

# Euclid's context principle

by Peter Simons

## **The context principle: foundation stone of modern analytic philosophy?**

According to Michael Dummett, modern analytic philosophy began when Gottlob Frege first formulated and applied the principle that the meaning of a word is to be asked after not in isolation, but only in the context of a sentence. Dummett writes:

What distinguishes analytical philosophy from other schools is the belief that a philosophical account of thought can be attained through a philosophical account of language, and can only be so attained. On this characterization, analytical philosophy was born when the 'linguistic turn' was taken; the first clear example known to me occurs in Frege's *Die Grundlagen der Arithmetik* of 1884. At a crucial point in the book, Frege raises the Kantian question, 'How are numbers given to us, granted that we have no idea or intuition of them?' His answer depends upon the celebrated context principle, which he had laid down in the Introduction as one of the fundamental methodological principles to be followed in the book.<sup>1</sup>

In the introduction to the *Grundlagen* Frege enunciates three principles which guided him. The second is the context principle (CP). In the original it runs:

nach der Bedeutung der Wörter muss im Satzzusammenhang,  
nicht in ihrer Vereinzelung gefragt werden.

Which Austin translates as:

never to ask for the meaning of a word in isolation, but only in  
the context of a proposition.<sup>2</sup>

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<sup>1</sup> Dummett 1993, 4-5.

<sup>2</sup> Frege 1953, xxii.

The context principle was taken up by Wittgenstein in the *Tractatus*: there he writes:

Nur der Satz hat Sinn; nur im Zusammenhang des Satzes hat ein Name Bedeutung.

Only a sentence has sense; only in the context of a sentence does a name have a meaning.<sup>3</sup>

While the exact significance of Wittgenstein's employment may differ from that in Frege, the principle is taken by Dummett and others to be extremely important, perhaps criterial, in fashioning the doctrines of early analytic philosophy.

There is a fairly uncontroversial point about CP, which is that it in some way privileges sentences in language and complete thoughts or judgements in mind over their constituents, terms or ideas. For much of the history of philosophy and logic it was customary to introduce terms first, then consider sentences as given by the combination of terms. Indeed in medieval terminology a sentence was known as a 'complex'. Only then was inference, the passage from sentences to sentences, considered. This follows the order of development of logical doctrines in Aristotle: first terms in the *Categories*, then their combination into sentences in *De interpretatione*, finally the interweaving of sentences in syllogistic reasoning in the *Prior Analytics*. From this point of view, putting sentences first and considering words as essentially components of sentences taken as primary looks revolutionary in Frege. In fact it is not. Kant had asserted the priority of judgements over ideas in the first *Critique*. The point is not subtle: it is either a truism or false. At any rate, Kant's point was widely accepted in the nineteenth century, including by such philosophers who influenced Frege as Trendelenburg and Lotze,<sup>4</sup> so in this regard Frege is no revolutionary.

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<sup>3</sup> Wittgenstein 1922, 3.3.

<sup>4</sup> Janssen 2012, 22–25.

So it seems that if Frege's use of CP is to constitute the crucial difference between pre-analytic and analytic philosophy, it must lie elsewhere. And indeed Frege's approach to the logical analysis of sentences is radically different from that of his predecessors, since he takes a sentence to involve the completion of a function by one or more arguments. CP is not irrelevant to this analysis, especially in the early Frege, but it is not entailed by a function–argument analysis of sentences, since the later Frege has a more rigid conception of the functional and non-functional parts of sentences in which CP plays no role at all. CP then hardly seems to bear the weight that Dummett gives it. The turn to language is indeed significant in Frege, but there had been linguistic philosophy before, not least among medieval logicians like Ockham. And if it was so central to Frege's philosophy of logic and language, why does it disappear after 1884? In Frege's magnum opus *Grundgesetze der Arithmetik* (1893/1903), where one would expect a crucial methodological principle to bear repetition, CP makes no appearance, something that mystifies Dummett.

The heavy-duty significance of CP for Dummett's Frege is connected with the way in which Frege explains the meanings of numerical terms. When we examine this more closely, I think we can give a much more lightweight account of CP.

### **A deflationary interpretation**

On my interpretation of CP, Frege is simply calling attention to the fact that sometimes, phrases or clauses containing words are logically more basic than the words themselves. I will explain why this is so in connection with Frege's example, the use of number words.

In *Grundlagen* Frege is trying to show that the concept of (cardinal) number can be defined logico-analytically. His chosen definition scheme is:

the number belonging to concept F is the extension of the concept ‘concept equinumerous with F’.<sup>5</sup>

The term ‘equinumerous’ (*gleichzahlig*) appears to presuppose the concept number (*Zahl*): ‘equinumerous’ means ‘equal in number’. So is Frege’s definition circular? If it were, his whole logicist programme would be vitiated, at least in this form. But it is not: logically, we define ‘number’ in terms of ‘equal in number’. It is to forestall the thought that circularity is involved that Frege cautions not to ask what ‘number’ means in isolation, get an answer, then plug this into the phrase ‘equal in number’ to get *its* meaning. The natural order from a sentence-building point of view, bottom-up, from parts to whole, gets the logical cart before the horse.

Here is a simpler example to make the point. Consider the word ‘sister’. It is a common noun designating certain females, viz. those with a full sibling, i.e. those whose parents together have some child other than them. The word ‘sister’ is included in the phrase ‘is a sister of’. If we consider the meaning of the word ‘sister’ in isolation, it appears to be logico-analytically more basic than the relational predicate ‘is a sister of’. But in fact the reverse is the case: someone is a sister if and only if she is a sister-of someone. The noun is derelativised from the predicate. Derelativisation is commonplace: not only the majority of kinship terms like ‘sister’, ‘grandparent’ and ‘cousin’, but legions of other nouns like ‘author’, ‘employer’, ‘assistant’, ‘manager’, ‘lover’, ‘donor’ etc. are derelativised. Likewise reflexivisation: the nouns ‘suicide’, ‘autodidact’ and ‘narcissist’ are all to be defined by a context in which the subject and object of a verb denote the same person, so expressible using a reflexive pronoun. Derelativisation and reflexivisation work because of the context principle.

So CP is still very important: it opens up the modern theory of definition and explication, in which terms of any syntactic category are defined in a sentential context. But it

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<sup>5</sup> Frege 1953, §§ 68ff.

does not first appear with Frege. Though not explicitly flagged as such, it is already in use in Euclid's *Elements*, which predate Frege's *Grundlagen* by some 2,200 years. To that we turn.

### **The Context of the Context Principle in Euclid**

The use of CP that I shall highlight (there may be others, but this is striking enough) is in Book V of the *Elements*, the famous Definition 5.<sup>6</sup> The context of this was the discovery of incommensurable quantities by the Pythagoreans. Two quantities  $A$  and  $B$  are *commensurable* (have a common measure) when there are multiples of  $A$  and  $B$  which are equal, i.e. for some natural numbers  $n$  and  $m$ ,  $n.A = m.B$ , or, when there is a quantity  $C$  which commonly measures both  $A$  and  $B$ , i.e. such that  $A = m.C$  and  $B = n.C$  (the latter entails the former). A Pythagorean, possibly Hippasus of Megapontum, discovered that the diagonal of a square is incommensurable with its side, which is easily proved by a *reductio* argument. This plunged into disarray their worldview that the world is organised according to the whole numbers and their ratios, and it is said that the scandalized Pythagoreans drowned the hapless discoverer.

The discovery of incommensurable magnitudes was the first foundational crisis of mathematics, and like all such crises, it required a considerable intellectual leap to overcome it and move forward. Definition 5 of Book V of the *Elements* is that leap. It is generally held that the author of this definition and the theory it opens up was Eudoxus of Cnidus (410 or 408 BC to 355 or 347 BC), a student of Plato and possibly teacher of Aristotle. Since Eudoxus' works are no longer extant, I shall continue to refer to the use of CP as 'Euclid's Context Principle', but bear in mind throughout that the responsibility for the intellectual great leap forward was very probably Eudoxus, and dates therefore from the early to mid-4<sup>th</sup> century.

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<sup>6</sup> Euclid 1956, 114.

***Elements* Book V, the first ten definitions**

This section is simply an extended quotation from Heath's translation,<sup>7</sup> with the defined terms highlighted.

1. A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.
2. The greater is a *multiple* of the less when it is measured by the less.
3. A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.
4. Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another.
5. Magnitudes are said to be in *the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.
6. Let magnitudes which have the same ratio be called *proportional*.
7. When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth.
8. A proportion in three terms is the least possible.
9. When three magnitudes are proportional, the first is said to have to the third the *duplicate ratio* of that which it has to the second.
10. When four magnitudes are continuously proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion.

**Scholium on Definitions 1-4**

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<sup>7</sup> Euclid 1956, 113-114.

Despite the brilliance of these definitions, they are logically defective. Definition 1 simply equates the undefined notions of being a part of and being measured by. Let us simply write

$$A | B$$

for ‘A is part of B’ or ‘A measures B’. The words give us an idea of what is meant. They also tell us in passing that if  $A | B$  then A is less than B. That notion too is undefined, so let us write

$$A < B$$

for it. The words ‘greater’ and ‘less’ tell us that we have to do with an ordering. We now know what principles govern an ordering: it is asymmetric and transitive, so let us interpolate them as axioms:

$$\begin{array}{l} \text{ASYM If } A < B \text{ then not } B < A \\ \text{TRANS If } A < B \text{ and } B < C \text{ then } A < C \end{array}$$

We now connect this with being part of by the principle

$$\text{PL} \quad \text{If } A | B \text{ then } A < B$$

which is implied by the formulation of Definition 1.

Not all magnitudes are of the same kind, as the third definition implies. For example mass and length are of different kinds, so no mass is greater or less than any length. Definitions 3 and 4 tell us something about what being of the same kind involves, namely being in a ratio of one to another. This is not yet very informative. What seems to be assumed is that two magnitudes of a kind are *comparable*. To compare magnitudes is to find them equal or one greater than the other. So two (distinct) comparable magnitudes must be such that one is greater than the other. Putting this together and allowing the case that  $A = B$  we say that A is comparable with B, written

$$A \sim B$$

if they are the same or one is greater than the other. This now is a genuine definition, so let's go ahead and define

$$\text{Def.}\sim \quad A \sim B \text{ iff (Df.) } A < B \text{ or } A = B \text{ or } B < A$$

We introduce also the converse relation to  $<$

$$\text{Def.}> \quad A > B \text{ iff (Df.) } B < A$$

It follows immediately from the definition that any two comparable magnitudes stand in a *linear* ordering, which satisfies the law of trichotomy.

Definition 4 is very interesting because it is widely held to entail the Archimedean Principle, namely that any two comparable magnitudes are in a finite ratio, that one is not infinitely greater than the other. This rules out infinities and infinitesimals. But as we have it so far this is not true. To see this, consider how we might express the idea in Definition 4 of one magnitude being such that it can be multiplied so as to be greater than another. One straightforward way to do this would be as follows

$$\text{MULT} \quad \text{If } A < B \text{ then for some } C, A \mid C \text{ and } B < C$$

Given the definition of comparability, if we assume MULT axiomatically it follows that

$$\text{COMP1} \quad \text{If } A \sim B \text{ then for some } C \text{ and } D, A \mid C \text{ and } B < C \text{ and } B \mid D \text{ and } A < D$$

which is what, according to Definition 4, 'having a ratio' consists in. It is usual in modern commentaries to express the multipliability using numerical multipliers, so that we have, using the obvious notation

COMP2 If  $A \sim B$  then for some natural numbers  $m$  and  $n$ :  $m.A > B$  and  $n.B > A$ .

From this the Archimedean Principle indeed follows. But notice that Euclid does not do this. Indeed he goes out of his way to avoid talking about numbers at all. The Definitions 9 and 10 start introducing different numerical multiples: duplicate, triplicate and so on, but we are still far from ‘multiplying by numbers’, either conceptually or notationally.

It is very likely that Eudoxus and Euclid implicitly understood ‘multiple’ to mean ‘finite multiple’, so the Archimedean Principle would follow. That it does not follow from COMP1 and MULT can be shown as follows. Imagine we have a series of magnitudes as follows: we have no zero, but a series corresponding to the positive real line  $\{x \in \mathbb{R}: x > 0\}$ . Call this  $M(1)$ . Now imagine a second copy of these where we deem all the new magnitudes to be greater than all the old. Call this  $M(2)$ . Then add in ascending sequence another copy for each natural number  $n$ . Then any two magnitudes in this infinite sequence of infinite series are comparable according to the principle

$A < B$  iff for some  $n$ ,  $A, B \in M(n)$  and  $A < B$  in the usual way in  $M(n)$ , or else for some  $m$  and  $n$ ,  $A \in M(m)$  and  $B \in M(n)$  and  $m < n$ .

We can multiply even the infinite magnitudes. Suppose  $A$  is in  $M(n)$  and is  $B$  far from the origin of this line. Then  $2.A$  is in  $M(2n)$  and is  $2.B$  far from its origin. And so on. Now let  $A$  be in  $M(1)$  and  $B$  in  $M(2)$ . No matter how large the integer  $k$  by which we multiply  $A$ ,  $k.A < B$ . So the model is not Archimedean, yet it satisfies the other principles, including COMP1.

Therefore I suggest we use not MULT but the stronger principle

FMULT If  $A < B$  then for some  $C$ ,  $A \mid C$  and  $B < C$ , and the  $X$  such that  $A \mid X$  and  $X < B$  are finite.

This employs a plural variable ‘X’. X are one or more magnitudes. It is not a set variable. To use sets to interpret Euclid would be totally anachronistic. Sets were not invented until the 19<sup>th</sup> century, whereas the Greeks used plurals. It also assumes we know what is meant by ‘finite’. While the Greeks knew and used the term, they did not know to define it rigorously: that had to wait until Dedekind in the 19<sup>th</sup> century.<sup>8</sup> If we now use COMP1 along with FMULT the Archimedean Principle indeed follows.

### **Scholium on Definition 5**

Modern commentators on Definition 5 tend to formulate it as follows:

Def.SR1 A to B has the same ratio as C to D iff (Df.) for all n and m:  $n.A > m.B$  iff  $n.C > m.D$  and  $n.A = m.B$  iff  $n.C = m.D$  and  $n.A < m.B$  iff  $n.C < m.D$ .

Putting it more succinctly: for all m and n,  $n.A > = < m.B$  according as  $n.C > = < m.D$ .

This, using natural numbers as multiples, stands to Euclid’s Definition 5 in the same way as COMP2 stands to COMP1: it is an overinterpretation. It does fit with the implicitly Archimedean understanding of being in a ratio, but again it does not match Euclid’s subtlety. He uses only the notion of *equimultiples* which can be defined (though he does not do so) without presupposing numbers at all. The ratios are then used to define twofold, threefold etc. as special cases. Let’s see this work.

Define what it is for B and D to be *equimultiples* of A and C respectively as follows:

Def.EM B and D are equimultiples of A and C respectively iff (Df.)  $A \mid B$  and  $C \mid D$  and the X such that  $A \mid X$  and  $X < B$  are in one-one correlation with the Y such that  $C \mid Y$  and  $Y < D$ .

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<sup>8</sup> Dedekind 1963, 64: “A system S is said to be *infinite* when it is similar to [i.e., equinumerous with – PS] a proper part of itself.”

This is not a definition to be found in Euclid: he simply assumes we know what equimultiples are, but this definition delivers the results he needs. Five remarks are in order about our definition. Firstly, it presupposes we can define what it is for two collections to be in one-one correlation. We can do this in second-order logic and it is standard fare. Secondly, the variables ‘X’ and ‘Y’ are again here understood not as set variables but as plural variables. Thirdly, because we are now assuming FMULT the X and Y in question are finite. Fourthly, notice that EM does not quantify over numbers or sets, but only magnitudes, singly or plurally. Finally, note that for this to work A and B must be of one family and C and D must also be of one family but there is no need for all to be of a single family. For example the ratio between a length of two metres and a length of six metres is the same ratio as that between a mass of five grams and a mass of fifteen grams.

Now we come to

Definition 5: A is in the same ratio to B as C is to D iff (Df.) for all X Y Z and W: if X and Z are equimultiples of A and C respectively and Y and W are equimultiples of B and D respectively then  $(X > Y \text{ and } Z > W)$  or  $(X = Y \text{ and } Z = W)$  or  $(X < Y \text{ and } Z < W)$ .

Modulo the modern prefaced universal quantifier, this is exactly Euclid’s Definition 5, so we are entitled to call it that. Again A and B must be of one family as must C and D, but there may be two families, and notice again that no notion of particular numbers is used at all, nor are numbers (of any sort) quantified over. If A is in the same ratio to B as C is to D we write

$$A : B :: C : D$$

It might be thought we can replace the middle sign ‘::’ by an identity or equality sign ‘=’, and this is often done, but it would be wrong, despite the use of the term ‘same’ in the defined term. In fact, to suppose we can put ‘=’ for ‘same’ is

precisely to ignore the context principle, since it amounts to treating the meaning of ‘same’ in isolation rather than in context, and Euclid does not do that. The complex  $- : - :: - : -$  is in fact simply a four-place or tetradic relational predicate, and in a less flexible notation would be written something like  $P(-, -, -, -)$ . We have chosen the more traditional notation because of its suggestive symmetry and the fact that it naturally corresponds to the standard way to read it: ‘A is to B as C is to D’, which itself is a four-place predication. In this it is unlike ‘ $A : B = C : D$ ’, which contains a two-place predicate (identity) conjoining two two-place functional expressions, ‘ $A : B$ ’ and ‘ $B : C$ ’, to be read as ‘the ratio of A to B’ and ‘the ratio of C to D’. We are heading in their direction, but not yet there, and to jump the gun is again precisely to slur over Euclid’s subtlety in the use of CP.

Nevertheless for ease of reference I shall call the four-place predicate ‘the relation P’. This is suggested by Euclid’s introduction of the synonym ‘proportional’ (*analogon*) in Definition 6: we could say ‘A and B are proportional to C and D’, and there is no suggestion of an equality in this wording. The logical properties of this relation are vital: it is a four-place equivalence relation, the tetradic analogue of the more familiar dyadic or two-place equivalence relations. What this amounts to in logic is expressible by the following principles

- PREFL If A and B are magnitudes in the same family then  $A : B :: A : B$
- PSYM If  $A : B :: C : D$  then  $C : D :: A : B$
- PTRANS If  $A : B :: C : D$  and  $C : D :: E : F$  then  $A : B :: E : F$
- PCNV If  $A : B :: C : D$  then  $B : A :: D : C$
- PUN For any magnitudes A and B,  $A : A :: B : B$

The fourth of these principles, pairwise conversion of the terms, is characteristic of four-place equivalences and obviously has no analogue among two-place equivalences. The final principle holds for any magnitudes, whether they are in the same family or not, and expresses the transfamilial idea that the ratio of *any* magnitude to itself is the same as that of

any other magnitude to itself: what we shall be able to call ‘unity’. So the ratio of a mass of five grams to itself is the same as the ratio of a length of ten metres to itself, for example.

### **Scholium on Definition 7**

Definition 7 is close to being a good definition, given what has gone so far. The only difficulty concerns how to understand ‘of the equimultiples’. Does it mean *any* equimultiples as in Definition 5, or *some* particular instance, ‘some equimultiples’? We shall adopt the latter interpretation, for reasons to be made clear. So it may be formulated pretty much along Euclid’s lines as

Definition 7: A has a greater ratio to B than C has to D iff (Df.) for some X Y Z and W: X and Z are equimultiples of A and C respectively and Y and W are equimultiples of B and D respectively and  $X > Y$  but not  $Z > W$ .

Let us then write  $A : B :: C : D$

To see that the use of the particular quantifier ‘some some’ is correct, note two things. Firstly, if the condition of the definiens in 7 is satisfied, then A and B are *not* proportional to C and D. Secondly, the universal does not apply, for Archimedean reasons. For example suppose A is 6g, B is 2g, C is 10m and D is 5m. Then A has a greater ratio to B than C has to D, but there are multiples m of A and C and n of B and D such that  $m.A > n.B$  and  $m.C > n.D$ , for example  $m = 25$  and  $n = 4$ .

What is striking about Definition 7 however is its behaviour with respect to proportionality. If A has a greater ratio to B than C has to D,  $A : B :: C : D$ , then for *any* magnitudes E, F, G and H such that  $A : B :: E : F$  and  $C : D :: G : H$ , we have that  $E : F :: G : H$ . The fact of one ratio’s being greater than another is invariant with respect to the equivalence relation of proportionality. Invariance is a crucial property of relations with respect to equivalences, since it

allows facts about them to as it were float free of their particular instances. This is familiar from two-place equivalences. If John is taller than Mary then anyone who is as tall as John is taller than anyone who is as tall as Mary. We can then abstract from the persons involved and simply talk about one height (John's) being greater than another (Mary's). This is what Definition 7 is doing: it is allowing the notion of 'greater' to be transported from comparisons between magnitudes to comparisons between ratios of magnitudes. In this way it is preparing us for the introduction of an arithmetic of ratios, one which works equally well whether the ratios are rational (fractional), i.e. expressible as a ratio of whole numbers, or irrational, like  $\sqrt{2}$ .

### **Definitions 9 and 10**

A proportion in three terms among A B and C exists when  $A:B :: B:C$ . For example A is 227 g, B is 681 g, and C is 2043 g. Or A is 32 km, B is 8 km and C is 2 km. A proportion in three terms exists only when all three magnitudes belong to the same family.

If  $A:B :: B:C$  then A to C has the *duplicate* ratio of A to B.

If  $A:B :: B:C$  and  $B:C :: C:D$  then A to D has the *triplicate* ratio of A to B.

If  $A_1:A_2 :: A_2:A_3$  and ... and  $A_{n-1}:A_n :: A_n:A_{n+1}$  then  $A_1$  to  $A_{n+1}$  has the *n-plicate* ratio of  $A_1$  to  $A_2$ .

So duplicate, triplicate, ... , n-plicate are defined without ever quantifying over numbers (but as here stated we do use schematic numerical indices.) Definition 10 is really schematic, indicated by Euclid's 'and so on'. All the 'plicate' predicates definable by Definitions 9 and 10 are also invariant under proportionality, and so are beginning to introduce an arithmetic into ratios. We shall leave the development here because enough has been done to establish our main claim.

### **Abstraction and the context principle**

When Frege introduced the equivalence relation of equinumerosity among concepts (it could also be defined among pluralities, but Frege chose not to go that way), it was for the sake of defining *objects* called ‘numbers’. The way Frege did this was by taking numbers to be extensions of certain concepts, or as he later put it, as value-ranges (Wertverläufe) of certain functions, the concepts. Transposing this into the idiom of classes, such as we find in Russell, this amounts to saying that numbers are certain classes. They are in fact equivalence classes under the equivalence relation of equinumerosity. So the number 2 was the equivalence class whose members are all pairs, or two-membered classes, number 3 was the class of all triples, or three-membered classes, and so on. Now as we know, Russell showed that Frege’s procedure led to inconsistency, but let us not worry about that here. The point is that Frege was intent on defining ‘number’ in terms of ‘has the same number as’, and for this to be convincing he needed to invoke the Context Principle.

In modern, so-called neo-logicist or neo-Fregean reconstructions of Frege’s ideas, the detour via classes is eliminated in favour of what I call *direct abstraction*. This follows some remarks of Frege earlier in *Grundlagen*.<sup>9</sup> Suppose F and G are concepts (or classes, we are not being fussy), then the direct abstraction of numbers from concepts under the equivalence of equinumerosity goes as follows

ENUM the number of Fs = the number of Gs iff there are as many Fs as Gs

(i.e. if the Fs are equinumerous with the Gs), an ‘abstraction equivalence’ which nowadays passes under the historically horrendously misleading name of ‘Hume’s Principle’.

Frege had his reasons, into which I shall not here delve, for not accepting this way of ‘defining’ numbers, and preferring the explicit definition via classes or value-ranges mentioned at the outset. No matter: whichever way one might

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<sup>9</sup> Frege 1953, §§ 62–65.

wish to work it, Euclid’s treatment of ratios can be given the same treatment. Taking the direct abstraction route, it would go

ERAT the ratio of A to B = the ratio of C to D iff  $A : B :: C : D$

As pointed out before, the right-hand side of this equivalence is a four-place predication, the left-hand side is a two-place predication. The ‘objects obtained’ via such an abstraction are the ratios themselves, in the same way as the objects obtained under ENUM are the (natural) numbers themselves.

Once we have ‘obtained’ the ratios in this way, we can proceed to build up a suitable vocabulary for comparing and operating on ratios, and this can be expanded to become an arithmetic. With strong enough assumptions about the existence of suitable magnitudes, it will give us the arithmetic of the positive real numbers, such as are given axiomatizations around the turn of the century by Otto Hölder and Edward Huntington.

Frege himself took the first steps towards treating the real numbers along similar lines, the difference being that he was planning to use his value-ranges (until stopped by Russell) and that because he was concerned to deal straight away with negative as well as positive numbers, his magnitudes are relations rather than properties. Methodologically however the difference is relatively minor. One could treat the negatives, as do Whitehead and Russell later,<sup>10</sup> as differences among absolute magnitudes, so that while the difference between 20g and 25g is +5g, the difference between 20g and 15g is –5g.

It is interesting that Euclid did *not* take the step of abstracting or reifying the numbers as ratios, but remained content with ratios as ratios-of pairs of magnitudes. Frege would have disliked this and he would have been wrong to do so. In Fregean terms the ratio between A and B would be the relational value-range  $\epsilon\alpha[A:B :: \epsilon:\alpha]$ , the extension of the relation ‘standing in the same ratio as A to B’. Whitehead and

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<sup>10</sup> Whitehead and Russell 1913, \*303, “Ratios”.

Russell give their definition of ratios in *Principia Mathematica* at \*303.01. It is fairly closely related to that of Euclid but here is not the place to go into that.

When Euclid gets around to natural numbers in Book VII of the *Elements*, he does not define them by abstraction or any other way, but simply offers an inadequate definition of numbers as ‘a multitude of units’. His treatment of multiples is very reminiscent of that of multiples of magnitudes in Book V. Frege severely criticised Euclid’s definition of a number as a multitude of units, but I think a lot more can be made of this definition than Frege allows, and ultimately I think number as a measure of the size (cardinality) of multitudes (of individuals) is a better account of the ontology of number than Frege’s own. However, that is another story.

### **The context principle in Euclid**

Let us sum up. The order in which Euclid introduces the definitions in Book V and the subsequent use made of them is, while not unflawed, intellectually masterful. It shows he has a fully-fledged *working* knowledge of the Context Principle. By comparison with Frege:

1. the analysis is carried out in a somewhat regimented natural language (as in Frege’s *Grundlagen*, not his *Grundgesetze*)
2. Euclid does not highlight the principle methodologically.
3. Euclid’s use of the theory is more complex and more sophisticated than Frege’s: ratios are harder to work with than counting numbers.
4. It is applied in a much more historically important case than Frege’s application to the logicist definition of natural numbers. Whereas Frege was formulating foundational principles for a long-established and well-understood part of mathematics, Euclid was both providing some foundational material and, more crucially, making the mathematical breakthrough required to allow mathematicians to work rigorously with incommensurables.

5. Euclid does not seek to reify ratios as entities in their own right: this is done (in a different way, using so-called Dedekind cuts) by Dedekind in 1872 (thought out in 1858).<sup>11</sup> The route Frege himself was taking towards the real numbers in 1903 was closer to Euclid than to Dedekind, and modulo the inconsistency one could graft a Fregean form of abstraction onto Euclid to ‘get’ the ratios (positive real numbers).

There remains a residual mystery, one which puzzled Dummett.<sup>12</sup> Why, if the context principle was so central to Frege, did he (apparently) abandon it in his *Grundgesetze der Arithmetik* of 1893/1903, which was after all his major work? Here Dummett’s heavy-duty interpretation of CP lets him down, whereas our deflationary one explains the matter perfectly. The difference between *Die Grundlagen der Arithmetik* of 1884 on the one hand and both *Begriffsschrift* of 1879 and *Grundgesetze* on the other is that whereas the two outer works use Frege’s formal language, *Grundlagen* is written in (beautiful) German prose. It was intended to be read by people with no logic, probably following highly sage advice to Frege by Carl Stumpf that his logical notation had put people off and they needed to be won over to his case by prose argument and exposition. So the definitions in the later, constructive part of *Grundlagen* are given in German, with just the odd variable thrown in. Frege introduces the concepts in the logical order he wants, but his formulation of them relies on vernacular ways of expressing the concepts involved, and it is because some of the later definitions are of concepts whose formulation uses words used in earlier definitions, words like ‘number’, that he is forced to caution that this does not vitiate the process by definitional circularity. That is the whole *raison d’être* of the context principle. But in *Begriffsschrift* and *Grundgesetze*, Frege does not need to genuflect before ordinary German. He can introduce what symbols he wants in what order he wants and endow them

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<sup>11</sup> ‘Continuity and Irrational Numbers’, in Dedekind 1963, 1–30.

<sup>12</sup> See e.g. Dummett 1981, Ch. 19, ‘The Context Principle’.

with the meaning he wants. And this is precisely what he does. The definition of the number of a concept in *Grundgesetze* is the sixth in that work: it defines the number of a value-range  $u$  as the class of value-ranges to which the objects in  $u$  can be put in one-one correlation. Frege does not bother to stop and define equinumerosity first, but he is using the means to do so, and had the development in *Grundgesetze* stuck more closely to the sequence in *Grundlagen* he would have done so before defining number. As it is, there is, astonishingly, no outright definition of ‘natural number’ in *Grundgesetze*. Frege knew how to express the relevant concept, so did not bother.

Like Frege after him, Euclid expresses himself in ordinary language, and so in order to introduce the crucial concept of same ratio with the rigour that he did, he is bound to be using the context principle. In fact it seems as if it was much second nature to do so that he did not bother to stop and explain to the reader what he was doing. Had he given a rigorous symbolic axiomatization of his theory of ratios, as we could and perhaps should, he could like Frege have introduced symbols in the order he wished, and dispensed with CP. That he did not do so was a sign of the time in which he was working, not of any intellectual shortcoming.

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