out of spin operators and consider the linearised conditions for conformal invariance (marginality). Then $dF = 0$, $d \star F = 0$ will follow (see, e.g., [29]) just like the usual transversality conditions on the graviton operator follow from the marginality conditions of the $h_{mn}(p)e^{ipx}\partial x^m \partial x^n$ vertex. On a curved 2d background these are equivalent to the decoupling of derivatives $\partial_a \rho$ of the conformal factor of the 2d metric (see, e.g., [30,31]). These conditions are stronger than just scale invariance which requires only "masslessness" $p^2 F(p) = 0$ or $\partial^2 F = 0$.

equations leads to $d\star d\star F+\star d\star dF+\ldots=0$ or $D^2F+\ldots=0$, where the leading term is the Hodge–de Rham operator.

Moreover, the same equations should follow also from the modified type II equations (3.4)–(3.9) or (3.17) (as, e.g., the ABF background that solves the modified equations should also be a solution of the scale invariance conditions). This should provide a non-trivial consistency check: after properly "squaring" (3.4)–(3.9) the dependence on the Z and I vectors in any candidate scale invariance equations should appear only through their sum $X=Z+I$ as in (2.1), (2.2).

Starting from the modified type II equations (3.4)–(3.9) (which include the standard type IIB supergravity equations as a special case (3.10), $I_m=0$), let us outline the derivation of the 2nd order equations for the R–R couplings that should be equivalent to the scale invariance conditions for $\mathcal{F}_n$ of the GS sigma model (1.2). To be a candidate for the scale invariance conditions these equations should have the following properties:

(i) vanish on the supergravity equations (2.1), (2.2), (2.11), (3.1)–(3.3) with $X=d\phi$, $Y=0$

(ii) depend on Z and I through $X=Z+I$

(iii) depend on X through Lie derivatives.[22]

Starting with the modified equations (3.17) and acting with $\star d\star$ on the first equation and $d\star$ on the second and then using the modified equations (as described in Appendix D) we arrive at the following equation, which satisfies the above properties

$$\begin{aligned}
&d\star d\star\mathcal{F}_{2n+1}+\star d\star d\mathcal{F}_{2n+1}+\tfrac{1}{4}R\wedge\mathcal{F}_{2n+1}-\tfrac{1}{8}\star(H_3\wedge\star H_3)\wedge\mathcal{F}_{2n+1}\\
&\quad-H_3\wedge\star(H_3\wedge\star\mathcal{F}_{2n+1})-\star(H_3\wedge\star(H_3\wedge\mathcal{F}_{2n+1}))\\
&\quad-d\star(H_3\wedge\star\mathcal{F}_{2n+3})-\star(H_3\wedge\star d\mathcal{F}_{2n+3})+\star d\star(H_3\wedge\mathcal{F}_{2n-1})+H_3\wedge\star d\star\mathcal{F}_{2n-1}\\
&=\mathcal{L}_X\mathcal{F}_{2n+1}+\star\mathcal{L}_X\star\mathcal{F}_{2n+1}-(\star d\star X)\wedge\mathcal{F}_{2n+1}+\beta^B\wedge\mathcal{F}_{2n-1}-\star(\beta^B\wedge\star\mathcal{F}_{2n+3})\ .
\end{aligned}\tag{4.2}$$

Here β^B is the 2-form analog of (2.2), i.e.

$$\beta^B\equiv\tfrac{1}{2}\star d\star H_3+\mathcal{K}=\star(X\wedge\star H_3)+dY\ .\tag{4.3}$$

This is then a candidate for the scale invariance equation for the R–R form $\mathcal{F}_{2n+1}$.

Using the identity

$$\begin{aligned}
&\star\mathcal{L}_X\star\mathcal{F}_{2n+1}=\mathcal{L}_X\mathcal{F}_{2n+1}+\star(d\star X)\wedge\mathcal{F}_{2n+1}+\beta^G\cdot\mathcal{F}_{2n+1}\ ,\\
&\beta^G\cdot\mathcal{F}_{2n+1}\equiv\sum_i\beta^G_{m_i n}\mathcal{F}_{m_1\ldots m_{i-1}}{}^{n}{}_{m_{i+1}\ldots m_{2n+1}}\ ,
\end{aligned}\tag{4.4}$$

where β^G_{mn} is defined in (2.1), we find that (4.2) becomes

$$\begin{aligned}
&d\star d\star\mathcal{F}_{2n+1}+\star d\star d\mathcal{F}_{2n+1}+\tfrac{1}{4}R\wedge\mathcal{F}_{2n+1}-\tfrac{1}{8}\star(H_3\wedge\star H_3)\wedge\mathcal{F}_{2n+1}\\
&\quad-H_3\wedge\star(H_3\wedge\star\mathcal{F}_{2n+1})-\star(H_3\wedge\star(H_3\wedge\mathcal{F}_{2n+1}))\\
&\quad-d\star(H_3\wedge\star\mathcal{F}_{2n+3})-\star(H_3\wedge\star d\mathcal{F}_{2n+3})+\star d\star(H_3\wedge\mathcal{F}_{2n-1})+H_3\wedge\star d\star\mathcal{F}_{2n-1}\\
&=2\mathcal{L}_X\mathcal{F}_{2n+1}+\beta^G\cdot\mathcal{F}_{2n+1}+\beta^B\wedge\mathcal{F}_{2n-1}-\star(\beta^B\wedge\star\mathcal{F}_{2n+3})\ .
\end{aligned}\tag{4.5}$$

[22] Moreover, since the R–R fields $\mathcal{F}$ are invariant under the isometries generated by I, their Lie derivatives along I vanish, and therefore the scale invariance equations in fact depend only on Z.

The dependence of these equations on X rather than separately on Z and I can be related to their close connection to the particular X-dependent combinations of the modified equations in (3.11), (3.12), (3.13), i.e. to (here $n \in \mathbb{Z}$ as in (3.17))

$$\begin{aligned}\Xi_{2n} &\equiv d\mathcal{F}_{2n-1} - X \wedge \mathcal{F}_{2n-1} + H_3 \wedge \mathcal{F}_{2n-3} \\ &\quad + (-1)^n \star (d\mathcal{F}_{9-2n} - X \wedge \mathcal{F}_{9-2n} + H_3 \wedge \mathcal{F}_{7-2n}) = 0 \,.\end{aligned} \tag{4.6}$$

We also define as in (1.11), (2.2)

$$\begin{aligned}\bar\beta^B &\equiv \tfrac{1}{2} \star d \star H_3 + \mathcal{K} - \star(X \wedge \star H_3) - dX = 0 \,, \\ \bar\beta^X &\equiv R - \tfrac{1}{2} \star (H_3 \wedge \star H_3) + 4 \star d \star X - 4 \star (X \wedge \star X) = 0 \,.\end{aligned} \tag{4.7}$$

Deconstructing the derivation in Appendix D, we find that the 2nd-order equation for the R–R fluxes (4.2) can also be written as

$$\begin{aligned}& d\Xi_{2n} - X \wedge \Xi_{2n} + H_3 \wedge \Xi_{2n-2} - \mathcal{F}_{2n-1} \wedge \bar\beta^B \\ &\quad + (-1)^n \star \big(d\Xi_{8-2n} - X \wedge \Xi_{8-2n} + H_3 \wedge \Xi_{6-2n} - \mathcal{F}_{7-2n} \wedge \bar\beta^B\big) + \tfrac{1}{4}\mathcal{F}_{2n+1} \wedge \bar\beta^X \\ &= 0 \,.\end{aligned} \tag{4.8}$$

Finally, let us present the explicit form of eq. (4.5) in components. For $\mathcal{F}_1$ we find

$$\begin{aligned}& D^2\mathcal{F}_m - R_{mn}\mathcal{F}^n + \tfrac{1}{4}(R - \tfrac{3}{4}H^2)\mathcal{F}_m \\ &\quad + \tfrac{1}{2} H^{pnk} H_{mpn}\mathcal{F}_k - \tfrac{1}{6} D_m H^{pnk}\mathcal{F}_{pnk} - \tfrac{1}{2} H^{pnk} D_p \mathcal{F}_{nkm} \\ &= 2(X^p D_p \mathcal{F}_m + D_m X^p \mathcal{F}_p) + \beta^G_{mn}\mathcal{F}^n - \tfrac{1}{2}\beta^B_{nk}\mathcal{F}^{nk}{}_m \,.\end{aligned} \tag{4.9}$$

Using the identity $D_{[m} H_{npk]} = 0$ the term $\frac{1}{6} D_m H^{pnk}\mathcal{F}_{pnk}$ in (4.9) can be replaced by $\frac{1}{2} D_p H_{mnk}\mathcal{F}^{pnk}$. The equation for $\mathcal{F}_3$ may be written as

$$\begin{aligned}& D^2\mathcal{F}_{nkm} - R_{a[n}\mathcal{F}^a{}_{km]} + R_{ab[nk}\mathcal{F}^{ab}{}_{m]} + \tfrac{1}{4}(R - \tfrac{3}{4}H^2)\mathcal{F}_{nkm} \\ &\quad + \tfrac{1}{2} H^{abc} H_{ab[n}\mathcal{F}_{km]c} - \tfrac{1}{2} H^{abc} H_{a[nk}\mathcal{F}_{m]bc} \\ &\quad + D^a H_{a[nk}\mathcal{F}_{m]} + H_{a[nk} D^a \mathcal{F}_{m]} - \mathcal{F}_a D^a H_{nkm} \\ &\quad - \tfrac{1}{6} D_{[n} H^{abc}\mathcal{F}_{km]abc} - \tfrac{1}{2} H^{abc} D_a \mathcal{F}_{bcnkm} \\ &= 2(X^a D_a \mathcal{F}_{nkm} + D_{[n} X^a \mathcal{F}_{km]a}) + \beta^G_{a[n}\mathcal{F}^a{}_{km]} + \beta^B_{[nk}\mathcal{F}_{m]} - \tfrac{1}{2}\beta^B_{ab}\mathcal{F}^{ab}{}_{nkm} \,,\end{aligned} \tag{4.10}$$

while the equation for $\mathcal{F}_5$ can be put into the form

$$\begin{aligned}& D^2\mathcal{F}_{ijklm} - R_{a[i}\mathcal{F}^a{}_{jklm]} + R_{ab[ij}\mathcal{F}^{ab}{}_{klm]} + \tfrac{1}{4}(R - \tfrac{3}{4}H^2)\mathcal{F}_{ijklm} \\ &\quad + \tfrac{1}{2} H^{abc} H_{ab[i}\mathcal{F}_{jklm]c} - \tfrac{1}{2} H^{abc} H_{a[ij}\mathcal{F}_{klm]bc} \\ &\quad + D^a H_{a[ij}\mathcal{F}_{klm]} + H_{a[ij} D^a \mathcal{F}_{klm]} - \mathcal{F}_{a[ij} D^a H_{klm]} \\ &\quad + \tfrac{1}{12}\varepsilon_{ijklmbdef}(D_a H^{abc}\mathcal{F}^{def} + H^{abc} D_a \mathcal{F}^{def} - \mathcal{F}^{abc} D_a H^{def}) = \\ &= 2(X^a D_a \mathcal{F}_{ijklm} + D_{[i} X^a \mathcal{F}_{jklm]a}) + \beta^G_{a[i}\mathcal{F}^a{}_{jklm]} + \beta^B_{[ij}\mathcal{F}_{klm]} \\ &\quad + \tfrac{1}{12}\varepsilon_{ijklmabcde}(\beta^B)^{ab}\mathcal{F}^{cde} \,.\end{aligned} \tag{4.11}$$

This expression is consistent with the self-duality of $\mathcal{F}_5$ (in particular, the third and forth lines are manifestly dual to each other).

These 2nd-order equations for $\mathcal{F}_1$, $\mathcal{F}_3$ and $\mathcal{F}_5$ exhibit obvious structural similarities. In particular, they contain the expected Hodge–de Rham operator terms and the vector X only enters through the reparametrisation terms as in (4.1). The β^G and β^B terms in these equations are defined as in (2.1), (2.2) but can also be replaced by expressions on the r.h.s. of (2.1), (2.2).

As we shall discuss in Appendix G, similar equations come out of the computation of the one-loop beta-functions for the R–R couplings in the GS sigma model (1.2).

5. Origin of modified equations: T-duality relation to type II equations for backgrounds with non-isometric linear dilaton

Given a scale invariant sigma model in flat 2d space T-duality in an isometric direction should also produce a scale invariant sigma model. Similarly, given a Weyl invariant sigma model on curved 2d space with all couplings including the dilaton being isometric the T-dual background should also be Weyl invariant (provided the dilaton transforms in the usual way [32,33]). As discussed in the introduction, in general this is not so if the dilaton is not isometric: T-duality will still preserve scale invariance but not Weyl invariance. Thus given a solution of type II supergravity equations which has linear non-isometric term in the dilaton its T-duality image will no longer solve the standard type II equations but will satisfy instead a modified set of type II equations as discussed above.

5.1. Simple examples

Here we shall make the origin of the modified equations explicit by showing that they represent the original type II equations for a solution with non-isometric linear dilaton, rewritten in terms of the fields of the T-dual background. To explain how this happens in simple terms let us first start with a bosonic background (1.8) with $A_\mu = 0$, i.e.

$$\hat{ds}^2 = e^{-2a(x)} d\hat{y}^2 + g_{\mu\nu}(x) dx^\mu dx^\nu , \qquad \hat{\phi} = -c\,\hat{y} + \varphi(x) - \tfrac{1}{2} a(x) . \tag{5.1}$$

Then the corresponding Weyl anomaly coefficients

$$\bar{\beta}^G_{mn} = \hat{R}_{mn} + 2\hat{D}_m \hat{D}_n \hat{\phi} , \qquad \bar{\beta}^\phi = \hat{R} + 4\hat{D}^2\hat{\phi} - 4\hat{G}^{mn}\partial_m\hat{\phi}\partial_n\hat{\phi} , \tag{5.2}$$

have the following components under the $\hat{x}^m = (\hat{y}, x^\mu)$ splitting of coordinates[23]

$$\bar{\beta}^G_{\mu\nu} = \mathrm{R}_{\mu\nu} - \partial_\mu a \partial_\nu a + 2D_\mu D_\nu \varphi , \qquad \bar{\beta}^G_{\hat{y}\hat{y}} = e^{-2a}(D^\mu D_\mu a - 2\partial^\mu a \partial_\mu \varphi) , \tag{5.3}$$

$$\bar{\beta}^G_{\hat{y}\mu} = -2c\,\partial_\mu a , \qquad \bar{\beta}^\phi = \mathrm{R} - \partial^\mu a \partial_\mu a + 4D^2\varphi - 4\partial^\mu\varphi\partial_\mu\varphi - 4c^2 e^{2a} . \tag{5.4}$$

We see that if $c = 0$, i.e. the dilaton is isometric, then the Weyl invariance conditions $\bar{\beta}^G_{\mu\nu} = 0$, $\bar{\beta}^\phi = 0$ are invariant under T-duality in y, i.e. under $\hat{a} = -a \to a$, $\hat{\phi} \to \hat{\phi} + a$ or $\varphi \to \varphi$. The $c = -\partial_{\hat{y}}\hat{\phi}$ dependent terms in (5.4) thus represent obstructions to mapping one Weyl invariant model to another. The T-dual metric then solves weaker, modified, equations

$$R_{mn} + D_m X_n + D_n X_m = 0 , \qquad \bar{\beta}^X = R + 4D^m X_m - 4X^m X_m = 0 , \tag{5.5}$$

with X_m being (cf. (1.10))

[23] Here $\mathrm{R}_{\mu\nu}$ is the Ricci tensor of $g_{\mu\nu}(x)$, see Appendix A.

$$X_\mu = Z_\mu = \partial_\mu \phi = \partial_\mu(\varphi + \tfrac{1}{2}a) \ , \qquad\qquad X_y = I_y = -G_{yy}\partial_{\hat y}\hat\phi = ce^{2a} \ . \tag{5.6}$$

Similar conclusions are reached in the case we have a non-diagonal metric in (1.8) (see the general discussion below). The presence of non-zero $\hat A_\mu$ is in fact necessary to have a solution of the Weyl invariance conditions when $c \neq 0$ (cf. (5.4)) and the target space should thus be at least 3-dimensional. An example of such a solution was found in [13]. It represents a limit of the background associated with the $SO(4)/SO(3)$ gWZW model, which has the following metric and dilaton [34]

$$\begin{aligned} \hat{ds}^2 &= dt^2 + \frac{\tan^2 t\, dp^2 + \cot^2 t\, dq^2}{1-p^2-q^2} \\ &= dt^2 + \cot^2 t\ (d\theta + \tan\psi \cot\theta d\psi)^2 + \frac{\tan^2 t}{\sin^2\theta} d\psi^2 \ , \end{aligned} \tag{5.7}$$

$$\hat\phi = -\ln\big(\sqrt{b^2-p^2-q^2}\sin 2t\big) = -\ln\big(\sin\theta\ \cos\psi\ \sin 2t\big) \ , \tag{5.8}$$

where $p = \sin\psi$, $q = \cos\theta\cos\psi$ and t, ψ, θ are angles of the coset parametrisation of the $SO(4)$ group element. This background (which solves the Weyl invariance condition $\bar\beta^G = 0$ with $\bar\beta^\phi =$ const) has no isometries. One option to generate an isometry is to set $t = iz$ and then shift z by an infinite constant. Doing so we get linear dilaton in z, but the z direction decouples in the metric. A non-trivial alternative is to set $\psi = i\hat y$ and to shift $\hat y$ by an infinite constant (which corresponds to infinite rescaling of p, q generating a scaling isometry in the metric (5.7)). The resulting background (we drop an infinite constant in the dilaton)

$$\hat{ds}^2 = dt^2 - \frac{\tan^2 t\, dp^2 + \cot^2 t\, dq^2}{p^2+q^2} = dt^2 + \cot^2 t\ (d\theta + \cot\theta\, d\hat y)^2 - \frac{\tan^2 t}{\sin^2\theta} d\hat y^2 \ , \tag{5.9}$$

$$\hat\phi = -\ln\big(\sqrt{p^2+q^2}\sin 2t\big) = -\hat y - \ln\big(\sin\theta\ \sin 2t\big) \ , \tag{5.10}$$

is therefore of the same type as in (1.8) and defines a conformal sigma model.[24] Similar higher dimensional backgrounds can be constructed starting from $SO(n)/SO(n-1)$ gWZW models with $n > 4$ [13].

T-dualising the metric (5.9) along $\hat y$ we get a (G, B) background that will solve the modified (G, B) equations (2.1), (2.2) with non-trivial $X_m = I_m + Z_m$, where $I^y = -\partial_{\hat y}\hat\phi$ and Z_m is given by (2.14), with $\phi = \hat\phi - \frac{1}{2}\log G_{\hat y\hat y}$. These modified equations will be the original Weyl invariance conditions rewritten in terms of the dual G and B-fields.

Given the 2d CFT in (5.9) with 2d stress-tensor defined taking into account the dilaton in (5.10), one may formally compactify $\hat y$ and ask if this CFT has T-duality as a symmetry of its spectrum. The answer appears to be no as the 2d stress-tensor will not be invariant under T-duality (i.e. mapping momentum into winding modes).[25] This is compatible with our expectation that formally T-dualising the metric (5.9) will not lead to a consistent CFT.

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[24] The central charge for this $d = 3$ conformal model is given by $c = d - \frac{3}{2}\alpha'(R - \frac{1}{12}H^2 + 4D^2\phi - 4D_m\phi D^m\phi) + \ldots = 3 - \frac{3}{2}\alpha' \times 12 + \ldots$. Here the scale of the space was set to one, so that α' is then the inverse of the WZW level k. This is in agreement with the usual count of the central charge for the $SO(4)/SO(3)$ gWZW model $c = 6k/(k+4) - 3k/(k+2) = 3 - 18/k + \ldots$, which should be unchanged in the coordinate limit leading from (5.7) to (5.9).

[25] Given a free compact scalar CFT $L = r^2(\partial\phi)^2$ with $\phi \equiv \phi + 2\pi$ the spectrum of dimensions of primary operators (like $e^{in\phi + im\tilde\phi}$, etc.) is T-duality symmetric. If one formally adds a linear dilaton term $q\int d^2z\sqrt{h}\, R^{(2)}\phi$, or equivalently modifies the 2d stress tensor by $q\partial^2\phi$ terms (which are invariant under shifts of ϕ and thus defined for a compact boson) then the T-duality symmetry of the spectrum is broken by extra terms $\sim qn$. The formal symmetry would be

The same conclusions are reached for type II solutions with a linear dilaton and non-zero R–R fluxes (with isometric G, B and $\mathcal{F}_n = e^{\phi} F_n$), such as the HT solution dual to ABF background in the $AdS_5 \times S^5$ case and its counterparts in the $AdS_2 \times S^2 \times T^6$ and $AdS_3 \times S^3 \times T^4$ cases discussed in Appendix F. Explicitly, in the case of a solution $(\hat{G}, \hat{B}, \hat{\mathcal{F}}_n, \hat{\phi})$ with several isometries broken only by the linear dilaton term

$$\hat{\phi} = \phi_0 - c_i \hat{y}^i + f(x^\mu) \,, \tag{5.11}$$

we will get a generalisation of the type II supergravity equations, depending on the following two vectors Z and I (cf. (2.6), (2.7), (2.14))

$$I = c_i G_{y_i y_i} dy^i \,, \qquad Z = d\phi + \iota_I B \,, \qquad \phi = f - \frac{1}{2} \sum_i \log \hat{G}_{\hat{y}_i \hat{y}_i} \,. \tag{5.12}$$

Here G and B are the background fields of the T-dual background.

Below we shall illustrate these relations in the general case with one isometry.

5.2. NS–NS sector

We consider the following two d-dimensional backgrounds (here we use K_μ instead of $\hat{A}_\mu$ in (1.8))

$$\begin{aligned} ds^2 &= e^{2a}(dy + A_\mu dx^\mu)^2 + g_{\mu\nu} dx^\mu dx^\nu \,, \\ B &= K_\nu (dy + \tfrac{1}{2} A_\mu dx^\mu) \wedge dx^\nu + \tfrac{1}{2} b_{\mu\nu} dx^\mu \wedge dx^\nu \,, \\ \phi &= -c\, y + \varphi + \tfrac{1}{2} a \,, \qquad I^y = \hat{c} \,, \qquad I^\mu = 0 \,, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \hat{ds}^2 &= e^{-2a}(d\hat{y} + K_\mu dx^\mu)^2 + g_{\mu\nu} dx^\mu dx^\nu \,, \\ \hat{B} &= A_\nu (d\hat{y} + \tfrac{1}{2} K_\mu dx^\mu) \wedge dx^\nu + \tfrac{1}{2} b_{\mu\nu} dx^\mu \wedge dx^\nu \,, \\ \hat{\phi} &= -\hat{c}\, \hat{y} + \varphi - \tfrac{1}{2} a \,, \qquad \hat{I}^{\hat{y}} = c \,, \qquad \hat{I}^\mu = 0 \,. \end{aligned} \tag{5.14}$$

Here y and $\hat{y}$ are the directions that are assumed to be (shift) isometries of their respective metrics and B-fields. We use the indices $\mu, \nu, \ldots = 1, \ldots, d-1$ and $m, n, \ldots = 1, \ldots, d$. For $c = \hat{c} = 0$ (5.13) and (5.14) are related by standard T-duality, such that φ is the analog of the duality-invariant dilaton field. Let us also define

$$\begin{aligned} Z &= d\phi + \iota_I B = -c\, dy + d\varphi + \tfrac{1}{2} da + \hat{c}\, K_\mu dx^\mu \,, \\ X &= Z + I = (-c + \hat{c}\, e^{2a}) dy + d\varphi + \tfrac{1}{2} da + (\hat{c}\, K_\mu + \hat{c}\, e^{2a} A_\mu) dx^\mu \,, \\ \hat{Z} &= d\hat{\phi} + \iota_{\hat{I}} \hat{B} = -\hat{c}\, d\hat{y} + d\varphi - \tfrac{1}{2} da + c\, A_\mu dx^\mu \,, \\ \hat{X} &= \hat{Z} + \hat{I} = (-\hat{c} + c\, e^{-2a}) d\hat{y} + d\varphi - \tfrac{1}{2} da + (c\, A_\mu + c\, e^{-2a} K_\mu) dx^\mu \,, \end{aligned} \tag{5.15}$$

where

$$Z \cdot I = \hat{Z} \cdot \hat{I} = -c\, \hat{c} \,. \tag{5.16}$$

restored in the “doubled” formulation if the linear dilaton term were given by $q\phi + \tilde{q}\tilde{\phi}$ where $\tilde{\phi}$ is the dual field (with $\frac{1}{\sqrt{r}} q \leftrightarrow \sqrt{\frac{r}{\sqrt{\alpha'}}}\, \tilde{q}$ under T-duality).

The two (G, B) backgrounds in (5.13) and (5.14) are T-dual to each other and thus for $c = \hat{c} = 0$ solve the equivalent Weyl invariance equations (see, e.g., [37] and the references therein). We will now show how this relation also extends to the more general case with linear dilatons.

Let us first consider the generalised dilaton equation

$$R - \tfrac{1}{12}H^2 + 4D^m X_m - 4X^m X_m = 0\ . \tag{5.17}$$

The question we want to address is: if the background (5.13) satisfies (5.17) does that imply that (5.14) satisfies (5.17). As the two backgrounds (5.13) and (5.14) are related by the obvious symmetry

$$a \to -a\ , \qquad A_\mu \leftrightarrow K_\mu\ , \qquad c \leftrightarrow \hat{c}\ , \qquad y \leftrightarrow \hat{y}\ , \tag{5.18}$$

it is sufficient to compute the left-hand side of (5.17) for (5.13) and check that it is invariant (or at least covariant) under (5.18).

For (5.13) we have

$$X_y = -c + \hat{c}\, e^{2a}\ , \qquad\qquad X_\mu = \partial_\mu \varphi + \tfrac{1}{2}\partial_\mu a + \hat{c}\, K_\mu + \hat{c}\, e^{2a} A_\mu\ . \tag{5.19}$$

It will also be useful to define the following objects

$$\begin{aligned} &F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\ , \qquad H_{\mu\nu} \equiv \partial_\mu K_\nu - \partial_\nu K_\mu\ , \\ &h_{\mu\nu\rho} \equiv (db + \tfrac{1}{2} A \wedge dK + \tfrac{1}{2} K \wedge dA)_{\mu\nu\rho}\ , \end{aligned} \tag{5.20}$$

where we observe that h is invariant under (5.18). Now using the dimensional reduction formulae in Appendix A we find

$$\begin{aligned} &R - \tfrac{1}{12}H^2 + 4D^m X_m - 4X^m X_m \\ &\quad = \mathrm{R} - \partial^\mu a \partial_\mu a - \tfrac{1}{12}h^2 - \tfrac{1}{4}e^{2a} F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{4}e^{-2a}H^{\mu\nu}H_{\mu\nu} + 4\nabla^\mu \partial_\mu \varphi - 4\partial^\mu \varphi \partial_\mu \varphi \\ &\qquad + 4c\, \nabla^\mu A_\mu + 4\hat{c}\, \nabla^\mu K_\mu - 8(c\, A^\mu + \hat{c}\, K^\mu)\partial_\mu \varphi \\ &\qquad - 4c^2(A^\mu A_\mu + e^{-2a}) - 4\hat{c}^2(K^\mu K_\mu + e^{2a}) + 8c\, \hat{c}\, (1 - A^\mu K_\mu)\ , \end{aligned} \tag{5.21}$$

which is indeed invariant under (5.18). Therefore, if (5.17) is satisfied for background (5.13) it is satisfied for background (5.14) and vice versa.[26]

Let us now turn to the modified metric and B-field equations to show that the two combinations appearing in (1.3) and (1.4)

$$R_{mn} - \tfrac{1}{4}H_{mpq}H_n{}^{pq} + D_m X_n + D_n X_m\ , \tag{5.22}$$

$$\tfrac{1}{2}D^p H_{mnp} - X^p H_{mnp} - D_m X_n + D_n X_m\ , \tag{5.23}$$

are covariant under the symmetry (5.18).[27] Then if they vanish for the background (5.13) this implies that they vanish for (5.14) and vice versa. As (5.22) is symmetric and (5.23) is antisymmetric, we may just consider their difference

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[26] This generalises the usual ($c = \hat{c} = 0$) discussion of the T-duality invariance of the string effective action (1.7) with $\sqrt{G}\, e^{-2\phi} = \sqrt{g}\, e^{-2\varphi}$.

[27] Here we assume Y_m in (1.4), (2.2) is equal to X_m in (1.3), (2.1) as is the case for the I-modified equations satisfied by the ABF background. More generally, given a scale invariant sigma model with an isometry and the G and B-field couplings satisfying (1.3), (1.4), its T-dual counterpart will also satisfy (1.3), (1.4) with the roles of X_m and Y_m interchanged.

$$C_{mn} \equiv R_{mn} - \tfrac{1}{4}H_{mpq}H_n{}^{pq} - \tfrac{1}{2}D^p H_{mnp} + 2D_m X_n + X^p H_{mnp} \,, \tag{5.24}$$

which we can decompose into a part independent of c and $\hat{c}$ ($C^{(0)}_{mn}$) and a part linear in c and $\hat{c}$ ($C^{(1)}_{mn}$) as

$$C_{mn} = C^{(0)}_{mn} + 2C^{(1)}_{mn} \,, \tag{5.25}$$

$$C^{(0)}_{mn} = R_{mn} - \tfrac{1}{4}H_{mpq}H_n{}^{pq} - \tfrac{1}{2}D^p H_{mnp} + 2D_m X^{(0)}_n + X^{(0)p} H_{mnp} \,, \tag{5.26}$$

$$C^{(1)}_{mn} = D_m X^{(1)}_n + \tfrac{1}{2}X^{(1)p} H_{mnp} \,. \tag{5.27}$$

Here we have used the fact that all the c and $\hat{c}$ dependence is contained in $X = X^{(0)} + X^{(1)}$, where $X^{(0)}$ is the c- and $\hat{c}$-independent and $X^{(1)}$ is the c- and $\hat{c}$-dependent part. Using the specific form of X for the background (5.13), as given in (5.19), we have

$$X^{(0)}_y = 0 \,, \quad X^{(0)}_\mu = \partial_\mu \varphi + \tfrac{1}{2}\partial_\mu a \,, \quad X^{(1)}_y = -c + \hat{c}\, e^{2a} \,, \quad X^{(1)}_\mu = \hat{c}\, K_\mu + \hat{c}\, e^{2a} A_\mu \,. \tag{5.28}$$

Using the formulae in Appendix A we find the following relations for $C^{(0)}_{mn}$ and $C^{(1)}_{mn}$ evaluated on the background (5.13)

$$\begin{aligned}
C^{(0)}_{yy} &= 2e^{2a}\partial^\mu \varphi \partial_\mu a - e^{2a}\nabla^\mu \partial_\mu a + \tfrac{1}{4}e^{4a}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}H_{\mu\nu}H^{\mu\nu} \,, \\
C^{(0)}_{y\mu} - \frac{G_{y\mu}}{G_{yy}}C^{(0)}_{yy} &= e^{2a}(\tfrac{1}{2}\nabla^\nu + \partial^\nu a - \partial^\nu \varphi)F_{\mu\nu} + \tfrac{1}{4}h_{\mu\nu\rho}H^{\nu\rho} \\
&\quad + (\tfrac{1}{2}\nabla^\nu - \partial^\nu a - \partial^\nu \varphi)H_{\mu\nu} + \tfrac{1}{4}e^{2a}h_{\mu\nu\rho}F^{\nu\rho} \,, \\
C^{(0)}_{\mu y} - \frac{G_{\mu y}}{G_{yy}}C^{(0)}_{yy} &= e^{2a}(\tfrac{1}{2}\nabla^\nu + \partial^\nu a - \partial^\nu \varphi)F_{\mu\nu} + \tfrac{1}{4}h_{\mu\nu\rho}H^{\nu\rho} \\
&\quad - (\tfrac{1}{2}\nabla^\nu - \partial^\nu a - \partial^\nu \varphi)H_{\mu\nu} - \tfrac{1}{4}e^{2a}h_{\mu\nu\rho}F^{\nu\rho} \,, \\
C^{(0)}_{\mu\nu} - \frac{G_{\mu y}}{G_{yy}}C^{(0)}_{y\nu} - \frac{G_{y\nu}}{G_{yy}}C^{(0)}_{\mu y} &+ \frac{G_{\mu y}}{G_{yy}}\frac{G_{y\nu}}{G_{yy}}C^{(0)}_{yy} \\
= \mathrm{R}_{\mu\nu} - \partial_\mu a \partial_\nu a &+ 2\nabla_\mu \partial_\nu \varphi - \tfrac{1}{2}e^{2a}F_{\mu\nu}F_\nu{}^\rho - \tfrac{1}{2}e^{-2a}H_{\mu\rho}H_\nu{}^\rho \\
- \tfrac{1}{2}\nabla^\rho h_{\mu\nu\rho} &- \tfrac{1}{4}h_{\mu\rho\sigma}h_\nu{}^{\rho\sigma} + h_{\mu\nu\rho}\partial^\rho \varphi \,,
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
C^{(1)}_{yy} &= e^{2a}(cA^\mu + \hat{c}\, K^\mu)\partial_\mu a \,, \\
C^{(1)}_{y\mu} - \frac{G_{y\mu}}{G_{yy}}C^{(1)}_{yy} &= -\tfrac{1}{2}(e^{2a}F_{\mu\nu} + H_{\mu\nu})(cA^\nu + \hat{c}\, K^\nu) + (c - \hat{c}\, e^{2a})\partial_\mu a \,, \\
C^{(1)}_{\mu y} - \frac{G_{\mu y}}{G_{yy}}C^{(1)}_{yy} &= -\tfrac{1}{2}(e^{2a}F_{\mu\nu} - H_{\mu\nu})(cA^\nu \hat{c}\, K^\nu) + (c + \hat{c}\, e^{2a})\partial_\mu a \,, \\
C^{(1)}_{\mu\nu} - \frac{G_{\mu y}}{G_{yy}}C^{(1)}_{y\nu} - \frac{G_{y\nu}}{G_{yy}}C^{(1)}_{\mu y} &+ \frac{G_{\mu y}}{G_{yy}}\frac{G_{y\nu}}{G_{yy}}C^{(1)}_{yy} \\
= \tfrac{1}{2}c(\nabla_\mu A_\nu + \nabla_\nu A_\mu) &+ \tfrac{1}{2}\hat{c}\, e^{2a}(\nabla_\mu A_\nu - \nabla_\nu A_\mu) \\
+ \tfrac{1}{2}\hat{c}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) &+ \tfrac{1}{2}ce^{-2a}(\nabla_\mu K_\nu - \nabla_\nu K_\mu) + \tfrac{1}{2}h_{\mu\nu\rho}(cA^\rho + \hat{c}\, K^\rho) \,.
\end{aligned} \tag{5.30}$$

Then using the map (5.18) between the backgrounds (5.13) and (5.14), we find the following relations for C_{mn}

$$\frac{C_{yy}}{G_{yy}} = -\frac{\hat{C}_{\hat{y}\hat{y}}}{\hat{G}_{\hat{y}\hat{y}}} , \qquad \frac{1}{\sqrt{G_{yy}}}\Big[C_{y\mu} - \frac{G_{y\mu}}{G_{yy}}C_{yy}\Big] = \frac{1}{\sqrt{\hat{G}_{\hat{y}\hat{y}}}}\Big[\hat{C}_{\hat{y}\mu} - \frac{\hat{G}_{\hat{y}\mu}}{\hat{G}_{\hat{y}\hat{y}}}\hat{C}_{\hat{y}\hat{y}}\Big] ,$$
$$\frac{1}{\sqrt{G_{yy}}}\Big[C_{\mu y} - \frac{G_{\mu y}}{G_{yy}}C_{yy}\Big] = -\frac{1}{\sqrt{\hat{G}_{\hat{y}\hat{y}}}}\Big[\hat{C}_{\mu\hat{y}} - \frac{\hat{G}_{\mu\hat{y}}}{\hat{G}_{\hat{y}\hat{y}}}\hat{C}_{\hat{y}\hat{y}}\Big] ,$$
$$\hat{C}_{\mu\nu} - \frac{G_{\mu y}}{G_{yy}}\hat{C}_{y\nu} - \frac{G_{y\nu}}{G_{yy}}\hat{C}_{\mu y} + \frac{G_{\mu y}}{G_{yy}}\frac{G_{y\nu}}{G_{yy}}\hat{C}_{yy} = \hat{C}_{\mu\nu} - \frac{\hat{G}_{\mu\hat{y}}}{\hat{G}_{\hat{y}\hat{y}}}\hat{C}_{\hat{y}\nu} - \frac{\hat{G}_{\hat{y}\nu}}{\hat{G}_{\hat{y}\hat{y}}}\hat{C}_{\mu\hat{y}} + \frac{\hat{G}_{\mu\hat{y}}}{\hat{G}_{\hat{y}\hat{y}}}\frac{\hat{G}_{\hat{y}\nu}}{\hat{G}_{\hat{y}\hat{y}}}\hat{C}_{\hat{y}\hat{y}} , \tag{5.31}$$

where the left-hand side is evaluated on (5.13) and the right-hand side on (5.14). From these equalities it follows that the vanishing of the tensors (5.22), (5.23) on the background (5.13) implies their vanishing also on the background (5.14), and in this sense are covariant under T-duality.

Let us briefly comment on the generalisation when $I^\mu = \hat{I}^\mu \neq 0$ in the backgrounds (5.13) and (5.14) (i.e., when there are extra isometries in x^μ directions). Running through the same analysis we find that the result still holds only if I^μ satisfies certain properties. In particular, the T-duality relation between the equations for (5.13) and (5.14) still holds if

$$I^\mu A_\mu = I^\mu K_\mu = 0 . \tag{5.32}$$

This requirement is also sufficient for the T-duality of the modified equations of motion for the R–R fields discussed in the following section to continue when $I^\mu = \hat{I}^\mu \neq 0$.

One can check that these relations are valid at each stage in the sequence of T-dualities required to transform from the supergravity HT solutions of [12] to the ABF background (B.1) and its $AdS_3 \times S^3$ and $AdS_2 \times S^2$ counterparts (F.4), (F.13).

5.3. R–R sector

Let us now consider the case of non-zero isometric R–R fields $\mathcal{F}_n$. The contribution of the R–R fields to the metric and B-field equations appears in the usual unmodified form and hence we can focus our attention on the modified equations of motion for the R–R fields (3.17). Written in terms of the forms $f_k \equiv e^{-a/2}\mathcal{F}_k$ these take the following form (dropping the distinction between y and $\hat{y}$)

$$\mathcal{E}_k \equiv df_k - Z' \wedge f_k + H_3 \wedge f_{k-2} - \hat{c}\,\iota_y f_{k+2} = 0 , \tag{5.33}$$
$$\hat{\mathcal{E}}_k \equiv d\hat{f}_k - \hat{Z}' \wedge \hat{f}_k + \hat{H}_3 \wedge \hat{f}_{k-2} - c\,\iota_y \hat{f}_{k+2} = 0 , \tag{5.34}$$

where we have introduced ($K = K_\mu dx^\mu$, $A = A_\mu dx^\mu$)

$$Z' = Z - \frac{1}{2}da = -c\,dy + d\varphi + \hat{c}\,K , \qquad \hat{Z}' \equiv \hat{Z} - \frac{1}{2}d\hat{a} = -\hat{c}\,dy + d\varphi + c\,A , \tag{5.35}$$

which are related to each other under T-duality as

$$\hat{Z}' = Z' + c(dy + A) - \hat{c}(dy + K) \equiv Z' + \delta Z . \tag{5.36}$$

Recall that the invariance of the R–R forms under the isometry along y requires

$$c\,\hat{c} = 0 . \tag{5.37}$$

This follows from the condition $I \cdot Z = 0$, which is implied by the invariance of R–R fields as $\mathcal{L}_I \mathcal{F}_k = (I \cdot Z)\mathcal{F}_k$.

We want to show that if f_k satisfies the equation $\mathcal{E}_k = 0$, then $\hat{f}_k$ satisfies $\hat{\mathcal{E}}_k = 0$. Taking into account the T-duality relations in Appendix A one finds

$$
\begin{aligned}
&-\hat{Z}' \wedge \hat{f}_k \\
&\quad = -(dy+K)\wedge Z' \wedge f_{k-1} + (dy+K)\wedge(dy+A)\wedge \iota_y(Z'\wedge f_{k-1}) - \iota_y(Z'\wedge f_{k+1}) \\
&\qquad + c(dy+K)\wedge(dy+A)\wedge f_{k-1} - cf_{k+1} - \delta Z \wedge(-(dy+K)\wedge f_{k-1} - \iota_y f_{k+1})\,, \\
&\hat{H}_3 \wedge \hat{f}_{k-2} \\
&\quad = (dy+K)\wedge H_3 \wedge f_{k-3} - (dy+K)\wedge(dy+A)\wedge \iota_y(H_3\wedge f_{k-3}) + \iota_y(H_3\wedge f_{k-1}) \\
&\qquad + (dy+K)\wedge(dy+A)\wedge \iota_y H_3 \wedge f_{k-3} - \iota_y H_3 \wedge f_{k-1} \\
&\qquad + ((dy+A)\wedge H_2 - (dy+K)\wedge F_2)\wedge(-(dy+K)\wedge f_{k-3} - \iota_y f_{k-1})\,, \\
&-c\,\iota_y \hat{f}_{k+2} \\
&\quad = -\hat{c}(dy+K)\wedge \iota_y f_{k+1} + \hat{c}(dy+K)\wedge \iota_y f_{k+1} + cf_{k+1} - c(dy+A)\wedge \iota_y f_{k+1}.
\end{aligned}
\tag{5.38}
$$

Here we used that $\delta Z \wedge (dy+K)\wedge(dy+A)\wedge \iota_y f_{k-1} = 0$ and $((dy+A)\wedge H_2 - (dy+K)\wedge F_2)\wedge(dy+K)\wedge(dy+A)\wedge \iota_y f_{k-1} = 0$. Further using that $\iota_y H_3 = -H_2$, one finds

$$
\begin{aligned}
\hat{\mathcal{E}}_k = {}& (dy+K)\wedge \mathcal{E}_{k-1} - (dy+K)\wedge(dy+A)\wedge \iota_y \mathcal{E}_{k-1} + \iota_y \mathcal{E}_{k+1} \\
& - H_2 \wedge f_{k-1} + H_2 \wedge (dy+A)\wedge \iota_y f_{k-1} - (dy+K)\wedge F_2 \wedge \iota_y f_{k-1} \\
& + c(dy+K)\wedge(dy+A)\wedge f_{k-1} - cf_{k+1} - \delta Z\wedge(-(dy+K)\wedge f_{k-1} - \iota_y f_{k+1}) \\
& - (dy+K)\wedge(dy+A)\wedge H_2 \wedge f_{k-3} + H_2 \wedge f_{k-1} \\
& + ((dy+A)\wedge H_2 - (dy+K)\wedge F_2)\wedge(-(dy+K)\wedge f_{k-3} - \iota_y f_{k-1}) \\
& + \hat{c}(dy+K)\wedge \iota_y f_{k+1} + cf_{k+1} - c(dy+A)\wedge \iota_y f_{k+1}\,.
\end{aligned}
\tag{5.39}
$$

If we set $\mathcal{E}_k = 0$ and $c = \hat{c} = 0$ we get

$$
\begin{aligned}
& H_2 \wedge (dy+A)\wedge \iota_y f_{k-1} - (dy+K)\wedge F_2 \wedge \iota_y f_{k-1} \\
&\quad - (dy+K)\wedge(dy+A)\wedge H_2 \wedge f_{k-3} \\
&\quad + ((dy+A)\wedge H_2 - (dy+K)\wedge F_2)\wedge(-(dy+K)\wedge f_{k-3} - \iota_y f_{k-1}) = 0\,,
\end{aligned}
\tag{5.40}
$$

as expected. It remains to consider the c and $\hat{c}$ dependent terms only

$$
\begin{aligned}
\hat{\mathcal{E}}_k = {}& (dy+K)\wedge \mathcal{E}_{k-1} - (dy+K)\wedge(dy+A)\wedge \iota_y \mathcal{E}_{k-1} + \iota_y \mathcal{E}_{k+1} \\
& + c(dy+K)\wedge(dy+A)\wedge f_{k-1} - cf_{k+1} \\
& - (c(dy+A) - \hat{c}(dy+K))\wedge(-(dy+K)\wedge f_{k-1} - \iota_y f_{k+1}) \\
& + \hat{c}(dy+K)\wedge \iota_y f_{k+1} + cf_{k+1} - c(dy+A)\wedge \iota_y f_{k+1} \\
= {}& (dy+K)\wedge \mathcal{E}_{k-1} - (dy+K)\wedge(dy+A)\wedge \iota_y \mathcal{E}_{k-1} + \iota_y \mathcal{E}_{k+1}\,.
\end{aligned}
\tag{5.41}
$$

Thus, if $\mathcal{E}_k = 0$ then $\hat{\mathcal{E}}_k = 0$, i.e. the backgrounds (5.13) and (5.14) supplemented by R–R fields have their corresponding modified equations mapped into each other by this generalised T-duality.

6. Concluding remarks

There are several open problems and puzzling questions. First, it remains unclear if the scale invariant but arguably not Weyl invariant η-model can still be used to define a critical superstring theory. This might be possible in view of the existence of the λ-model [24] which is classically related to the η-model by the Poisson–Lie duality combined with an analytic continuation of the deformation parameter, and for which there is a candidate supergravity solution [25] (i.e. it should represent a Weyl invariant sigma model). In fact, a special limit [13] of this solution should be essentially equivalent to the HT solution [12].[28] Thus if the classical Poisson–Lie duality relation [26] between the η-model and λ-model [27,12] extends to the full quantum level there may be a way to associate a string theory to the ABF background. This might also require increasing the number of 2d fields (such as in a doubled or phase space formulation). Indeed, already at the classical level, establishing the connection between the two models calls for the use of the phase space formalism. The quantum η-model defined in terms of an extended number of fields (including, e.g., analogs of 2d gauge fields of the gWZW part of λ-model) may then be Weyl invariant, and integrating out extra fields might produce the GS action corresponding to the ABF background plus extra non-local terms required for restoring its Weyl invariance.

As we have seen above, the fact that the HT background solves the type IIB equations implies that the T-dual ABF background should satisfy the I-modified type II equations. These explicitly depend on the isometry vector I, whose origin can be traced to the presence of the linear term in the dilaton of the HT solution. One can ask whether these I-modified equations are Lagrangian, i.e. if they can be derived from the action principle. Answering this question may require the introduction of R–R potentials and understanding whether one should treat the vector I as an external source or as an auxiliary field with no physical degrees of freedom. In view of our analysis of T-duality in section 5, it would be interesting to know if there exist more general I-modified equations that are compatible with T-duality and have $c\,\hat{c} \neq 0$ in (5.16). One would also like to understand how the usual action of the T-duality group $O(d,d)$ is modified.

In the present work we discussed only the I-modified equations for bosonic fields. It is an interesting question how the equations for the fermionic fields of type II theory are modified. Furthermore, if the I-modification destroys the local supersymmetry of type II theory one may ask if there is still any (hidden) symmetry of the I-modified equations for bosonic and fermionic fields.

To better understand the nature of the ABF background it would be important to derive the quartic fermionic action for the η-model of [1] and to show that the I-modified equations indeed follow from the κ-symmetry [1] of this action. Starting with the standard GS action for the HT solution [12] (which, as was mentioned in the Introduction, is invariant under shifts of the 6 isometric coordinates) and performing the T-dualities one will get θ^4 and higher terms in the η-model GS action depending on the vectors I and Z. These will originate from the dilaton, $\theta^4 \partial\phi$, etc., terms in the HT GS action. The resulting η-model action should still be invariant under the κ-symmetry defined in [1], however it is then probable that the structure of these transformations will deviate from those of the usual GS action.

[28] The need for T-duality in order to relate the HT solution to the ABF background can be understood from the two facts: that the λ-model is a deformation of the non-abelian T-dual of the $AdS_5 \times S^5$ sigma model and that in the limit of [13], which enhances the Cartan directions making them the isometries, the non-abelian T-duality along these isometric directions turns into the standard abelian one.

The knowledge of the quartic fermionic action should also enable one to perform the full computation of one-loop divergences of the η-model in the R–R sector (completing our discussion in Appendix G) and hence check the agreement between the 2nd-order equations for R–R fields derived from the modified type II equations with the scale invariance beta-functions for $\mathcal{F}_n$.

It would also be important to attempt a direct analysis of the Weyl invariance conditions, which should lead to 1st-order conditions for R–R strengths equivalent to type II supergravity equations. More generally, one may study the one-loop renormalisation of a generic κ-symmetric sigma model with 8 bosonic and 8 fermionic degrees of freedom, and classify interaction terms for which the corresponding model is either conformal or scale invariant only. It is possible that the class of conformally invariant models may be bigger than just the usual type II GS superstring sigma models.

Finally, it would be interesting to perform a similar analysis for the deformations of $AdS_n \times S^n$ backgrounds constructed from other solutions of the modified classical YB equation [1,40], or solutions of the classical (non-modified) YB equation, see, e.g., [41,42]. In the latter case many of the resulting metrics and B-fields can be completed to full type II supergravity solutions, however it remains to verify that these completions are indeed realised by the supercoset action. Indeed, the analysis of [7] has shown that the large $\varkappa$-limit of the η-model does not coincide with the $AdS_5 \times S^5$ mirror sigma model [43] even though the bosonic part of the model does.

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Appendix A. Conventions and some standard relations

A.1. Conventions for forms

We have for any m-form Y and n-form Z on a manifold of dimension d

$$\begin{aligned}
&Z = \tfrac{1}{n!} Z_{i_1\cdots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \ , \qquad (\star Z)_{i_1\cdots i_{d-n}} = \tfrac{1}{n!} \varepsilon_{i_1\cdots i_{d-n} j_1 \cdots j_n} Z^{j_1\cdots j_n} \ , \\
&\iota_I Z = \tfrac{1}{(n-1)!} I^p Z_{p i_2\cdots i_n} dx^{i_2} \wedge \cdots \wedge dx^{i_n} \ , \qquad (Y\wedge Z)_{i_1\cdots i_m j_1\cdots j_n} = Y_{[i_1\cdots i_m} Z_{j_1\cdots j_n]} \ , \\
&Y \wedge Z = \tfrac{1}{m!n!} Y_{i_1\cdots i_m} Z_{j_1\cdots j_n} dx^{i_1} \wedge \cdots \wedge dx^{i_m} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_n} \ ,
\end{aligned} \tag{A.1}$$

where the antisymmetrisation is understood as

$$Y_{[i_1\cdots i_m} Z_{i_{m+1}\cdots i_{m+n}]} dx^{i_1} \wedge \cdots \wedge dx^{i_{m+n}} = \tfrac{(m+n)!}{m!n!} Y_{i_1\cdots i_m} Z_{i_{m+1}\cdots i_{m+n}} dx^{i_1} \wedge \cdots \wedge dx^{i_{m+n}} \ . \tag{A.2}$$

In d dimensions with Lorentzian signature we have

$$\star^2 Z_n = (-1)^{dn+n+1} Z_n \ , \qquad [\star(Y_m \wedge \star Z_n)]_{i_1\cdots i_{n-m}} = \frac{(-1)^{nd+n+1}}{m!} Y^{j_1\cdots j_m} Z_{i_1\cdots i_{n-m} j_1\cdots j_m} \ . \tag{A.3}$$

In particular for $m=1$ and even d one has

$$\star(I \wedge \star Z_n) = \iota_I Z_n \ . \tag{A.4}$$

A.2. Dimensional reduction formulae

Let us take the metric and B-field as in (5.13)

$$\begin{aligned} ds^2 &= e^{2a}(dy + A_\mu dx^\mu)^2 + g_{\mu\nu} dx^\mu dx^\nu \ , \\ B &= K_\nu (dy + \tfrac{1}{2} A_\mu dx^\mu) \wedge dx^\nu + \tfrac{1}{2} b_{\mu\nu} dx^\mu \wedge dx^\nu \ , \end{aligned} \tag{A.5}$$

where y is an isometric direction. It is useful to define

$$\begin{aligned} &F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \ , \qquad H_{\mu\nu} \equiv \partial_\mu K_\nu - \partial_\nu K_\mu \ , \\ &h_{\mu\nu\rho} \equiv (db + \tfrac{1}{2} A \wedge dK + \tfrac{1}{2} K \wedge dA)_{\mu\nu\rho} \ . \end{aligned} \tag{A.6}$$

We can now write the various d-dimensional quantities appearing in the modified type II equations in terms of $(d-1)$-dimensional ones as follows

$$\begin{aligned} &G^{yy} = e^{-2a} + A^2 \ , \qquad G^{y\mu} = -A^\mu \ , \qquad G^{\mu\nu} = g^{\mu\nu} \ , \\ &H_{y\mu\nu} = -H_{\mu\nu} \ , \qquad H_{\mu\nu\rho} = h_{\mu\nu\rho} - (A \wedge dK)_{\mu\nu\rho} \ , \end{aligned} \tag{A.7}$$

$$\begin{aligned} &\Gamma^y_{yy} = e^{2a} A^\mu \partial_\mu a \ , \qquad \Gamma^\mu_{yy} = -e^{2a} \partial^\mu a \ , \\ &\Gamma^y_{y\mu} = \partial_\mu a + e^{2a} A_\mu A^\nu \partial_\nu a + \tfrac{1}{2} e^{2a} A^\nu F_{\nu\mu} \ , \\ &\Gamma^y_{\mu\nu} = \tfrac{1}{2}(\nabla_\mu A_\nu + \nabla_\nu A_\mu) + A_\nu \partial_\mu a + A_\mu \partial_\nu a + e^{2a} A_\mu A_\nu A^\rho \partial_\rho a \\ &\qquad + \tfrac{1}{2} e^{2a} A^\rho (A_\mu F_{\rho\nu} + A_\nu F_{\rho\mu}) \ , \\ &\Gamma^\mu_{y\nu} = -e^{2a} A_\nu \partial^\mu a - \tfrac{1}{2} e^{2a} F^\mu{}_\nu \ , \qquad \Gamma^m_{ym} = 0 \ , \qquad \Gamma^m_{\mu m} = \partial_\mu a + \gamma^\nu_{\mu\nu} \ , \\ &\Gamma^\rho_{\mu\nu} = \gamma^\rho_{\mu\nu} - e^{2a} A_\mu A_\nu \partial^\rho a - \tfrac{1}{2} e^{2a}(A_\mu F^\rho{}_\nu + A_\nu F^\rho{}_\mu) = \gamma^\rho_{\mu\nu} + \delta\Gamma^\rho_{\mu\nu} \ , \end{aligned} \tag{A.8}$$

$$\begin{aligned} &R_{yy} = -e^{2a} \nabla^\mu \partial_\mu a - e^{2a} \partial^\mu a \partial_\mu a + \tfrac{1}{4} e^{4a} F_{\mu\nu} F^{\mu\nu} \ , \\ &R_{y\mu} = R_{yy} A_\mu - \tfrac{3}{2} e^{2a} \partial^\nu a F_{\nu\mu} - \tfrac{1}{2} e^{2a} \nabla^\nu F_{\nu\mu} \ , \\ &R_{\mu\nu} = \mathrm{R}_{\mu\nu} - \nabla_\mu \partial_\nu a - \partial_\mu a \partial_\nu a + A_\mu R_{y\nu} + A_\nu R_{y\mu} - A_\mu A_\nu R_{yy} - \tfrac{1}{2} e^{2a} F_\mu{}^\rho F_{\nu\rho} \ , \\ &R = \mathrm{R} - \tfrac{1}{4} e^{2a} F^{\mu\nu} F_{\mu\nu} - 2\partial^\mu a \partial_\mu a - 2 D^\mu \partial_\mu a \ , \end{aligned} \tag{A.9}$$

$$\begin{aligned} &H_y{}^{kl} H_{ykl} = H^{\mu\nu} H_{\mu\nu} \ , \qquad H_y{}^{kl} H_{\mu kl} = -H^{\nu\rho} h_{\mu\nu\rho} + A_\mu H^{\nu\rho} H_{\nu\rho} \ , \\ &H_\mu{}^{kl} H_{\nu kl} = h_\mu{}^{\rho\sigma} h_{\nu\rho\sigma} - h_\mu{}^{\rho\sigma} A_\nu H_{\rho\sigma} - h_\nu{}^{\rho\sigma} A_\mu H_{\rho\sigma} + A_\mu A_\nu H^{\rho\sigma} H_{\rho\sigma} \\ &\qquad + 2 e^{-2a} H_{\mu\rho} H_\nu{}^\rho \ , \\ &H^{\mu\nu\rho} H_{\mu\nu\rho} = h^{\mu\nu\rho} h_{\mu\nu\rho} + 3 e^{-2a} H^{\mu\nu} H_{\mu\nu} \ , \end{aligned} \tag{A.10}$$

$$\begin{aligned} &D^k H_{\mu yk} = \nabla^\nu H_{\mu\nu} - H_{\mu\nu} \partial^\nu a + \tfrac{1}{2} e^{2a} F^{\nu\rho} h_{\mu\nu\rho} \ , \\ &D^k H_{\mu\nu k} = \nabla^\rho h_{\mu\nu\rho} + (h_{\mu\nu\rho} + A_\mu H_{\nu\rho} - A_\nu H_{\mu\rho}) \partial^\rho a - e^{2a} F^{\rho\sigma} A_{[\mu} h_{\nu]\rho\sigma} \\ &\qquad - 2 A_{[\mu} \nabla^\rho H_{\nu]\rho} \ . \end{aligned} \tag{A.11}$$

Also, for a vector $X = X_y dy + X_\mu dx^\mu$ we have

$$
\begin{aligned}
D^m X_m &= \nabla^\mu X_\mu - X_y \nabla^\mu A_\mu - X_y A^\mu \partial_\mu a + X^\mu \partial_\mu a - A^\mu \partial_\mu X_y \, , \\
X^m X_m &= e^{-2a} X_y^2 + X_y^2 A^\mu A_\mu - 2 X_y A^\mu X_\mu + X^\mu X_\mu \, ,
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
D_y X_y &= e^{2a} X^\mu \partial_\mu a - e^{2a} X_y A^\mu \partial_\mu a \, , \\
D_y X_\mu &= \tfrac{1}{2} e^{2a} (-F_{\mu\nu} + 2\partial_\nu a A_\mu)(X^\nu - X_y A^\nu) - X_y \partial_\mu a \, , \\
D_\mu X_y &= \tfrac{1}{2} e^{2a} (-F_{\mu\nu} + 2\partial_\nu a A_\mu)(X^\nu - X_y A^\nu) - X_y \partial_\mu a + \partial_\mu X_y \, , \\
D_\mu X_\nu &= \nabla_\mu X_\nu - X_y A_\mu \partial_\nu a - X_y A_\nu \partial_\mu a - \tfrac{1}{2} X_y \nabla_\mu A_\nu - \tfrac{1}{2} X_y \nabla_\nu A_\mu \\
&\quad - \tfrac{1}{2} e^{2a} (A_\mu F_{\nu\rho} + A_\nu F_{\mu\rho} - 2\partial_\rho a A_\mu A_\nu)(X^\rho - X_y A^\rho) \, .
\end{aligned}
\tag{A.13}
$$

Here $Q_{[\mu\nu]} \equiv \frac{1}{2}(Q_{\mu\nu} - Q_{\nu\mu})$, ∇_μ is the covariant derivative with respect to the $(d-1)$-dimensional metric $g_{\mu\nu}$ with connection $\gamma^\lambda_{\mu\nu}$, and $\mathrm{R}_{\mu\nu}$ and R are the $(d-1)$-dimensional Ricci tensor and scalar respectively.

A.3. T-duality rules

Let us consider two isometric backgrounds related by T-duality, with the fields of the dual background denoted with hats. The metric and B-field will be taken in the form of (5.13), and we will also consider the isometric dilaton ϕ and the R–R field strengths $\mathcal{F}_k \equiv e^\phi F_k$ of type II theory

$$
\begin{aligned}
ds^2 &= e^{2a}(dy + A_\mu dx^\mu)^2 + g_{\mu\nu} dx^\mu dx^\nu \, , \qquad \phi \, , \qquad \mathcal{F}_k \, , \\
B &= K_\nu (dy + \tfrac{1}{2} A_\mu dx^\mu) \wedge dx^\nu + \tfrac{1}{2} b_{\mu\nu} dx^\mu \wedge dx^\nu \, ,
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
d\hat{s}^2 &= e^{2\hat{a}}(d\hat{y} + \hat{A}_\mu dx^\mu)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu \, , \qquad \hat{\phi} \, , \qquad \hat{\mathcal{F}}_k \, , \\
\hat{B} &= \hat{K}_\nu (d\hat{y} + \tfrac{1}{2} \hat{A}_\mu dx^\mu) \wedge dx^\nu + \tfrac{1}{2} \hat{b}_{\mu\nu} dx^\mu \wedge dx^\nu \, .
\end{aligned}
\tag{A.15}
$$

The T-duality rules for the NS–NS fields are (see (A.6))

$$
\begin{aligned}
a &= -\hat{a} \, , \quad A_\mu = \hat{K}_\mu \, , \quad g_{\mu\nu} = \hat{g}_{\mu\nu} \, , \quad b_{\mu\nu} = \hat{b}_{\mu\nu} \, , \quad \phi = \hat{\phi} - \hat{a} = \hat{\phi} + a \, , \\
K_\mu &= \hat{A}_\mu \, , \quad F_{\mu\nu} = \hat{H}_{\mu\nu} \, , \quad H_{\mu\nu} = \hat{F}_{\mu\nu} \, , \quad h_{\mu\nu\rho} = \hat{h}_{\mu\nu\rho} \, .
\end{aligned}
\tag{A.16}
$$

In terms of the forms $A = A_\mu dx^\mu$, $K = K_\mu dx^\mu$, $H_2 = dK$, $H_3 = dB$, and the corresponding hatted ones, one has[29]

$$
\begin{aligned}
H_3 &= \hat{H}_3 + (dy + \hat{A}) \wedge \hat{H}_2 - (dy + \hat{K}) \wedge \hat{F}_2 \, , \\
\hat{H}_3 &= H_3 + (dy + A) \wedge H_2 - (dy + K) \wedge F_2 \, .
\end{aligned}
\tag{A.17}
$$

To write the T-duality rules for the R–R fields it is convenient to introduce

$$
f_k \equiv e^{-a/2} \mathcal{F}_k = e^{\phi - a/2} F_k \, , \qquad \hat{f}_k \equiv e^{-\hat{a}/2} \hat{\mathcal{F}}_k = e^{\hat{\phi} - \hat{a}/2} \hat{F}_k \, .
\tag{A.18}
$$

Then

$$
\hat{f}_k = -(dy + K) \wedge f_{k-1} + (dy + K) \wedge (dy + A) \wedge \iota_y f_{k-1} - \iota_y f_{k+1} \, ,
\tag{A.19}
$$

29 Here for notational simplicity we use the same y for the isometric direction and its dual – whether it is y or $\hat{y}$ is clear from context.

where $\iota_y f_k \equiv \iota_{I_y} f_k$, $I_y^m = \delta_y^m$. Also using the assumption of invariance of the R–R forms under the isometry, $\mathcal{L}_{I_y} f^{(k)} = 0$, one has

$$
\begin{aligned}
\iota_y \hat{f}_k &= -f_{k-1} + (dy + A) \wedge \iota_y f_{k-1} \;, \\
d\hat{f}_k &= (dy + K) \wedge df_{k-1} - (dy + K) \wedge (dy + A) \wedge \iota_y df_{k-1} + \iota_y df_{k+1} \\
&\quad - H_2 \wedge f_{k-1} + H_2 \wedge (dy + A) \wedge \iota_y f_{k-1} - (dy + K) \wedge F_2 \wedge \iota_y f_{k-1} \;.
\end{aligned}
\tag{A.20}
$$

Appendix B. ABF background and T-dual HT solution

The ABF background [6,7] represents the couplings in the η-deformed $AdS_5 \times S^5$ action [1] expanded to quadratic order in fermions and formally identified with a GS action. This background for the type IIB fields $(G, B, \mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5)$ (but not the dilaton which cannot be extracted from the DMV action, and, in fact, does not exist) is given by

$$
\begin{aligned}
ds^2 &= -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1^2 \\
&\quad + \frac{d\zeta^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \rho^2\sin^2\zeta\, d\psi_2^2 \\
&\quad + \frac{1-r^2}{1+\varkappa^2 r^2}d\varphi^2 + \frac{dr^2}{(1+\varkappa^2 r^2)(1-r^2)} + \frac{r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi}d\phi_1^2 \\
&\quad + \frac{d\xi^2}{1+\varkappa^2 r^4\sin^2\xi} + r^2\sin^2\xi\, d\phi_2^2 \;, \\
B &= \frac{\varkappa\rho^4\sin\zeta\cos\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1\wedge d\zeta - \frac{\varkappa r^4\sin\xi\cos\xi}{1+\varkappa^2 r^4\sin^2\xi}d\phi_1\wedge d\xi \;, \\
\mathcal{F}_1 &= \varkappa^2 \mathrm{F}\Big[\rho^4\sin^2\zeta\, d\psi_2 - r^4\sin^2\xi\, d\phi_2\Big] \;, \\
\mathcal{F}_3 &= \varkappa\,\mathrm{F}\Big[\frac{\rho^3\sin^2\zeta}{1-\varkappa^2\rho^2}dt\wedge d\psi_2\wedge d\rho + \frac{r^3\sin^2\xi}{1+\varkappa^2 r^2}d\varphi\wedge d\phi_2\wedge dr \\
&\quad + \frac{\rho^4\sin\zeta\cos\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_2\wedge d\psi_1\wedge d\zeta + \frac{r^4\sin\xi\cos\xi}{1+\varkappa^2 r^4\sin^2\xi}d\phi_2\wedge d\phi_1\wedge d\xi \\
&\quad + \frac{\varkappa^2\rho r^4\sin^2\xi}{1-\varkappa^2\rho^2}dt\wedge d\rho\wedge d\phi_2 - \frac{\varkappa^2\rho^4 r\sin^2\zeta}{1+\varkappa^2 r^2}d\psi_2\wedge d\varphi\wedge dr \\
&\quad + \frac{\varkappa^2\rho^4 r^4\sin\zeta\cos\zeta\sin^2\xi}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1\wedge d\zeta\wedge d\phi_2 \\
&\quad + \frac{\varkappa^2\rho^4 r^4\sin^2\zeta\sin\xi\cos\xi}{1+\varkappa^2 r^4\sin^2\xi}d\psi_2\wedge d\phi_1\wedge d\xi\Big] \;, \\
\mathcal{F}_5 &= \mathrm{F}\Big[\frac{\rho^3\sin\zeta\cos\zeta}{(1-\varkappa^2\rho^2)(1+\varkappa^2\rho^4\sin^2\zeta)}dt\wedge d\psi_2\wedge d\psi_1\wedge d\zeta\wedge d\rho \\
&\quad - \frac{r^3\sin\xi\cos\xi}{(1+\varkappa^2 r^2)(1+\varkappa^2 r^4\sin^2\xi)}d\varphi\wedge d\phi_2\wedge d\phi_1\wedge d\xi\wedge dr
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\varkappa^2 \rho r}{(1-\varkappa^2\rho^2)(1+\varkappa^2 r^2)}\\
&\times(\rho^2 \sin^2\zeta\, dt\wedge d\psi_2\wedge d\rho\wedge d\varphi\wedge dr + r^2\sin^2\xi\, dt\wedge d\rho\wedge d\varphi\wedge d\phi_2\wedge dr)\\
&+\frac{\varkappa^2\rho^4 r^4 \sin\zeta\cos\zeta\sin\xi\cos\xi}{(1+\varkappa^2\rho^4\sin^2\zeta)(1+\varkappa^2 r^4\sin^2\xi)}\\
&\times(d\psi_2\wedge d\psi_1\wedge d\zeta\wedge d\phi_1\wedge d\xi - d\psi_1\wedge d\zeta\wedge d\phi_2\wedge d\phi_1\wedge d\xi)\\
&+\frac{\varkappa^2\rho r^4\sin\xi\cos\xi}{(1-\varkappa^2\rho^2)(1+\varkappa^2 r^4\sin^2\xi)}\\
&\times(\rho^2\sin^2\zeta\, dt\wedge d\psi_2\wedge d\rho\wedge d\phi_1\wedge d\xi - dt\wedge d\rho\wedge d\phi_2\wedge d\phi_1\wedge d\xi)\\
&-\frac{\varkappa^2\rho^4 r\sin\zeta\cos\zeta}{(1+\varkappa^2 r^2)(1+\varkappa^2\rho^4\sin^2\zeta)}\\
&\times(r^2\sin^2\xi\, d\psi_1\wedge d\zeta\wedge d\varphi\wedge d\phi_2\wedge dr + d\psi_2\wedge d\psi_1\wedge d\zeta\wedge d\varphi\wedge dr)\\
&-\frac{\varkappa^4\rho^5 r^4\sin\zeta\cos\zeta\sin^2\xi}{(1-\varkappa^2\rho^2)(1+\varkappa^2\rho^4\sin^2\zeta)} dt\wedge d\psi_1\wedge d\zeta\wedge d\rho\wedge d\phi_2\\
&-\frac{\varkappa^4\rho^4 r^5\sin^2\zeta\sin\xi\cos\xi}{(1+\varkappa^2 r^2)(1+\varkappa^2 r^4\sin^2\xi)} d\psi_2\wedge d\varphi\wedge d\phi_1\wedge d\xi\wedge dr\Big],
\end{aligned}
$$

$$
\mathrm{F}\equiv\frac{4\sqrt{1+\varkappa^2}}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2\rho^4\sin^2\zeta}\sqrt{1+\varkappa^2 r^2}\sqrt{1+\varkappa^2 r^4\sin^2\xi}}\,. \tag{B.1}
$$

Here $\varkappa=\frac{2\eta}{1-\eta^2}$ is a continuous deformation parameter of the η-model: $\varkappa=0$ corresponds to the standard $AdS_5\times S^5$ solution [28].

$ds^2\equiv G_{mn}(x)dx^m dx^n$ defines the 10d metric G and the sign of B-field is chosen as in [6], i.e. it corresponds to the sign in (1.2). $\mathcal{F}_k\equiv e^{\phi}F_k$ are effective R–R k-form strengths of type IIB theory that appear in the GS action. The self-duality equation satisfied by the R–R 5-form is

$$
F_{mnpqr}=\tfrac{1}{5!}\varepsilon_{mnpqrstuvw}F^{stuvw}\,,\quad \varepsilon_{mnpqrstuvw}\equiv\sqrt{G}\,\varepsilon_{mnpqrstuvw}\,,\quad G=|\det G_{mn}|\,, \tag{B.2}
$$

where we order the coordinates as $x^m=(t,\psi_2,\psi_1,\zeta,\rho,\varphi,\phi_2,\phi_1,\xi,r)$ and take $\epsilon_{t\psi_2\psi_1\zeta\rho\varphi\phi_2\phi_1\xi r}=-1$.

As found in [12], there exists an exact solution of the standard type IIB supergravity equations that is T-dual to the ABF background (provided we ignore the dilaton transformation). This HT background has the following explicit form[30]

$$
\begin{aligned}
\hat{ds}^2 = &-\frac{1-\varkappa^2\rho^2}{1+\rho^2}d\hat{t}^2+\frac{d\rho^2}{(1+\rho^2)(1-\varkappa^2\rho^2)}+\frac{d\hat{\psi}_1^2}{\rho^2\cos^2\zeta}+(\rho\, d\zeta+\varkappa\rho\tan\zeta\, d\hat{\psi}_1)^2\\
&+\frac{d\hat{\psi}_2^2}{\rho^2\sin^2\zeta}+\frac{1+\varkappa^2 r^2}{1-r^2}d\hat{\varphi}^2+\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)}+\frac{d\hat{\phi}_1^2}{r^2\cos^2\xi}
\end{aligned}
$$

[30] Here we have redefined $\hat{\phi}_2\to-\hat{\phi}_2$ compared to [12] to account for the opposite definition we use for the Hodge dual. Also, recall that to perform the T-duality in t we first analytically continue to Euclidean time, then T-dualise and finally continue back.

$$
\begin{aligned}
&\quad + (r\,d\xi - \varkappa r \tan\xi\, d\hat{\phi}_1)^2 + \frac{d\hat{\phi}_2^2}{r^2\sin^2\xi}\,,\\
&\hat{B} = 0\,, \qquad \hat{\mathcal{F}}_1 = \hat{\mathcal{F}}_3 = 0\,,\\
&\hat{\mathcal{F}}_5 = \frac{4i\sqrt{1+\varkappa^2}}{\sqrt{1+\rho^2}\sqrt{1-r^2}}\Big[(d\hat{t} + \frac{\varkappa\rho d\rho}{1-\varkappa^2\rho^2}) \wedge \frac{d\hat{\psi}_2}{\rho\sin\zeta} \wedge \frac{d\hat{\psi}_1}{\rho\cos\zeta} \wedge (r d\xi - \varkappa r\tan\xi\, d\hat{\phi}_1)\\
&\quad \wedge (\frac{dr}{1+\varkappa^2 r^2} + \varkappa r d\hat{\varphi}) - (d\hat{\varphi} - \frac{\varkappa r dr}{1+\varkappa^2 r^2}) \wedge \frac{d\hat{\phi}_2}{r\sin\xi} \wedge \frac{d\hat{\phi}_1}{r\cos\xi}\\
&\quad \wedge (\rho d\zeta + \varkappa\rho\tan\zeta\, d\hat{\psi}_1) \wedge (\frac{d\rho}{1-\varkappa^2 r^2} + \varkappa\rho d\hat{t})\Big]\\
&\hat{\phi} = \phi_0 - 4\varkappa(\hat{t} - \hat{\varphi}) - 2\varkappa(\hat{\psi}_1 - \hat{\phi}_1) + \log\frac{(1-\varkappa^2\rho^2)^2(1+\varkappa^2 r^2)^2}{\rho^2 r^2\sqrt{1+\rho^2}\sqrt{1-r^2}\sin 2\zeta\,\sin 2\xi}\,.
\end{aligned}
\tag{B.3}
$$

When written in terms of the following "boosted"/"rotated" vielbein basis

$$
\begin{aligned}
&e^0 = \tfrac{1}{\sqrt{1+\rho^2}}\big(d\hat{t} + \tfrac{\varkappa\rho d\rho}{1-\varkappa^2\rho^2}\big)\,, \qquad e^1 = \tfrac{d\hat{\psi}_2}{\rho\sin\zeta}\,, \qquad e^2 = \tfrac{d\hat{\psi}_1}{\rho\cos\zeta}\,,\\
&e^3 = \rho\, d\zeta + \varkappa\rho\tan\zeta\, d\hat{\psi}_1\,, \qquad e^4 = \tfrac{1}{\sqrt{1+\rho^2}}\big(\tfrac{d\rho}{1-\varkappa^2\rho^2} + \varkappa\rho d\hat{t}\big)\,,\\
&e^5 = \tfrac{1}{\sqrt{1-r^2}}\big(d\hat{\varphi} - \tfrac{\varkappa r dr}{1+\varkappa^2 r^2}\big)\,, \qquad e^6 = \tfrac{d\hat{\phi}_2}{r\sin\xi}\,, \qquad e^7 = \tfrac{d\hat{\phi}_1}{r\cos\xi}\,,\\
&e^8 = r\, d\xi - \varkappa r\tan\xi\, d\hat{\phi}_1\,, \qquad e^9 = \tfrac{1}{\sqrt{1-r^2}}\big(\tfrac{dr}{1+\varkappa^2 r^2} + \varkappa r d\hat{\varphi}\big)\,,
\end{aligned}
\tag{B.4}
$$

the metric and $\hat{\mathcal{F}}_5$ in (B.3) take the following remarkably simple form [12]

$$
\hat{ds}^2 = \eta_{MN} e^M e^N\,, \qquad \hat{\mathcal{F}}_5 = 4i\sqrt{1+\varkappa^2}\big(e^0\wedge e^1\wedge e^2\wedge e^8\wedge e^9 - e^3\wedge e^4\wedge e^5\wedge e^6\wedge e^7\big)\,,
\tag{B.5}
$$

where $M, N = 0, \ldots, 9$ are flat tangent-space indices, and η_{MN} is the Minkowski metric.

Appendix C. Conservation of R–R stress tensor and dilaton beta function identity

Given a Weyl invariant sigma model the dilaton beta function $\bar{\beta}^\phi$ in (1.6) represents a natural definition of the central charge: it appears as the coefficient of the $R^{(2)}$-term in the expectation value of the trace of the stress tensor on a curved 2d background [20,21], and for this reason must be a constant [9].[31]

In the case of the ABF background we found an analog of the dilaton beta-function

$$
\bar{\beta}^X \equiv R - \tfrac{1}{12} H_{nkl}H^{nkl} + 4D_n X^n - 4X^2\,,
\tag{C.1}
$$

and the equation $\bar{\beta}^X = 0$ was used in section 2 to determine the isometric part I of the diffeomorphism vector X. In this appendix we reverse the logic and show that the modified type II equations for the NS–NS and R–R fields with the same vector X implies the constancy of $\bar{\beta}^X$. In other words, on the equations of motion (2.1)–(2.4), (3.4)–(3.9) governed by the vector (2.9) we have the dilaton beta-function identity $\partial_m \bar{\beta}^X = 0$.

[31] The one-loop equation $\partial_m \bar{\beta}^\phi = 0$ is a special case of the Curci–Paffuti identity [22] that extends to higher loops.

To proceed, we first need to derive the conservation law for the R–R stress tensor $\mathcal{T}_{mn}$ in (2.3) that should hold on the R–R equations of motion. First, consider

$$(D^n - 2Z^n)(\mathcal{F}_m \mathcal{F}_n) = -(d\mathcal{F}_1)_{mn}\mathcal{F}^n + \tfrac{1}{2}D_m(\mathcal{F}_n\mathcal{F}^n) + \mathcal{F}_m(D^n - Z^n)\mathcal{F}_n - \mathcal{F}_m Z^n \mathcal{F}_n \,. \tag{C.2}$$

Now using (3.4) and (3.5), we find

$$(D^n - 2Z^n)(\mathcal{F}_m \mathcal{F}_n) = \tfrac{1}{2}(D_m - 2Z_m)(\mathcal{F}_n\mathcal{F}^n) - I^p(\mathcal{F}^n\mathcal{F}_{mnp}) + \tfrac{1}{6}\mathcal{F}_m H^{abc}\mathcal{F}_{abc} \,. \tag{C.3}$$

Next, we have

$$\begin{aligned}(D^n - 2Z^n)(\mathcal{F}_{mpq}\mathcal{F}_n{}^{pq}) = &-\tfrac{1}{3}(d\mathcal{F}_3)_{mnpq}\mathcal{F}^{npq} + \tfrac{1}{6}D_m(\mathcal{F}_{npq}\mathcal{F}^{npq}) \\ &+ \mathcal{F}_m{}^{pq}(D^n - Z^n)\mathcal{F}_{npq} - \mathcal{F}_m{}^{pq}Z^n\mathcal{F}_{npq} \,,\end{aligned} \tag{C.4}$$

such that using (3.7) and (3.6), we obtain

$$\begin{aligned}(D^n - 2Z^n)(\mathcal{F}_{mpq}\mathcal{F}_n{}^{pq}) = &\tfrac{1}{6}(D_m - 2Z_m)(\mathcal{F}_{npq}\mathcal{F}^{npq}) - \tfrac{1}{3}\mathcal{F}_m H^{abc}\mathcal{F}_{abc} + H_{mpq}\mathcal{F}_n\mathcal{F}^{npq} \\ &+ \tfrac{1}{6}H^{abc}\mathcal{F}_m{}^{pq}\mathcal{F}_{pqabc} - 2I^p(\mathcal{F}^n\mathcal{F}_{mnp}) - \tfrac{1}{3}I^p\mathcal{F}^{abc}\mathcal{F}_{mabcp} \,.\end{aligned} \tag{C.5}$$

Finally, we need

$$\begin{aligned}(D^n - 2Z^n)(\mathcal{F}_{mpqrs}\mathcal{F}_n{}^{pqrs}) = &-\tfrac{1}{5}(d\mathcal{F}_5)_{mnpqrs}\mathcal{F}^{npqrs} \\ &+ \mathcal{F}_m{}^{pqrs}(D^n - Z^n)\mathcal{F}_{npqrs} - Z^n\mathcal{F}_n{}^{pqrs}\mathcal{F}_{mpqrs} \\ = &\tfrac{1}{5}(H_3 \wedge \mathcal{F}_3)_{mnpqrs}\mathcal{F}^{npqrs} + \tfrac{1}{30}\varepsilon_{mabcdnpqrs}I^a\mathcal{F}^{bcd}\mathcal{F}^{npqrs} \\ &- \tfrac{1}{36}\mathcal{F}_m{}^{pqrs}\varepsilon_{pqrsabcde}H^{abc}\mathcal{F}^{def} - 4I^p\mathcal{F}^{abc}\mathcal{F}_{mabcp} \,,\end{aligned} \tag{C.6}$$

where we have used (3.8) and (3.9) and that $\mathcal{F}_5^2 = 0$. Taking into account the self-duality of $\mathcal{F}_5$, which also implies that $\mathcal{F}_m{}^{pqrs}\varepsilon_{pqrsabcde} = -24 g_{m[a}\mathcal{F}_{bcdef]}$, we find

$$\begin{aligned}&(D^n - 2Z^n)(\mathcal{F}_{mpqrs}\mathcal{F}_n{}^{pqrs}) \\ &\quad = 4H_m{}^{np}\mathcal{F}^{abc}\mathcal{F}_{npabc} - 4\mathcal{F}_m{}^{np}H^{abc}\mathcal{F}_{npabc} - 8I^p\mathcal{F}^{abc}\mathcal{F}_{mabcp} \,.\end{aligned} \tag{C.7}$$

Combining (C.3), (C.5), (C.7) we find the following conservation law for the stress tensor $\mathcal{T}_{mn}$

$$(D^n - 2Z^n)\mathcal{T}_{mn} = 2\mathcal{K}_{mn}I^n + \tfrac{1}{2}H_{mkn}\mathcal{K}^{kn} \,, \tag{C.8}$$

where $\mathcal{K}_{mn}$ is defined in (2.4). We would like to rewrite this formula in terms of $X = Z + I$. We have

$$(D^n - 2X^n)\mathcal{T}_{mn} = 2(\mathcal{K}_{mn} - \mathcal{T}_{mn})I^n + \tfrac{1}{2}H_{mkn}\mathcal{K}^{kn} \,. \tag{C.9}$$

Further, we use (2.1) and (2.2) (with $Y = X$) to write

$$\begin{aligned}(\mathcal{K}_{mn} - \mathcal{T}_{mn})I^n = &-\tfrac{1}{2}D^k H_{kmn}I^n + Z^k H_{kmn}I^n + (D_m I_n - D_n I_m)I^n - R_{mn}I^n \\ &+ \tfrac{1}{4}H_m{}^{kl}H_{nkl}I^n - (D_m Z_n + D_n Z_m)I^n \,.\end{aligned} \tag{C.10}$$

Notice that due to the properties of I_m in (2.6) one has $[D_n, D_m]I^n = R_{mn}I^n = -D^n D_n I_m$, which implies the following identity

$$R_{mn} I^n = \tfrac{1}{2} D^n (D_m I_n - D_n I_m) \,. \tag{C.11}$$

Then

$$(\mathcal{K}_{mn} - \mathcal{T}_{mn}) I^n = -\tfrac{1}{2} D^k (H_{kmn} I^n) - \tfrac{1}{2} H_{mkn} D^k I^n + Z^k H_{kmn} I^n + \tfrac{1}{4} H_m{}^{kl} H_{kln} I^n \\ + (D_m I_n - D_n I_m) I^n - \tfrac{1}{2} D^n (D_m I_n - D_n I_m) - (D_m Z_n + D_n Z_m) I^n \,. \tag{C.12}$$

Now using (2.13), we obtain

$$(\mathcal{K}_{mn} - \mathcal{T}_{mn}) I^n = -\tfrac{1}{2} D^n (D_m Z_n - D_n Z_m) - \tfrac{1}{2} D^n (D_m I_n - D_n I_m) \\ - \tfrac{1}{2} H_{mkn} D^k I^n - \tfrac{1}{2} H_{mkn} D^k Z^n \\ + Z^n (D_m Z_n - D_n Z_m) + I^n (D_m I_n - D_n I_m) - (D_m Z_n + D_n Z_m) I^n \,. \tag{C.13}$$

Taking into account that

$$-(D_m Z_n + D_n Z_m) I^n = (D_m Z_n - D_n Z_m) I^n - 2 D_m Z_n I^n \\ = (D_m Z_n - D_n Z_m) I^n + 2 Z^n D_m I_n \\ = I^n (D_m Z_n - D_n Z_m) + Z^n (D_m I_n - D_n I_m) \,, \tag{C.14}$$

we find

$$(\mathcal{K}_{mn} - \mathcal{T}_{mn}) I^n = -\tfrac{1}{2} (D^n - 2X^n)(D_m X_n - D_n X_m) - \tfrac{1}{2} H_{mkn} D^k X^n \,. \tag{C.15}$$

Thus, the conservation law (C.8) acquires the following form depending only on the vector X

$$(D^n - 2X^n) \mathcal{T}_{mn} = \tfrac{1}{2} H_{mkn} \mathcal{K}^{kn} - (D^n - 2X^n)(D_m X_n - D_n X_m) - H_{mkn} D^k X^n \,. \tag{C.16}$$

Here, using (2.2), the tensor $\mathcal{K}^{kn}$ can be eliminated such that the r.h.s. of (C.16) is written solely in terms of H_3 and X.

Now we ready to show the constancy of $\bar{\beta}^X$. We have from (C.1)

$$\partial_m \bar{\beta}^X = 2 D^n R_{mn} - \tfrac{1}{6} H^{nkl} D_{[m} H_{nkl]} - \tfrac{1}{2} H^{nkl} D_n H_{mkl} + 4 D_m D_n X^n - 8 X^n D_m X_n \,. \tag{C.17}$$

Since $D_{[m} H_{nkl]} = 0$ this can be rewritten as

$$\partial_m \bar{\beta}^X = 2 D^n \big(R_{mn} - \tfrac{1}{4} H_{mkl} H_n{}^{kl} \big) - 4 X^n R_{mn} \\ + \tfrac{1}{2} D_n H^{nkl} H_{mkl} + 4 D^n D_m X_n - 8 X^n D_m X_n \,. \tag{C.18}$$

Furthermore, using

$$4 D^n D_m X_n = 2 D^n (D_m X_n + D_m X_n) - 2 D^n (D_n X_m - D_m X_n) \,, \\ -8 X^n D_m X_n = -4 X^n (D_m X_n + D_n X_m) + 4 X^n (D_n X_m - D_m X_n) \,, \tag{C.19}$$

we may combine the terms in (C.18) as

$$\partial_m \bar{\beta}^X = 2 (D^n - 2X^n) \big(R_{mn} - \tfrac{1}{4} H_{mkl} H_n{}^{kl} + D_m X_n + D_n X_m \big) \\ + \big(\tfrac{1}{2} D_n H^{nkl} - X_n H^{nkl} \big) H_{mkl} - 2 (D^n - 2X^n)(D_n X_m - D_m X_n) \,. \tag{C.20}$$

Finally, using eq. (2.1) we have

$$\partial_m \bar{\beta}^X = 2(D^n - 2X^n)\mathcal{T}_{mn} \\ - H_{mkn}\mathcal{K}^{kn} + H_{mkn} D^k X^n + 2(D^n - 2X^n)(D_m X_n - D_n X_m) = 0 \,, \tag{C.21}$$

where the r.h.s. vanishes due to the conservation law (C.16). This proves that $\bar{\beta}^X$ is a constant (actually zero) on the modified equations of motion. The same is then true also in the spacial case of the standard type IIB supergravity equations (i.e. in the limit (3.10)) with the R–R strengths non-zero.[32]

Appendix D. Derivation of second-order equations for R–R strengths from modified type II equations

Here we present the derivation of the 2nd-order equations for the R–R field strengths, which, as discussed in section 4, are candidates for the scale invariance conditions of the GS sigma model, starting with the modified type II equations (3.4)–(3.9) or (3.17), i.e. ($n \in \mathbb{Z}$)

$$d\mathcal{F}_{2n+1} - Z \wedge \mathcal{F}_{2n+1} + H_3 \wedge \mathcal{F}_{2n-1} - \star(I \wedge \star\mathcal{F}_{2n+3}) = 0 \,, \\ d \star \mathcal{F}_{2n+1} - Z \wedge \star\mathcal{F}_{2n+1} - H_3 \wedge \star\mathcal{F}_{2n+3} + \star(I \wedge \mathcal{F}_{2n-1}) = 0 \,. \tag{D.1}$$

Our aim is to derive (4.2). Acting on the first equation by $\star d\star$ and on the second equation by $d\star$ we get

$$\star d \star d\mathcal{F}_{2n+1} - \star d \star (Z \wedge \mathcal{F}_{2n+1}) + \star d \star (H_3 \wedge \mathcal{F}_{2n-1}) + \star d(I \wedge \star\mathcal{F}_{2n+3}) = 0 \,, \\ d \star d \star \mathcal{F}_{2n+1} - d \star (Z \wedge \star\mathcal{F}_{2n+1}) - d \star (H_3 \wedge \star\mathcal{F}_{2n+3}) - d(I \wedge \mathcal{F}_{2n-1}) = 0 \,. \tag{D.2}$$

Taking the sum of these equations and using $\star(I \wedge \star Z_n) = \iota_I Z_n$, we find

$$\star d \star d\mathcal{F}_{2n+1} + d \star d \star \mathcal{F}_{2n+1} - \star \mathcal{L}_X \star \mathcal{F}_{2n+1} - \mathcal{L}_X \mathcal{F}_{2n+1} \\ + \star d \star (H_3 \wedge \mathcal{F}_{2n-1}) - d \star (H_3 \wedge \star\mathcal{F}_{2n+3}) + \star \iota_Z\, d(\star\mathcal{F}_{2n+1}) + \iota_Z\, d\mathcal{F}_{2n+1} \\ + \star d(I \wedge \star\mathcal{F}_{2n+3}) - d(I \wedge \mathcal{F}_{2n-1}) + 2I \cdot Z\mathcal{F}_{2n+1} = 0 \,, \tag{D.3}$$

where we have used (3.18): $\mathcal{L}_Z\mathcal{F}_{2n+1} = \mathcal{L}_X\mathcal{F}_{2n+1} - \mathcal{L}_I\mathcal{F}_{2n+1} = \mathcal{L}_X\mathcal{F}_{2n+1} - (I \cdot Z)\mathcal{F}_{2n+1}$.

The terms on the first line are the same as in (4.2), so we consider the last line in (D.3)

$$\star \iota_Z\, d \star \mathcal{F}_{2n+1} + \iota_Z\, d\mathcal{F}_{2n+1} + \star d(I \wedge \star\mathcal{F}_{2n+3}) - d(I \wedge \mathcal{F}_{2n-1}) + 2I \cdot Z\mathcal{F}_{2n+1} \\ = \star \iota_Z(Z \wedge \star\mathcal{F}_{2n+1} + H_3 \wedge \star\mathcal{F}_{2n+3} - \iota_I \star \mathcal{F}_{2n-1}) + \iota_Z(Z \wedge \mathcal{F}_{2n+1} - H_3 \wedge \mathcal{F}_{2n-1} \\ + \iota_I\mathcal{F}_{2n+3}) + \star(dI \wedge \star\mathcal{F}_{2n+3}) - dI \wedge \mathcal{F}_{2n-1} - \star(I \wedge d \star \mathcal{F}_{2n+3}) + I \wedge d\mathcal{F}_{2n-1} \\ + 2I \cdot Z\mathcal{F}_{2n+1} \\ = \star(dI \wedge \star\mathcal{F}_{2n+3}) + \star(\iota_Z(H_3) \wedge \star\mathcal{F}_{2n+3}) - dI \wedge \mathcal{F}_{2n-1} - \iota_Z(H_3) \wedge \mathcal{F}_{2n-1} \\ - \star(H_3 \wedge \iota_Z(\star\mathcal{F}_{2n+3})) + \star \iota_Z(Z \wedge \star\mathcal{F}_{2n+1} - \iota_I \star \mathcal{F}_{2n-1})$$

[32] While expected, this was not explicitly shown before in the literature. This provides a consistency check of the equivalence of the supergravity equations of motion with the sigma model Weyl invariance conditions.

$$+ H_3 \wedge \iota_Z \mathcal{F}_{2n-1} + \iota_Z(Z \wedge \mathcal{F}_{2n+1} + \iota_I \mathcal{F}_{2n+3}) - \star(I \wedge d \star \mathcal{F}_{2n+3}) + I \wedge d\mathcal{F}_{2n-1}$$
$$+ 2I \cdot Z\mathcal{F}_{2n+1} \, . \tag{D.4}$$

Now we use (2.2) with $Y = X$ or

$$dI + \iota_Z H_3 = \beta^B \, , \tag{D.5}$$

to get

$$\begin{aligned}
&\star(\beta^B \wedge \star\mathcal{F}_{2n+3}) - \beta^B \wedge \mathcal{F}_{2n-1} \\
&\quad - \star(H_3 \wedge \iota_Z(\star\mathcal{F}_{2n+3})) + \star\iota_Z(Z \wedge \star\mathcal{F}_{2n+1} - \iota_I \star \mathcal{F}_{2n-1}) \\
&\quad + H_3 \wedge \iota_Z\mathcal{F}_{2n-1} + \iota_Z(Z \wedge \mathcal{F}_{2n+1} + \iota_I\mathcal{F}_{2n+3}) - \star(I \wedge d \star \mathcal{F}_{2n+3}) + I \wedge d\mathcal{F}_{2n-1} \\
&\quad + 2I \cdot Z\mathcal{F}_{2n+1} \, .
\end{aligned} \tag{D.6}$$

The two terms on the first line are the same as in (4.2). To derive the remaining terms of (4.2), we use the relations

$$\begin{aligned}
-\star(I \wedge d \star \mathcal{F}_{2n+3}) &= -\star(H_3 \wedge \star d\mathcal{F}_{2n+3}) - \star(H_3 \wedge \star(H_3 \wedge \mathcal{F}_{2n+1})) \\
&\quad + \star(H_3 \wedge \star(Z \wedge \mathcal{F}_{2n+3})) - \star(I \wedge Z \wedge \star\mathcal{F}_{2n+3}) \\
&\quad + \star(I \wedge \star(I \wedge F_{2n+1})) \, , \\
I \wedge d\mathcal{F}_{2n-1} &= H_3 \wedge \star d \star \mathcal{F}_{2n-1} - H_3 \wedge \star(H_3 \wedge \star\mathcal{F}_{2n+1}) \\
&\quad - H_3 \wedge \star(Z \wedge \star\mathcal{F}_{2n-1}) + I \wedge Z \wedge \mathcal{F}_{2n-1} + I \wedge \star(I \wedge \star\mathcal{F}_{2n+1}) \, ,
\end{aligned} \tag{D.7}$$

which transform the last two lines of (D.6) into

$$\begin{aligned}
&-\star(H_3 \wedge \star d\mathcal{F}_{2n+3}) - \star(H_3 \wedge \star(H_3 \wedge \mathcal{F}_{2n+1})) + H_3 \wedge \star d \star \mathcal{F}_{2n-1} \\
&\quad - H_3 \wedge \star(H_3 \wedge \star\mathcal{F}_{2n+1}) + \star(I \wedge Z \wedge \star\mathcal{F}_{2n+3}) - \star(I \wedge \star(I \wedge F_{2n+1})) + I \wedge Z \wedge \mathcal{F}_{2n-1} \\
&\quad + I \wedge \star(I \wedge \star\mathcal{F}_{2n+1}) + \star\iota_Z(Z \wedge \star\mathcal{F}_{2n+1} - \iota_I \star \mathcal{F}_{2n-1}) + \iota_Z(Z \wedge \mathcal{F}_{2n+1} + \iota_I\mathcal{F}_{2n+3}) \\
&\quad + 2I \cdot Z\mathcal{F}_{2n+1} \, .
\end{aligned} \tag{D.8}$$

Now using the identities

$$\begin{aligned}
&\star(I \wedge Z \wedge \star\mathcal{F}_{2n+3}) = \iota_Z\iota_I\mathcal{F}_{2n+3} \, , \qquad \star\iota_Z\iota_I \star \mathcal{F}_{2n-1} = I \wedge Z \wedge \mathcal{F}_{2n-1} \, , \\
&\star(I \wedge \star(I \wedge F_{2n+1})) = \iota_I(I \wedge \mathcal{F}_{2n+1}) \, , \\
&\star\, \iota_I(I \wedge \star\mathcal{F}_{2n+1}) = I \wedge \star(I \wedge \star\mathcal{F}_{2n+1}) = I \wedge \iota_I F_{2n+1} \, ,
\end{aligned} \tag{D.9}$$

one finds

$$\begin{aligned}
&-\star(H_3 \wedge \star d\mathcal{F}_{2n+3}) - \star(H_3 \wedge \star(H_3 \wedge \mathcal{F}_{2n+1})) + H_3 \wedge \star d \star \mathcal{F}_{2n-1} \\
&\qquad - H_3 \wedge \star(H_3 \wedge \star\mathcal{F}_{2n+1}) + \iota_X(X)\mathcal{F}_{2n+1} \\
&\quad = -\star(H_3 \wedge \star d\mathcal{F}_{2n+3}) - \star(H_3 \wedge \star(H_3 \wedge \mathcal{F}_{2n+1})) + H_3 \wedge \star d \star \mathcal{F}_{2n-1} \\
&\qquad - H_3 \wedge \star(H_3 \wedge \star\mathcal{F}_{2n+1}) + (\tfrac{1}{4}R - \tfrac{1}{8} \star (H_3 \wedge \star H_3))\mathcal{F}_{2n+1} + (\star d \star X)\mathcal{F}_{2n+1} \, .
\end{aligned} \tag{D.10}$$

This leads precisely to (4.2).

Appendix E. Derivation of "Bianchi identity" for Z

Here we observe that the modified "Bianchi identity for the dilaton" (2.13) that holds for the ABF background may be derived more generally from the Bianchi equations for $\mathcal{F}_k$, the invariance of the R–R fields under the isometry $\mathcal{L}_I\mathcal{F}_k = 0$, the conditions $\mathcal{F}_1 \wedge \mathcal{F}_3 \neq 0$, $\mathcal{F}_1 \wedge \mathcal{F}_5 \neq 0$ or $\mathcal{F}_3 \wedge \mathcal{F}_5 \neq 0$ and the condition $\iota_I\mathcal{F}_1 = 0$. Starting from (see (3.4)–(3.9))

$$d\mathcal{F}_1 - Z \wedge \mathcal{F}_1 - \iota_I\mathcal{F}_3 = 0\,, \tag{E.1}$$

$$d\mathcal{F}_3 - Z \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1 - \iota_I\mathcal{F}_5 = 0\,, \tag{E.2}$$

we take the differential of (E.1) and use (E.2) to give[33]

$$\begin{aligned}
&-dZ \wedge \mathcal{F}_1 + Z \wedge d\mathcal{F}_1 - d(\iota_I\mathcal{F}_3) = -dZ \wedge \mathcal{F}_1 + Z \wedge \iota_I\mathcal{F}_3 + \iota_I d\mathcal{F}_3 \\
&\quad = -dZ \wedge \mathcal{F}_1 + Z \wedge \iota_I\mathcal{F}_3 + \iota_I(Z \wedge \mathcal{F}_3) - \iota_I(H \wedge \mathcal{F}_1) \\
&\quad = -dZ \wedge \mathcal{F}_1 + \iota_I Z \wedge \mathcal{F}_3 - \iota_I H \wedge \mathcal{F}_1 = -(dZ + \iota_I H) \wedge \mathcal{F}_1 = 0\,.
\end{aligned} \tag{E.3}$$

Thus

$$dZ + \iota_I H \sim \mathcal{F}_1\,. \tag{E.4}$$

A similar analysis of the Bianchi equations for $\mathcal{F}_3$ and $\mathcal{F}_5$ gives

$$(dZ + \iota_I H) \wedge \mathcal{F}_3 = 0\,, \qquad (dZ + \iota_I H_3) \wedge \mathcal{F}_5 = 0\,. \tag{E.5}$$

Thus if $\mathcal{F}_1 \wedge \mathcal{F}_3 \neq 0$, $\mathcal{F}_1 \wedge \mathcal{F}_5 \neq 0$ or $\mathcal{F}_3 \wedge \mathcal{F}_5 \neq 0$ then $dZ + \iota_I H = 0$.

Appendix F. Deformed $AdS_3 \times S^3$ and $AdS_2 \times S^2$ cases

In the deformed $AdS_3 \times S^3$ case the (complete) T-dual HT background [12] consists of just the metric, dilaton and a single R–R 3-form flux, and therefore has a simple embedding into Type IIB supergravity – one just needs to add 4 extra toroidal dimensions. Explicitly, this background which is T-dual to the η-deformed $AdS_3 \times S^3$ background (cf. [35,36]) is given by

$$\begin{aligned}
\hat{ds}^2 &= -\frac{1-\varkappa^2\rho^2}{1+\rho^2}d\hat{t}^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{d\hat{\psi}_1^2}{\rho^2} \\
&\quad + \frac{1+\varkappa^2 r^2}{1-r^2}d\hat{\varphi}^2 + \frac{dr^2}{(1+\varkappa^2 r^2)(1-r^2)} + \frac{d\hat{\phi}_1^2}{r^2} + dx_a dx_a\,, \\
\hat{B} &= 0\,, \qquad \hat{\mathcal{F}}_1 = \hat{\mathcal{F}}_5 = 0\,, \\
\hat{\mathcal{F}}_3 &= \frac{2i\sqrt{1+\varkappa^2}}{\sqrt{1+\rho^2}\sqrt{1-r^2}}\Big[(d\hat{t} + \frac{\varkappa\rho d\rho}{1-\varkappa^2\rho^2}) \wedge \frac{d\hat{\psi}_1}{\rho} \wedge (\frac{dr}{1+\varkappa^2 r^2} + \varkappa r d\hat{\varphi}) \\
&\qquad + (d\hat{\varphi} - \frac{\varkappa r dr}{1+\varkappa^2 r^2}) \wedge \frac{d\hat{\phi}_1}{r} \wedge (\frac{d\rho}{1-\varkappa^2\rho^2} + \varkappa\rho d\hat{t})\Big]\,,
\end{aligned}$$

[33] Note that here we use the condition $\iota_I\mathcal{F}_1 = 0$, which if $dZ \neq 0$ follows from (E.1) after acting on it with ι_I

$$\iota_I d\mathcal{F}_1 + Z\iota_I\mathcal{F}_1 = 0 \quad \Rightarrow \quad d\,\iota_I\mathcal{F}_1 - Z\,\iota_I\mathcal{F}_1 = 0\,,$$

where we have used $\iota_I Z = 0$ and $\mathcal{L}_I\mathcal{F}_1 = 0$. We see that if $\iota_I\mathcal{F}_1 \neq 0$ then $Z = d\log\iota_I\mathcal{F}_1$, which contradicts our assumption.

$$\hat{\phi} = \phi_0 - 2\varkappa(\hat{t} + \hat{\varphi}) + \log \frac{(1 - \varkappa^2\rho^2)(1 + \varkappa^2 r^2)}{\rho r\sqrt{1 + \rho^2}\sqrt{1 - r^2}} . \tag{F.1}$$

When written in terms of the "boosted"/"rotated" vielbein basis [12]

$$\begin{aligned} e^0 &= \tfrac{1}{\sqrt{1+\rho^2}}\big(d\hat{t} + \tfrac{\varkappa\rho d\rho}{1-\varkappa^2\rho^2}\big) , \qquad e^1 = \tfrac{d\hat{\psi}_1}{\rho} , \qquad e^2 = \tfrac{1}{\sqrt{1+\rho^2}}\big(\tfrac{d\rho}{1-\varkappa^2\rho^2} + \varkappa\rho d\hat{t}\big) , \\ e^3 &= \tfrac{1}{\sqrt{1-r^2}}\big(d\hat{\varphi} - \tfrac{\varkappa r dr}{1+\varkappa^2 r^2}\big) , \qquad e^4 = \tfrac{d\hat{\phi}_1}{r} , \qquad e^5 = \tfrac{1}{\sqrt{1-r^2}}\big(\tfrac{dr}{1+\varkappa^2 r^2} + \varkappa r d\hat{\varphi}\big) , \end{aligned} \tag{F.2}$$

the metric and $\hat{\mathcal{F}}_3$ take the following simple form (cf. (B.5))

$$\hat{ds}^2 = \eta_{MN}\, e^M e^N + dx_a dx_a , \qquad \hat{\mathcal{F}}_3 = 2i\sqrt{1 + \varkappa^2}\big(e^0 \wedge e^1 \wedge e^5 + e^2 \wedge e^3 \wedge e^4\big) . \tag{F.3}$$

As in the $AdS_5 \times S^5$ case, the dilaton and the R–R flux F_3 depend on the isometric directions of the metric, but this dependence is such that $e^\phi F = \mathcal{F}$ is invariant under the isometries. Therefore, we can formally T-dualise the metric and $\hat{\mathcal{F}}$ to find the following analog of the ABF background (cf. (B.1))

$$\begin{aligned} ds^2 &= -\frac{1 + \rho^2}{1 - \varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1 - \varkappa^2\rho^2)(1 + \rho^2)} + \rho^2 d\psi_1^2 + \frac{1 - r^2}{1 + \varkappa^2 r^2}d\varphi^2 \\ &\quad + \frac{dr^2}{(1 + \varkappa^2 r^2)(1 - r^2)} + r^2 d\phi_1^2 + dx_a dx_a , \\ B &= 0 , \\ \mathcal{F}_1 &= \varkappa\, \mathrm{F}\Big[\rho^2 d\psi_1 + r^2 d\phi_1\Big] , \\ \mathcal{F}_3 &= \mathrm{F}\Big[\frac{\rho}{1 - \varkappa^2\rho^2}(dt \wedge d\psi_1 \wedge d\rho + \varkappa^2 r^2\, dt \wedge d\phi_1 \wedge d\rho) \\ &\quad - \frac{r}{1 + \varkappa^2 r^2}(d\varphi \wedge d\phi_1 \wedge dr - \varkappa^2\rho^2\, d\varphi \wedge d\psi_1 \wedge dr)\Big] , \\ \mathcal{F}_5 &= \varkappa\, \mathrm{F}\Big[\frac{\rho r}{(1 - \varkappa^2\rho^2)(1 + \varkappa^2 r^2)}(dt \wedge d\rho \wedge d\varphi \wedge d\phi_1 \wedge dr - dt \wedge d\psi_1 \wedge d\rho \wedge d\varphi \wedge dr) \\ &\quad - (\rho^2 d\psi_1 + r^2 d\phi_1) \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\Big] , \\ \mathrm{F} &\equiv \frac{2\sqrt{1 + \varkappa^2}}{\sqrt{1 - \varkappa^2\rho^2}\sqrt{1 + \varkappa^2 r^2}} . \end{aligned} \tag{F.4}$$

As in the $AdS_5 \times S^5$ case, it turns out that there exist vectors X and Y such that the scale invariance conditions for the metric and B-field (2.1), (2.2) are satisfied (cf. (2.5), (2.8))

$$\begin{aligned} X = X_m dx^m &= c_0\frac{1 + \rho^2}{1 - \varkappa^2\rho^2}dt + c_1\rho^2 d\psi_1 - \frac{\varkappa^2\rho}{1 - \varkappa^2\rho^2}d\rho \\ &\quad + c_2\frac{1 - r^2}{1 + \varkappa^2 r^2}d\varphi + c_3 r^2 d\phi_1 + \frac{\varkappa^2 r}{1 + \varkappa^2 r^2}dr + k_a dx^a , \end{aligned} \tag{F.5}$$

$$Y = Y_m dx^m = 2\varkappa\frac{1 + \rho^2}{1 - \varkappa^2\rho^2}dt - \frac{\varkappa^2\rho}{1 - \varkappa^2\rho^2}d\rho + 2\varkappa\frac{1 - r^2}{1 + \varkappa^2 r^2}d\varphi + \frac{\varkappa^2 r}{1 + \varkappa^2 r^2}dr . \tag{F.6}$$

The parameters c_i and k_a are eight arbitrary constants parametrising the Killing vector part of X_m, while Y is defined up to a total derivative. As in the $AdS_5 \times S^5$ case, we may split the vector

X into two parts: I, containing the 8 commuting Killing vectors, and Z, which contains the rest. If we fix the constants c_i and k_a as

$$c_0 = c_2 = 2\varkappa\ , \qquad c_1 = c_3 = k_a = 0\ , \tag{F.7}$$

so that $Y_m = X_m$ then the equations (2.10), (2.11), (2.13), (3.4)–(3.9) are all satisfied, and hence the background (F.4) solves the same system of equations as the ABF background (B.1). Finally, we find ϕ in (2.14) is given by

$$\phi = \tfrac{1}{2}\log(1-\varkappa^2\rho^2)(1+\varkappa^2 r^2)\ . \tag{F.8}$$

For the deformed $AdS_2 \times S^2$ case, the T-dual HT background [12] consists of just the metric, dilaton and a single R–R 2-form flux. It can be again embedded into Type IIB supergravity by adding 6-torus T^6 and combining the 2-form with the holomorphic 3-form on T^6 to give a self-dual 5-form:

$$\begin{aligned}
\hat{ds}^2 &= -\frac{1-\varkappa^2\rho^2}{1+\rho^2}d\hat{t}^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{1+\varkappa^2 r^2}{1-r^2}d\hat{\varphi}^2 + \frac{dr^2}{(1+\varkappa^2 r^2)(1-r^2)} \\
&\qquad + dx_a dx_a\ , \\
\hat{B} &= 0\ , \qquad \hat{\mathcal{F}}_1 = \hat{\mathcal{F}}_3 = 0\ , \\
\hat{\mathcal{F}}_5 &= \frac{i\sqrt{1+\varkappa^2}}{\sqrt{2}\sqrt{1+\rho^2}\sqrt{1-r^2}}\Big[(d\hat{t} + \frac{\varkappa\rho d\rho}{1-\varkappa^2\rho^2})\wedge(\frac{dr}{1+\varkappa^2 r^2} + \varkappa r d\hat{\varphi})\wedge(\omega_r + \omega_i) \\
&\qquad + (d\hat{\varphi} - \frac{\varkappa r dr}{1+\varkappa^2 r^2})\wedge(\frac{d\rho}{1-\varkappa^2\rho^2} + \varkappa\rho d\hat{t})\wedge(\omega_r - \omega_i)\Big]\ , \\
\hat{\phi} &= \phi_0 - \varkappa(\hat{t} + \hat{\varphi}) + \log\frac{(1-\varkappa^2\rho^2)(1+\varkappa^2 r^2)}{\sqrt{1+\rho^2}\sqrt{1-r^2}}\ ,
\end{aligned} \tag{F.9}$$

where ω_r and ω_i are the real and imaginary parts of the holomorphic 3-form on T^6, e.g.,

$$\begin{aligned}
\omega_r &= dx^1\wedge dx^3\wedge dx^5 - dx^1\wedge dx^4\wedge dx^6 - dx^2\wedge dx^3\wedge dx^6 - dx^2\wedge dx^4\wedge dx^5\ , \\
\omega_i &= dx^2\wedge dx^4\wedge dx^6 - dx^2\wedge dx^3\wedge dx^5 - dx^1\wedge dx^4\wedge dx^5 - dx^1\wedge dx^3\wedge dx^6\ .
\end{aligned} \tag{F.10}$$

As in the $AdS_5 \times S^5$ and $AdS_3 \times S^3$ cases, when written in terms of the "boosted"/"rotated" vielbein basis [12]

$$\begin{aligned}
e^0 &= \tfrac{1}{\sqrt{1+\rho^2}}\big(d\hat{t} + \tfrac{\varkappa\rho d\rho}{1-\varkappa^2\rho^2}\big)\ , \qquad e^1 = \tfrac{1}{\sqrt{1+\rho^2}}\big(\tfrac{d\rho}{1-\varkappa^2\rho^2} + \varkappa\rho d\hat{t}\big)\ , \\
e^2 &= \tfrac{1}{\sqrt{1-r^2}}\big(d\hat{\varphi} - \tfrac{\varkappa r dr}{1+\varkappa^2 r^2}\big)\ , \qquad e^3 = \tfrac{1}{\sqrt{1-r^2}}\big(\tfrac{dr}{1+\varkappa^2 r^2} + \varkappa r d\hat{\varphi}\big)\ ,
\end{aligned} \tag{F.11}$$

the metric and $\hat{\mathcal{F}}_5$ have take the following simple form

$$\begin{aligned}
\hat{ds}^2 &= \eta_{MN} e^M e^N + dx_a dx_a\ , \\
\hat{\mathcal{F}}_5 &= \tfrac{i}{\sqrt{2}}\sqrt{1+\varkappa^2}\big[e^0\wedge e^3\wedge(\omega_r+\omega_i) - e^1\wedge e^2\wedge(\omega_r - \omega_i)\big]\ .
\end{aligned} \tag{F.12}$$

Applying T-duality to the metric and $\hat{\mathcal{F}}$ gives the analog of ABF background for the $AdS_2 \times S^2$ η-model

$$
\begin{aligned}
ds^2 =& -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{1-r^2}{1+\varkappa^2 r^2}d\varphi^2 \\
&+ \frac{dr^2}{(1+\varkappa^2 r^2)(1-r^2)} + dx_a dx_a \ , \\
B =& \, 0 \ , \qquad \mathcal{F}_1 = 0 \ , \\
\mathcal{F}_3 =& \, \tfrac{1}{2}\varkappa\, \mathrm{F}\Big[(-\rho + r)\omega_r + (\rho + r)\omega_i\Big] \ , \\
\mathcal{F}_5 =& \, \tfrac{1}{2}\mathrm{F}\Big[\frac{1-\varkappa^2\rho r}{1-\varkappa^2\rho^2}dt\wedge d\rho\wedge\omega_r - \frac{1+\varkappa^2\rho r}{1-\varkappa^2\rho^2}dt\wedge d\rho\wedge\omega_i + \frac{1+\varkappa^2\rho r}{1+\varkappa^2 r^2}d\varphi\wedge dr\wedge\omega_r \\
&+ \frac{1-\varkappa^2\rho r}{1+\varkappa^2 r^2}d\varphi\wedge dr\wedge\omega_i\Big] \ , \\
\mathrm{F} \equiv& \, \frac{\sqrt{2}\sqrt{1+\varkappa^2}}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}} \ .
\end{aligned}
\tag{F.13}
$$

Here again the scale invariance conditions for the metric and B-field (2.1), (2.2) are satisfied provided

$$
X = X_m dx^m = c_0 \frac{1+\rho^2}{1-\varkappa^2\rho^2}dt + c_1\frac{1-r^2}{1+\varkappa^2 r^2}d\varphi + k_a dx^a \ , \tag{F.14}
$$

$$
Y = Y_m dx^m = \varkappa\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt + \varkappa\frac{1-r^2}{1+\varkappa^2 r^2}d\varphi \ . \tag{F.15}
$$

The parameters c_i and k_a are eight arbitrary constants parametrising the Killing vector part of X_m, while Y is defined up to a total derivative. Here X_m is given just by I_m (i.e. the sum of commuting Killing vectors) and thus $Z_m = 0$. If we fix the constants c_i and k_a as

$$
c_0 = c_1 = \varkappa \ , \qquad k_a = 0 \ , \tag{F.16}
$$

so that $Y_m = X_m$ then the equations (2.10), (2.11), (2.13), (3.4)–(3.9) are all satisfied, i.e. the background (F.13) solves the same system of equations as the ABF background (B.1) in the $AdS_5 \times S^5$ case. Finally, here we find that ϕ in (2.14) is given by

$$
\phi = 0 \ . \tag{F.17}
$$

As in the $AdS_5 \times S^5$ case, the coefficients in (F.7), (F.16) are equal to (minus) the corresponding coefficients of the isometric coordinates in the linear terms of the dual dilatons $\hat{\phi}$ of the T-dual HT backgrounds [12]. Furthermore, the “dilatons” ϕ in (F.8), (F.17) are again found by applying the standard T-duality rules to the remaining parts (depending only on non-isometric coordinates) of the dilatons $\hat{\phi}$ of the T-dual solutions. Therefore, these examples also fit into the general picture described in section the main text.

Appendix G. Second-order equations for R–R fields from scale invariance conditions for type II GS sigma model

In this appendix we shall expand on the discussion in section 4 and explain how the 2nd-order equations for the R–R couplings $\mathcal{F}$ such as (4.9)–(4.11) can emerge as the one-loop conditions of scale invariance (UV finiteness) of the GS sigma model (1.2). While we will not compute the beta-functions for R–R couplings in full, our aim will be to illustrate how the relevant structures come out of logarithmically divergent parts of the corresponding one-loop Feynman graphs.[34]

We shall consider the type IIB GS sigma model [5] with couplings representing a generic type IIB superspace background subject to constraints required for κ-symmetry: we will assume κ-symmetry to be able to gauge fix it but otherwise will keep the R–R fields unconstrained. The GS sigma model action expanded in powers of fermions may be written as (see, e.g., [14,8], cf. (1.2))[35]

$$L_{GS} = L_b + L_{2f} + L_{4f} + \dots \ , \tag{G.1}$$

$$L_b = \tfrac{1}{2}\gamma^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} - \tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu} \ , \tag{G.2}$$

$$L_{2f} = i(\gamma^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}s^{IJ})\bar\theta^I e^a_\alpha \Gamma_a \mathrm{D}^{JK}_\beta \theta^K \ , \qquad e^a_\alpha = e^a_\mu(x)\partial_\alpha x^\mu \ , \tag{G.3}$$

$$\begin{aligned}\mathrm{D}_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}) - \tfrac{1}{8}s_3 H_{ab\mu}\Gamma^{ab} + \tfrac{1}{8}e^\phi\big[\not{F}_{(1)}s_0 + \not{F}_{(3)}s_1 + \tfrac{1}{2}\not{F}_{(5)}s_0\big]\Gamma_\mu \\ &= \mathcal{D}_\mu + \tfrac{1}{8}e^\phi\big[\not{F}_{(1)}s_0 + \not{F}_{(3)}s_1 + \tfrac{1}{2}\not{F}_{(5)}s_0\big]\Gamma_\mu \ , \qquad \mathrm{D}_\alpha = \partial_\alpha x^\mu \mathrm{D}_\mu \ ,\end{aligned} \tag{G.4}$$

$$L_{4f} = K^{\alpha\beta}_{IJKLXY}\bar\theta^I M^X_\alpha \theta^J \bar\theta^K N^Y_\beta \theta^L \ . \tag{G.5}$$

In (G.5) the indices X and Y stand for multi-indices of the same type as the one carried by the fermions. The 2×2 matrices appearing in L_{2f} are $s \equiv s_3 = \sigma_3$, $s_1 = \sigma_1$, $s_0 = i\sigma_2$. The R–R couplings are

$$e^\phi \not{F}_n = \tfrac{1}{n!}e^\phi F_{a_1\dots a_n}\Gamma^{a_1\dots a_n} \equiv \tfrac{1}{n!}\mathcal{F}_{a_1\dots a_n}\Gamma^{a_1\dots a_n} \ , \tag{G.6}$$

where F_n are not required a priori to be field strengths.

We shall first fix the κ-symmetry gauge $\theta^1 = \theta^2$ and also consider flat 2d space (or, equivalently, fix the conformal gauge for 2d diffeomorphisms) and then expand (x,θ) near some background values $(\bar x, \Theta)$. The aim will then be to compute the one-loop UV divergences that renormalise the R–R couplings in the quadratic fermionic term (G.3), i.e. ($\bar{\mathcal{F}} = \mathcal{F}(\bar x)$)

$$\bar L_{2f} = \tfrac{1}{4}\epsilon^{\alpha\beta}\bar\Theta \bar e^a_\alpha \Gamma_a \bar H_{abc}\bar e^c_\beta \Gamma^{ab}\Theta + \bar L^{\mathcal{F}}_{2f} \tag{G.7}$$

$$\bar L^{\mathcal{F}}_{2f} = \tfrac{1}{4}\eta^{\alpha\beta}\bar\Theta \bar e^a_\alpha \Gamma_a \Sigma_e \bar e^b_\beta \Gamma_b \Theta + \tfrac{1}{4}\epsilon^{\alpha\beta}\bar\Theta \bar e^a_\alpha \Gamma_a \Sigma_o \bar e^b_\beta \Gamma_b \Theta \ , \tag{G.8}$$

$$\Sigma_e = \bar{\not{\mathcal{F}}}_{(3)} \ , \qquad \Sigma_o = \bar{\not{\mathcal{F}}}_{(1)} + \tfrac{1}{2}\bar{\not{\mathcal{F}}}_{(5)} \ , \tag{G.9}$$

$$\delta L_{2f} = \epsilon^{\alpha\beta}\bar\Theta O^H_{\alpha\beta}\Theta + \delta L^{\mathcal{F}}_{2f} \tag{G.10}$$

$$\delta L^{\mathcal{F}}_{2f} = \eta^{\alpha\beta}\bar\Theta E_{\alpha\beta}\Theta + \epsilon^{\alpha\beta}\bar\Theta O_{\alpha\beta}\Theta \ . \tag{G.11}$$

Here the classical term $\bar L^{\mathcal{F}}_{2f}$ and the expected divergent term $\delta\bar L^{\mathcal{F}}_{2f}$ are decomposed into parity-even and parity-odd parts containing the linearly-independent combinations of antisymmetrised products of Dirac matrices. The combinations E and O should then represent the R–R beta-functions that should be set to zero modulo use of equations of motion on $(\bar x, \Theta)$ or modulo target space (super)reparametrisations. A further contribution to the two-fermion divergence is the first

[34] Previous studies of the UV finiteness conditions of the GS string [10,11] did not include R–R couplings, but special cases of $AdS_5\times S^5$ [38] and pp-wave backgrounds [39] were explicitly discussed. The vanishing of the beta-functions for the R–R couplings was not checked as the fermionic coordinate was assumed to have trivial background.

[35] In this appendix we use $\alpha,\beta,\gamma,\dots$ for 2d indices, with $\gamma^{\alpha\beta} \equiv \sqrt{h}h^{\alpha\beta}$. $\mu,\nu,\dots$ are 10d coordinate indices, and $a,b,c,\dots$ are tangent space indices with $G_{\mu\nu} = e^a_\mu e^b_\mu \eta_{ab}$. The indices $I,J,K = 1,2$ label two MW spinors of type IIB action.

term in δL_{2f}; it should be proportional to the NS–NS fields (vielbein and H) beta-functions and thus should contain terms independent of the R–R fields.

Introducing the fluctuations (ξ^μ, θ) around $(\bar{x}^\mu, \Theta)$ as

$$x^\mu \to \bar{x}^\mu + \pi^\mu(\xi) , \qquad \theta \to \Theta + \theta , \tag{G.12}$$

the standard relations of the bosonic normal coordinate expansion are

$$\begin{aligned}
\partial_\alpha(\bar{x}^\mu + \pi^\mu) &= \partial_\alpha \bar{x}^\mu + \nabla_\alpha \xi^\mu + \tfrac{1}{3} R^\mu{}_{\lambda\sigma\nu} \partial_\alpha \bar{x}^\nu \xi^\lambda \xi^\sigma + \mathcal{O}(\xi^3) , \\
\partial_\alpha(\bar{x}^\mu + \pi^\mu) e^a_\mu &= \zeta^a_\alpha + \nabla_\alpha \xi^a + \tfrac{1}{2} R^a{}_{bcd} \zeta^b_\alpha \xi^c \xi^d + \mathcal{O}(\xi^3) , \qquad \zeta^a_\alpha \equiv \partial_\alpha \bar{x}^\mu \bar{e}^a_\mu ,
\end{aligned} \tag{G.13}$$

$$\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + \tfrac{1}{3} R_{\mu\lambda\sigma\nu} \xi^\lambda \xi^\sigma + \mathcal{O}(\xi^3) , \\
e^a_\mu &= \bar{e}^a_\mu + \frac{1}{6} R^a{}_{\lambda\sigma\mu} \xi^\lambda \xi^\sigma + \mathcal{O}(\xi^3) , \\
\omega_\mu{}^a{}_b &= \bar{\omega}_\mu{}^a{}_b + \tfrac{1}{2} \xi^\nu R^a{}_{b\nu\mu} + \tfrac{1}{3} \xi^\nu \xi^\rho \nabla_\rho R^a{}_{b\nu\mu} + \mathcal{O}(\xi^3) .
\end{aligned} \tag{G.14}$$

The normal coordinate expansion of the R–R tensor fields ($\bar{\mathcal{F}} \equiv \mathcal{F}(\bar{x})$)

$$\begin{aligned}
\mathcal{F}_{\mu_1 \ldots \mu_n} &= \bar{\mathcal{F}}_{\mu_1 \ldots \mu_n} + \xi^\nu \nabla_\nu \bar{\mathcal{F}}_{\mu_1 \ldots \mu_n} \\
&\quad + \frac{1}{2} \xi^\mu \xi^\nu \big(\nabla_\mu \nabla_\nu \bar{\mathcal{F}}_{\mu_1 \ldots \mu_n} + \frac{1}{3} \sum_{j=1}^n \bar{\mathcal{F}}_{\mu_1 \ldots \sigma_j \ldots \mu_n} R^{\sigma_j}{}_{\mu\nu\mu_j} \big) + \mathcal{O}(\xi^3)
\end{aligned}$$

takes a simpler form using tangent space indices:

$$\mathcal{F}_{a_1 \ldots a_n} = \bar{\mathcal{F}}_{a_1 \ldots a_n} + \xi^\nu \nabla_\nu \bar{\mathcal{F}}_{a_1 \ldots a_n} + \tfrac{1}{2} \xi^\mu \xi^\nu \nabla_\mu \nabla_\nu \bar{\mathcal{F}}_{a_1 \ldots a_n} + \mathcal{O}(\xi^3) . \tag{G.15}$$

Note that the beta-functions for the couplings $\mathcal{F}_{a_1 \ldots a_n}$ and $\mathcal{F}_{\mu_1 \ldots \mu_n}$ are related by extra terms involving beta-functions of vielbein e^a_μ or the metric $G_{\mu\nu}$; this is related to the presence of β^G terms in (4.5) or (4.9)–(4.11).

The expanded Lagrangian (G.1) has the following structure:

$$L = L_b + L_f^{\xi\xi} + L_f^{\xi\theta} + L_f^{\theta\theta} + \ldots , \tag{G.16}$$

$$L_b = \tfrac{1}{2} \eta^{\alpha\beta} \nabla_\alpha \xi^a \nabla_\beta \xi^b \eta_{ab} + \nabla_\alpha \xi^a \xi^b U^\alpha_{ab} + \tfrac{1}{2} \xi^a \xi^b X_{ab} , \tag{G.17}$$

$$L_f^{\xi\xi} = \tfrac{1}{2} \nabla_\alpha \xi^a \nabla_\beta \xi^b C^{\alpha\beta}_{ab} + \nabla_\alpha \xi^a \xi^b C^\alpha_{ab} + \tfrac{1}{2} \xi^a \xi^b C_{ab} , \tag{G.18}$$

$$L_f^{\xi\theta} = \nabla_\alpha \xi^a \bar{\Psi}^{\alpha\beta}_a \mathcal{D}_\beta \theta + \nabla_\alpha \xi^a \bar{\Psi}^\alpha_a \theta + \xi^a \bar{\Psi}_a \theta , \tag{G.19}$$

$$L_f^{\theta\theta} = i \bar{\theta} \rho^\alpha \mathcal{D}_\alpha \theta + \bar{\theta} Y_{0F} \theta + \bar{\theta} Y_{2f} \theta , \tag{G.20}$$

where Y_{0F} and Y_{2f} contain zero and two background fermions Θ, respectively. We have also defined $\rho_\alpha \equiv \zeta^a_\alpha \Gamma_a$. It will be sufficient to further assume that the induced metric is trivial, i.e. $G_{\mu\nu}(\bar{x}) \partial_\alpha \bar{x}^\mu \partial_\beta \bar{x}^\nu = \eta_{\alpha\beta}$ and $\rho_{(\alpha} \rho_{\beta)} = \eta_{\alpha\beta}$.

The explicit form of the quadratic terms in (G.16) is ($L_f = L_{2f} + L_{4f} + \ldots$, see (G.3),(G.5))

$$\begin{aligned}
L_b &= \tfrac{1}{2} \eta^{\alpha\beta} \Big[\nabla_\alpha \xi^a \nabla_\beta \xi^b \eta_{ab} + R_{acdb} \zeta^a_\alpha \zeta^b_\beta \xi^c \xi^d \Big] \\
&\quad + \frac{1}{2} \epsilon^{\alpha\beta} \Big[\zeta^a_\alpha \nabla_\beta \xi^b \xi^c H_{abc} + \frac{1}{2} \zeta^a_\alpha \zeta^b_\beta \xi^d \xi^c \, \nabla_d H_{abc} \Big] \\
-i L^{\xi\xi}_{2f} &= \tfrac{1}{12} \xi^c \xi^d (R^a{}_{cde} \zeta^e_\alpha \zeta^b_\beta + R^b{}_{cde} \zeta^a_\alpha \zeta^e_\beta)(\eta^{\alpha\beta} \bar{\Theta}^I \Gamma_a \Sigma_e \Gamma_b \Theta - \epsilon^{\alpha\beta} \bar{\Theta}^I \Gamma_a \Sigma_o \Gamma_b \Theta)
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{4}\nabla_\alpha\xi^a\nabla_\beta\xi^b(\eta^{\alpha\beta}\bar\Theta^I\Gamma_a\Sigma_e\Gamma_b\Theta-\epsilon^{\alpha\beta}\bar\Theta^I\Gamma_a\Sigma_o\Gamma_b\Theta)\\
&+\tfrac{1}{4}(\nabla_\alpha\xi^a\zeta^b_\beta+\zeta^a_\alpha\nabla_\beta\xi^b)\xi^d(\eta^{\alpha\beta}\bar\Theta^I\Gamma_a\nabla_d\Sigma_e\Gamma_b\Theta-\epsilon^{\alpha\beta}\bar\Theta^I\Gamma_a\nabla_d\Sigma_o\Gamma_b\Theta)\\
&+\tfrac{1}{24}\xi^c\xi^d(\zeta^f_\alpha\zeta^b_\beta R_{fcd}{}^a+\zeta^f_\alpha\zeta^a_\beta R_{fcd}{}^b)(\eta^{\alpha\beta}\bar\Theta^I\Gamma_a\Sigma_e\Gamma_b\Theta-\epsilon^{\alpha\beta}\bar\Theta^I\Gamma_a\Sigma_o\Gamma_b\Theta)\\
&+\tfrac{1}{8}\zeta^a_\alpha\zeta^b_\beta\xi^c\xi^d(\eta^{\alpha\beta}\bar\Theta^I\Gamma_a\nabla_c\nabla_d\Sigma_e\Gamma_b\Theta-\epsilon^{\alpha\beta}\bar\Theta^I\Gamma_a\nabla_c\nabla_d\Sigma_o\Gamma_b\Theta)\ ,\\
-iL^{\xi\theta}_{2f}&=4\eta^{\alpha\beta}\nabla_\alpha\xi^a\bar\Theta\Gamma_a\mathcal{D}_\beta\theta+\tfrac{1}{2}\eta^{\alpha\beta}\zeta^a_\alpha\zeta^c_\beta\xi^bR^{de}{}_{bc}\bar\Theta\Gamma_a\Gamma_{de}\theta\\
&+\tfrac{1}{4}\epsilon^{\alpha\beta}(\zeta^a_\alpha\nabla_\beta\xi^c+\nabla_\alpha\xi^a\zeta^c_\beta)(\bar\Theta\Gamma_aH_{cde}\Gamma^{de}\theta+\bar\theta\Gamma_aH_{cde}\Gamma^{de}\Theta)\\
&+\eta^{\alpha\beta}(\zeta^a_\alpha\nabla_\beta\xi^c+\nabla_\alpha\xi^a\zeta^c_\beta)(\bar\Theta\Gamma_a\Sigma_e\Gamma_c\theta+\bar\theta\Gamma_a\Sigma_e\Gamma_c\Theta)\\
&-\epsilon^{\alpha\beta}(\zeta^a_\alpha\nabla_\beta\xi^c+\nabla_\alpha\xi^a\zeta^c_\beta)(\bar\Theta\Gamma_a\Sigma_o\Gamma_c\theta+\bar\theta\Gamma_a\Sigma_o\Gamma_c\Theta)\\
&+\eta^{\alpha\beta}\zeta^a_\alpha\zeta^c_\beta\xi^d(\bar\Theta\Gamma_a\nabla_d\Sigma_e\Gamma_c\theta+\bar\theta\Gamma_a\nabla_d\Sigma_e\Gamma_c\Theta)\\
&-\epsilon^{\alpha\beta}\zeta^a_\alpha\zeta^c_\beta\xi^d(\bar\Theta\Gamma_a\nabla_d\Sigma_o\Gamma_c\theta+\bar\theta\Gamma_a\nabla_d\Sigma_o\Gamma_c\Theta)\ ,\\
-iL^{\theta\theta}_{2f}&=2\eta^{\alpha\beta}\zeta^a_\alpha\zeta^b_\beta\bar\theta\Gamma_a\bar{\mathcal{D}}_b\theta+\tfrac{1}{4}\epsilon^{\alpha\beta}\zeta^a_\alpha\zeta^b_\beta\bar\theta\Gamma_aH_{bcd}\Gamma^{cd}\theta\\
&+\tfrac{1}{4}\eta^{\alpha\beta}\zeta^a_\alpha\zeta^b_\beta\bar\theta\Gamma_a\Sigma_e\Gamma_b\theta-\tfrac{1}{4}\epsilon^{\alpha\beta}\zeta^a_\alpha\zeta^b_\beta\bar\theta\Gamma_a\Sigma_o\Gamma_b\theta\\
L^{\theta\theta}_{4f}&=\mathrm{K}^{\alpha\beta}_{XY}(\bar\Theta M^X_\alpha\Theta\bar\theta N^Y_\beta\theta+\bar\Theta M^X_\alpha\theta\bar\Theta N^Y_\beta\theta+\bar\Theta M^X_\alpha\theta\bar\theta N^Y_\beta\Theta\\
&+\bar\theta M^X_\alpha\Theta\bar\Theta N^Y_\beta\theta+\bar\theta M^X_\alpha\Theta\bar\theta N^Y_\beta\Theta+\bar\theta M^X_\alpha\theta\bar\Theta N^Y_\beta\Theta)\ ,\\
\mathrm{K}^{\alpha\beta}_{XY}&=\sum_{IJKL}K^{\alpha\beta}_{IJKLXY}
\end{aligned}
\tag{G.21}
$$

Thus the matrix coefficients appearing in (G.16)–(G.20) are

$$
\begin{aligned}
U^\alpha_{ab}&=\tfrac{1}{2}\epsilon^{\alpha\beta}\zeta^c_\beta H_{abc}\ ,\\
X_{ab}&=\eta^{\alpha\beta}\zeta^c_\alpha\zeta^d_\beta R_{cabd}+\tfrac{1}{4}\epsilon^{\alpha\beta}\zeta^c_\alpha\zeta^d_\beta(\nabla_aH_{bcd}+\nabla_bH_{acd})\ ,\\
C^{\alpha\beta}_{ab}&=\tfrac{1}{4}i(\eta^{\alpha\beta}\bar\Theta^I\Gamma_a\Sigma_e\Gamma_b\Theta-\epsilon^{\alpha\beta}\bar\Theta^I\Gamma_a\Sigma_o\Gamma_b\Theta)\ ,\qquad\text{etc.}
\end{aligned}
\tag{G.22}
$$

It is straightforward to find the UV-divergent term (G.11) for the general Lagrangian (G.16). It receives contributions from Feynman graphs with one C or one Y_{2f} vertex (each containing two background fermions) and from Feynman graphs with two vertices of the type Ψ (each containing a single background fermion). The result has the form

$$
\delta L^{\mathcal{F}}_{2f}=\delta L_1+\delta L_2+\delta L_3+\delta L_4+\delta L_5\ ,
\tag{G.23}
$$

where δL_1 contains one vertex from $L^{\xi\xi}_f$ and any number of vertices from L_B, δL_5 contains Y_{2f}, δL_2 contains two vertices from $L^{\xi\theta}_f$, δL_3 contains two vertices from $L^{\xi\theta}_f$ and one from L_B, δL_4 contains two vertices from $L^{\xi\theta}_f$ and more than one from L_B. Explicitly,

$$
\begin{aligned}
\delta L_1&=\big(-\tfrac{1}{2}C_{ab}\eta^{ab}+\tfrac{1}{4}\eta_{\beta\gamma}\,\mathrm{Tr}[C^{\beta\gamma}X]-\tfrac{1}{4}\eta_{\beta\gamma}\,\mathrm{Tr}[(U^\beta-U^{\beta T})(C^\gamma-C^{\gamma T})]\big)I_0\\
&+\tfrac{1}{8}\,\mathrm{Tr}[(U^\alpha+U^{\alpha T})(\eta_{\alpha\beta}\partial_\gamma-\eta_{\gamma\beta}\partial_\alpha-\eta_{\gamma\alpha}\partial_\beta)C^{\beta\gamma}]\,I_0\ ,
\end{aligned}
\tag{G.24}
$$

$$
\begin{aligned}
\delta L_2&=\tfrac{1}{4}\eta_{\alpha\beta}\bar\Psi^\alpha_aY_{0F}\Psi^\beta_b\eta^{ab}I_0+\tfrac{1}{8}\zeta^{\gamma c}\bar\Psi^\alpha_a\Gamma_c(\eta_{\alpha\beta}\partial_\gamma-\eta_{\gamma\beta}\partial_\alpha-\eta_{\gamma\alpha}\partial_\beta)\Psi^\beta_b\eta^{ab}I_0\\
&+\tfrac{1}{2}\bar\Psi^{\alpha\beta}_aI_{\alpha\beta\gamma\delta}(\partial)\Psi^{\gamma\delta}_b\\
&-\tfrac{1}{2}\eta_{\alpha\beta}\bar\Psi^{\alpha\beta}_aY_{0F}\Psi_b\eta^{ab}I_0+\tfrac{1}{4}\zeta^\gamma_c\bar\Psi^{\alpha\beta}_a\Gamma^c(\eta_{\alpha\beta}\partial_\gamma-\eta_{\alpha\gamma}\partial_\beta-\eta_{\beta\gamma}\partial_\alpha)\Psi_b\eta^{ab}I_0
\end{aligned}
$$

$$
\begin{aligned}
&+\bar\Psi^{\alpha\beta}_a I_{\alpha\beta\gamma}(\partial)\Psi^\gamma_b\eta^{ab}+\tfrac12\zeta^c_\alpha\bar\Psi^\alpha_a\Gamma_c\Psi_b\eta^{ab}I_0\,, &\text{(G.25)}\\
\delta L_3 =\ &\tfrac18\zeta^{\gamma c}\bar\Psi^\alpha_a\Gamma_c\Psi^\beta_b(U^\gamma-U^{\gamma T})^{ab}(\eta_{\alpha\beta}\eta_{\gamma\delta}+\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})I_0\\
&+\tfrac1{16}\zeta^{\gamma c}\bar\Psi^\alpha_a\Gamma_c\Psi^\beta_b(U^\delta-U^{\delta T})^{ab}(\eta_{\alpha\beta}\eta_{\gamma\delta}+\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})I_0\\
&+\tfrac12\bar\Psi(x_1)^{\alpha\beta}_a I_{\alpha\beta\gamma\delta}(\partial_1,\partial_2)\Psi(x_2)^{\gamma\delta}_b x^{ab}\\
&+\tfrac12\bar\Psi(x_1)^{\alpha\beta}_a(I_{\alpha\beta\gamma\delta\rho}(\partial_1,\partial_2)U^\rho-I_{\alpha\delta\gamma\beta\rho}(\partial_2,\partial_1)U^{\rho T})^{ab}\Psi(x_2)^{\gamma\delta}_b\\
&+\tfrac18\zeta^{c\gamma}\bar\Psi^{\alpha\beta}_a\Gamma_c\Psi_b(U^\delta-U^{\delta T})^{ab}(\eta_{\alpha\beta}\eta_{\gamma\delta}+\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})I_0\\
&+\tfrac14\zeta^{\gamma c}\bar\Psi^{\alpha\beta}_a\Gamma_c\Psi^\delta_b x^{ab}(\eta_{\alpha\beta}\eta_{\gamma\delta}+\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})I_0\\
&+\bar\Psi(x_1)^{\alpha\beta}_a(I_{\alpha\beta\gamma\rho}(\partial_1,\partial_2)U^\rho-I_{\gamma\beta\alpha\rho}(\partial_2,\partial_1)U^{\rho T})^{ab}\Psi(x_2)^\gamma_b\,, &\text{(G.26)}\\
\delta L_4 =\ &\tfrac12\zeta^{\gamma c}\bar\Psi^{\alpha\beta}_a\Gamma_c\Psi^\delta_b(U^\rho-U^{\rho T})^{ad}(U^\xi-U^{\xi T})^{db}I_{\alpha\beta\gamma\delta\rho\xi}\\
&+\tfrac12\zeta^{\gamma c}\bar\Psi^{\alpha\beta}_a\Gamma_c\Psi^{\delta\rho}_b[(U^\xi-U^{\xi T})^{ad}X^{db}+X^{ad}(U^\rho-U^{\rho T})^{db}]I_{\alpha\beta\gamma\delta\rho\xi}\\
&+\tfrac12\bar\Psi^{\alpha\beta}_a(x_1)I_{\alpha\beta\gamma\delta}(\partial_1,\partial_2,U)^{ab}\Psi^{\delta\rho}_b(x_2)\\
&+\tfrac12\zeta^{\gamma c}\bar\Psi^{\alpha\beta}_a\Gamma_c\Psi^{\delta\sigma}_b(U^\rho-U^{\rho T})^{ad}(U^\xi-U^{\xi T})^{de}(U^\zeta-U^{\zeta T})^{eb}I_{\alpha\beta\gamma\delta\rho\xi\sigma\zeta}\,, &\text{(G.27)}\\
\delta L_5 =\ &\tfrac12\,\mathrm{Tr}[Y_{0F}Y_{2f}]I_0\,. &\text{(G.28)}
\end{aligned}
$$

The standard dimensional regularisation integrals used to derived these expressions are ($d = 2-\epsilon$)

$$
\begin{aligned}
I_0 &= \int\frac{d^dl}{l^2}\,, \qquad I^{(0)}_2=\int d^dl\frac{l_\alpha l_\beta}{l^2(l+p)^2}=\tfrac12\eta_{\alpha\beta}I_0+\text{finite}\,,\\
I^{(1)}_2 &= \int d^dl\frac{l_\alpha l_\beta l_\gamma}{l^2(l+p)^2}=-\tfrac14(\eta_{\alpha\beta}p_\gamma+\eta_{\alpha\gamma}p_\beta+\eta_{\beta\gamma}p_\alpha)I_0+\text{finite}\,,\\
I^{(0)}_3 &= \int d^dl\frac{l_\alpha l_\beta l_\gamma l_\delta}{l^2(l+p)^2(l+q)^2}=\tfrac18H_{\alpha\beta\gamma\delta}I_0+\text{finite}\,,\\
I_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} &= \int d^dl\frac{l_{\alpha_1}l_{\alpha_2}l_{\alpha_3}l_{\alpha_4}l_{\alpha_5}l_{\alpha_6}}{(l^2)^4}=\tfrac1{48}H_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}I_0+\text{finite}\,,\\
I_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8} &= \int d^dl\frac{l_{\alpha_1}l_{\alpha_2}l_{\alpha_3}l_{\alpha_4}l_{\alpha_5}l_{\alpha_6}l_{\alpha_7}l_{\alpha_8}}{(l^2)^5}=\tfrac1{384}H_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8}I_0+\text{finite}\,. \qquad \text{(G.29)}
\end{aligned}
$$

The tensors H are given iteratively by:

$$
\begin{aligned}
H_{\alpha_1\alpha_2\alpha_3\alpha_4} &= \eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4}+\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}\\
H_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} &= \eta_{\alpha_1\alpha_2}H_{\alpha_3\alpha_4\alpha_5\alpha_6}+\eta_{\alpha_1\alpha_3}H_{\alpha_2\alpha_4\alpha_5\alpha_6}+\eta_{\alpha_1\alpha_4}H_{\alpha_2\alpha_3\alpha_5\alpha_6}\\
&\quad+\eta_{\alpha_1\alpha_5}H_{\alpha_2\alpha_3\alpha_4\alpha_6}+\eta_{\alpha_1\alpha_6}H_{\alpha_2\alpha_3\alpha_4\alpha_5}\\
H_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8} &= \eta_{\alpha_1\alpha_2}H_{\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8}+\eta_{\alpha_1\alpha_3}H_{\alpha_2\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8}+\eta_{\alpha_1\alpha_4}H_{\alpha_2\alpha_3\alpha_5\alpha_6\alpha_7\alpha_8}\\
&\quad+\eta_{\alpha_1\alpha_5}H_{\alpha_2\alpha_3\alpha_4\alpha_6\alpha_7\alpha_8}+\eta_{\alpha_1\alpha_6}H_{\alpha_2\alpha_3\alpha_4\alpha_5\alpha_7\alpha_8}+\eta_{\alpha_1\alpha_7}H_{\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_8}\\
&\quad+\eta_{\alpha_1\alpha_8}H_{\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7}\,. \qquad \text{(G.30)}
\end{aligned}
$$

Some integrals which lead to derivatives acting on the background-dependent fermion mass term Y_{0F} were left unevaluated:

$$
\begin{aligned}
I_{\alpha\beta\gamma}(\partial) &= \int d^dl\, l_\beta(l+i\partial)_\alpha(l+i\partial)_\gamma\ \frac{1}{\not l+Y_{0F}}\frac{1}{(l+i\partial)^2}\,,\\
I_{\alpha\beta\gamma\delta}(\partial) &= \int d^dl\, l_\alpha l_\gamma(l+i\partial)_\beta(l+i\partial)_\delta\ \frac{1}{\not l+Y_{0F}}\frac{1}{(l+i\partial)^2}\,,\\
I_{\alpha\beta\gamma\rho}(\partial_1,\partial_1,\partial_2) &= \int d^dl\, l_\beta(l+i\partial_1)_\alpha(l+i\partial_1)_\rho(l+i\partial_2)_\gamma\ \frac{1}{\not l+Y_{0F}}\frac{1}{(l+i\partial_1)^2}\frac{1}{(l+i\partial_2)^2}
\end{aligned}
$$

$$I_{\alpha\beta\gamma\delta}(\partial_1,\partial_2) = \int d^dl\, l_\beta l_\gamma (l+i\partial_1)_\alpha (l+i\partial_2)_\delta \,\frac{1}{\not{l}+Y_{0F}}\frac{1}{(l+i\partial_1)^2}\,\frac{1}{(l+i\partial_2)^2}$$

$$I_{\alpha\beta\gamma\delta\rho}(\partial_1,\partial_2) = \int d^dl\, l_\beta l_\gamma (l+i\partial_1)_\beta (l+i\partial_1)_\rho (l+i\partial_2)_\delta \,\frac{1}{\not{l}+Y_{0F}}\frac{1}{(l+i\partial_1)^2}\frac{1}{(l+i\partial_2)^2}$$

$$\begin{aligned} I_{\alpha\beta\gamma\delta}(\partial_1,\partial_2,U)^{ab} &= \int d^dl\,\frac{l_\beta l_\gamma}{\not{l}+Y_{0F}}(l+i\partial_1)_\alpha(l+i\partial_2)_\delta(-(l+i\partial_1)_\rho U(x_3)^\rho \\ &\quad + (l+i\partial_{1+3})_\rho U(x_3)^\rho)^{ad}((l+i\partial_{2+3})_\rho U(x_3)^\rho - (l+i\partial_2)_\rho U(x_3)^\rho)^{db}\,. \end{aligned} \tag{G.31}$$

The same applies to integrals that lead to derivatives acting on two of the three vertices.

Using the coefficients (G.22) extracted from the expanded Lagrangian (G.21), one finds that the divergent term δL_1 in (G.24) is given by

$$\begin{aligned} \delta L_1 = \Big[&\tfrac{i}{8}\zeta_\alpha^a\zeta_\beta^b(\eta^{\alpha\beta}\bar\Theta\Gamma_a\nabla^2\Sigma_e\Gamma_b\Theta - \epsilon^{\alpha\beta}\bar\Theta\Gamma_a\nabla^2\Sigma_o\Gamma_b\Theta) \\ &-\tfrac{i}{8}\zeta_\alpha^d\zeta_\beta^f\left(R_d{}^a\delta_f^b + \delta_d^a R_f{}^b - R_d{}^{ab}{}_f - R_d{}^{ba}{}_f\right)(\eta^{\alpha\beta}\bar\Theta\Gamma_a\Sigma_e\Gamma_b\Theta - \epsilon^{\alpha\beta}\bar\Theta\Gamma_a\Sigma_o\Gamma_b\Theta) \\ &- \tfrac{i}{16}\zeta_\alpha^d\zeta_\beta^f(\nabla^a H_{df}{}^b + \nabla^b H_{df}{}^a)(\eta^{\alpha\beta}\bar\Theta\Gamma_a\Sigma_o\Gamma_b\Theta - \epsilon^{\alpha\beta}\bar\Theta\Gamma_a\Sigma_e\Gamma_b\Theta) \\ &- \tfrac{i}{2}\zeta_\alpha^a\zeta_\beta^b H_a{}^{df}\,(\eta^{\alpha\beta}\bar\Theta\Gamma_d\nabla_f\Sigma_e\Gamma_b\Theta - \epsilon^{\alpha\beta}\bar\Theta\Gamma_d\nabla_f\Sigma_o\Gamma_b\Theta)\Big]I_0\,, \end{aligned} \tag{G.32}$$

where Σ_o, Σ_e were defined in (G.9) and $I_0 \sim \frac{1}{\epsilon}$ is the UV pole factor.

Comparing (G.32) to the corresponding terms in the classical action (G.8) one can read off the contributions to the beta-functions for the R–R couplings. Projecting onto the independent set of Dirac matrices $\Gamma_{a_1\ldots a_n}$ we indeed observe the presence of the Hodge–de Rham operator terms as in (4.9)–(4.11). There are also similar terms depending on the H_3 field strength and its derivatives. The UV singular terms in $\delta L_2, \delta L_3, \delta L_4$ in (G.25)–(G.27) containing a single factor of $\Psi_a^{\alpha\beta}$ will have a similar structure. The first term in δL_4 contains two factors of H and one of R–R field and should account for all such terms in eqs. (4.9)–(4.11).

There are apparently also other UV singular terms that do not appear in (4.9)–(4.11): terms containing two $\Psi_a^{\alpha\beta}$ factors are independent of the R–R fields and contain only the H_3 strength and factors of the curvature tensor. We expect such terms to combine into the beta-function of the NS–NS fields entering the couplings (G.10) and thus yield the same scale invariance conditions as in eqs. (2.1), (2.2). Moreover, all terms in δL_2, δL_3 and δL_4 which do not contain $\Psi_a^{\alpha\beta}$ are bilinear in R–R fields and all terms in δL_3 contain at least one additional factor of either H_3 flux or the curvature tensor. We expect such terms to cancel or to vanish upon use of the NS–NS scale invariance conditions (2.1), (2.2).

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