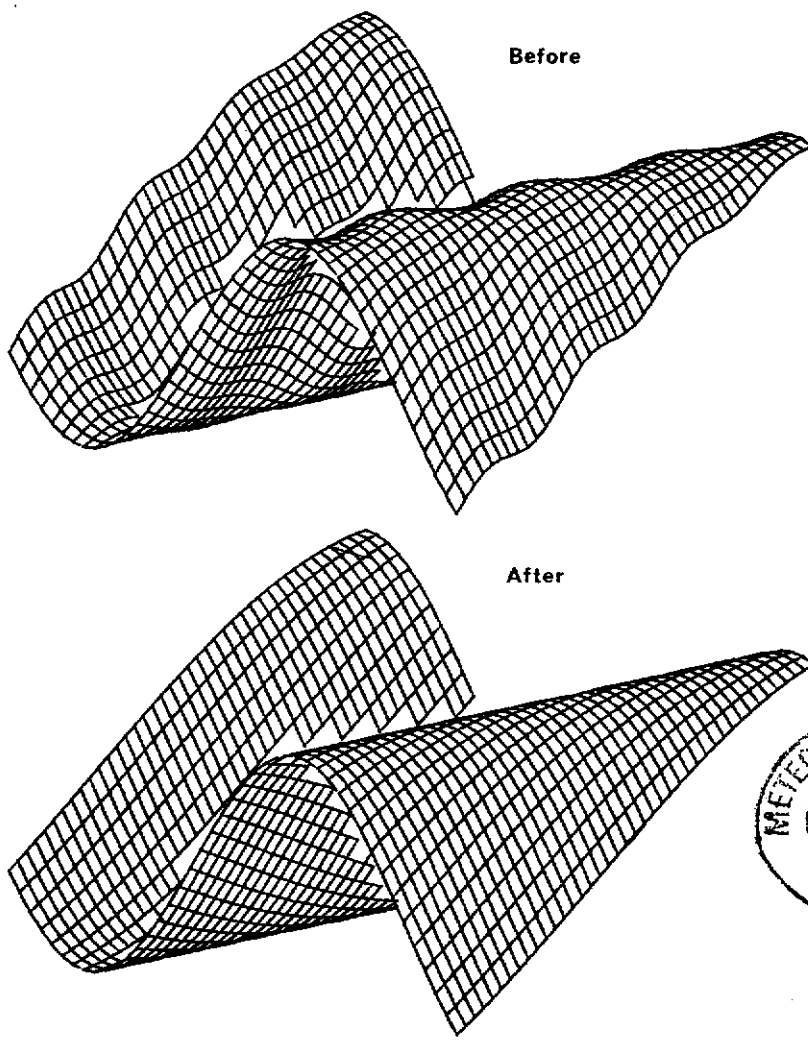
METEOROLOGICAL SERVICE

TECHNICAL NOTE No. 45

INITIALIZATION USING LAPLACE TRANSFORMS



METEOROLOGICAL SERVICE LIBRARY
13 FEB 1984
DUBLIN
20707

BY
PETER LYNCH, M.Sc., Ph.D.

U.D.C. 551. 509. 313

GLASNEVIN HILL, DUBLIN 9
JANUARY 1984

Price: £2·20

INITIALIZATION USING LAPLACE TRANSFORMS

BY

PETER LYNCH

Irish Meteorological Service

JANUARY, 1984

Abstract. The initialization of limited area models is complicated by the difficulty of determining the linear normal modes and of allowing for general boundary conditions. A new method of initialization is devised, which does not require explicit knowledge of the normal modes. The method is based on a filtering procedure which uses a modified inverse Laplace transform. The efficacy of the method is demonstrated by application to a one dimensional model, and the rationale for application to a general forecasting model is discussed. The method is closely related to the nonlinear normal mode method of initialization.

TABLE OF CONTENTS

CHAPTER 1. REVIEW OF INITIALIZATION METHODS 1

1·1 *The Whys and Wherefores* 1

1·2 *Some Initialization Methods* 3

1·3 *The Situation with Regard to Limited Area Modelling* 6

1·4 *A New Approach: The Laplace Transform Technique* 7

CHAPTER 2. THEORETICAL DISCUSSION 9

2·1 *Filtering with Laplace Transforms* 9

2·2 *Separation of Scales in a General Nonlinear System* 11

2·3 *Application to Initialization* 13

2·4 *Relationship to Normal Mode Initialization* 14

CHAPTER 3 APPLICATION TO A ONE-DIMENSIONAL MODEL 15

3·1 *Formulation of the Method* 15

3·2 *Computational Results* 19

CHAPTER 4 PRINCIPLES OF APPLICATION TO A GENERAL FORECASTING MODEL 26

CHAPTER 5 SUMMARY 29

REFERENCES 30

1. Review of Initialization Methods

1·1 *The Whys and Wherefores*

Consider a tide-gauge in open water. The gauge will measure the slow changes between low and high water, which have periods of about twelve hours. It will also register the changes due to sea and swell, which have periods of less than a minute, but whose amplitude may equal or exceed the tidal range. Clearly, the instantaneous readings of the gauge cannot be used for tidal analysis until the 'noise' due to the short-period waves has been removed in some way. The problem arises because the water level is governed by processes with widely differing time-scales.

The primitive equations used for numerical weather prediction also have solutions with different time-scales. The solutions of meteorological significance are low frequency, rotational motions with phase speeds of the order of ten metres per second, and small divergence. There are also very fast gravity inertia wave solutions, with phase speeds of hundreds of metres per second and large divergence; these do not interact strongly with the rotational motions, and are generally regarded as noise.

A subtle state of balance exists in the atmosphere between the wind and pressure fields, ensuring that the gravity inertia waves have much smaller amplitude than the rotational part of the flow. However, when we make a forecast starting from analysed mass and wind fields large high frequency oscillations occur; these are due to slight imbalances in the initial data resulting from observational and other errors and from imperfections in the forecast model.

Figure 1·1 shows a trace of the surface pressure at Ocean Weather Ship 'Charlie' (52.5°N, 35.5°W) for the 12 hour period starting at 00Z, 17 Jan, 1984. The forecast pressure at a nearby point for the same period is also shown. Clearly, the forecast has unrealistic high frequency oscillations with amplitudes of several *mbar*. These are due to a lack of balance in the initial fields resulting in the presence of spuriously large gravity waves.

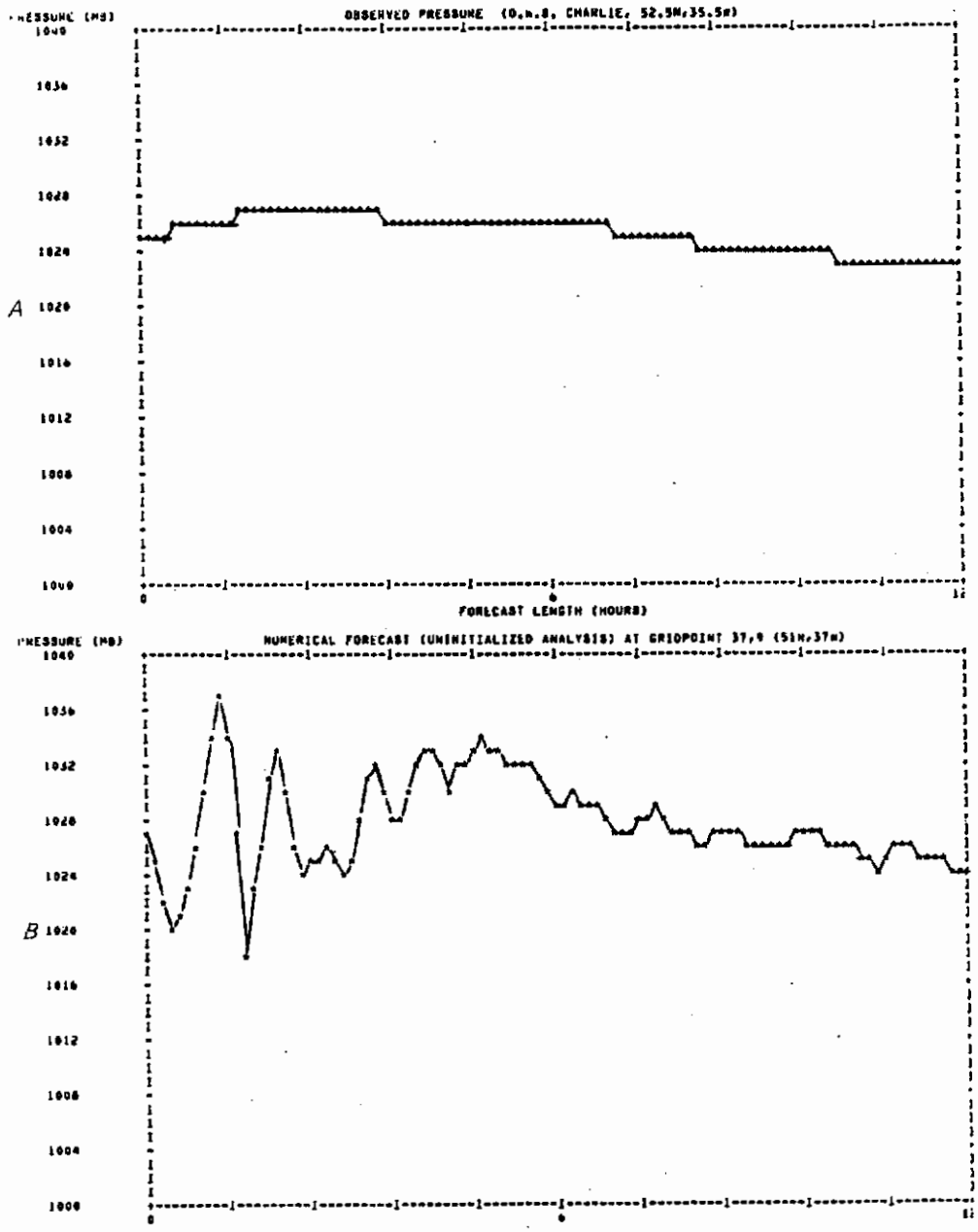


Figure 1.1: (A) Observed surface pressure at Ocean Weather Ship 'Charlie' (52.5°N, 35.5°W) for the 12 hour period from 00Z, 17 Jan 1984. (B) Forecast surface pressure at a nearby point (51°N, 37°W) for the same period.

One of the long-standing problems in numerical weather prediction has been to overcome the problems associated with high frequency motions. *The principal goal of initialization is to define the initial fields in such a way that the gravity inertia waves remain small throughout the forecast.* If this is not done the spurious oscillations which occur in the forecast can lead to various problems, as outlined below.

(i) Data-Assimilation: When an objective analysis is performed new observations are checked against a preliminary field which is normally a short range forecast. If this is noisy, due to the presence of unrealistically large gravity inertia waves, good observations may be rejected or erroneous ones accepted. The resulting degraded analysis deteriorates the ensuing forecast, and the situation worsens as the data assimilation cycle continues.

(ii) Precipitation: Since the gravity inertia waves have large divergence they give rise to unrealistic vertical velocity fields. Since the moisture field depends crucially on vertical motions, hopelessly inaccurate precipitation forecasts can result if the gravity inertia waves are not controlled.

(iii) Advection: The advection process, whereby momentum, energy and moisture are carried along in the flow by the velocity fields, is crucial to the dynamics of the large scale, low frequency meteorological motions. Since this is a slow process, a long timestep should be adequate to represent it accurately. However, if the velocity fields are corrupted by large gravity inertia waves the use of long timesteps may result in unacceptable smoothing or distortion of the meteorological fields. Thus, although, for example, semi-Lagrangian advection schemes provide an unconditionally stable method of integrating the advection process, permitting unrestricted timesteps, their maximum benefits appear to be dependent upon proper initialization of the fields.

1·2 *Some Initialization Methods*

In the early days of numerical weather prediction forecasts were made using governing equations which were modified in such a way that they

filtered out gravity inertia waves altogether. Therefore, the initialization problem did not arise. However, the approximations made in deriving the filtered equations are not always valid, and the primitive equations are now generally used.

The first efforts to control the high frequency motions consisted of defining various diagnostic relationships between the mass and wind fields: the geostrophic relation; the linear balance equation; the nonlinear balance equation (Hinkelmann, 1951; Charney, 1955; Phillips, 1960). These *static initialization* methods reduce the noise to varying degrees, but are unsatisfactory for a number of reasons: they discard useful observational information; the fields must be modified artificially for reasons of mathematical expediency; the remaining noise is still a problem.

A method of analysis based on a variational technique was developed by Sasaki (1958). This method permitted mutual adjustment between the mass and wind fields subject to some constraint, such as that they satisfy the balance equation. The method has been extended in various ways, and applied to the initialization problem with partial success in supressing the gravity wave noise.

An alternative approach to balancing the fields is to use the forecast model to integrate in a foreward-backward cycle with a damping time-scheme (Miyakoda and Moyer, 1968). This is known as *dynamic initialization*. It has not been found to be very satisfactory: it damps the meteorological modes along with the noise; it is computationally expensive; no irreversible processes can be included because of the backward integrations.

Recently, methods of initialization based upon the linear normal modes of the forecast model have been proposed. The first approach was to analyse the initial conditions into their slow rotational and fast gravity inertia components and to set to zero the coefficients of the latter (Dickinson and Williamson, 1972). This was partially successful but because of the nonlinearity of the equations the high frequency modes are rapidly re-excited (Figure 1·2(a)). A more sophisticated approach is to set the initial rate of change of the fast modes to zero; this method is called *nonlinear normal mode*

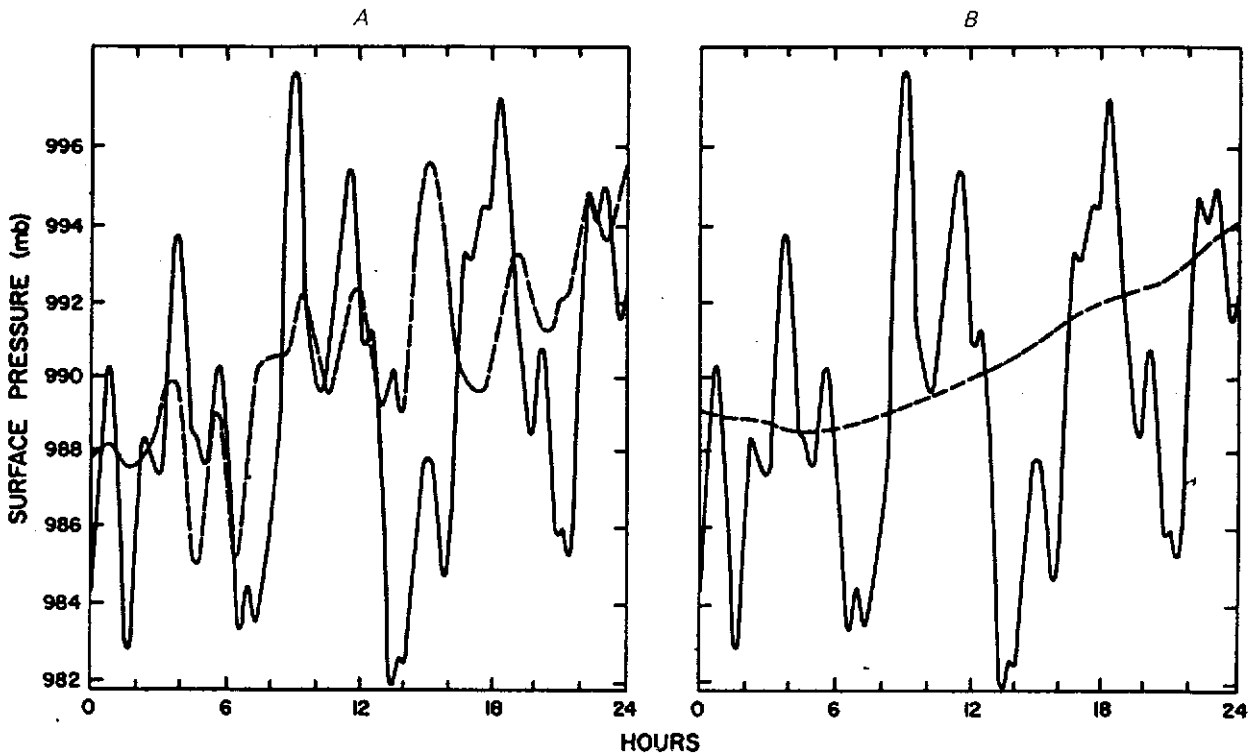


Figure 1.2: (A) Surface pressure traces at a particular gridpoint as forecast by the ECMWF model with no initialization (solid) and linear initialization (dashed). (B) Pressure traces for no initialization (solid) and nonlinear initialization (dashed). [After Temperton and Williamson, 1979]

initialization (Baer, 1977; Machenhauer, 1977). It has been found to be superior to the simpler, linear method (Figure 1·2(b)), and is currently regarded as the most satisfactory initialization method, despite certain disadvantages. The method has been reviewed comprehensively by Daley (1981).

Another recently proposed scheme is the bounded derivative method (Kreiss, 1979, 1980; Browning *et al.*, 1980). If the evolution of the fields is slow then their time derivatives of several orders must be small. This condition is used to derive diagnostic relationships between the variables. The method is thus an extension of the classical static initialization methods described above (and appears to have the same defects). A comparison between the nonlinear normal mode and bounded derivative methods in a simple context has been made by Kasahara (1982).

1·3 *The Situation with Regard to Limited Area Modelling*

To apply the nonlinear normal mode method we must know the linear normal modes of the model equations. For a global domain this is no particular problem: the equations can be linearized in such a way that the spatial variable dependencies are separable and the horizontal boundary conditions are those of periodicity or regularity of the solution. The vertical structure equation is easily solved for simple mean temperature profiles.

For forecasts over a limited area the situation is dramatically different. Here the dependent variables are specified on the boundary of the forecast domain and are allowed to vary in time. It is difficult, if not impossible, to define the normal modes in a completely general way. If the area is of an irregular shape the horizontal variables may not be separable. Non-separability also results if the Coriolis parameter is a function of both variables, as in the case of a rotated latitude/longitude grid whose poles are not on the earth's axis.

Some progress has been made recently in adapting the normal mode approach to limited area modelling. Briere (1982) has applied the method to a fine-mesh model in cartesian coordinates with constant boundary conditions. Bijlsma and Hafkenscheid (1983) have incorporated the effects of sphericity,

and their method apparently allows for variable boundary conditions (although the results they show seem to hold the boundary values constant). Both applications have been successful in reducing the noise in short range forecasts. However, in each case it has been necessary to make simplifying assumptions in order to derive the linear normal modes: the analysis has been done with a constant Coriolis parameter. It is not obvious if these assumptions can be relaxed. Nor is it obvious whether or not they have a degrading effect on the specification of the initial fields. Ballish (1979) has demonstrated that failure to include the β-term in the derivation of the linear normal modes may lead to noise problems. Thus, a totally satisfactory resolution of the problem of initialization for limited area models is still outstanding.

1·4 *A Novel Approach: The Laplace Transform Technique*

In this report the following question is addressed: Given a nonlinear prediction model, how can the initial conditions be specified in such a way that no component of the flow will evolve with high frequency gravity inertia type timescales? A general technique is developed which provides the answer to this question. The technique involves the use of a filtering method based on a modification of the inversion formula for the Laplace transform. This filter is used in an iterative procedure which is applied to a general nonlinear system with multiple timescales. *Provided the procedure converges*, the result is a function which (i) satisfies the system equations and (ii) evolves slowly in time. The initial value of this function provides the answer to the question posed above.

The practical implementation of the procedure necessitates some approximations (a) to reduce the computational effort and (b) to avoid numerical instabilities associated with the Laplace transform inversion. If the nonlinear terms are small (a necessary condition for convergence anyway, and true for atmospheric motions) these approximations are quite acceptable, and lead to a simple initialization procedure. The procedure is seen to bear a close relationship to the linear and nonlinear normal mode initialization methods. A point of vital *practical* importance is that the new technique does *not* require

knowledge of the linear normal modes of the system. Thus, it can be applied more generally than the previously proposed methods.

The feasability of the Laplace transform technique is demonstrated by application to a simple one-dimensional model similar to that used by Baer (1977). A model run, starting from geostrophically balanced initial conditions, results in large amplitude, high frequency oscillations. These oscillations are most clearly seen in the evolution of the divergent kinetic energy. Application of the first approximant of the new technique (equivalent to linear normal mode initialization) reduces this noise considerably, but some small oscillations remain. The nonlinear procedure, with just one iteration, leads to virtually total extinction of the noise. These results are in complete agreement with those obtained by Baer (1977) with his technique, and demonstrate that the present method is an effective means of controlling the high frequency oscillations in this simple primitive equation model.

The application of the technique in the context of a general baroclinic (limited area or global) primitive equation model is discussed. The proposed method is essentially the same as that used in the case of normal mode initialization: the vertical and horizontal structures are separated, and the horizontal structure functions are initialized separately for each vertical mode or equivalent depth. The technique would appear to be as flexible as normal mode methods in its ability to handle orography, model physics, *etc.* It is more flexible insofar as it can also be applied to a limited area model, where the horizontal variables need not be separable, where the boundary conditions may vary in time and where the linear normal modes are unknown.

2. Theoretical Discussion.

We consider the following question: given a nonlinear system with multiple timescales, how can the initial conditions be specified in such a way that the rapidly varying components of the solution are completely supressed *for all time*? We formulate a general procedure which, given convergence, provides a solution satisfying the original nonlinear equation and evolving slowly in time. It is shown how this procedure may be applied in an approximate form to specify initial conditions which guarantee a slowly evolving solution.

2·1 *Filtering with Laplace Transforms*

Consider a function $f(t)$ with components of various frequencies

$$f(t) = \sum_{n=1}^{\infty} a_n \exp(i\omega_n t) \tag{1}$$

Suppose we wish to isolate that part of f which varies with frequency less than γ. Knowing the expansion (1), it is trivial to set a_n equal to zero whenever $|\omega_n| > \gamma$; the remaining components provide the solution. Now suppose that $f(t)$ is unknown but that its Laplace transform is given by

$$\hat{f}(s) = \sum_{n=1}^{\infty} a_n / (s - i\omega_n) \tag{2}$$

The filtering of f can again be done by setting $a_n = 0$ when $|\omega_n| > \gamma$, and inverting the sum of the remaining terms.

Next, suppose we know $\hat{f}(s)$ to be of the form (2) but do not know the precise values (a_n, ω_n). The original function $f(t)$ may be obtained from the inverse transform

$$f(t) = L^{-1}\{\hat{f}\} = \frac{1}{2\pi i}\int_C e^{st}\hat{f}(s)\cdot ds \tag{3}$$

where C is a line parallel to the imaginary axis and to the right of all

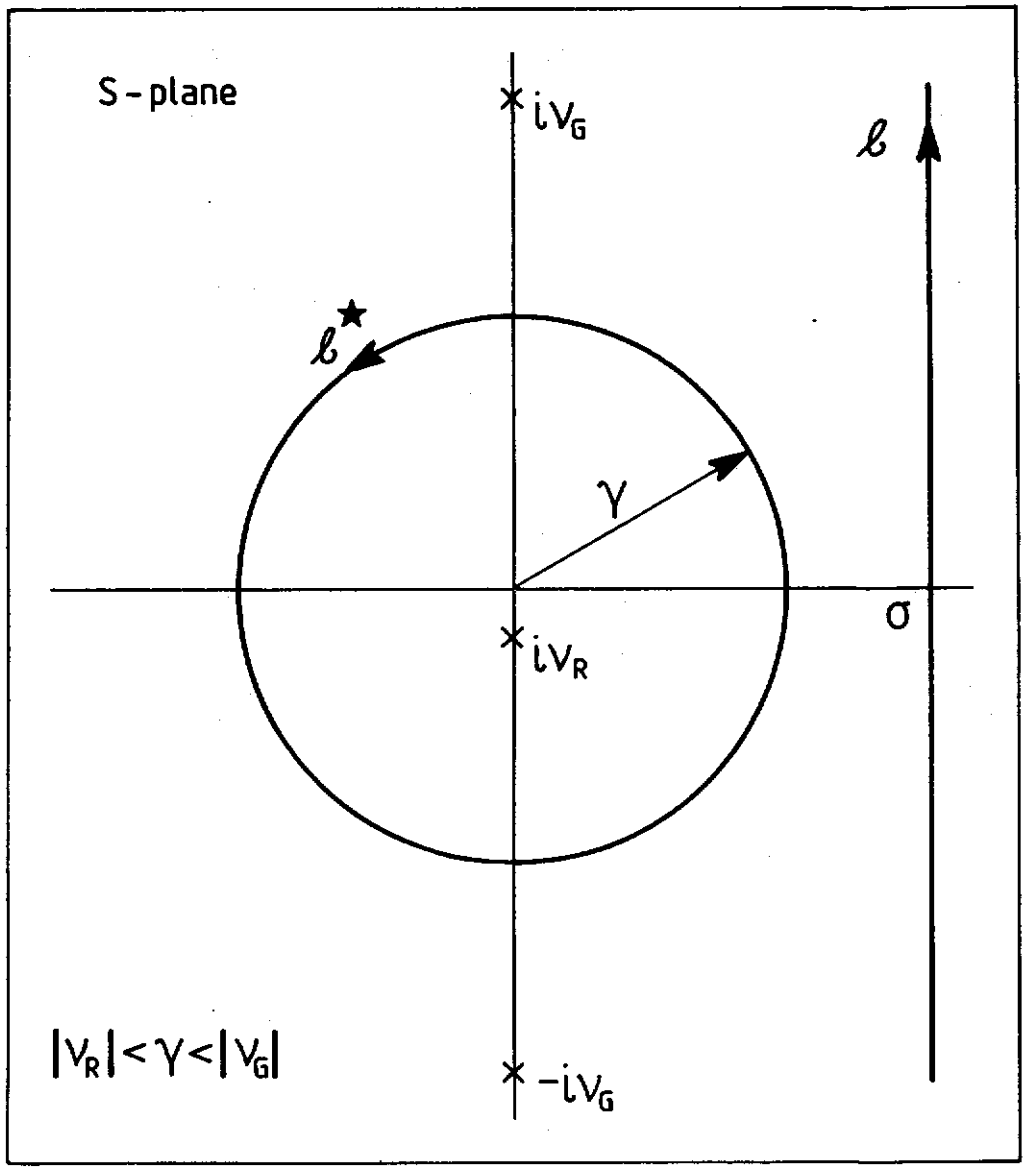


Figure 2.1: Contours in the s-plane used for the regular and modified inverse Laplace transform. The value of γ is chosen to separate the rotational frequencies (ν_R) and the gravity wave frequencies (ν_G).

singularities of $\hat{f}$. The contribution to this integral from the pole of $\hat{f}(s)$ at $i\omega_n$ is just $a_n \exp(i\omega_n t)$. Without knowing (a_n, ω_n), we can still eliminate all components with frequencies greater than γ by changing the contour C in (3) to a circle C^* of radius γ centred at the origin of the s-plane (see Figure 2·1):

$$f^*(t) = L^* \{ \hat{f} \} = \frac{1}{2\pi i} \oint_{C^*} e^{st} \hat{f}(s) \cdot ds \tag{4}$$

The function f^* is just the sum of the components resulting from those poles which fall within C^*, *i.e.* the components with frequency less than γ. This is exactly what is required. Thus, knowledge of the transformed function on the contour C^* is sufficient to perform the desired filtering.

2·2 *Separation of Scales in a General Nonlinear System*

Consider a system whose state at time t is specified by the vector X(t). For example, X might comprise the values of the dependent variables at all the gridpoints of a forecast model. We assume that the evolution of X is governed by a nonlinear equation of the form

$$\frac{dX}{dt} + LX + N(X) = 0 \tag{5}$$

where L is a constant linear operator and N is a nonlinear vector function of X. If the system is in the state X_0 at $t=0$ then the Laplace transform of this equation may be written

$$M\hat{X} + \hat{N} = X_0 \tag{6}$$

where we define the matrix $M = M(s) = (sI + L)$, with I the identity matrix.

If the nonlinear terms in (6) are ignored the transformed solution is just

$$\hat{X} = M^{-1} X_0. \tag{7}$$

Inverting this we recover the solution $X(t)$. Suppose we are interested only in the slowly evolving part of X, *i.e.* that part which oscillates with frequency less than γ. Then we may apply the operator $\mathcal{L}^*$ to obtain

$$X^*(t) = \mathcal{L}^*\{\hat{X}(s)\} \tag{8}$$

This will have the desired property. Alternatively, if we define

$$X^*(0) = \mathcal{L}^*\{\hat{X}(s)\}\Big|_{t=0} \tag{9}$$

and use this as the initial state, the linear evolution of the system will again contain only low frequency components.

If $X^*(0)$ is used as the initial state for the nonlinear equations (5) the high frequency components will be re-excited immediately by the nonlinear term $N(X)$. We wish to avoid this happening by modifying the initial state. We can solve (6) for $\hat{X}$ by an iterative procedure analogous to that used by Mechenhauer (1977). We assume that the nonlinear terms are small and take as the first approximant the linear solution

$$\hat{X}_1 = M^{-1} X_0 \tag{10}$$

The first estimate of the slowly varying solution is then

$$X_1^* = \mathcal{L}^*(\hat{X}_1) \tag{11}$$

When this is known the nonlinear term $N_1 = N(X_1^*)$ can be obtained and its transform evaluated. Further approximations are given by the iterative procedure

$$\hat{X}_{n+1} = M^{-1}[X_n^*(0) - \hat{N}_n] \tag{12}$$

$$X_{n+1}^* = \mathcal{L}^*(\hat{X}_{n+1}) \tag{13}$$

Suppose the procedure converges: then we can write

$$\hat{X}_\infty = M^{-1}[X^*_\infty(0)-\hat{N}_\infty] \quad ; \quad X^*_\infty = \mathcal{L}^*(\hat{X}_\infty) \tag{14}$$

which, when inverted, shows that the vector $X^*=X^*_\infty$ has the following properties: (a) it satisfies the original equation (5); (b) it contains only slowly evolving components. This is exactly the solution which was desired. We note in passing that, if $\mathcal{L}^{-1}$ is used instead of $\mathcal{L}^*$ above, the full solution to the nonlinear problem is obtained.

2·3 *Application to Initialization*

The procedure developed above is difficult to apply in practice for two reasons. First, it involves the calculation of the Laplace transform of the nonlinear term at each iteration, which implies a formidable ammount of work. Second, the Laplace transform is notoriously difficult to invert numerically, because of the factor exp(st); this also applies to the modified inverse transform, $\mathcal{L}^*$. We circumvent these problems by assuming that the nonlinear term varies so slowly that it may be considered constant and its transform approximated by

$$\hat{N}_n = N(X^*_n(0))/s \tag{15}$$

Furthermore, since we are interested in the appropriate initial values to use, the operator $\mathcal{L}^*$ need only be evaluated at time $t=0$. The approximate procedure is outlined below:

First approximent: $$\hat{X}_1 = M^{-1} X_0. \tag{16}$$

$$X^*_1(0) = \mathcal{L}^*\{\hat{X}_1\}\Big|_{t=0} \tag{17}$$

Iterative procedure: $$\hat{N}_n = N(X^*_n(0))/s \tag{18}$$

$$\hat{X}_{n+1} = M^{-1}[X_n^*(0)-\hat{N}_n] \tag{19}$$

$$X_{n+1}^*(0) = \mathcal{L}^*\{\hat{X}_{n+1}\}\Big|_{t=0} \tag{20}$$

The final iteration of equation (20) gives us the required initial conditions. In practice it is found that one or two iterations are sufficient to reduce the amplitude of the high frequency components to a negligible level. Note that the matrix M may be pre-calculated, inverted and stored for a set of values of s lying on C^*; thus, the method is reasonably economical.

2·4 *Relationship to Normal Mode Initialization*

From the definition (4) of the modified inverse Laplace transform, it is clear that the first approximant in the above technique is equivalent to linear normal mode initialization. That is, application of (16), (17) with an appropriate value of γ has the same effect as spectral resolution into normal modes followed by removal of all modes with frequency greater than γ. Note that if the equations have been linearized about a general mean flow $\bar{u}=\bar{u}(\phi,Z)$ its effects are felt by the linear normal modes. The eigenvalues may be complex and there may be a continuous spectrum in the case of a varying mean flow.

The full nonlinear technique yields a solution which evolves slowly for all time. The essential approximation in the practical implementation of the method is the assumption that the nonlinear terms evolve so slowly that they may be approximated by their initial values (see equation (15)). This is the same as the assumption made by Machenhauer (1977) (see his equations (14), (15)). It would thus appear that the method proposed here is closely related to that of Machenhauer. The formal relationship between the present method and those of Machenhauer and of Baer and Tribbia (1977) will be considered elsewhere.

3. Application to a One-dimensional Model

3·1 *Formulation of the Method*

In order to test the feasability of the Laplace transform method in the simplest context it has been used to initialize the data for a one-dimensional model. The model is similar to that used by Beer (1977) and a full description is given in Lynch (1984). The basic equations are

$$\zeta_t + (u\zeta)_x + f\delta + \beta v = 0 \tag{21}$$

$$\delta_t + (u\delta)_x - f\zeta + \beta u' + \phi_{xx} = 0 \tag{22}$$

$$\phi_t + (u\phi)_x - f\bar{u}v + \bar{\phi}\delta = 0 \tag{23}$$

Here x is distance eastward, t is time, $u = \bar{u} + u'$ is the zonal velocity with $\bar{u}$ the constant mean windspeed, v the northward velocity, and $\phi = \bar{\phi} + \phi'$ the geopotential. The vorticity and divergence are $\zeta = v_x$ and $\delta = u_x$. The Coriolis parameter f and its meridional derivitive β are assumed constant.

The above system is easily shown to have normal mode solutions of two kinds: the slow Rossby waves have frequencies given approximately by

$$\nu = -(\beta/k)/[1 + f^2/k^2\bar{\phi}]$$

where $k = (2\pi/L)$ is the wavenumber and L the wavelength. They have small but nonzero divergence. There are also fast gravity inertia waves whose approximate frequencies are

$$\nu = \pm\sqrt{(\bar{\phi}k^2 + f^2)}$$

These divergent waves travel in both directions. For typical parameter values the two wave types are clearly distinguished by the magnitudes of their

frequencies.

Energy equations are derived in the usual manner (see *e.g.* Pedlosky, 1979). The rate of change of eddy kinetic *plus* available potential energy is given by

$$\frac{d}{dt}\int \left[\tfrac{1}{2}\rho(u'^2+v^2)\Phi+\tfrac{1}{2}\rho\Phi'^2 \right] dx = -\int \left[\rho v[\tfrac{1}{2}(u'^2+v^2)+\Phi']\frac{\partial\bar{\Phi}}{\partial y} \right] dx \quad (24)$$

Clearly, if the mean flow vanishes ($\bar{u}=\bar{\Phi}_y=0$) the total eddy energy remains constant. The eddy kinetic energy can be split into contributions due to the rotational and divergent motions:

$$K = K_\psi+K_\chi \quad ; \quad K_\psi = \int \tfrac{1}{2}\rho v^2\Phi\, dx \quad ; \quad K_\chi = \int \tfrac{1}{2}\rho u'^2\Phi\, dx.$$

The values of these are calculated at each timestep and give valuable information about the dynamics of the motion being considered.

In order to clarify the relative magnitude of the various terms in the equations of motion it is convenient to nondimensionalize the equations by defining characteristic scales for length, time and velocity. It is also convenient numerically to have the principal terms of order unity. We introduce length and velocity scales L and V and scale time by f^{-1} and geopotential by fLV. We define some nondimensional combinations:

$$Ro = (V/fL) \; ; \; R_\beta = (\beta L/f) \sim (L/a) \; ; \; R_F = \bar{\Phi}/(fL)^2 = (L_R/L)^2.$$

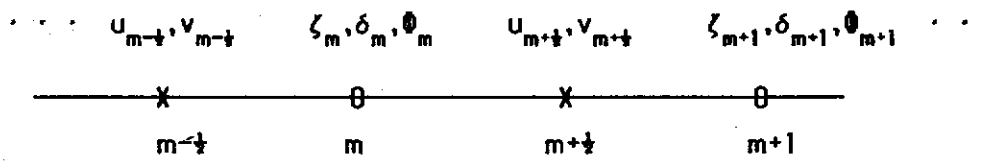
Here Ro is the Rossby number; R_β is a measure of the importance of the β-effect, determined by the scale of the motion; R_F is the reciprocal of the Froude number, and relates the length scale of the motion to the *Rossby radius of deformation*, $L_R = \sqrt{\bar{\Phi}}/f$. The equations of motion, (21), (22) and (23), may now be written in nondimensional form

$$\zeta_t + Ro(u\zeta)_x + \delta + R_\beta v = 0 \quad (25)$$

$$\delta_t + Ro(u\delta)_x - \zeta + R_\beta u' + \Phi_{xx} = 0 \tag{26}$$

$$\Phi_t + Ro(u\Phi)_x - Rou_0 v + R_F\delta = 0 \tag{27}$$

The relationship between the velocities (u,v) and the prognostic variables (ζ,δ) suggests that we specify them at alternate points of a grid staggered in space. The velocities are specified at "half-points" and the vorticity, divergence and geopotential at "whole-points":



Quantities not available directly are obtained by averaging. We define some finite difference operators in the usual way:

$$(q_m)_x = (q_{m+\frac{1}{2}} - q_{m-\frac{1}{2}})/\Delta x \quad ; \quad (\overline{q_m}) = \tfrac{1}{2}(q_{m-\frac{1}{2}} + q_{m+\frac{1}{2}})$$

A consistent (spatial) finite difference approximation to the equations may now be written as follows:

$$\frac{\partial}{\partial t}(v_m)_x + (u_m)_x + R_\beta(\overline{v_m}) = -Ro((u\overline{v_x})_m)_x \tag{28}$$

$$\frac{\partial}{\partial t}(u_m)_x - (v_m)_x + R_\beta(\overline{u'_m}) + \Phi_{xx} = -Ro((u\overline{u_x})_m)_x \tag{29}$$

$$\frac{\partial}{\partial t}(\Phi_m) - Rou_0(\overline{v_m}) + R_F(u_m)_x = -Ro((u\overline{\Phi})_m)_x \tag{30}$$

All dependent quantities are assumed to have period $(N\cdot\Delta x)$. The state of the system at any time is completely defined by the vector

$$X = (u_{\frac{1}{2}}, v_{\frac{1}{2}}, \Phi_1, \cdots, u_{m-\frac{1}{2}}, v_{m-\frac{1}{2}}, \Phi_m, \cdots, u_{N-\frac{1}{2}}, v_{N-\frac{1}{2}}, \Phi_N)$$

We assume in the following that the mean flow u_0 vanishes; this assumption is made to simplify the discussion and can easily be relaxed.

From the initial velocities and geopotential we assemble the vector

$$X_0 = (\zeta_1^0, \delta_1^0, \Phi_1^0, \cdots, \zeta_m^0, \delta_m^0, \Phi_m^0, \cdots, \zeta_N^0, \delta_N^0, \Phi_N^0)$$

where $\zeta_m^0 = (v_{m+\frac{1}{2}}^0 - u_{m-\frac{1}{2}}^0)/\Delta x$ and similarly for δ_m^0. The equations (28), (29) and (30) are now Laplace-transformed to give

$$s(\hat{v}_m)_x + (\hat{u}_m)_x + R_\beta(\hat{\overline{v_m}}) = \zeta_m^0 - L\{Ro((u\overline{v_x})_m)_x\} \tag{31}$$

$$s(\hat{u}_m)_x - (\hat{v}_m)_x + R_\beta(\hat{\overline{u^*_m}}) + \hat{\Phi}_{xx} = \delta_m^0 - L\{Ro((u\overline{u_x})_m)_x\} \tag{32}$$

$$s(\hat{\Phi}_m) + R_F(\hat{u}_m)_x = \Phi_m^0 - L\{Ro((u\overline{\Phi})_m)_x\} \tag{33}$$

(The Laplace transform of f is denoted by $\hat{f}$). To cast this system into matrix form let us define some 3x3-matrices as follows:

$$D = \begin{bmatrix} (-s/\Delta x + \frac{1}{2}R_\beta) & -1/\Delta x & 0 \\ 1/\Delta x & (-s/\Delta x + \frac{1}{2}R_\beta) & -2/\Delta x^2 \\ 0 & -R_F/\Delta x & s \end{bmatrix}$$

$$R = \begin{bmatrix} (s/\Delta x + \frac{1}{2}R_\beta) & 1/\Delta x & 0 \\ -1/\Delta x & (s/\Delta x + \frac{1}{2}R_\beta) & 1/\Delta x^2 \\ 0 & R_F/\Delta x & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\Delta x^2 \\ 0 & 0 & 0 \end{bmatrix}$$

We assemble these into the periodic, block tri-diagonal matrix of order 3N:

$$M = \begin{bmatrix} D & R & 0 & \cdots & L \\ L & D & R & \cdots & 0 \\ 0 & L & D & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ R & 0 & 0 & \cdots & D \end{bmatrix}$$

The system of equations (31) – (33) may now be written in the form

$$M\hat{X} = X_0 - \hat{N}(X) \tag{34}$$

Here the nonlinear terms have been collected into $\hat{N}$, which is a transformed nonlinear vector function of the state vector X. Equation (34) is formally identical to equation (6) in section 2·2 and the method of initialization described there can now be applied to the present model.

3·2 *Computational Results*

A number of model runs were carried out, with various parameter values and initial conditions. For all runs described here the nondimensional numbers had the following values:

$$Ro = 10^{-1} \quad ; \quad R_\beta = 1.6\times10^{-1} \quad ; \quad R_F = 10.$$

The channel length was $L = 10^7\,m.$, where $N_x\Delta x = L$ with $N_x = 20$ and $\Delta x = 500\,km.$ *One day* forecasts were of duration $N_T\Delta t = 10^5\,sec.$ (= 27.8 hours) where $N_T = 1000$ and $\Delta t = 100\,sec.$ An Adams-Bashforth timestepping scheme was used. For the given parameters the maximum Rossby wave frequency and minimum gravity wave frequency can be calculated, and have the (nondimensional) values:

$$|\,\nu_R\,|_{max} = 0.203 \quad ; \quad |\,\nu_G\,|_{min} = 2.224.$$

Any value of γ lying between these values should serve to separate the timescales. The value $\gamma = 1$ was chosen; the matrices M(s) were calculated for

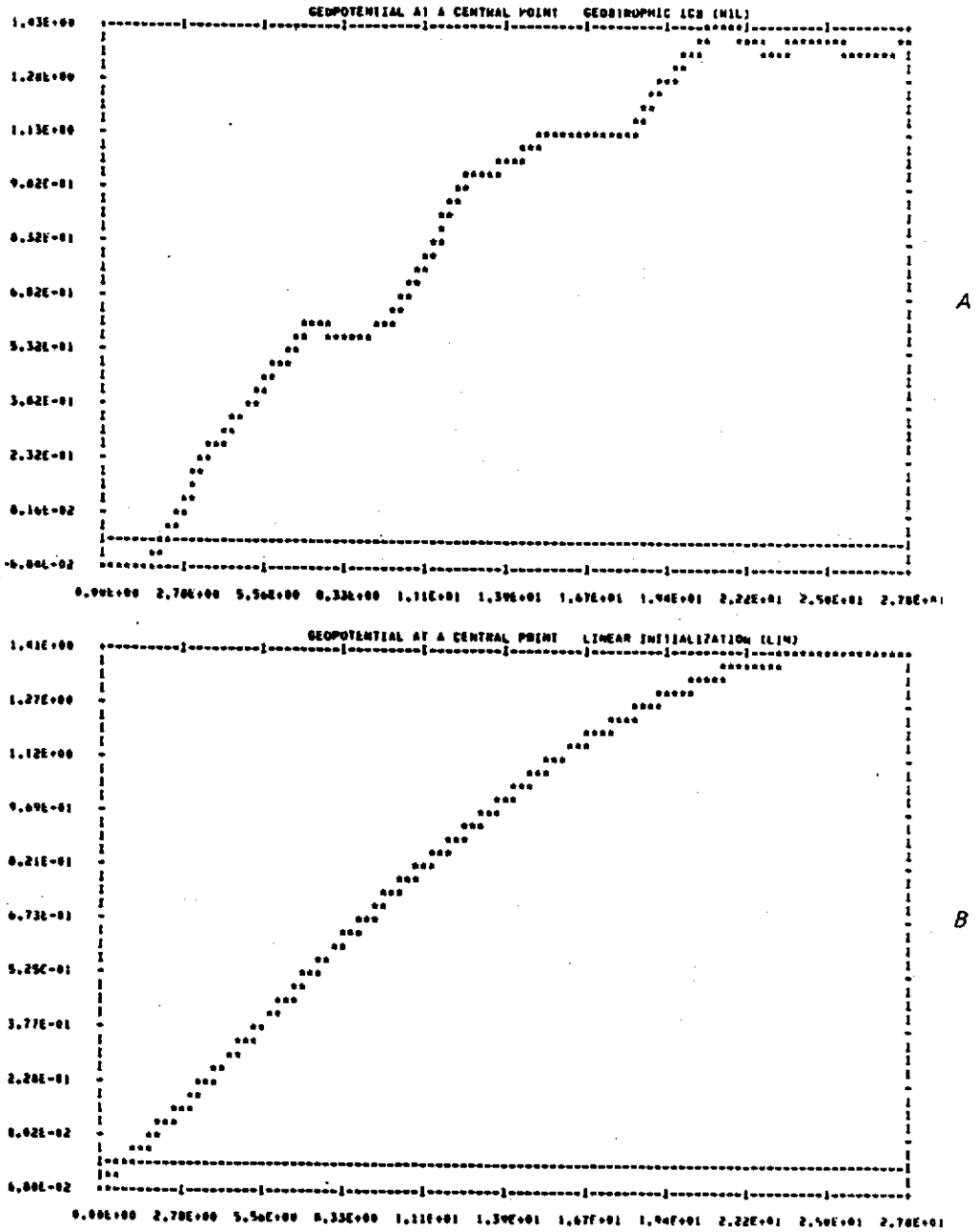


Figure 3.1: Evolution of the geopotential for a *one day* forecast starting from: (A) Geostrophic initial winds (NIL); (B) Linearly initialized fields (LIN).

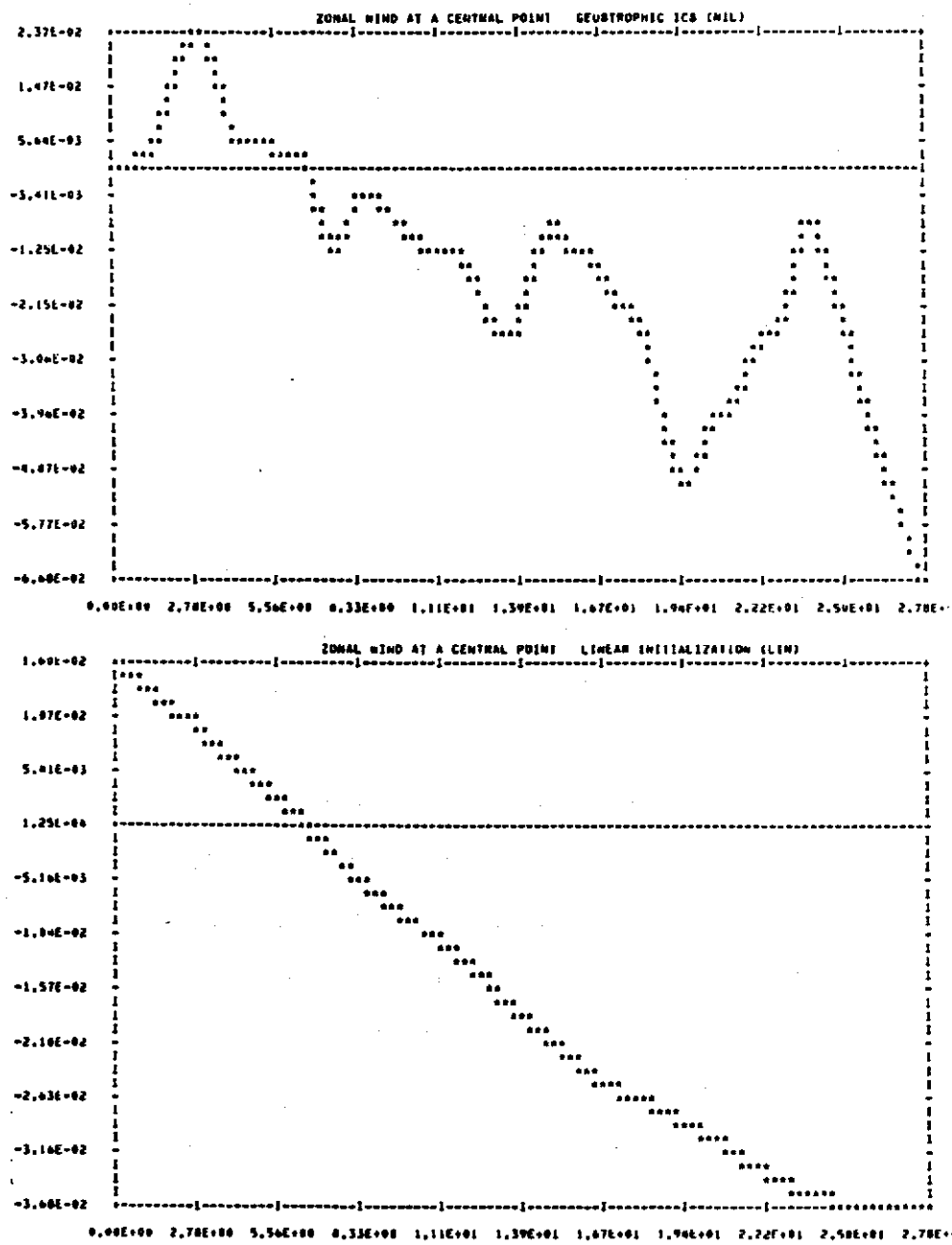


Figure 3.2: Evolution of the zonal wind for a *one day* forecast starting from: (A) Geostrophic initial winds (NIL); (B) Linearly initialized fields (LIN).

24 points evenly spaced around the unit circle.

The initial conditions were defined by setting

$$\Phi^0_m = \sum_{\ell=1}^{10} \cos\left[\ell\left(\frac{m\Delta x}{L}\right) + \phi_\ell\right]$$

where the phases ϕ_ℓ were chosen randomly, and deriving the geostrophic winds

$$v^0_{m-\frac{1}{2}} = (\Phi^0_m - \Phi^0_{m-1})/\Delta x \quad ; \quad u^0_{m-\frac{1}{2}} = 0.$$

We consider three different cases of initialization as follows:

NIL:	No Initialization	Geostrophic IC's as above
LIN:	Linear Initialization	IC's from (17)
NLI:	Nonlinear Initialization	IC's from (20), one iteration.

The final values of the v and Φ fields were very similar in all three cases. However, without initialization (NIL) the time evolution at a central point had small, high frequency fluctuations superimposed on the slow development. With linear initialization (LIN) these disappeared. The evolution of Φ is shown in Figure 3·1 for the two cases. The zonal wind field evolved noisily before, but smoothly after initialization (Figure 3·2). The divergence field behaved in a similar manner (not shown).

Since the gravity waves have relatively large divergence, we would expect the divergent kinetic energy to be a good indicator of their presence (Baer, 1977). In figure 3·3 the divergent energy, K_χ, as a function of time is shown for the case of geostrophic initial winds (NIL) and linear initialization (LIN). The oscillations in K_χ are extremely large before initialization; the linear initialization reduces them dramatically, but does not remove them completely. Figure 3·3(b) is reproduced in figure 3·4(a) with a much expanded vertical scale. The gravity oscillation remaining after linear initialization (LIN) are clear to see. In figure 3·4(b) we show the evolution of K_χ, on the same scale, after a single nonlinear iteration of the initialization (NLI). The gravity wave noise is

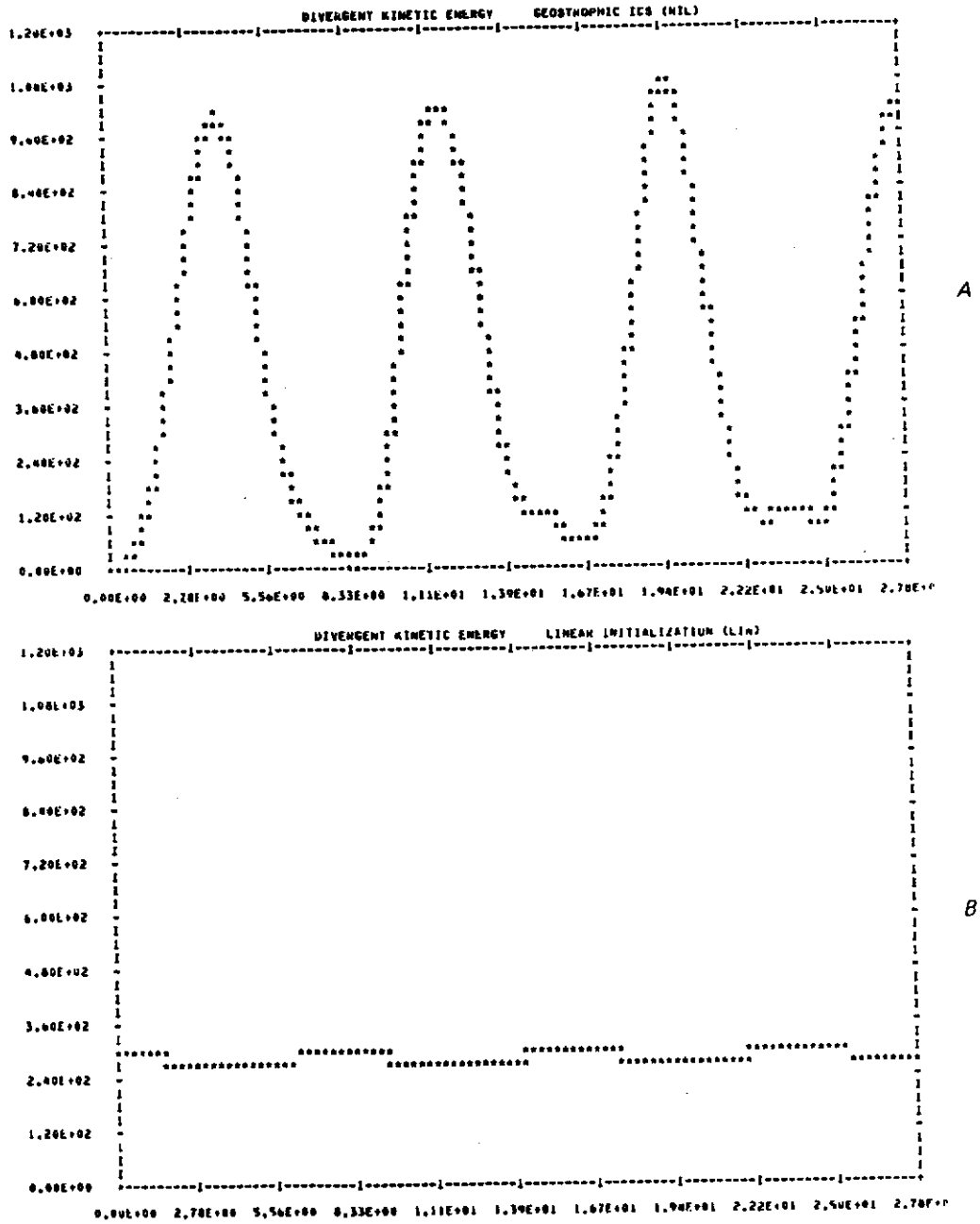


Figure 3.3: Plot of the divergent kinetic energy *versus* time starting from: (A) Geostrophic initial winds (NIL); (B) Linearly initialized fields (LIN).

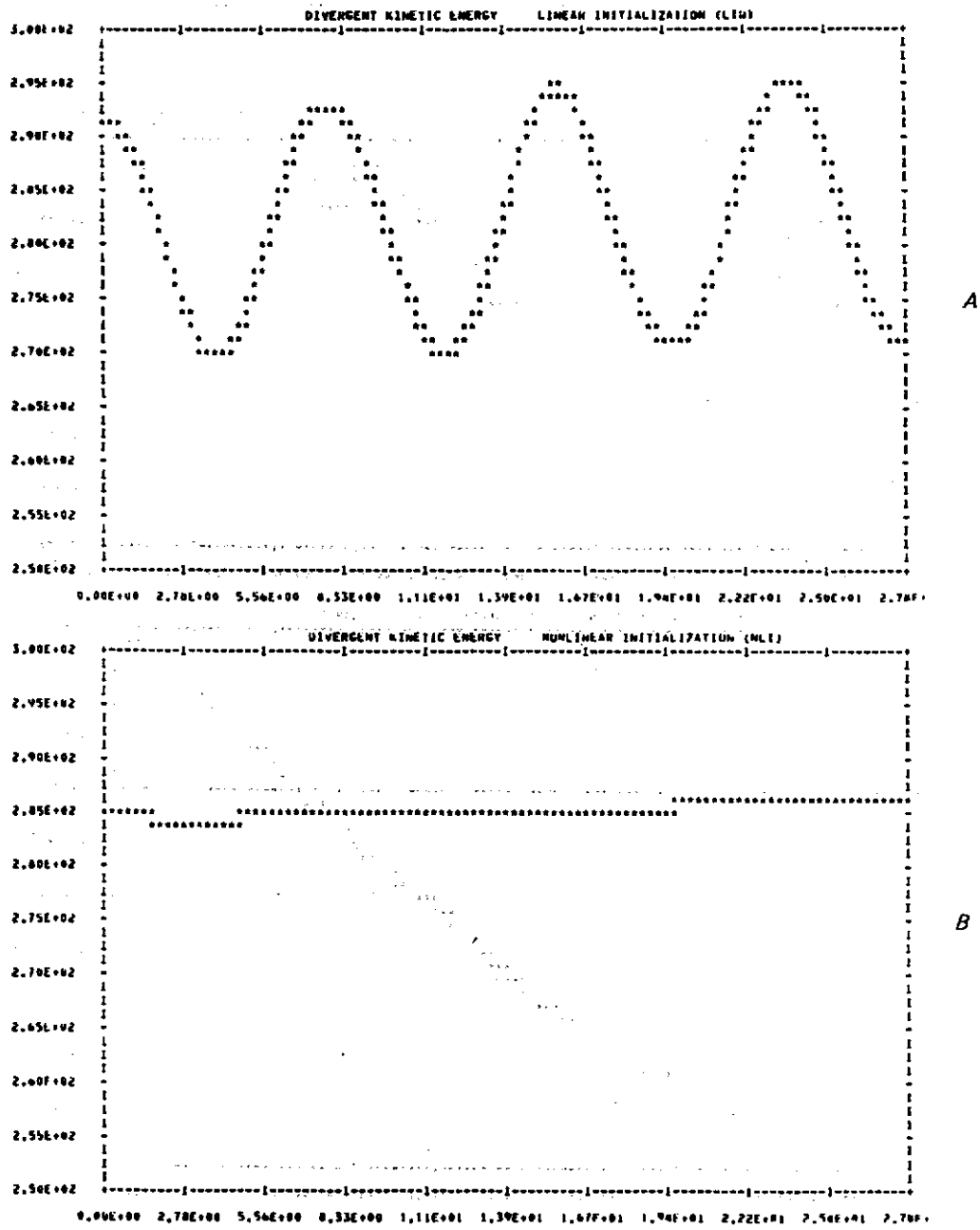


Figure 3.4: Plot of the divergent kinetic energy *versus* time starting from: (A) Linearly initialized fields (LIN); (B) Nonlinearly initialized fields, one nonlinear iteration (NLI).

almost completely eliminated. This result is in full agreement with the results of Beer (1977) for his nonlinear normal mode method, and demonstrates the effectiveness of the present method in dealing with the problem of high frequency oscillations.

Several other model runs confirmed the efficacy of the method in controlling the gravity wave noise. A number of *ten day* (10^6 sec.) forecasts showed that there is no tendency for the noise to return during this period.

In all cases the convergence was very rapid, with little change after the first nonlinear iteration. A small residue of noise is not removed by further iterations; presumably, this residue is associated with the approximation (15). Modification of (15) to allow $N(X_n^*)$ to vary linearly in time did not lead to any significant improvement. However, the residue is too small to cause any practical problems.

4. Principles of Application to a General Forecasting Model

The application of the Laplace transform technique to a one dimensional model was straightforward. For a general baroclinic model the state vector might contain, typically, about ten thousand elements. The inversion of the resulting gigantic matrices M(s) is an almost impossible task on present-day computers. Therefore, the problem must be formulated so as to produce matrices of manageable size.

Let us assume that initial fields are required for a limited area model with vertical coordinate $\sigma = p/p_s$, where p_s is the surface pressure. The vertical structure can be seperated out if we linearize about a motionless state with mean temperature $\bar{T} = \bar{T}(\sigma)$. To perform the separation we introduce a new dependent variable, $P = \Phi' + R\bar{T}\ln p_s$, where Φ' is the perturbation geopotential (see, *e.g.*, Kasahara and Puri, 1981). If the model has K levels, the vertical structure equation yields K eigenvalues, or equivalent depths, and K corresponding eigenfunctions, $\Phi_r(\sigma_k)$. For each equivalent depth, the horizontal structure is governed by a set of three equations for u, v and P which are of the form of the Laplace tidal equations.

For global and many limited area models the horizontal structure is separable into dependencies upon longitude λ and latitude ϕ. However, for a transformed longitude/latitude grid, whose poles are removed from the earth's axis, the Coriolis parameter is a function of both horizontal variables, x and y, and separation of variables no longer obtains. We consider a method of applying the Laplace transform initialization technique in this case.

The Laplace transform of the horizontal equations (for a given equivalent depth, D, may be written:

$$
\begin{aligned}
s\hat{u} - f\hat{v} + \hat{P}_x &= u_0 \\
s\hat{v} + f\hat{u} + \hat{P}_y &= v_0 \qquad (35) \\
\hat{u}_x + \hat{v}_y + s\epsilon\hat{P} &= P_0
\end{aligned}
$$

(where $\epsilon = (2\Omega a)^2/gD$). Let the domain be discretized into $I \times J$ points and derivatives approximated by finite differences in the usual way. For simplicity, we consider a non-staggered grid where all prognostic variables are specified at all gridpoints. The values of $\hat{u}$, $\hat{v}$ and $\hat{P}$ on a single row (excluding end-points) are assembled in a vector

$$\hat{X}_j = (\hat{u}_{2j}, \hat{v}_{2j}, \hat{P}_{2j}, \cdots, \hat{u}_{Ij}, \hat{v}_{Ij}, P_{Ij}, \cdots, \hat{u}_{I-1j}, \hat{v}_{I-1j}, \hat{P}_{I-1j})$$

where $\hat{u}_{ij} = \hat{u}(x_i, y_j)$, *etc.* The system (35) may now be written formally as

$$A_j\hat{X}_{j-1} + B_j\hat{X}_j + C_j\hat{X}_{j+1} = D_j \tag{36}$$

Where A_j, B_j, C_j and D_j are tridiagonal matrices and a column vector whose elements depend upon the values of the coefficients of the equation. The lateral boundary values u_{1j}, u_{Ij}, *etc.* occur in the vector D_j.

Lindzen and Kuo (1969) have described a reliable direct method for solving systems of the form (36). We introduce a set of intermediate matrices and vectors, α_j and β_j, such that

$$\hat{X}_j = \alpha_j\hat{X}_{j+1} + \beta_j \tag{37}$$

When (37) is used to eliminate $\hat{X}_{j-1}$ from (36) a solution for α_j and β_j is apparent in the form

$$\begin{aligned} \alpha_j &= -(M_j)^{-1}C_j \\ \beta_j &= -(M_j)^{-1}(A_j\beta_{j-1} - D_j) \end{aligned} \tag{38}$$

Where $M_j = (A_j\alpha_{j-1} + B_j)$. The boundary conditions for $j = 1$ are used to obtain α_1, β_1 and from (38) we then get α_j, β_j. The boundary conditions at $j = J$ give us $\hat{X}_J$ and the solution $\hat{X}_j$ is obtained from (37). The crucial point is that the matrices α_j are independent of the boundary values and of the forcing terms;

therefore, they can be calculated *once for all* and stored.

The first step in the initialization is now performed. The operator $\mathcal{L}^*$, with a suitably chosen value of τ, is applied to $\hat{X}_j$ and the new initial values are obtained by inverting the vertical separation of variables

$$(X^*(0))_{ijk} = \sum_{r=1}^{K} \mathcal{L}^*(\hat{X}_j)_i \Big|_{t=0} \cdot \Phi_r(\sigma_k)$$

For linear initialization these are the required starting values.

The nonlinear terms can be calculated using the new values. Usually, this is equivalent to performing a single model timestep. We then form the combination

$$[\ X^*(0) - \mathbf{N}(X^*(0))/s\ ]$$

and perform the vertical separation of variables again (assuming for simplicity that the eigenfunctions Φ_k are orthogonal; see Kasahara and Shigehisa, 1983). We can now calculate updated values for the vector $\mathbf{D}_j$ in (36). The system (36) is solved as before and the cycle repeated for the required number of iterations. Note that the matrices α_j do not change, so no further matrix inversion is required.

Since the operator $\mathcal{L}^*$ is applied separately for each vertical mode a different value of τ (the cutoff frequency) can be chosen in each case. Furthermore, not all the vertical modes need be initialized.

Although the above method seems to be computationally feasable, there may be heavy storage requirements for the matrices α_j They are of order $3(I-1)$ and there are J of them for each of K vertical modes. Thus, if S values of s are chosen on the inversion contour $\mathcal{C}^*$ the total storage is about $9I^2JKS$. Assuming $S \doteq I$ this is of the same order as the requirement to store the normal modes when the horizontal structure is nonseparable. Separability reduces the requirement by a factor I.

5. Summary

The new method of initialization developed in this report has been seen to be very effective in controlling the gravity wave noise in a one dimensional model. The methodology for applying the method to a more general model has been outlined. The method is closely related to the nonlinear normal mode method of Machenhauer (1977), but has the advantage that it does not require knowledge of the model linear normal modes. This allows it to be applied in cases where the horizontal structure of the linear equations is nonseparable.

The method allows for incorporation of orography and model physics in the same way as the normal mode method. The boundary conditions may be specified in a completely general way. The method is computationally feasable, but has heavy peripheral storage requirements in the nonseparable case (this is also true for the normal mode approach).

It is intended that the Laplace transform method be applied in the context of an operational baroclinic limited area model. The results of this application will be the subject of a future report.

REFERENCES

Baer, F., 1977: Adjustment of initial conditions required to supress gravity oscillations in nonlinear flows. *Beitr. Phys. Atmos.*, **50**, 350-366.

Baer, F. and J.J. Tribbia, 1977: On complete filtering of gravity modes through nonlinear initialization. *Mon. Weather Rev.*, **105**, 1536-1539.

Bellish, B., 1979: Comparison of some nonlinear initialization techniques. *Preprints of the fourth conference on numerical weather prediction*, Amercian Meteorological Society, Boston, pp9-12.

Bijlsma, S.J. and L.M. Hefkenscheid, 1983: A nonlinear normal mode initialization method for a limited area model. Royal Netherlands Meteorological Institute, De Bilt.

Briere, S., 1982: Nonlinear normal mode initialization of a limited area model. *Mon. Weather Rev.*, **110**, 1166-1186.

Browning, G., A. Kasahara and H.-O. Kreiss, 1980: Initialization of the primitive equations by the bounded derivative method. *J. Atmos. Sci.*, **37**, 1424-1436.

Charney, J., 1955: The use of the primitive equations of motion in numerical prediction. *Tellus*, **7**, 22-26.

Daley, R., 1981: Normal mode initialization. *Rev. Geophys. Space Phys.*, **19**, 450-468.

Dickinson, R. and D. Williamson, 1972: Free oscillations of a discrete stratified fluid with application to numerical weather prediction. *J. Atmos. Sci.*, **29**, 623-640.

Hinkelmann, K. von, 1951: Der Mechanismus des meteorologischen Ia"rmes. *Tellus*, **3**, 285-296.

Kasahara, A., 1982: Nonlinear normal mode initialization and the bounded derivative method. *Rev. Geophys. Space Phys.*, **20**, 385-397.

Kasahara, A. and K. Puri, 1981: Spectral representation of three dimensional data by expansion in normal mode functions. *Mon. Weather Rev.*, **109**, 37-51.

Kasahara, A. and Y. Shigehisa, 1983: Orthogonal vertical modes of a vertically staggered discretized atmospheric model. *Mon. Weather Rev.*, **111**, 1724-1735.

Kreiss, H.-O., 1979: Problems with different timescales for ordinary differential equations. SIAM *J. Numer. Anal.*, **16**, 980-998.

Kreiss, H.-O., 1979: Problems with different timescales for partial differential equations. *Comm. Pure Appl. Maths.*, **33**, 399-439.

Lindzen, R.S. and H.L. Kuo, 1969: A reliable method for the numerical integration of a large class of ordinary and partial differential equations. *Mon. Wea. Rev.*, **97**, 732-734.

Lynch, P., 1984: Dynamo - A one dimensional primitive equation model. Tech. Note No. 44, Irish Meteorological Service, Dublin.

Machenhauer, B., 1977: On the dynamics of gravity oscillations in a shallow water model, with applications to normal mode initialization. *Beitr. Phys. Atmos.*, **50**, 253-271.

Miyakoda, K. and R. Moyer, 1968: A method of initialization for dynamical weather forecasting. *Tellus*, **20**, 115-128.

Pedlosky, J., 1979: *Geophysical Fluid Dynamics.* Springer-Verlag, New York, Heidelberg, Berlin. 624pp.

Phillips, N.A., 1960: On the problem of initial data for the primitive equations. *Tellus*, **12**, 121-126.

Sasaki, Y.,1958: An objective analysis based on the variational method. *J. Meteor. Soc. Japan*, **36**, 77-88.

Temperton, C. and D. Williamson, 1979: Normal mode initialization for a multilevel gridpoint model. *Tech. Rep. No. 11*, European Centre for Medium range Weather Forecasts, Reading, England, 91pp.