

# Plurality and Unity



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*Logic, Philosophy, and Linguistics*

EDITED BY

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**OXFORD**  
UNIVERSITY PRESS

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Great Clarendon Street, Oxford, OX2 6DP,  
United Kingdom

Oxford University Press is a department of the University of Oxford.  
It furthers the University's objective of excellence in research, scholarship,  
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First Edition published in 2016

Impression: 1

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Published in the United States of America by Oxford University Press  
198 Madison Avenue, New York, NY 10016, United States of America

British Library Cataloguing in Publication Data

Data available

Library of Congress Control Number: 2015944345

ISBN 978-0-19-871632-7

Printed and bound by

CPI Group (UK) Ltd, Croydon, CR0 4YY

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## 4

# The Ontology and Logic of Higher-Order Multitudes

*Peter Simons*

## 1 The variety of collective objects

There are many genera of collective objects and as yet nothing approaching a systematic taxonomy of them. A collective object is anything which is a plurality. A plurality is something that has more than one object as a *member*. For example, a string quartet is four string players playing together: two violinists, a violist, and a cellist. Each of them is a member of the quartet. But not just any four such string players make up a string quartet. They have to play together to do that. Once string quartets are institutionalized, it becomes possible for a quartet *qua* institution to survive the replacement of members; for example, the Hungarian Quartet had, at different times, ten members, while the Borodin Quartet has had to date fourteen members. The Amadeus Quartet by contrast always had the same four members.

Many collectives have variable membership, for example a club, a university, a battalion. Collectives which never change their members are temporally rigid. Collectives which essentially have just the members they do have are modally rigid. Clubs etc. are neither modally nor temporally rigid, while the prime numbers form a modally rigid collective. Many collectives have vague membership conditions; for example, it is vague which individual organisms belong to a given species, or which people are in a country at a particular instant. However, many collectives have perfectly definite membership. We shall be dealing with such collectives. For such a collective *c* and any object *a* it is determinately true or determinately false whether *a* is a member of *c*. Thus it is determinately true that Norbert Brainin was a member of the Amadeus Quartet, and determinately false that four is one of the prime numbers.

## 2 Higher-order collectives

A collective all of whose members are individuals is a *first-order* collective. The Amadeus Quartet is a first-order collective, since its members are all individual people. On the other hand, there are collectives whose members are themselves collectives, wholly or in

part. They are *higher-order* collectives. Take the collection of all string quartets who have recorded for Deutsche Grammophon since 1945. They include (among others) the Amadeus Quartet, the Emerson Quartet, and the Hagen Quartett. This collection does not have individual people as members, but string quartets, which do have individual people as members. So it is a second-order collective. Now consider the collections of string quartets that have recorded for other record labels, such as the Juilliard and Tokyo Quartets (etc.) for CBS, the Tokyo and Kolisch Quartets (etc.) for RCA, the Budapest and Artemis Quartets (etc.) for EMI, etc. This is a collection of collections of collections of people, and so third-order. Notice the duplication: the Tokyo Quartet have recorded for both CBS and RCA and so appear twice. This is perfectly all right.

Some higher-order collectives are even institutionalized. The FISP—Fédération Internationale des Sociétés de Philosophie—has as its members various national philosophy societies, and so is second-order. Likewise FIFA—the Fédération Internationale de Football Association—has 209 national football associations as its members. Since these are typically composed of football clubs, FIFA is probably third-order. Other international sporting bodies, such as the International Association of Athletics Federations (212 members), the Fédération Equestre Internationale (132) and the International Cricket Council (106) are similar. One could imagine these international federations combining to form a yet higher-order federation, the Fédération Internationale des Fédérations Internationales des Sports (FIFIS).

How high such a hierarchy could go is not obvious. But we can demonstrate that there is no obvious upper limit to the hierarchy of orders with a geometric example. Imagine three small circular dots arranged at the corners of an equilateral triangle. Let the distance between the dots' centers be  $d$ . Imagine three such arrangements so arranged that they form overall a larger equilateral triangle of side  $3d$  with a distance of  $d$  between the innermost dots along each side. Then imagine three of these arrangements also in a larger equilateral triangle with side length  $9d$  and a distance of  $3d$  between the innermost dots on each side. Iterate. At each level of iteration we generate a new figure composed of three figures of the preceding type. At the  $n$ th level we need  $3^n$  dots to make a figure of side length  $3^{n-1}d$ . These dots of course compose a single complex first-order figure, but we also have a figure composed of three figures which are composed of three figures which are etc., so the dots are also the ultimate components of an  $n$ th-order collective. Since there is no theoretical limit to  $n$ , only practical and physical ones, it follows that physically recognizable and distinctive  $n$ th-order collectives are physically realizable for very large  $n$ .

### 3 Order, rank, width

Collections can be mixed in order, for example, we can consider a society which has both individuals and clubs as members, or the collection: Sviatoslav Richter and the Borodin Quartet. But at the bottom are always individuals, singular things that do not have members, though they may and typically do have *parts*. But the part-relation and

the member-relation, though formally fairly similar, are distinct.<sup>1</sup> The *rank* of a collection may be taken as the order of its highest-ranked member, plus 1, counting individuals as having rank 0. So the rank of the Borodin is 1, like its order, while the rank of Richter + the Borodin is 2. A collective is *homogeneous* if all its members have the same rank, and is *fully homogeneous* if all its members are homogeneous, and so on right down to the original members (*Urelemente*), that is, entities which have no members. In a fully homogeneous collective, rank and order coincide. A first-order collective is perforce fully homogeneous.

Order and rank constitute one dimension of variation of collectives, the *height* dimension. The number of members constitutes another: the *width* dimension. For example the *n*th order triangular figure of the last section has rank (= order, as it is fully homogeneous) *n*, and width 3, whereas the complex figure of just its dots has rank 1 and width 3<sup>*n*</sup>.

#### 4 Multiple collectives from the same urelements

In the Book of Genesis it is described how Noah and his family survived the flood:

And Noah went in, and his sons, and his wife, and his sons' wives with him, into the ark, because of the waters of the flood (*Genesis* 7, 7)

The order of listing of these people is not adventitious: it reflects the social superiority of males over females and then of older over younger in that ancient society. In the Bible, Noah's sons are named as Shem, Ham, and Japheth. His wife and daughters-in-law are not there named, but the Hebrew *Book of Jubilees* names them as respectively Emzara, Sedeqetelabab, Ne'elatama'uk and 'Adataneses. I shall abbreviate these eight names as, in order: N S H J E Q K and A. Now let us look at the different ways in which these eight people, the Flood Survivors, are grouped. One is obviously simply: these eight survivors. A second is the division by men and women. The third is by generation. A fourth is to group them into the four married couples. Then there is the Biblical listing, which lumps the sons together, the sons' wives together, and keeps Noah and Emzara as two separate individuals. Borrowing the notation of set theory for our purposes, these are

1. {N,E,S,H,J,Q,K,A}
2. {{N,S,H,J},{E,Q,K,A}}
3. {{N,E},{S,H,J,Q,K,A}}
4. {{N,E},{S,Q},{H,K},{J,A}}
5. {N,{S,H,J},E,{Q,K,A}}

Now I shall use an alternative and less fussy notation, grouping letters together to represent collectives of first-order and separating higher orders by spaces and numbers of

<sup>1</sup> Simons 1983.

vertical lines. As in standard set-theoretical notation the order of listing in any one list is of no consequence. The same collections then look like

1. NSHJEQKA
2. NSJH EQKA
3. NE SHJQKA
4. NE SQ HK JA
5. N SHJ E QKA

Noticeable about these groupings is the fact that they all partition the eight survivors, each survivor occurring only once and each occurring once in a subgroup. But now consider these five groupings of the eight into collectives of second rank. The first three are (more or less) natural groupings (according to survival, gender, and generation); the last two being more obviously social, depending on the existence of marriage as an institution, and on the two hierarchies of sex and generation in the final case. Of course, many other partitions are possible. But if we group the first three groupings together as “natural” and the second two as “social” and then group these two third-rank collectives together, we get the fourth-rank collective

NSHJEQKA | NSJH EQKA | NE SHJQKA || NE SQ HK JA | N SHJ E QKA

And in this case it is notable that individuals are listed or “occur” more than once. This too is perfectly all right.

## 5 Multitudes

We have used simple listing to indicate collectives by naming their members. Strictly speaking this pre-empts a distinction we shall now make. This is between collectives which are in some way *structured*, in that their existence in some way depends on the relations between and among the members, and *unstructured* ones which lack such a requirement. Take the example of the Amadeus Quartet again. Its four members, Norbert Brainin, Siegmund Nissel, Peter Schidlof, and Martin Lovett, only composed a string quartet in virtue of playing their various instruments together in the performance of musical works. They were not a string quartet all their lives, and would not have been one had they not played together. But these four would have been the four human beings they were whether or not they had played together, whether or not they had met, or even all grown up. An unstructured collective has no further requirement for its existence than that its various members exist. They need not exist together in space or time, need not interact, need have nothing in common beyond all existing. I call such collectives *multitudes*. The term is chosen to be different from others commonly used for collections such as “class,” “group,” and “set.” All of these have the advantage of being monosyllabic, but they all have far too many other uses and connotations to be employed happily here.

A multitude is a mere plurality: it is just such and such objects, no more, no less. We can designate multitudes in English and other languages by means of lists. I call such expressions *list terms*. The terms “Whitehead and Russell” and “Shem, Ham, and Japheth” are list terms, and name a pair of men and three men respectively. By contrast, descriptive terms like “the authors of *Principia Mathematica*” and “the sons of Noah” denote the same individuals, but as falling under a certain description. Had other authors than our two Cambridge men authored *Principia Mathematica*, for example if Wiener, Wittgenstein, and Ramsey had also taken part in the writing, “the authors of *Principia Mathematica*” would have denoted a different collection of men. So, descriptive plural terms are typically not modally rigid.

Any collection is intimately associated with at least one multitude. If it is a collection whose members do not change, the multitude just is those members. But remember, a multitude must have just those members, whereas the collection, depending on kind, need not have just those members, and indeed need not exist even though the multitude does, since its additional existence conditions may not hold. The Amadeus String Quartet is different from the four-person multitude Brainin, Nissel, Schidlof, and Lovett. A collection whose members change over time is associated with several multitudes. At each time at which the collection exists, its members at that time comprise the multitude in question. The multitude that is associated with a collection (at a time) we call the collection’s *membership* (at that time). So a collection which changes its members has more than one membership successively.

In cases where we have a higher-order collection, the members of the top level will typically not be multitudes, but other collections. The members of FIFA, for example, are national football associations, and these are not multitudes. But if we take every collection that is in a higher-order collection right back down to the originating individuals, these determine a multitude whose memberships going down to the individuals correspond one-to-one with the members of the collection at every level. We may call this the *transitive membership* of the higher-order collection, and it is a higher-order multitude. It was in fact the transitive memberships that we picked out with our notation in the Noah’s family example.

So multitudes are plural objects that are purely extensional, in that multitudes of necessity are identical when and only when they have the same members. A multitude has its members essentially: it could not exist were any of them not to exist, and if they all exist, so must it. Or rather, *they*. For everything we have said so far applies also to sets as standardly understood. But multitudes are not sets, and it is important to spell out why.

## 6 Why multitudes are not sets

Sets are abstract individuals: even if the elements of a set are concrete, the set is abstract. A multitude (with one class of exceptions to be noted below) is not an individual but

precisely a many, and if its members are concrete, so is it. A multitude whose members are located is located where its members are; its location is the sum of the locations of its members; its causal powers are the sum of those of its members.

There is a null set, but there is no null multitude. A multitude is its several members: nothing can be the several of nothing. In standard set theory the object  $x$  and its singleton set  $\{x\}$  are distinct entities, whereas a multitude of just one thing simply is that one thing, whether that thing is an individual or itself a multitude. In other words, there are no singletons. Finally, despite what set theorists might like to think, there is no necessity, given some objects, say Whitehead and Russell, that the set of them, here  $\{\text{Whitehead, Russell}\}$ , exists. One can without contradiction accept that both Whitehead and Russell exist and yet deny that their pair set exists. To do so is basically to reject set theory as a piece of ontology. It is an ontological stance that perhaps not many would dare to adopt, mindful of the success of set theory in mathematics. It is however a stance that as a nominalist, a denier of abstract objects, I dare to adopt. In so doing I may be extreme or foolish, but not self-contradictory. But to accept that Whitehead exists and Russell exists and yet deny that Whitehead and Russell, the pair, the two men, exist, is not extreme or foolish: it is self-contradictory. It takes away with one act of rejection what has been accepted in two acts of acceptance, and is just as bad as accepting Russell and then rejecting him, or accepting Whitehead and Russell but rejecting Russell: it is a particular species of self-contradiction peculiar to the institution of plurality.

One way of declaring the ontological “innocence” of certain objects, given other objects, is to say that they represent “no addition of being.” So those who claim that mereology is ontologically innocent say that given two individuals  $a$  and  $b$ , their mereological sum  $a + b$  is no addition of being. Such talk is slippery at best and inconsistent at worst. There are two things it can sensibly mean. One is that the sum doesn’t exist at all. But then why were we talking about “it” in the first place? The straightforward thing to say in that case would be that there is no such sum. The other thing it might mean is that it is just one of the things to which we are already committed in saying that  $a$  and  $b$  exist. But the things we are committed to are just  $a$  and  $b$ , and unless they are identical or one is part of the other,  $a + b$  is by definition something new, so after all an addition to being. David Lewis, who proclaimed the ontological innocence of mereology, tries to escape from this trap by saying that  $a + b$  is nothing other than  $a$  and  $b$ : they are it and it is them.<sup>2</sup> This is Baxter’s thesis of composition as identity.<sup>3</sup> But it is a leap from the frying pan into the fire, because it is saying of several things that they are one thing, and that is a plain contradiction. It is consistent to hold that  $a$  exists and  $b$  exists, and that  $a + b$  does not exist; it is inconsistent to hold that  $a$  exists and  $b$  exists, and that not both  $a$  and  $b$  exist.

Given two individuals  $a$  and  $b$ , the pair  $a$  and  $b$  has to exist. There is no option. God could not create  $a$  and create  $b$ , and still retain the option of not creating  $a$  and  $b$ . They are an addition to being in the sense that *they* are not either of them:  $a$  and  $b$  are not  $a$

<sup>2</sup> Lewis 1991, 81–7.

<sup>3</sup> Baxter 1988.

and they are not  $b$ : they are precisely both of them. But nothing further is required for them to exist than for each of the members to exist. They come automatically, as part of the package. In this way they are unlike sets or mereological sums, for whose existence an additional assumption is required.

Nominalists can accept multitudes, of first or higher order, because provided the urelements are concrete, so are all multitudes generated from them. But there is one kind of nominalist who will be unhappy with higher-order multitudes, and that is a Goodmanian one. Goodman proclaimed the principle, “No distinction of entities without distinction of content.”<sup>4</sup> If, as Goodman does, we believe in full extensional mereology, then given any individuals  $a$  and  $b$  we have their mereological sum  $a + b$ . If we now believe in the pair multitude  $ab$  as something distinct from  $a$ ,  $b$  and  $a + b$ , then we have apparently violated Goodman’s maxim. And even if we leave aside mereological sums, with such second-order multitudes as  $ab | cd$  as distinct from  $ac | bd$ , we have again gone against it. That indeed was one motive for my own earlier rejection of higher-order multitudes.<sup>5</sup> Now whether or not one can make sensible sense of Goodman’s maxim,<sup>6</sup> it now appears to me that the arguments for higher-order multitudes show that it cannot be cited as evidence against their existence without simply begging the question.

## 7 The need for a logic of multitudes

The theory of multitudes is not like set theory or mereology, an additional piece of ontology. It is simply logic. That it has not hitherto been standardly incorporated into logic is due to the prejudice in favor of the singular, ushered in by Frege’s treatment of names. This contrasts with the previous two thousand years of logic, where terms denoting more than one individual were treated on a par with those denoting just one individual. Indeed, because Aristotle considered science to concern not individuals but kinds, singular terms were excluded from proper logic. Early symbolic logicians such as Boole, Peirce, and Schröder used term variables not confined to the singular, and their logic of classes can be seen simply as a logic of multitudes. In twentieth-century logic the tradition was continued by Leśniewski, whose logical system called “ontology” contains a theory of multitudes in the first-order fragment that Śłupecki called “Leśniewski’s calculus of names.” The theory is magnificent, but it does not have multitudes beyond the first order, so even without moving into the logic of predicates, it does not capture all there is to say about multitudes. In Leśniewski the standard primitive predicate is “ $\varepsilon$ ,” which can be read “is one of.” The lack of hierarchy in Leśniewski’s ontology comes out in the theorem that this predicate is transitive:  $a \varepsilon b \wedge b \varepsilon c \rightarrow a \varepsilon c$ . For a logic of multitudes to overcome this restriction it must widen the “is one of” predicate to encompass cases where  $a$  is one of  $b$ ,  $b$  is one of  $c$ , but  $a$  is not one of  $c$ . For example, Ringo is one of the Beatles, the Beatles are one of the most successful

<sup>4</sup> Goodman 1972, 159–60.

<sup>5</sup> Simons 1982, 191–2.

<sup>6</sup> See Oliver 1993.

rock bands, but it is not the case that Ringo is one of the most successful rock bands, because he's not a rock band at all. So we need an "is one of" predicate which reflects this. For this purpose, and to distinguish our theory from Leśniewski's and from set theory, we shall use the letter eta, "η."

We quickly get into a grammatical mess when trying to employ the standard singular/plural distinction in talking about higher-order multitudes. The reason is that languages such as English, which employ the singular/plural distinction, use it to cope with first-order multitudes and the difference between one and many at that level. All ways of talking about higher-order multitudes, where a many is one of a higher many, get into grammatical difficulties because all (or nearly all) languages lack super-plurals, or plu plurals. But logically, this is not the point. If it is legitimate to accept that plurals designate collections, including multitudes, in the first place, then similar considerations apply to superplurals, and so to higher-order collections and multitudes, at any level.<sup>7</sup>

## 8 Three possible approaches to multitude logic

One intuitively appealing way to think of the logic of multitudes is to think of it as formally what we get from set theory by eliminating the empty set, all singletons, and all sets whose transitive membership contains the empty set or a singleton. Thus for example where  $a$  and  $b$  are individuals, the sets  $\{\emptyset, \{a,b\}\}$ ,  $\{\{a,b\}\}$  and  $\{\{a\}, \{a,b\}\}$  have no multitude counterparts, whereas the sets  $\{a,b\}$ ,  $\{a, \{a,b\}\}$  and  $\{a, b, \{a,b\}\}$  do have multitude counterparts, namely  $ab$ ,  $a ab$  and  $a b ab$  respectively. The justification for the last is this:  $a$ ,  $b$  and  $ab$  are three things: the first two are two individuals, the third is a multitude. So there is a multitude of which these are all the members, just as there is if we replace  $ab$  by anything else that is not  $a$  and not  $b$ .

This last form of justification is theoretically simple and attractive. It basically says that given any entities whatsoever, there is a multitude whose members are just those entities. When spelled out in terms of the conditions governing such entities, it is a comprehension principle. One consequence of it is the existence of a universal multitude  $V$ : the multitude of everything there is. The problem with this principle is that if carelessly formulated and applied, it leads to inconsistencies, including Russell's Paradox and Cantor's Paradox. In set theory there have been three major ways to avoid such paradoxes. The standard one, taken by Zermelo–Fraenkel set theory and its variants, has been to reject a comprehension principle leading to  $V$  and other "big" sets. In the standard cumulative theory ZFC, ZF with Foundation (Regularity), sets are built up from below by existential construction principles, starting from urelements: entities without elements. In pure set theory the only urelement is the null set  $\emptyset$ . In multitude

<sup>7</sup> Linnebo 2012, which is carefully and generally favorable to the idea of higher-order collectives, adds linguistic evidence from Icelandic numerals that the plu plural idea is not completely absent from syntax.

theory there is no empty multitude, so we would need other urelements to get things started; though, of course, to preserve ontological neutrality we cannot postulate their existence as an axiom. The advantages of such a logic of multitudes along ZFF lines would be familiarity and the presumption of consistency. The disadvantage would be that the attractive comprehension idea is rejected and that the choice of conditional existence principles has an air of ad hocery about it.

A potential solution to this is to allow that some multitudes are too large to themselves be elements. This is the idea of sets and proper classes, as developed successively by von Neumann, Bernays, and Gödel, and hence called NBG theory.<sup>8</sup> The comprehension principle could be retained, but would need to be restricted so that only “small” multitudes, those like sets, are elements. There would be a universal multitude, but it (and equisized multitudes) could not be members. This attractive option is the one closest to the ideas of Cantor, who distinguished between sets (*Mengen*), which are multitudes (*Vielheiten*) that can consistently be “thought together,” and multitudes which are absolutely inconsistent in that the assumption that they are sets leads to contradiction.<sup>9</sup>

The third and least popular way to steer clear of inconsistency has been to allow comprehension, but to restrict it. In Quine’s New Foundations set theory—hence called NF<sup>10</sup>—this is accomplished by requiring the condition employed in any instance of the comprehension principle to be *stratified*, this being a requirement that the free variables in the comprehension scheme be such that they can be consistently assigned types, subformulas  $x = y$  being such that  $x$  and  $y$  have the same type and subformulas  $x \in y$  being such that the type of  $y$  is one higher than that of  $x$ . This allows a universal set  $V$  since the defining formula in  $V = \{x: x = x\}$  is stratified.

Quine’s theory has the advantage of simplicity and the disadvantage of unfamiliarity. It is not known whether NF is consistent, and further NF entails the falsity of the Axiom of Choice, which I consider a logical truth. However, any multitude equivalent would reject  $\emptyset$  and conditionally employ other urelements, and it is known that NFU—that is NF with urelements—is consistent relative to simple type theory and elementary number theory and consistent with the Axiom of Choice.<sup>11</sup> So NFU + Choice + Infinity would be the nearest set-theoretic equivalent to this approach.<sup>12</sup>

At this time it is unclear to me which of these approaches is the correct one. Pending further investigation therefore, in what follows I shall outline the basic principles underlying any of the three approaches, attempting to isolate a common core dealing with the matters of individuality, existence, and identity. My purpose here is not to provide a foundation for mathematics, but to understand what higher-order multitudes are, and for this purpose the question of how strong the conditional existence principles should be is a distraction.

<sup>8</sup> Vide Gödel 1940.    <sup>9</sup> Cantor 1932, 443 f.    <sup>10</sup> Quine 1937.    <sup>11</sup> Jensen 1969.

<sup>12</sup> Despite the best partisan efforts of Forster 1995 and Holmes 1998, NFU still looks a much less straightforward environment than ZF or NBG, at least to my otherwise willing understanding.

## 9 Ontological intuitions about multitudes

Ontological intuitions are to be satisfied by a formal system for multitudes:

- The logic is to be compatible with the existence of any number of individuals, including one, none, finitely many, and infinitely many. This is the requirement of ontological neutrality.
- Difference drives everything: individuals  $a$  and  $b$  are two because they are different. Likewise given four individuals  $a$ ,  $b$ ,  $c$ , and  $d$ , the multitudes  $ab$  and  $cd$  are different, and these two are different from the two multitudes  $ac$  and  $bd$ . Higher-order multitudes result from ramifying this generating principle.
- Multitudes of individuals, including the multitude of all individuals, are unproblematic.
- There is no empty or null multitude.
- To be is to be one or more things.
- Every (particular) thing in the theory is either an individual or a plurality.
- A plurality is a multitude (thing) with more than one member.
- An individual is a thing that has no other member than itself. It is an urelement.
- All pluralities come from individuals. Without individuals there are no multitudes. There is no “pure multitude theory” as there is a pure set theory. We do not extract something from nothing.
- Pluralities are extensional. A plurality is completely determined by its members. Distinct pluralities cannot have the same members. An individual is automatically extensional since its only member is itself.
- Given any thing  $a$  and any thing  $b$  that is not one of  $a$  (not a member of  $a$ ), the multitude whose members are those of  $a$  together with  $b$  is a new multitude.
- Given any multitudes, the multitude obtained by merging all their members is a multitude.

## 10 Logic of multitudes: common core

In all formulas we understand apparently free variables  $a$ ,  $b$ ,  $c$  etc. as being tacitly quantified by a universal quantifier having the whole formula as scope. We understand the quantifiers as in Leśniewski in an unrestricted way, which is to say that we allow empty terms, and if a formula  $\dots a \dots$  is true when the term “ $a$ ” is empty, we are nevertheless entitled to infer the particularization  $\exists x \ulcorner \dots x \dots \urcorner$ . In all our quantified formulas, upper corners mark quantifier scope. The axioms are added to second-order predicate calculus. Names and nominal variables form a unified semantic category.

*Primitive predicates*

Identity:      Symbol “=”    Form  $a = b$

Membership:    Symbol “ $\eta$ ”    Form  $a \eta b$

*Regulative axioms*

Identity

$$a = b \leftrightarrow \forall \varphi \lceil \varphi(a) \leftrightarrow \varphi(b) \rceil$$

Extensionality

$$\forall c \lceil c \eta a \leftrightarrow c \eta b \rceil \rightarrow a = b$$

Existence

$$a \eta b \rightarrow \exists c \lceil c \eta a \rceil$$

Anti-symmetry

$$a \eta b \wedge b \eta a \rightarrow a = b$$

Individuality

$$a \eta a \rightarrow . b \eta a \rightarrow a \eta b$$

Supplementation

$$a \eta b \wedge b \neq a \rightarrow \exists c \lceil c \eta b \wedge c \neq a \rceil$$

Regularity

$$\exists b \lceil b \eta a \wedge c \eta a \wedge b \neq c \rceil \rightarrow \exists b \lceil b \eta a \wedge \sim \exists c \lceil c \eta b \wedge c \neq b \wedge c \eta a \wedge c \neq b \rceil \rceil$$

Remark 1: None of these entails the existence of anything.

Remark 2: The form of the regularity axiom differs from that of standard set theory in that it requires the multitude in question to be a plurality: individuals are self-membered and so falsify the consequent.

*Special multitudes*

Universe of all Individuals

$$\exists a \lceil \forall b \lceil b \eta a \leftrightarrow b \eta b \rceil \rceil$$

Remark: This axiom is still true if nothing exists.

Null

$$\exists a \lceil \forall b \lceil b \eta a \leftrightarrow b \eta b \wedge \sim (b \eta b) \rceil \rceil$$

Remark: Because of the meaning of “ $\exists$ ” in our system this emphatically does not entail the existence of a “null multitude.”

Useful definitions are:

Existent

$$Ea :: \leftrightarrow \exists b \lceil b \eta a \rceil$$

Individual

$$Ia :: \leftrightarrow \exists b \lceil b \eta a \rceil \wedge \forall c \lceil c \eta a \rightarrow a \eta c \rceil$$

Plurality

$$Pa :: \leftrightarrow \exists bc \ulcorner b \eta a \wedge c \eta a \wedge b \neq c \urcorner$$

Term

$$Ta :: \leftrightarrow \exists b \ulcorner a \eta b \urcorner$$

Myriad

$$Ma :: \leftrightarrow \exists b \ulcorner b \eta a \urcorner \wedge \sim \exists c \ulcorner a \eta c \urcorner$$

A *term*<sup>13</sup> is an entity that is a member. In an NBG style of multitude theory it is not coextensive with *existent*. Avoiding standard terminology, we call multitudes which are too “large” to be members *myriads*. (A plurality which is not a myriad could be called an *ensemble*, in which case terms comprise individuals and ensembles.)

These axioms are consistent because they are satisfied in a domain of one individual. But they do not automatically give us higher-order multitudes, being satisfied when “ $\eta$ ” is interpreted as Leśniewski’s singular inclusion functor “ $\varepsilon$ ,” where all multitudes are first-order. To distinguish the theory from that of Leśniewski and ensure there are higher-order multitudes, provided of course that there are at least two individuals, we need more.

## 11 Additional principles

Pair

$$\exists a \ulcorner \forall b \ulcorner b \eta a \leftrightarrow Tc \wedge Td \wedge c \neq d \wedge (b = c \vee b = d) \urcorner \urcorner$$

Given two or more individuals, this gives us higher-order multitudes. Suppose  $a$ ,  $b$ , and  $c$  are individuals. Then we have the two multitudes  $ab$  and  $bc$  by two applications of this axiom, and a third gives us the second-order multitude  $ab bc$ . By repeated application, designations for multitudes of any finite order can be constructed.

Adjoint

$$\exists a \ulcorner \forall b \ulcorner b \eta a \leftrightarrow Tb \wedge Tc \wedge Ed \wedge \sim c \eta d \wedge (b \eta d \vee b = c) \urcorner \urcorner$$

This allows us to adjoin a non-member of a multitude to that multitude. In conjunction with Pair, it allows us to form multitudes with any finite number of terms. The requirement that  $b$  and  $c$  be terms (and not just existents) is there in case we wish to employ this axiom in an NBG style theory. If  $d$  is a term, adjoining a term  $c$  to it gives another term. If  $d$  is a myriad, adjoining a term  $c$  to it gives another myriad. We cannot form a pair of two different myriads, for then they would not be myriads.

Union

$$\exists d \ulcorner \forall b \ulcorner b \eta d \leftrightarrow \exists c \ulcorner c \eta a \wedge b \eta c \urcorner \urcorner$$

<sup>13</sup> The expression is taken from Russell 1903, 43.

This intuitive axiom gives us a multitudes whose members are all the members of the members of  $a$ . It is thus of one order lower than  $a$ , unless  $a$  is already a first-order multitude, in which case  $d = a$ . For example if  $a$  is the second-order multitude  $efg\ eg$ , its union is the first-order multitude  $efg$ , while if  $a$  is the third-order multitude  $efg\ |eg$ , its union is the four-membered second-order multitude  $efg\ eg$ .

It is tempting to essay stronger axioms, allowing power multitudes, an axiom of infinity, and axiom of choice, and so on, and to go as far as possible towards full comprehension. But for current purposes, these suffice. I cannot stress enough that the logic of multitudes is not conceived like set theory as a non-logical theory added on to logic, but as itself part of a universal logic. The relative paucity of stable intuitions about higher-order multitudes means that the correct principles for such a logic are presently not clear. The precedents of set theory are only helpful up to a point. Because of the lack of singletons and a null multitude (the definable empty name does not name a null multitude, it precisely names nothing at all) standard results and conceptions from set theory, whether of the ZF or the NF sort, are often not applicable.<sup>14</sup> So the logic remains work in progress. However, even in this partially inchoate state, multitude theory can be of philosophical use.

## 12 An application: concrete models

One of the things we have been told many times is that in order to provide an adequate semantics for logic, in particular higher-order logic, but also for first-order predicate logic, we need to avail of abstract entities such as functions or sets. This is distressing for a nominalist, who wishes to get by without any abstract entities. A sensible nominalist will not wish to reject powerful logic, however, so appears to be in a cleft stick: either use the logic without making use of its semantic motivation, or embrace some kind of platonism, at least pragmatically. The first option was taken by one hardened nominalist, Leśniewski, while the second was reluctantly followed by his one-time student and would-be nominalist, Tarski. Neither is comfortable. Since Gödel, we have known that a proof-theoretic explication of logical consequence is inherently weaker than the arguably correct semantic account. But it would be so nice to have a semantic account of logical consequence without those sets and functions.

Multitude theory offers a way out. Recall that a multitude is concrete if its members are. So let us be nominalists about individuals and see how multitudes can deliver us entities with which to ply our semantics. For concreteness we stick to first-order logic: the adaptation to higher orders is relatively routine. The tricky part of providing a semantics for first-order predicate logic is what to do about polyadic predicates.<sup>15</sup> The

<sup>14</sup> Several of the notable model-theoretic results affecting NF and NFU turn on the distinction between  $x$  and  $\{x\}$ , which we simply lack. See Holmes 2012, Section 6.

<sup>15</sup> It is this that vitiates George Boolos's otherwise praiseworthy attempt to take the platonism out of the semantics of logic: the monadic part is no problem, but he simply helps himself to ordered pairs. Cf. Boolos 1985.

typical semantic value of a polyadic predicate in an interpretation is a set of  $n$ -tuples. An  $n$ -tuple may be construed in several ways. One is to use sets of sets, using the Kuratowski trick for representing ordered pairs, and then iterating in some way. Another is to treat an  $n$ -tuple as an  $n$ -place relation, as do Whitehead and Russell for the case of the ordered pair. Yet another way is to take an  $n$ -tuple as a function from the numbers  $\{1, \dots, n\}$  into the domain. All these use abstract entities of one sort or another.

Two features of  $n$ -tuples need to be reproduced: the ordering of the elements, and the repetition of elements at different places. If there are no repetitions, then a higher-order multiple capable of representing  $\langle a, b, c \rangle$  is  $a\ ab\ abc$ . The occupant of the first position is uniquely determined by  $a$ 's occurrence in each of the members,  $b$ 's second position by its occurring in all but one, and  $c$ 's third and final position by its occurring only once. The device works obviously by extension for any number of positions. But how can we represent repetition without collapsing the tuple onto something different? We cannot do it simply by direct substitution, since, for example,  $\langle a, b, a \rangle$  would be indistinguishable from  $\langle a, b \rangle$ . But we can explicitly represent the substitution by another multitude. So  $\langle a, b, a \rangle$  can be represented by  $a\ ab\ abc\ ca$ , the last pair showing that  $a$  substitutes in the third or  $c$  place. Multiple repetitions are done the same way, for example the quintuple  $\langle a, b, a, c, b \rangle$  can be represented by  $a\ ab\ abd\ abdc\ abdce\ ad\ be$ . This uniquely represents the correct quintuple with repetitions. Of course there is more than one way to represent it, depending on the choice of the "spare" objects  $d$  and  $e$ , but we can either pick objects not in use in the domain or, more cleanly, pick two at random (distinct from  $a$ ,  $b$ , and  $c$  of course) but make sure there are no equivalent multitudes using other objects. Either way, we can represent our tuples, and putting a multitude of these together gives us a representation every bit as good as a set of tuples from the domain. It will be a multitude three orders higher than  $a$ ,  $b$  etc., but that is fine. The other abstract object usually invoked in formal semantics will be an interpretation function, or a satisfaction sequence, as in Tarski. The interpretation is a matter of correlating expressions with semantic values and that can also be expressed by multitudes, some of whose members are the expressions in question and others of which are the representations of the semantic values in question.

It is freely granted that these are representational tricks. But the more standard and familiar use of sets is no different. No one who thinks about it for a moment imagines that the meaning or referent of the term "loves" in English is an abstract object, a set of ordered pairs of people, sometimes with repetitions (those loving themselves). The set is there to do a semantic job, namely to provide a model or representation. When it comes to that job, any adequate representation is as good as any other. The exercise is not to provide the real meaning or referent, but something that can drive the semantics. Only the familiarity of the set-theoretical representation may dull one into thinking it is somehow more "real" or natural. And since multitude theory is nominalistically acceptable, given only concrete individuals to start with, we can furnish logic of first or higher orders with a nominalistically acceptable account of logical truth and logical consequence. The only occasion on which the representation falters is on a domain of a

single individual, since this generates no multitudes. But it can be dealt with by special clauses, as can the even more extreme empty domain. For finite type theory we would need to have multitudes of transfinite order. There appears to be no reason to disallow transfinite orders, but that brings in a raft of other considerations which it would be best to consider another time.

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