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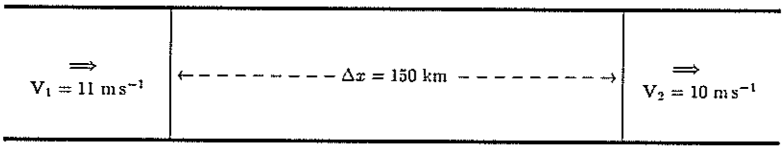
The Irish Meteorological Service

Historical Note No. 5

Max Margules and his Tendency Equation

by

Peter Lynch



- $p_0 = 10^5$ Pa
- $\Delta x = 1.5 \times 10^5$ m
- $\Delta V = V_2 - V_1 = -1\ \mathrm{m\,s^{-1}}$
- $\delta = \nabla \cdot V = \dfrac{\Delta V}{\Delta x} = -6.67 \times 10^{-6}\ \mathrm{s^{-1}}$
- $\dfrac{\partial p_0}{\partial t} = -\int_0^{p_0} \nabla \cdot V\, dp = -p_0 \nabla \cdot V = 0.667\ \mathrm{Pa\,s^{-1}}$

0.667 Pa s^{-1} = 144 hPa in 6 hours!

Including English and German versions of the paper

On the Relationship between Barometric Variations and the Continuity Equation

by Max Margules

Translated by Klara Finkele and Peter Lynch

October, 2001

Met Éireann, Glasnevin Hill, Dublin 9, Ireland

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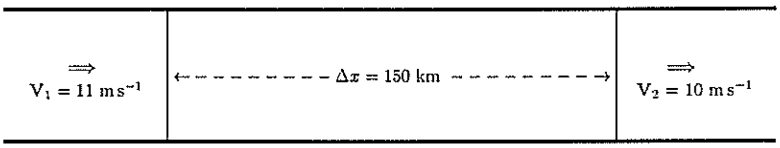

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from the original German version

Über die Beziehung zwischen Barometerschwankungen und Kontinuitätsgleichung

by Max Margules

Translated by Klara Finkele and Peter Lynch

(A PostScript version of this paper is available on request from Peter.Lynch@met.ie)

Met Éireann, Glasnevin Hill, Dublin 9, Ireland

October, 2001

Max Margules (1856-1920)

Photograph supplied by Christa Hammerl, from the archives of Zentralanstalt für Meteorologie und Geodynamik, Wien

Max Margules and his Tendency Equation

Peter Lynch

The paper appearing below in translation was published in the Boltzmann Festschrift in 1904. In this short paper, originally entitled *Über die Beziehung zwischen Barometerschwankungen und Kontinuitätsgleichung*, Margules examines the relationship between the continuity equation and changes in surface pressure. He concludes that any attempt to derive a reliable estimate of synoptic-scale changes in pressure, using the continuity equation alone, is doomed to failure.

1 Margules' Tendency Paper

Margules begins with the hydrostatic approximation that the pressure at a point is determined by the mass of air above that point (his Assumption (A)). Then the surface pressure is given by $p_0 = \int_0^\infty g\mu\, dz$, where μ is the density. He introduces vertically averaged velocities, which we may define as

$$\mathfrak{u} = \frac{1}{p_0}\int_0^{p_0} u\, dp\,; \qquad \mathfrak{v} = \frac{1}{p_0}\int_0^{p_0} v\, dp\,.$$

He then integrates the continuity equation from the earth's surface, assumed flat, to a height h. If h is assumed large, we arrive at his equation (2) which may be written

$$\frac{\partial p_0}{\partial t} + \nabla\cdot p_0\mathfrak{V} = 0\,, \tag{1}$$

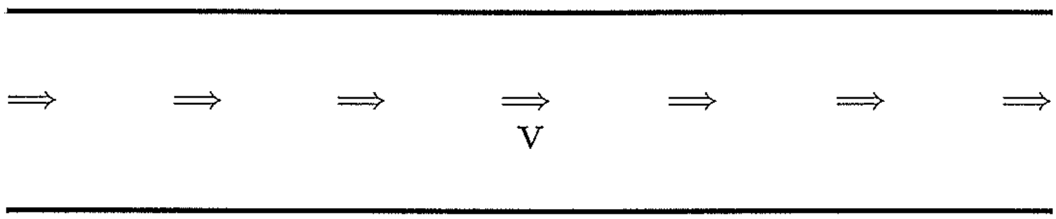
where $\mathfrak{V} = (\mathfrak{u}, \mathfrak{v})$ is the vertically averaged horizontal velocity vector. We note that this is equivalent to the more familiar form

$$\frac{\partial p_0}{\partial t} + \int_0^{p_0} \nabla\cdot\mathbf{V}\, dp = 0\,, \tag{2}$$

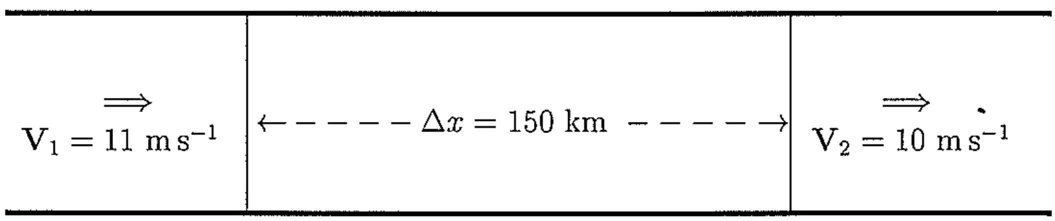
where $\mathbf{V}$ is the horizontal velocity. This is the familiar tendency equation, which we may justifiably call Margules' Equation (Wallace and Hobbs, 1977).

The tendency equation gives us a means of calculating changes in the surface pressure, given the vertical distribution of the wind velocity $\mathbf{V}$. The central result of Margules (1904) is that this procedure is fraught with difficulty. He cites, as motivation for his study, earlier publications of Exner and Trabert which addressed the problem of predicting pressure changes over a day with a view to forecasting the weather. He demonstrates that, if an accurate pressure tendency is to be obtained, the winds must be known to a precision quite beyond what is practically feasible. The ineluctable conclusion is that Margules' Equation alone does not provide a useful means of calculating synoptic pressure changes.

Margules considers various simple cases in which the tendency equation can be easily solved for changes in pressure. We give one example here, based on his special case (2_1). Let us consider uniform zonal flow in a channel



We assume that the velocity $\mathbf{V} = 10\,\mathrm{m\,s^{-1}}$ is constant with height and that the surface pressure is uniform, $p_0 = 10^5\,\mathrm{Pa}$. Clearly, the divergence vanishes identically, so the tendency equation implies that the pressure remains constant. Now let us assume that the velocity is measured at two points, one 150 km downstream from the other. If the observation at the first point overestimates the true value by 10%, we obtain $\mathbf{V}_1 = 11\,\mathrm{m\,s^{-1}}$ and $\mathbf{V}_2 = 10\,\mathrm{m\,s^{-1}}$:



Such a magnitude of the error is typical of realistic observations. We next estimate the divergence in the box between the two points by a finite difference ratio:

$$\nabla \cdot \mathbf{V} \approx \frac{\Delta \mathbf{V}}{\Delta x} = \frac{-1\,\mathrm{m\,s^{-1}}}{1.5 \times 10^5\,\mathrm{m}} = -6.67 \times 10^{-6}\ \mathrm{s^{-1}}$$

But now Margules' Equation (2) gives, for the tendency,

$$\frac{\partial p_0}{\partial t} = -p_0 \nabla \cdot \mathbf{V} = 0.667\,\mathrm{Pa\ s^{-1}}$$

so the pressure in the box should rise by two-thirds of a Pascal every second. Perhaps this seems a small value, but recall the song from *The Pajama Game*:

> Seven and a half cents doesn't buy a heck-of-a-lot,
> Seven and a half cents doesn't mean a thing.
> But give it to me every hour, forty hours of every week,
> That's enough for me to be livin' like a king.

If this tendency is sustained over a long period, the resulting pressure rise is dramatic:

$$\boxed{0.667\,\mathrm{Pa\ s^{-1}} = 144\ \mathrm{hPa\ in\ 6\ hours!}} \qquad (3)$$

The implication is clear: if we apply the tendency equation over a synoptic period, the resulting pressure change may be utterly unrealistic.

Astute readers will be reminded of Richardson's calculated pressure change, of 145 hPa in six hours, almost identical to the value in (3). Richardson (1922) used Margules' Tendency Equation, in the form

$$\frac{\partial p_0}{\partial t} = -g \int_0^\infty \nabla \cdot \mu \mathbf{V}\, dz = 0\,,$$

to obtain this value, so it is hardly surprising that his forecast was unrealistic.

There is no reason to believe that Richardson was aware of Margules' paper; certainly, he makes no reference to it in his book. Its contents were summarized by Exner in his textbook *Dynamische Meteorologie*, which Richardson does cite, but without explicit reference to the relevant section. Since this book was published in 1917, Richardson could not have seen it until his return to Britain after the First World War, and after his trial forecast had been completed. Richardson ascribed the difficulties with his predicted tendency to spurious values of divergence arising from errors in the wind observations. This explanation, while incomplete, is consistent with the analysis of Margules. Had Richardson been aware at an earlier stage of Margules' results, he might well have decided not to proceed with his trial forecast, or sought a radically different approach (Platzman, 1967).

On the final page of his short paper, Margules investigated pressure changes due solely to vertical motion. He considered a column of air extending from the surface to a fixed height h and assumed that all horizontal fluxes vanished. He found that a persistent downward velocity of $1\,\mathrm{cm\,s^{-1}}$ at 10 km — roughly, tropopause height — would cause a pressure drop of 1 mm Hg at that height, or about 8 hPa in six hours. Thus, small persistent vertical velocities can result in large pressure changes. In general, there are both horizontal and vertical fluxes and it is impossible to determine the vertical velocity from the continuity equation alone.

2 Max Margules (1856–1920)

Many outstanding scientists were active in meteorological studies in Austria in the period 1890–1925, and great progress was made in dynamic and synoptic meteorology and in climatology during this time. Amongst the most important members of this 'Vienna School' were Julius Hann, Josef Pernter, Wilhelm Trabert, Felix Exner, Wilhelm Schmidt, Heinrich Ficker, Albert Defant and, of course, Max Margules. The Austrian Central Institute for Meteorology and Geodynamics (ZAMG) recently celebrated its 150th anniversary, in conjunction with which a beautiful book has been produced (Hammerl, *et al.*, 2001) containing contributions on the work of the Vienna School and on the many scientists who worked there (see Davies, 2001; Fortak, 2001; Pichler, 2001).

Margules, one of the founders of dynamical meteorology, was unquestionably a brilliant theoretician, the true value of whose work was adequately appreciated only after his death. The present biographical sketch is based on Khrgian (1959), Kutzbach (1979) and Gold (1920), and on several articles in Hammerl (2001). Margules was born in the town of Brody, in western Ukraine, in 1856. He studied mathematics and physics at Vienna University, and among his teachers was Ludwig Boltzmann.[1] After a two-year spell as a Volunteer at the Meteorological Institute in Vienna, Margules went to Berlin University in 1879. He returned to Vienna University the following year as a lecturer in physics. In 1882 he rejoined the Meteorological Institute as an Assistant, and continued to work there for 24 years.

[1] Indeed, the paper appearing below was first published in the Festschrift on the occasion of Boltzmann's 60th birthday.

Margules studied the diurnal and semi-diurnal variations in atmospheric pressure due to solar radiative forcing, analyzing the Laplace tidal equations and deriving two species of solutions, which he called 'Wellen erster Art' and 'Wellen zweiter Art' (Margules, 1893). This was the first identification of the distinct types of waves now known as inertia-gravity waves and rotational waves. This work, soon followed by Hough's closely related but independent study (Hough, 1898), foreshadowed the studies of atmospheric planetary waves some thirty years later and its full significance was appreciated only after the insights of Rossby (1939) and Haurwitz (1940).

Margules turned next to the study of the source of energy of storms. He demonstrated that the available potential energy associated with horizontal temperature contrasts within a cyclone was, if converted to kinetic energy, sufficient to explain the observed winds. In the course of this work, he derived an expression for the slope of inclination of the boundary between two air masses, a formula which bears his name and is occasionally found in modern textbooks. On the basis of observations carried out at Vienna and Bratislava on 3 December, 1899, Margules showed that surfaces of separation between distinct air masses actually exist in the atmosphere (Kutzbach, 1979). This work overturned the convective theory of cyclones and adumbrated the frontal theory which emerged about a decade later.

Margules was an introverted and lonely man, who never married and worked in isolation, not collaborating with other scientists. His published work was often abstruse and inaccessible and as a result was undervalued. It is fair to say that he was far ahead of his time. Felix Exner, a friend and colleague was one of the few who understood and appreciated Margules research. Margules was disappointed and disillusioned at the lack of recognition of his work and retired from the Meteorological Institute in 1906, aged only fifty, on a modest pension. After retirement he turned his back on meteorology and spent his energy exclusively on chemical studies. His last meteorological publication (Margules, 1906) opened with the surprisingly personal remark "Circumstances, on which I cannot elaborate here, compel me to write down this paper and supplement in haste, and to bid farewell to meteorology" (Fortak, 2001).

In a moving appreciation of Margules shortly after his death, Exner (1920) wrote that Austria had lost an outstanding scientist and that, taking a broad perspective, Margules could be accurately described as one of the first ever theoretical meteorologists. His altogether tragic fate greatly saddened Exner, who described him as one of the loneliest men he had ever known. Gold wrote in an obituary (1920) that "meteorology lost him some fifteen years ago, and is forever the poorer for a loss which one feels might and ought to have been prevented". However, Exner and other colleagues had tried their best to prevent this loss, making repeated offers of help, which Margules resolutely resisted. The value of his pension was severely eroded during the First World War so that his 400 crowns per month was worth about one Euro, insufficient for more than the most meagre survival. He was awarded the Hann Medal by the Austrian Meteorological Society in 1919 but declined the honorarium which accompanied it, with the stoical comment "I would accept the offer if it could be of help but, as things stand, the Institute could put the money to better use" (Hammerl, *et al.*, 2001, page 134). Margules died of starvation just one year later.

In 1987 the computer building of ZAMG was completely renovated and named Max-Margules-Haus. This is most appropriate: Margules' 1904 paper played an important rôle in helping us to understand the problems of numerically integrating the primitive equations used in numerical weather prediction today. In his day, Margules considered that any attempt to predict the evolution of atmospheric flow, that is, to forecast the weather, was premature and indeed futile. He was quoted by Exner as saying that forecasting was "immoral and damaging to the character of a meteorologist". The paper appearing in translation here gives some insight into why Margules held this view.

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Acknowledgement

Many thanks to Michael Goesch of Deutscher Wetterdienst for providing a copy of Margules' original paper from the Boltzmann Festschrift. Klara Finkele, Met Éireann, undertook the bulk of the work in translating the German text. The photograph of Margules was kindly supplied by Christa Hammerl, ZAMG.

On the Relationship between Barometric Variations and the Continuity Equation

Max Margules
Vienna.

Translated by Klara Finkele and Peter Lynch

We denote by p_0 and p_h the pressure at two points in a vertical line, at the surface and at height h. We assume that the pressure difference, whether the air is in motion or stationary, is given by

$$\text{(A)} \qquad \mathfrak{p} = p_0 - p_h = \int_0^h g\mu \, dz \,,$$

where μ is density at height z. Because of the short distance, we assume that the acceleration due to gravity g is constant (Assumption B) and also that the earth's surface is an x–y plane (Assumption C).

The continuity equation is written

$$\frac{\partial \mu}{\partial t} + \frac{\partial(\mu u)}{\partial x} + \frac{\partial(\mu v)}{\partial y} + \frac{\partial(\mu w)}{\partial z} = 0$$

(where u, v are the horizontal velocity components, w the vertical velocity component and t the time). Multiplying by $g\,dz$ and vertical integration from the surface to height h, and introducing the quantities $\mathfrak{u}, \mathfrak{v}$ defined by

$$\mathfrak{p}\mathfrak{u} = \int_0^h g\mu u \, dz \,, \qquad \mathfrak{p}\mathfrak{v} = \int_0^h g\mu v \, dz \,,$$

we obtain the equation for the temporal variation of $\mathfrak{p}$:

$$\text{(1)} \qquad \frac{\partial \mathfrak{p}}{\partial t} + \frac{\partial(\mathfrak{p}\mathfrak{u})}{\partial x} + \frac{\partial(\mathfrak{p}\mathfrak{v})}{\partial y} + g\mu_h w_h = 0 \,.$$

This holds for all h. For very large height h we introduce the assumptions

$$\text{(D)} \quad p_h = 0\,, \qquad \text{(E)} \quad \mu_h w_h = 0\,.$$

Then $\mathfrak{p} = p_0$ is the pressure at the surface, and $\mathfrak{u}, \mathfrak{v}$ are the average horizontal wind components in a column of unit area at location (x, y) at time t. The average is formed by weighting (u, v) at every level in the column with the density at that level. It follows that

$$\frac{\partial p_0}{\partial t} + \frac{\partial (p_0 \mathfrak{u})}{\partial x} + \frac{\partial (p_0 \mathfrak{v})}{\partial y} = 0\,, \tag{2}$$

an equation of the same form as the continuity equation for horizontal motion. It can also be written as

$$\frac{\partial p_0}{\partial t} = -\frac{1}{\delta n} \frac{\partial (p_0 \cdot \mathfrak{c}\delta n)}{\partial s}\,. \tag{2*}$$

where

$\mathfrak{c}$ is the resultant of $\mathfrak{u}$ and $\mathfrak{v}$ [actually, the magnitude of $(\mathfrak{u}, \mathfrak{v})$]

s is the curve whose tangent is, at every point, in the direction of $\mathfrak{c}$ [actually, the direction of $(\mathfrak{u}, \mathfrak{v})$] at time t

δn is the [infinitesimal] normal distance between s and a specific neighboring curve s' of the same type.

The tendency of p_0 depends on the spatial difference of $\mathfrak{c}$, δn and p_0 along the curve s. The influence of each individual factor is as follows:

p_0 and δn spatially constant on s: $\quad (2_1) \quad \dfrac{\partial p_0}{\partial t} = -p_0 \dfrac{\partial \mathfrak{c}}{\partial s}$

p_0 and $\mathfrak{c}$ spatially constant on s: $\quad (2_2) \quad \dfrac{\partial p_0}{\partial t} = -\dfrac{p_0 \mathfrak{c}}{\delta n} \cdot \dfrac{\partial \delta n}{\partial s}$

$\mathfrak{c}$ and δn spatially constant on s: $\quad (2_3) \quad \dfrac{\partial p_0}{\partial t} = -\mathfrak{c} \dfrac{\partial p_0}{\partial s}\,.$

Following the assumption (2_1), the s-lines are parallel and coincide locally with the surface isobars, or they are in an area of uniform pressure.

To determine the value of $\partial \mathfrak{c}/\partial s$ which occurs under typical changes of pressure, we postulate that the barometer rises by 1 mm in one hour and fix $p_0 = 760\,\text{mm Hg}$; then

$$\frac{\partial \mathfrak{c}}{\partial s}\left[= -\frac{1}{p_0}\frac{\partial p_0}{\partial t}\right] = -\frac{1}{760}\cdot\frac{1}{3600\,\text{s}} = -3.65\times 10^{-7}\,\text{s}^{-1}\,,$$

and with $ds = 10^5\,\text{m}$ we get $d\mathfrak{c} \approx -0.04\,\text{m}\,\text{s}^{-1}$. If the resulting velocity at a point on the s-line is greater by 0.04 m/s than at a point 100 km downstream on the same s-line and the gradient is constant, then the barometric pressure increases along the whole distance by 1 mm per hour.

(At the same time, the velocity of the wind can vary from 0 to $40\,\text{m}\,\text{s}^{-1}$, from different directions, at different heights. How accurate must the wind conditions be known if the continuity equation is used to determine if the barometer is going to rise or fall.) [Margules poses a question here—without a question-mark—but provides no answer.]

Eqn. (2_2) determines the pressure tendency which is caused by the divergence of s-lines at constant $\mathfrak{c}$ in an area of uniform pressure. It can be written as

$$\frac{1}{p_0}\frac{\partial p_0}{\partial t}\left[= -\mathfrak{c}\left(\frac{1}{\delta n}\frac{\partial \delta n}{\partial s}\right)\right] = -\mathfrak{c}\frac{\partial \alpha}{\partial n}\,,$$

were α denotes the angle between a given direction on the plane and the tangent on s. Using the above equation and setting $\mathfrak{c} = 1\,\text{m}\,\text{s}^{-1}$ and $dn = 1\,\text{km}$, it follows that $d\alpha = -1.26'$ [minutes of arc corresponding to $d\alpha = -3.65\times 10^{-4}$ radians]

Eqn. (2_3) holds for parallel s-lines and with $\mathfrak{c}$ constant along every s-line. The resulting velocity can be a function of the parameters of the family of s-lines [*i.e.*, can vary from one s-line to another]. A pressure change occurs where the direction of $\mathfrak{c}$ is different from that of the isobar at the surface; because of the difference in p_0, the incoming and outgoing air masses in δn, $\delta n'$ [*i.e.*, normals between s and s' at two different points] are different. If $\mathfrak{c}$ is constant in time, then (2_3) has the general integral

$$p_0 = f(s - \mathfrak{c}t)$$

and if, in addition, $\mathfrak{c}$ assumes the same value in the entire region, there is a parallel displacement of the isobaric system in the s-direction with velocity $\mathfrak{c}$. Similar displacements occur which do not necessarily require that $\mathfrak{c}$ be constant

The pressure tendency at the surface is completely determined if p_0, $\mathfrak{u}$ and $\mathfrak{v}$ are known as functions of position; however, $\mathfrak{u}$ and $\mathfrak{v}$ are not uniquely determined by p_0 and $\partial p_0/\partial t$. For every family of [curves] s_j, one can select a $\mathfrak{c}_j$ so that along every curve [the product] $p_0 \cdot \mathfrak{c}_j \cdot \delta n_j$ is contant; this component gives no contribution to $\partial p_0/\partial t$ [see (2*)]. If the continuity equation alone is used, then all non-contributing components of $\mathfrak{c}$ to the pressure tendency can be neglected. [For example,] the expression $(\bar{p}/p_0)\mathfrak{c}$ is such a component, where $\bar{p}$ denotes constant pressure (say, standard pressure), corresponding to parallel displacement with constant $\mathfrak{c}$. The same displacement of the isobars as obtained above with (2_3) is also obtained from (2*) with parallel s [curves], with the resulting velocity

$$\mathfrak{c}' = \mathfrak{c}\left(1 - \frac{\bar{p}}{p_0}\right) .$$

The velocity distribution is now similar to that of a travelling wave: motion opposite to the propagation direction at locations of low pressure, and motion in the same direction as propagation at locations of high pressure; and $\mathfrak{c}'$ is small compared to the propagation velocity $\mathfrak{c}$. The same displacement of the isobar system can also be produced in infinitely many different ways.

It is not expected that the continuity equation by itself can lead far. The motivation for collecting these considerations was given by two publications[1] which attempted to find a connection between this equation and certain hypotheses so that one could predict the pressure fluctuations or the weather during a day. This depends very much on the hypotheses, which are not discussed here.

Of the assumptions introduced above, (B) and (C) are for convenience only, and they could be disregarded. Assumption (A) has the following implication: It is very probable[2] that the static pressure difference does not deviate more than 1 mm Hg from the true value $p_0 - p_h$, even at the largest height difference; at least, not persisting for a [full] day. The changes of p_0 or $\mathfrak{p}$ often reach 10 to 20 mm Hg at the same time. Considering large [scale] changes, (A) can be assumed to be reasonably accurate. Assumptions (D) and (E) can be omitted if one confines attention to Equation (1). In this case, however, a large portion of the changes in $\mathfrak{p}$ are caused by inflow and outflow of air at the upper surface.

[1]Felix M. Exner, Sitzungsber. d. k. Akad. d. Wissensch. zu Wien. **111**. p. 707. 1902; W. Trabert, Meteorolog. Zeitschr. **38**, p. 231. 1903.

[2]A. Sprung, Lehrb. d. Meteorologie, p. 160. Hamburg, 1885.

Assuming that the change is caused by vertical motion only, then

$$\mu_h w_h = -\frac{1}{g}\frac{\partial \mathfrak{p}}{\partial t},$$

and a rise of $\mathfrak{p}$ by 1 mm Hg/hour corresponds to[3]

$$\left[-\frac{1}{g}\frac{\partial \mathfrak{p}}{\partial t} =\right] \qquad -\frac{10\,333}{760}\cdot\frac{1}{3600} = -0.00378\ \mathrm{kg\,m^{-2}\,s^{-1}}\,.$$

At the heights of 10, 20 [and] 30 km, the density is

$$0.42\,,\quad 0.089\,,\quad 0.0067\ \mathrm{kg\,m^{-3}}\,;$$

then w_h has the values

$$-0.009\,,\quad -0.042\,,\quad -0.56\ \mathrm{m\,s^{-1}}\,,$$

which are small downward velocities.

Large changes in $\mathfrak{p}$ are caused by persistent small differences in horizontal air inflow or outflow, and also by small values of the vertical component of velocity. The effect of these two [factors] can be cancelled in different ways to produce an unchanged $\mathfrak{p}$; from the continuity equation alone, it is not possible to determine if rising or sinking air motion will occur.

In equation (1) it is assumed that steady distributed sources and sinks occur only at the upper boundary surface. If air (water vapour) leaves the lower surface or is absorbed, then the appropriate term has to be added. If condensation of vapour takes place, then there are sinks within the air mass too, and a term must be added to the left hand side of equation (1) to account for the weight of condensed mass in the unit column between 0 and h per unit time.

(Submitted 26 September, 1903)

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[3][Margules does not explain the origin of the factor 10 333 here. We recall that $\frac{4}{3}\times 760 = 1013.33$ is an excellent approximation to the pressure in hectoPascals corresponding to 760 mm Hg. Taking the acceleration of gravity to be $g = 9.8066\,\mathrm{m\,s^{-2}}$, we find that $100\times 1013.33/g = 10\,333\,\mathrm{kg\,m^{-2}}$.]

74. Über die Beziehung zwischen Barometerschwankungen und Kontinuitätsgleichung.

Von **Max Margules** in Wien.

Wir bezeichnen mit $p_0\,p_h$ den Luftdruck in zwei Punkten der Vertikalen am Boden und in der Höhe h, nehmen an, daß die Differenz in bewegter gleichwie in ruhender Luft gegeben ist durch

$$\text{(A)} \qquad \mathfrak{p} = p_0 - p_h = \int_0^h g\,\mu\,dz,$$

worin μ die Dichte in der Höhe z bedeutet. Der Kürze wegen setzen wir die Schwerebeschleunigung g konstant (Annahme B) und betrachten den Boden als Ebene xy (Annahme C).

Aus der Kontinuitätsgleichung

$$\frac{\partial \mu}{\partial t} + \frac{\partial (\mu u)}{\partial x} + \frac{\partial (\mu v)}{\partial y} + \frac{\partial (\mu w)}{\partial z} = 0$$

(u, v horizontale Geschwindigkeitskomponenten, w vertikale, t Zeit) erhält man, wenn man den Faktor $g\,dz$ beifügt und über das Höhenintervall o bis h integriert; ferner $\mathfrak{u}$, $\mathfrak{v}$ einführt, durch die Definitionsgleichungen

$$\mathfrak{p}\,\mathfrak{u} = \int_0^h g\,\mu\,u\,dz, \qquad \mathfrak{p}\,\mathfrak{v} = \int_0^h g\,\mu\,v\,dz$$

die Gleichung für die zeitliche Änderung von $\mathfrak{p}$

$$\text{(1)} \qquad \frac{\partial \mathfrak{p}}{\partial t} + \frac{\partial (\mathfrak{p}\,\mathfrak{u})}{\partial x} + \frac{\partial (\mathfrak{p}\,\mathfrak{v})}{\partial y} + g\,\mu_h\,w_h = 0.$$

Dies gilt allgemein für jedes h. Für eine sehr große Höhe wollen wir die Annahmen einführen

$$\text{(D)}\ \ p_h = 0, \qquad \text{(E)}\ \ \mu_h\,w_h = 0.$$

Dann ist $\mathfrak{p} = p_0$ der Druck am Boden, $\mathfrak{u}$, $\mathfrak{v}$ sind die mittleren horizontalen Geschwindigkeitskomponenten in der

Einheitssäule im Ort xy zur Zeit t. Das Mittel ist derart gebildet, daß die u, v jeder Schicht der Säule mit dem Gewicht derselben Schicht genommen sind. Es folgt

$$\frac{\partial p_0}{\partial t} + \frac{\partial (p_0 \mathfrak{u})}{\partial x} + \frac{\partial (p_0 \mathfrak{v})}{\partial y} = 0, \tag{2}$$

eine Gleichung von derselben Form wie die Kontinuitätsgleichung der ebenen Bewegung. Man kann dafür auch schreiben, wenn

$\mathfrak{c}$ die Resultante von $\mathfrak{u}$, $\mathfrak{v}$ ist,

s eine Kurve, deren Tangente im Sinn des wachsenden Bogens in jedem Punkte die Richtung des $\mathfrak{c}$ zur Zeit t angibt,

δn das Normalenstück zwischen s und einer bestimmten Nachbarkurve s' derselben Art.

$$\frac{\partial p_0}{\partial t} = -\frac{1}{\delta n} \frac{\partial (p_0 \cdot \mathfrak{c}\, \delta n)}{\partial s}. \tag{2*}$$

Die zeitliche Änderung des p_0 hängt ab von den örtlichen Unterschieden der $\mathfrak{c}$, δn, p_0 längs der Linie s. Den Einfluß jedes einzelnen Faktors kann man leicht angeben:

Wenn längs s örtlich konstant sind

p_0 und δn $\quad (2_1)\ \dfrac{\partial p_0}{\partial t} = -p_0 \dfrac{\partial \mathfrak{c}}{\partial s}$

p_0 und $\mathfrak{c}$ $\quad (2_2)\ \dfrac{\partial p_0}{\partial t} = -\dfrac{p_0 \mathfrak{c}}{\delta n} \cdot \dfrac{\partial \delta n}{\partial s}$

$\mathfrak{c}$ und δn $\quad (2_3)\ \dfrac{\partial p_0}{\partial t} = -\mathfrak{c} \dfrac{\partial p_0}{\partial s}$.

Nach den Voraussetzungen von (2_1) sind die s-Linien parallel und fallen in der Nähe des Beobachtungsortes mit den Isobaren am Boden zusammen, oder sie liegen in einem Gebiet gleichen Druckes. Um den Wert $\partial \mathfrak{c}/\partial s$ zu bestimmen, der bei Druckänderungen nicht zu seltener Art eintritt, postulieren wir, daß das Barometer um 1 mm in der Stunde steigt und setzen $p_0 = 760$ mm Hg; dann ist

$$\frac{\partial \mathfrak{c}}{\partial s} = -\frac{1}{760} \cdot \frac{1}{3600\,\text{sec}} = -10^{-7} . 3{,}65\,(\text{sec}^{-1})$$

$ds = 10^5$ m gesetzt gibt $d\mathfrak{c} = -0{,}04\ \text{m} . \text{sec}^{-1}$. Wenn die resultierende Geschwindigkeit in einem Punkte der s-Linie beständig um 0,04 m/sec größer ist, als in einem 100 km stromabwärts entfernten Punkt derselben Linie und der Abfall gleichmäßig, steigt das Barometer auf der ganzen Strecke um 1 mm in der Stunde.

(In derselben Zeit kann die Geschwindigkeit des Windes von verschiedener Richtung, in verschiedenen Höhen 0 bis 40 m/sec betragen. Wie genau müßte man den Zustand kennen, um aus der Kontinuitätsgleichung anzugeben, ob in der nächsten Stunde das Barometer steigen oder fallen wird.)

(2_2) gibt die zeitliche Druckänderung an, die durch Divergenz der s-Linien bei konstantem $\mathfrak{c}$ in einem Gebiete gleichen Druckes eintritt. Man kann dafür auch schreiben

$$\frac{1}{p_0}\frac{\partial p_0}{\partial t} = -\mathfrak{c}\frac{\partial \alpha}{\partial n},$$

wenn α den Winkel zwischen einer festen Richtung in der Ebene und der Tangente an s bezeichnet. Mit dem Postulat wie oben und mit $\mathfrak{c} = 1$ m/sec erhält man für $dn = 1$ km, $d\alpha = -1{,}26$ Minuten.

(2_3) gilt für parallele s-Linien und längs jeder s konstantes $\mathfrak{c}$. Dabei kann die resultierende Geschwindigkeit eine Funktion des Parameters der s-Schar sein. Druckänderung tritt ein, wo die Richtung des $\mathfrak{c}$ von der Isobare am Boden abweicht; wegen der Unterschiede von p_0 sind die bei δn, $\delta n'$ ein- und austretenden Luftmassen verschieden. Wenn $\mathfrak{c}$ zeitlich konstant ist, hat (2_3) das allgemeine Integral

$$p_0 = f(s - \mathfrak{c}t)$$

und wenn noch $\mathfrak{c}$ im ganzen Gebiet den gleichen Wert hat, bedeutet das eine Parallelverschiebung des Isobarensystems in der s-Richtung mit der Geschwindigkeit $\mathfrak{c}$. Ähnliche Verschiebungen kommen vor; sie müssen nicht notwendig durch konstantes $\mathfrak{c}$ entstehen.

Die zeitliche Druckänderung am Boden ist vollständig bestimmt, wenn man p_0, $\mathfrak{u}$, $\mathfrak{v}$ als Funktionen des Ortes kennt; $\mathfrak{u}$, $\mathfrak{v}$ sind aber aus p_0 und $\partial p_0/\partial t$ nicht eindeutig abzuleiten. Zu einer s_j-Schar läßt sich das zugehörige $\mathfrak{c}_j$ so wählen, daß längs jeder Kurve $p_0 \cdot \mathfrak{c}_j \cdot \delta n_j$ konstant ist; das gibt keinen Beitrag zu $\partial p_0/\partial t$. Man darf, soweit die Kontinuitätsgleichung allein gebraucht wird, alle für die zeitliche Druckänderung unwirksamen Teile von $\mathfrak{c}$ weglassen. Ein solcher ist bei der Parallelverschiebung durch konstantes $\mathfrak{c}$ der Ausdruck $(\bar{p}/p_0)\mathfrak{c}$, wenn $\bar{p}$ einen konstanten Druck bezeichnet, sagen wir den normalen. Dieselbe Ortsveränderung der Isobaren wie zuvor mit (2_3) erhält man auch bei parallelen s aus (2^*) mit der resultierenden Geschwindigkeit

$$\mathfrak{c}' = \mathfrak{c}\left(1 - \frac{\bar{p}}{p_0}\right).$$

Die Geschwindigkeitsverteilung ist jetzt ähnlich der in einer fortschreitenden Welle, Bewegung gegen die Fortpflanzungsrichtung in den Orten niedrigen Druckes mit der Fortpflanzung in jenen hohen Druckes und $\mathfrak{c}'$ klein im Vergleich mit der Geschwindigkeit des Fortschreitens $\mathfrak{c}$. — Dieselbe Verschiebung des Isobarensystems kann noch auf unendlich viele andere Arten entstehen.

Man erwartet nicht, daß die Kontinuitätsgleichung allein weit führt. Den Anlaß, diese Erwägungen zusammenzustellen, geben zwei Publikationen [1]), in denen der Versuch gemacht wird, aus jener Gleichung in Verbindung mit gewissen Hypothesen die in einem Tage stattfindende Druckänderung bzw. das im Laufe des Tages eintretende Wetter vorauszusehen. Dabei kommt es sehr auf die Hypothesen an, die hier nicht diskutiert werden.

Von den Annahmen, die oben eingeführt wurden, dienen (B, C) nur zur Bequemlichkeit und sind entbehrlich. (A) ist so gemeint: Es ist sehr wahrscheinlich [2]), daß die statische Druckdifferenz von dem wahren Wert $p_0 - p_h$ um nicht mehr als 1 mm Hg abweicht, auch bei dem größten Höhenunterschied; mindestens nicht andauernd während eines Tages. Die Änderungen von p_0 oder $\mathfrak{p}$ erreichen nicht selten in der gleichen Zeit 10 bis 20 mm Hg. Wenn man große Schwankungen betrachtet, kann man (A) als angenähert richtig benutzen.

(D) und (E) entfallen, wenn man bei der Gleichung (1) bleibt. Dann wird aber das Ein- und Ausströmen der Luft an der oberen Fläche einen großen Teil der Schwankung von $\mathfrak{p}$ bewirken können.

1) Felix M. Exner, Sitzungsber. d. k. Akad. d. Wissensch. zu Wien. 111. p. 707. 1902; W. Trabert, Meteorolog. Zeitschr. 38. p. 231. 1903.

2) A. Sprung, Lehrb. d. Meteorologie p. 160. Hamburg 1885.

Nimmt man an, daß die Änderung durch die vertikale Bewegung allein entsteht, so hat man

$$\mu_h w_h = -\frac{1}{g}\frac{\partial \mathfrak{p}}{\partial t},$$

für das Steigen von $\mathfrak{p}$ um 1 mm Hg in der Stunde

$$-\frac{10\,333}{760}\cdot\frac{1}{3600} = -0{,}00378\ (\text{kg}\,.\,\text{m}^{-2}\text{sec}^{-1}).$$

In den Höhen von 10, 20, 30 km die Dichte

$$0{,}42, \quad 0{,}089, \quad 0{,}0067\ (\text{kg}\,.\,\text{m}^{-3})$$

gesetzt, erhält man für w_h die Werte

$$-0{,}009, \quad -0{,}042, \quad -0.56\ (\text{m}\,.\,\text{sec}^{-1}),$$

Geschwindigkeiten abwärts von geringem Betrag.

Große Änderungen von $\mathfrak{p}$ entstehen durch andauernde kleine Unterschiede der horizontalen Luftzufuhr und Abfuhr, auch durch kleine Werte der vertikalen Komponente. Bei ungeändertem $\mathfrak{p}$ können die Wirkungen beider sich auf verschiedene Art aufheben; aus der Kontinuitätsgleichung allein läßt sich nicht bestimmen, ob eine aufsteigende oder sinkende Luftbewegung eintritt.

In der Gleichung (1) sind stetig verteilte Quellen und Senken nur an der oberen Grenzfläche angenommen. Wenn Luft (Dampf) am Boden austritt oder absorbiert wird, hat man den bezüglichen Ausdruck hinzuzufügen. Findet Kondensation des Dampfes statt, so gibt es Senken auch im Innern der Luftmasse, und der Gleichung (1) ist auf der linken Seite ein Glied anzufügen, welches das Gewicht der in der Zeiteinheit in der Einheitssäule zwischen 0 und h kondensierten Masse angibt.

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