

Neutral kaon mixing beyond the standard model with $n_f = 2 + 1$ chiral fermions

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We compute the hadronic matrix elements of the four-quark operators needed for the study of $K^0 - \bar{K}^0$ mixing beyond the Standard Model (SM). We use $n_f = 2 + 1$ flavours of domain-wall fermion (DWF) which exhibit good chiral-flavour symmetry. The renormalization is performed non-perturbatively through the RI-MOM scheme and our results are converted perturbatively to $\overline{\text{MS}}$. The computation is performed on a single lattice spacing $a \sim 0.086$ fm with a lightest unitary pion mass of 290 MeV. The various systematic errors, including the discretisation effects, are estimated and discussed. Our results confirm a previous quenched study, where large ratios of non-SM to SM matrix elements were obtained.

Introduction.— Recent progress achieved by the lattice community is greatly improving our theoretical understanding of CP violation in kaon decays. The experimentally well-measured parameters ε_K and ε'_K , which quantify indirect and direct CP violation in $K \rightarrow \pi\pi$ can be confronted with a theoretical computation of the $K \rightarrow (\pi\pi)_{I=0,2}$ amplitudes and of neutral kaon mixing, providing the non-perturbative effects are correctly accounted for. The first direct and realistic computation of the $K \rightarrow (\pi\pi)_{I=2}$ amplitude has been recently performed [1, 2]. A complete computation of the $\Delta I = 1/2$ amplitude is still missing, but important work is being performed in that direction [3]. The situation is much more favourable for neutral kaon oscillations: the bag parameter B_K which describes the long-distance contributions to neutral kaon mixing in the SM, is now computed with a precision of a few percent (see e.g. [4–6]) for recent unquenched determinations). By combining B_K with the experimental value of ε_K , one obtains important constraints on the free parameters associated with quark flavour mixing (see [7] for a recent pedagogical review). In principle, the same techniques can be applied for beyond the Standard Model (BSM) theories (see for example [8–17]), but not much is known concerning the long-distance contributions of the non-standard operators beyond the quenched approximation.

In this letter, we present the first realistic computation of the matrix elements of neutral kaon mixing beyond the standard model. Previous studies were either preliminary [18, 19] or suffered from the quenched approximation [20, 21]. Since a noticeable disagreement was observed between [20] and [21] it is important to repeat this computation in a more realistic framework [43].

Formalism.— In the SM, neutral kaon mixing is dominated by box diagrams as in figure 1. By performing an operator product expansion, one factorises the long-distance effects in the weak matrix element (WME)

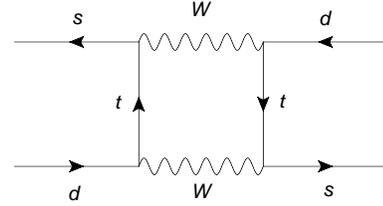


FIG. 1: Diagram contributing to $K^0 - \bar{K}^0$ mixing in the SM.

$\langle \bar{K}^0 | O_1^{\Delta s=2} | K^0 \rangle$ of the four-quark operator $O_1^{\Delta s=2}$ given by (α and β are colour indices)

$$O_1^{\Delta s=2} = (\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) d_\alpha) (\bar{s}_\beta \gamma_\mu (1 - \gamma_5) d_\beta). \quad (1)$$

Only one four-quark operator appears in the SM because neutral kaon mixing occurs under W-boson exchange, implying a “vector-axial” Dirac structure. Since this four-quark operator is invariant under Fierz re-arrangement, the two different colour structures (mixed and unmixed) are equivalent.

We also seek to understand whether new physics beyond the standard model could play a detectable role in kaon CP violation. Assuming that this new physics is perturbative, we might also describe this by contributions to the effective Hamiltonian, and other four quark operators are induced by such extensions of the SM. It is conventional to introduce the so-called SUSY basis $O_{i=1..5}^{\Delta s=2}$: in addition to the SM operator $O_1^{\Delta s=2}$, we define [22] [44]

$$\begin{aligned} O_2^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\alpha) (\bar{s}_\beta (1 - \gamma_5) d_\beta), \\ O_3^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\beta) (\bar{s}_\beta (1 - \gamma_5) d_\alpha), \\ O_4^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\alpha) (\bar{s}_\beta (1 + \gamma_5) d_\beta), \\ O_5^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\beta) (\bar{s}_\beta (1 + \gamma_5) d_\alpha). \end{aligned} \quad (2)$$

These four-quark operators appear in the generic $\Delta s = 2$ Hamiltonian

$$H^{\Delta s=2} = \sum_{i=1}^5 C_i(\mu) O_i^{\Delta s=2}(\mu), \quad (3)$$

where μ is a renormalization scale. The Wilson coefficients C_i , which encode the short-distance effects, depend on the new physics model under consideration. The long-distance effects are factorised into the matrix elements of the four-quark operators $O_i^{\Delta S=2}$. Lattice QCD offers a unique opportunity to quantify these effects in a model-independent way.

In phenomenological applications, by combining our results for the BSM matrix elements with experimental observables (typically the mass difference $\Delta M_K = m_{K_L} - m_{K_S}$ and ϵ_K), one obtains important constraints on the model under consideration. The Wilson coefficients and the bare matrix elements have to be converted into a common scheme, at a given scale μ (typically $\overline{\text{MS}}$ at a scale of 2 or 3 GeV). In the SM case ($i = 1$) it is conventional to introduce the kaon bag parameter B_K , which measures the deviation of the SM matrix element from its vacuum saturation approximation (VSA)

$$B_K = \frac{\langle \bar{K}^0 | O_1 | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2}. \quad (4)$$

Where the normalisation for the decay constant is such that $f_{K^-} = 156.1 \text{ MeV}$. Several normalisations for the BSM operators have been proposed in the literature (see for example [20]), in this work we follow [21] and define the ratios

$$R_i^{\text{BSM}} = \left[\frac{f_K^2}{m_K^2} \right]_{\text{expt}} \left[\frac{m_P^2 \langle \bar{P} | O_i | P \rangle}{f_P^2 \langle \bar{P} | O_1 | P \rangle} \right]_{\text{latt}}, \quad (5)$$

where P is a pseudo-scalar particle of mass m_P and decay constant f_P . The term $[\]_{\text{latt}}$ is obtained from our lattice simulations for different values of m_P .

As advocated in [21], there are various reasons to choose this normalisation rather than, for example, the standard bag parameters. In particular, we expect some systematic errors to cancel in the ratio of the bare WMEs, there is no need to introduce the light quark masses and the partial conservation of the axial current. Furthermore, the mass factors in eq.(5) compensate the leading-order chiral behaviour of the WMEs, making all of the chiral extrapolations smoother. Finally, at the physical point $P = K^0$, the R_i^{BSM} 's give directly the ratios of the non-SM to SM contributions.

The R_i^{BSM} 's will be the main result of this work, but for completeness we will also give the BSM bag parametrisation, defined as (where $N_{2,\dots,5} = \frac{5}{3}, -\frac{1}{3}, -2, -\frac{2}{3}$) [23],

$$B_i = - \frac{\langle \bar{K}^0 | O_i | K^0 \rangle}{N_i \langle \bar{K}^0 | \bar{s} \gamma_5 d | 0 \rangle \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle}, \quad i = 2, \dots, 5. \quad (6)$$

Computation details. — This computation is performed on the finer of the two ensembles described in detail in [4, 24]. We use $32^3 \times 64 \times 16$ (the last number corresponds to the length of the fifth dimension of the domain-wall action in lattice units) Iwasaki gauge configurations

with an inverse lattice spacing $a^{-1} = 2.28(3) \text{ GeV}$, corresponding to $a \sim 0.086 \text{ fm}$. The quarks are described by the domain-wall action [25–27], both in the valence and in the sea sectors. We have three different values of the light sea quark mass $am_{\text{light}}^{\text{sea}} = 0.004, 0.006, 0.008$ corresponding to unitary pion masses of approximately 290, 340 and 390 MeV respectively. The simulated strange sea quark mass is $am_{\text{strange}}^{\text{sea}} = 0.03$, which is close to its physical value $0.0273(7)$ [24]. For the main results of this work, we consider only unitary light quarks, $am_{\text{light}}^{\text{valence}} = am_{\text{light}}^{\text{sea}}$, whereas for the physical strange we interpolate between the unitary ($am_{\text{strange}}^{\text{valence}} = am_{\text{strange}}^{\text{sea}} = 0.03$) and the partially quenched ($am_{\text{strange}}^{\text{valence}} = 0.025$) data.

To extract the bare matrix elements we follow [4, 24], where the procedure for the evaluation of the two-point functions and of the three point function of the SM operator has been explained in detail. In particular, we have used Coulomb gauge fixed wall sources to obtain very good statistical precision. From the measurement of B_K , the generalisation to the BSM operators is straightforward. We define the three point function

$$c_i = \langle \bar{P}(t_f) O_{i=2,\dots,5}^{\Delta S=2}(t) \bar{P}(t_i) \rangle. \quad (7)$$

From the asymptotic Euclidean time behaviour of the ratios of three point-functions c_i/c_1 (we fit these ratios to a constant in the interval $t/a = [12, 52]$ [45]) we obtain the bare matrix elements of the four-quark operators normalised by the SM one: $[\langle \bar{P} | O_i^{\Delta S=2} | P \rangle / \langle \bar{P} | O_1^{\Delta S=2} | P \rangle]_{\text{bare}}$. In figure 2, we show the corresponding plateaux obtained for our lightest unitary kaon $am_{\text{light}}^{\text{sea}} = am_{\text{light}}^{\text{valence}} = 0.004$, $am_{\text{strange}}^{\text{sea}} = am_{\text{strange}}^{\text{valence}} = 0.03$.

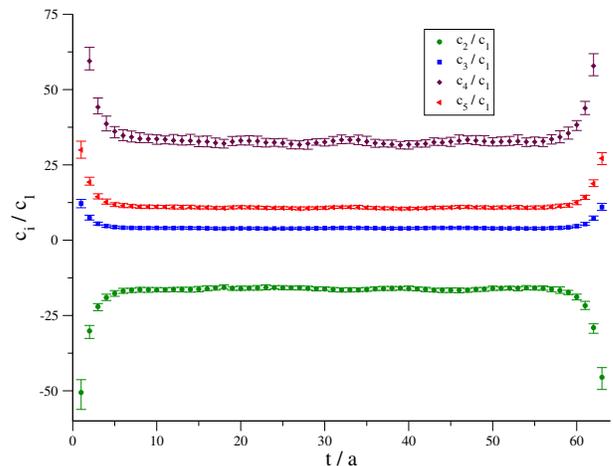


FIG. 2: Ratios of the bare three point functions from which we extract R_i^{BSM} . Results are shown for our lightest simulated unitary kaon.

Renormalization. — The four-quark operators given in eq. (1-2) mix under renormalization. In a scheme which

preserves chiral symmetry, the mixing pattern is given by the transformation properties of these operators under chiral rotations $SU(3)_L \times SU(3)_R$. The SM operator $O_1^{\Delta s=2}$ belongs to a $(27, 1)$ irreducible representation of $SU(3)_L \times SU(3)_R$ and renormalizes multiplicatively. The BSM operators fall into two categories: $O_2^{\Delta s=2}$ and $O_3^{\Delta s=2}$ transform like $(6, \bar{6})$ and mix together. Likewise $O_4^{\Delta s=2}$ and $O_5^{\Delta s=2}$ belong to $(8, 8)$ and also mix. If chiral symmetry is realised, the five-by-five renormalization matrix Z_{ij} is block diagonal: a single factor for the $(27, 1)$ operator and two 2×2 matrices for the BSM operators. Because we work with the domain-wall fermions formulation, in which the explicit breaking of chiral symmetry can be made arbitrary small (and in practice numerically irrelevant), we expect to find this continuum pattern, up to small discretisation effects.

We perform the renormalization of the four-quark operators $O_i^{\Delta s=2}$ non-perturbatively in the RI-MOM scheme [28]. We compute the forward, amputated, vertex functions of the relevant operators between external quark states in the lattice Landau gauge for a given set of momenta. As a renormalization condition, we impose these to be equal to their tree-level values once projected onto their colour-Dirac structures and extrapolated to the chiral limit. By using both momentum sources [29] and partially twisted boundary conditions, we obtain smooth functions of the external momentum with very good statistical accuracy [46]. Although we have performed this computation in various intermediate schemes (including the non-exceptional (γ_μ, γ_μ) -scheme introduced in [4]) we quote here the results obtained via the RI-MOM scheme because only in this case are the conversion factors to $\overline{\text{MS}}$ (computed in continuum perturbation theory) available for the whole set of operators. In such a scheme, the presence of exceptional channels enhances the Goldstone pole contaminations [18, 30], which have then been subtracted explicitly [31]. We choose to impose the renormalization conditions at $\mu = 3 \text{ GeV}$. At this scale, perturbation theory (PT) converges rather quickly ($\alpha_s \sim 0.25$), the chiral symmetry breaking effects are small, and we still have good control on the discretisation effects [4, 32]. Matching at this scale is important for our error budget since the perturbative conversion factors between the lattice schemes and $\overline{\text{MS}}$ of the four-quark operators are only known at one-loop (as discussed later, matching to PT is actually one of our dominant sources of systematic error). We observe that the effects of chiral symmetry breaking are not completely negligible, even at 3 GeV [33]. Therefore we must assess a systematic error to the mixing of operators of different chirality (see next section). We have checked that in a non-exceptional scheme this small chirally forbidden mixing is strongly reduced and becomes numerically irrelevant at 3 GeV [18, 33]. Thus we conclude that this effect is a physical manifestation of the infrared behaviour of the exceptional intermediate scheme. The results for

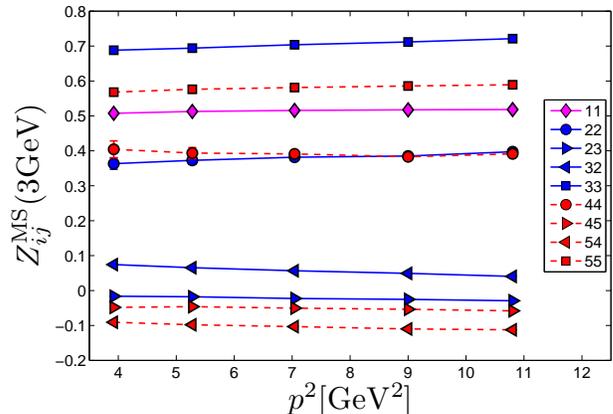


FIG. 3: Renormalization factors of the four-quark operators. At each value of the simulated momentum p , we run to the scale of 3 GeV and convert to $\overline{\text{MS}}$. The remaining scale dependence can be imputed to the truncation of the perturbative expansion. We show only the Z -factors allowed by chiral symmetry.

the chirally allowed renormalization factors $Z_{ij}^{\overline{\text{MS}}}(3\text{GeV})$ are shown in figure 3. They relate the bare four-quark operators to the renormalized ones through the usual relation (Z_q is the renormalization factor of the quark wave function and cancels in the ratios)

$$O_i^{\Delta s=2, \overline{\text{MS}}}(3\text{GeV}) = \frac{Z_{ij}^{\overline{\text{MS}}}(3\text{GeV})}{Z_q^2} O_j^{\Delta s=2, \text{bare}} \quad (8)$$

Physical results and error estimation.— Once the bare ratios have been renormalized, we extrapolate them to the physical kaon mass. We have explored different strategies, such as a simple polynomial form or next-to-leading order chiral perturbation theory (ChPT) predictions [47]. The value of the simulated unitary strange mass differs somewhat from the physical one, therefore we perform an interpolation using a partially quenched strange quark. We find that the R^{BSM} 's exhibit a very mild quark mass dependence (see figure 4 [48]), therefore we take the results obtained with the polynomial Ansatz as our central values [49].

Our final results are the R_i^{BSM} quoted in $\overline{\text{MS}}$ at 3 GeV given in table I. For completeness, we also convert these to the BSM bag parameters, using eq. (6) [50]. We also note that, using the same framework, the SM contribution is found to be $B_1 = B_K = 0.517(4)_{\text{stat}}$ in the $\overline{\text{MS}}$ scheme at 3 GeV, whereas a continuum value of $0.529(5)_{\text{stat}}(19)_{\text{syst}}$ was quoted in [4]. The difference comes from the fact that a different intermediate scheme was used in [4] (such a difference is accounted for in our estimation of the systematic errors). From the same reference, the discretisation effects for B_K on this lattice are seen to be of the order of 1.5%. Since we have only one lattice spacing for the BSM ratios, we make the assump-

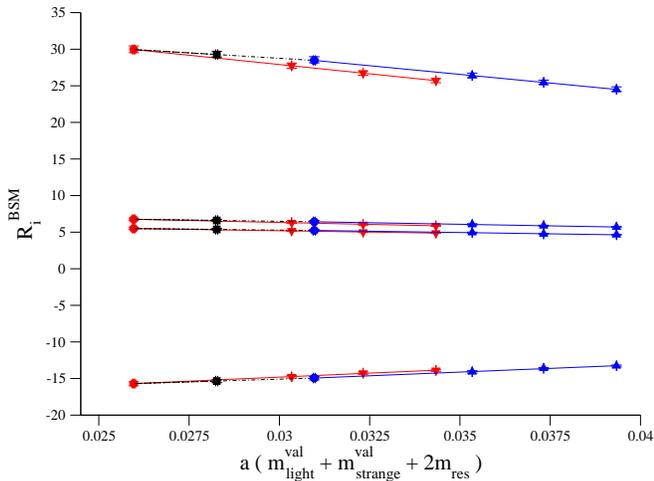


FIG. 4: Renormalized BSM ratios R_i^{BSM} in function of the bare valence quark mass. We show the three unitary light quarks for both the unitary strange (upward blue triangles) and the partially quenched strange (downward blue triangles), together with their extrapolations in the light sector (blue and red circles) and their interpolation to the physical kaon mass (black squares).

tion that the discretisation errors are of the same size as those affecting B_K , and estimate a 1.5% error to all the different operators. This might appear like a crude estimate, but this effect is expected to be sub-dominant compared to other sources of systematic errors. The next systematic error (called “extr.”) represents the spread of the results obtained from different extrapolation strategies to the physical point. The systematic error associated with the non-perturbative renormalization (NPR) has been estimated from the breaking of chiral symmetry. The mixing of the $(6, \bar{6})$ with the $(8, 8)$ operators is forbidden by chiral symmetry, but likely to be enhanced by the exchange of light pseudo-scalar particles. As the matrix element of $O_4^{\Delta s=2}$ is numerically large, this non-physical mixing has effect on $O_2^{\Delta s=2}$ and $O_3^{\Delta s=2}$ of the order of 8 – 9%. As discussed in the previous section, this unwanted infrared effect is absent if a non-exceptional scheme is used. The last error we quote (“PT”) arises from the matching between the intermediate RI-MOM scheme and $\overline{\text{MS}}$, which is performed at one-loop order in perturbation theory [34, 35] in the three-flavour theory. The associated error is obtained by taking half the difference between the leading order and the next to leading order result [51]. We note that this error is one of the dominant ones in our budget, and we expect this error to be reduced by an important factor if a non-exceptional scheme were used, since the latter are known to converge faster in perturbation theory. We neglect the finite volume effects which have been found to be small in [4], as one can expect from the value of $m_\pi L \sim 4$ for our lightest

i	R_i^{BSM}	B_i	stat.	discr.	extr.	NPR	PT	total
2	-15.3(1.7)	0.43 (5)	1.3	1.5	4.0	9.4	4.7	11.3
3	5.4(0.6)	0.75 (9)	2.0	1.5	3.9	7.8	7.6	12.0
4	29.3(2.9)	0.69 (7)	1.3	1.5	4.1	3.0	8.2	9.8
5	6.6(0.9)	0.47 (6)	2.1	1.5	3.8	3.2	12.6	13.8

TABLE I: Final results of this work: the first two columns show the ratios R_i^{BSM} and the corresponding bag parameters B_i in $\overline{\text{MS}}$ at 3 GeV, together with their total error, combining systematics and statistics. In the remaining columns, we give our error budget for the R^{BSM} , detailing the contributions in percentage of the different sources of systematics (see text for more details).

pion mass $m_\pi \sim 290$ MeV.

Conclusions.— We have computed the electroweak matrix elements of the $\Delta s = 2$ four-quark operators which contribute to neutral kaon mixing beyond the SM. Our work improves on the the previous studies by using both dynamical and chiral fermions. We confirm the effect seen in a previous quenched computation [21], where a large enhancement of the non-standard matrix elements were observed. The errors quoted in this work are of the order of 10%. We note that the main limitation of this study comes from the lack of matching factors between non-exceptional renormalization schemes (such as SMOM) and $\overline{\text{MS}}$. Once these factors are available, we expect to reach a precision better than 5%. We also plan to utilise another lattice spacing in order to have a better handle on the discretisation effects.

Appendix

We have computed the non-perturbative scale evolution of the R^{BSM} ’s between 3 and 2 GeV, and then converted the results to $\overline{\text{MS}}$ using one-loop perturbation theory [34, 35]:

$$U^{\overline{\text{MS}}}(2 \text{ GeV}, 3 \text{ GeV}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.87 & 0.02 & 0 & 0 \\ 0 & 0.09 & 1.09 & 0 & 0 \\ 0 & 0 & 0 & 0.86 & -0.01 \\ 0 & 0 & 0 & -0.03 & 0.98 \end{pmatrix}. \quad (9)$$

Our conventions are such that

$$R^{\text{BSM}}(2 \text{ GeV}) = U^{\overline{\text{MS}}}(2 \text{ GeV}, 3 \text{ GeV}) R^{\text{BSM}}(3 \text{ GeV}). \quad (10)$$

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- [43] The European twisted mass collaboration has recently reported on a similar study [36]
- [44] We discard the parity odd operators since they are not relevant in the present case.
- [45] From figure 2 we deduce that we have reached the asymptotic region in this range for each operator insertion.
- [46] More details about the computation of the renormalization factors can be found in [4, 33]
- [47] The functional forms of the B parameters predicted by ChPT can be found in [37], by staggered ChPT in [38], and by heavy-meson ChPT in [39]
- [48] Because of a small explicit chiral symmetry breaking due to the finite fifth dimension of the domain-wall action, a residual mass m_{res} has to be added to the bare quark mass; on this lattice $am_{\text{res}} \sim 10^{-4}$ [24].
- [49] Our strategy to obtain the values of the bare quark masses at the physical point differs slightly from the one presented in [24] but we find compatible results.
- [50] The details of the analysis of the two-point functions needed for the B-parameters can be found in [24].
- [51] To obtain α_s at 3 GeV in the three-flavour theory, we start from $\alpha_s(M_Z) = 0.1184$ [40], we use the four-loop running [41, 42] to compute the scale evolution down to the charm mass, while changing the number of flavours when crossing a threshold, and then run up to 3 GeV in the three-flavour theory.