

Competitive Two-Person Percentile Game Theory with Direct Consideration of Payoff Matrices

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CONSIDERED is the case of two persons who have finite numbers of strategies and who act competitively toward each other. Separately and independently, each player selects one of his strategies. The payoffs to a player for the possible strategy combinations can be conveniently expressed in matrix form, where the rows are his strategies and the columns the strategies of the other player. Both players have knowledge of both payoff matrices.

A player uses a mixed strategy when he assigns probabilities (that sum to unity) to his possible strategies and randomly selects one of them according to these probabilities. When at least one player uses a randomly chosen strategy, the payoff to each player is a random variable.

The allowable payoffs can be of a very general nature. In fact, some or all payoff "values" need not even be numbers. For example, some payoffs may designate categories. Also, the payoffs that are numbers need not be expressible in a common unit or even satisfy numerical operations. However, consideration is restricted to situations where, within each matrix, the payoffs can be ranked according to increasing desirability (with equal desirability possible) separately by each player. These rankings can be arbitrarily different and a player does not necessarily know the rankings used by the other player. However, game solutions are more easily interpreted when the players are in agreement on the rankings (such as when an objectively determined ranking occurs for the payoffs of each matrix; refs. [1] and [2] represent examples).

In many cases, the players may be satisfied with a "middle of the road" viewpoint toward game solutions. Use of a median criterion has this aspect. That is, both players adopt a 50-percentile criterion and, using the rankings of payoffs, apply it to each matrix. Then, according to his ordering, a largest desirability level $p_i = p_i(1/2)$ occurs in the payoff matrix for player i , ($i = 1, 2$), such that, when acting protectively, he can assure himself a payoff of desirability at least p_i with probability at least $1/2$. Also, according to the ordering by player i , a smallest desirability level $p'_j = p'_j(1/2)$ occurs in the matrix for the other player (called player j) such that player i , when acting vindictively, can assure with probability at least $1/2$ that the payoff to player j has desirability at most p'_j (to player i). These results, in less generality and using two slightly different methods, were initially developed in [1] and [2].

For competitive behaviour by the players, a median optimum solution occurs for a player when, according to his rankings and the median criterion, he can be simultaneously protective and vindictive. Then, the situation is said to be one-player-median competitive (OPMC) for this player, and can occur for him when

it does not happen for the other player. If a game is OPMC for both players, it is said to be median competitive. This material on median optimum solutions was initially given, with less generality, in [2].

There are several reasons why player i might want a percentile criterion that differs from the median criterion. First, he may want more, or less, assurance than is provided by use of the 50-percentile. Second, the 50-percentile may not be achievable, in a protective and/or a vindictive sense. That is, when acting protectively player i can assure with probability greater than $1/2$ that he receives at least p_i , and/or when acting vindictively he can assure with probability greater than $1/2$ that player j receives at most p'_j . Then, use of a percentile exceeding the 50-percentile, rather than the 50-percentile, would seem preferable for player i . Third, special characteristics of the payoff matrices sometimes can be exploited when the desirability levels of payoffs are of a quantitative nature. As an example, increasing the percentile value substantially may not result in much change in the largest desirability level that player i can assure himself or the smallest level that he can impose on the other player. As another example, a small decrease in the percentile might result in a substantial increase in the desirability level that player i can assure himself and/or a substantial decrease in the level that he can impose on the other player.

This paper extends the previous median results for competitive players (with direct consideration of the payoff matrices) to the case where player i selects a percentile criterion $100\alpha_i$, ($i = 1, 2$), with $0 < \alpha_i \leq 1$. In general, and according to his ordering, a largest desirability level occurs among the payoffs to player i such that, when acting protectively, he can assure himself at least this level with probability at least α_i . Also, according to the ordering by player i , a smallest desirability level $p'_j(\alpha_i)$ occurs in the matrix for player j such that, when acting vindictively, player i can assure with probability at least α_i that player j receives a payoff with at most this desirability level (as ranked by player i).

For specified α_i and competitive behaviour, a game has an overall optimum solution for player i when he can be simultaneously α_i -protective and α_i -vindictive. A game with this property is said to be one-player- α_i competitive ($OP\alpha_i C$) for player i . An optimum strategy for player i in an $OP\alpha_i C$ game is said to be α_i -optimum for him. A game is both-player- α_1 - α_2 -competitive if, and only if, it is $OP\alpha_1 C$ for player 1 and also $OP\alpha_2 C$ for player 2. That is, the game is α_1 -optimum for player 1 and also α_2 -optimum for player 2.

For player i acting protectively, α_i has an achievable value when player i cannot assure himself at least $p_i(\alpha_i)$ with probability exceeding α_i . For player i acting vindictively, α_i has an achievable value when player i cannot assure that player j receives at most $p'_j(\alpha_i)$ with probability greater than α_i . Identification of achievable α_i for $OP\alpha_i C$ games is discussed later. Use of achievable α_i seems highly desirable.

These percentile results have the same desirable application properties that occur for the corresponding median results. That is, knowledge of the ranking of payoffs within each matrix, and of the "values" of the payoffs corresponding to

the desirability levels $p_i(\alpha_i)$, $p'_i(\alpha_j)$ in the matrix for player i , ($i = 1, 2$), is sufficient for application. Moreover, the locations of payoffs with levels $p_i(\alpha_i)$, $p'_i(\alpha_j)$ in the matrix for player i are determined by the orderings within this matrix. Thus, much of the effort that could be needed for evaluation of payoffs is avoided. For example, suppose that each player has 500 (pure) strategies, which is not large for nontrivial practical situations. Then, each matrix contains 250,000 payoffs. Two orderings of 250,000 payoffs and evaluation of several payoffs ordinarily requires a very small fraction of the effort needed to evaluate 500,000 payoffs.

The concept of a competitive game was introduced in [1]. Some extensions of this concept (for example, to individual players) are considered in this paper. Also, generation of $OP\alpha_i C$ games by one-player games of extended competitive types is considered.

The next section contains a statement of the results for protective players. This is followed by a section with a statement of the results for vindictive players. The next to last section is devoted to statement of the results for games where an overall optimum solution exists for one or both players. The extensions of competitive games are considered in this section. The final section contains an outline of the basis for the stated results.

Protective Results

Suppose that player i is acting protectively. Determination of the payoff, or payoffs, that correspond to $p_i(\alpha_i)$ is considered first. This determination is based on a marking of payoffs in the matrix for player i (according to his ranking for that matrix). That is, the position(s) in his matrix of the payoff(s) with largest desirability level are marked first. This marking, according to decreasing desirability level, is continued until the first time that player i can assure a payoff of the marked set with probability at least α_i . Then, $p_i(\alpha_i)$ is the desirability level of the payoff(s) marked last. An optimum α_i -protective strategy for player i can then be determined on the basis of the payoffs whose desirability level is at least $p_i(\alpha_i)$.

In more detail, consider a method for determination of $p_i(\alpha_i)$. The cases of $\alpha_i \leq 1/2$ and $\alpha_i > 1/2$ are treated separately, with $\alpha_i \leq 1/2$ occurring first. The initial step consists in continuing the marking of the matrix for player i , according to decreasing desirability level, until the first time that marks in all columns are obtainable from at most $1/\alpha_i$ rows. Player i can always assure a marked payoff with probability at least α_i when this situation occurs. Then, remove the mark(s) for the payoff(s) with the smallest desirability level and determine whether a payoff of the remaining marked set can be assured with probability at least α_i by player i .

To make this determination, replace the (remaining) marked payoffs in the matrix for player i by unity and the unmarked payoffs by zero. Consider the resulting matrix of ones and zeros to be for player i in a zero-sum game with an expected-value basis. Player i can assure a payoff of the marked set with probability at least α_i if, and only if, the value of this game (to player i) is at least α_i .

When player i cannot assure a payoff of the remaining marked set with

probability at least α_i , then $p_i(\alpha_i)$ is the desirability level of the payoff(s) with marking(s) removed. Otherwise, remove the mark(s) for the payoff(s) with the smallest of the remaining desirability levels and determine whether a payoff of the resulting marked set can be assured with probability at least α_i . If not, $p_i(\alpha_i)$ is the desirability level of the payoff(s) with marking(s) removed last.

Otherwise (for this case of $\alpha_i \leq 1/2$), continue removing marks, according to smallest desirability level, until the first time a payoff of the resulting marked set cannot be assured with probability at least α_i . Then, $P_i(\alpha_i)$ is the desirability level of the payoff(s) with marking(s) removed last.

Now consider determination of $P_i(\alpha_i)$ for the case of $\alpha_i > 1/2$. Continue the marking, according to decreasing desirability level, until the first time that at least $(1-\alpha_i)^{-1}$ columns are needed to obtain unmarked payoffs in all rows. Player i can assure a marked payoff with probability at most α_i , but ordinarily near α_i , when this situation occurs. Next determine whether a payoff of this marked set can be assured by player i with probability at least α_i .

To make the determination, replace the marked payoffs by unity and the other payoffs by zero. Consider the resulting matrix to be for player i in a zero-sum game with an expected-value basis. Player i can assure a payoff of the marked set with probability at least α_i if, and only if, the game value (to him) is at least α_i .

If the game value is at least α_i , $P_i(\alpha_i)$ is the smallest of the desirability levels for the marked payoffs. Otherwise, also mark the position(s) of the payoff(s) with the largest desirability level among the remaining unmarked payoffs and determine whether a payoff of the marked set can be assured with probability at least α_i . If so, $P_i(\alpha_i)$ is the desirability level for the payoff(s) marked last.

Otherwise (for this case of $\alpha_i > 1/2$), continue marking the positions of the payoffs according to decreasing desirability level until the first time that a payoff of the marked set can be assured with probability at least α_i . Then, $P_i(\alpha_i)$ is the desirability level for the payoff(s) marked last.

Finally, consider determination of an optimum α_i -protective strategy for player i . In the matrix for player i , let all payoffs with desirability level at least $P_i(x_i)$ be replaced by unity while all others are replaced by zero. Consider the resulting matrix to be for player i in a zero-sum game with an expected-value basis. An optimum strategy for player i in this game is an optimum α_i -protective strategy for him.

Vindictive Results

Now suppose that player i is acting vindictively (toward player j). Determination of the payoff(s) corresponding to $P'_j(\alpha_i)$, in the matrix for player j , is considered first. This determination is based on a marking of payoffs in the matrix for player j (according to the ranking by player i for that matrix). That is, the position(s) of the payoff(s) with smallest desirability level are marked first. This marking, according to increasing desirability level, is continued until the first time that player i can assure a payoff (to player j) of the marked set with probability at least α_i . Then $P'_j(\alpha_i)$ is the desirability level of the payoff(s) marked last.

First, consider detailed determination of $P'_j(\alpha_i)$ for the case of $\alpha_i \leq 1/2$. The initial step is to continue marking of the payoff matrix for player j , according to increasing desirability level, until the first time that marks in all rows are obtainable from at most $1/\alpha_i$ columns. Player i can assure a marked payoff with probability at least α_i when this situation occurs. Then remove the mark(s) for the payoff(s) with the largest desirability level and determine whether player i can assure a payoff of the remaining marked set with probability at least α_i .

To make this determination, replace the marked payoffs by zero and the unmarked payoffs by unity. Consider the resulting matrix to be for player j in a zero-sum game with an expected-value basis. Player i can assure a payoff of the marked set with probability at least α_i if, and only if, the value of this game, to player j , is at most α_i .

When player i cannot assure a payoff of the remaining marked set with probability at least α_i , then $P'_j(\alpha_i)$ is the desirability level of the payoff(s) with marking(s) removed. Otherwise, remove the mark(s) for the payoff(s) with the largest of the remaining desirability levels and determine whether player i can assure a payoff of the remaining set with probability at least α_i . If not, $P'_j(\alpha_i)$ is the desirability level of the payoff(s) with marking(s) removed last. Otherwise (for this case of $\alpha_i \leq 1/2$), continue removing marks, according to largest desirability level, until the first time that player i cannot assure a payoff of the resulting marked set with probability at least α_i . Then, $P'_j(\alpha_i)$ is the desirability level of the payoff(s) with marking(s) removed last.

Next, consider determination of $P'_j(\alpha_i)$ for the case of $\alpha_i > 1/2$. Continue the marking (of the matrix for player j), according to increasing desirability level, until the first time that at least $(1-\alpha_i)^{-1}$ rows are needed to obtain unmarked payoffs in all columns. Player i can assure a marked value with probability at most α_i , but usually near α_i , when this situation occurs. Next, determine whether player i can assure a payoff (to player j) of the marked set with probability at least α_i .

To make the determination, replace the marked payoffs by zero and the unmarked payoffs by unity. The resulting matrix is considered to be for player j in a zero-sum game with an expected-value basis. Player i can assure a payoff (to player j) of the marked set with probability at least α_i if, and only if, the value of the game, to player j , is at most α_i .

If the game value is at most α_i , $P'_j(\alpha_i)$ is the largest of the desirability levels for the marked payoffs. Otherwise, also mark the position(s) of the payoff(s) with the smallest desirability level among the remaining unmarked payoffs and determine whether player i can assure a payoff of the unmarked set with probability at least α_i . If so, $P'_j(\alpha_i)$ is the largest of the desirability levels for the payoffs in the resulting marked set. Otherwise (for this case of $\alpha_i > 1/2$), continue marking the positions of the payoffs according to increasing desirability level until the first time that player i can assure a payoff of the marked set with probability at least α_i . Then, $P'_j(\alpha_i)$ is the desirability level for the payoff(s) marked last (and the largest of the desirability levels for the payoffs in the resulting marked set).

Finally, consider determination of an optimum α_i -vindictive strategy for player i . In the payoff matrix for player j , let all payoffs with desirability level at most $P'_j(\alpha_i)$ be replaced by zero and all others replaced by unity. The resulting matrix is considered to be for player j in a zero-sum game with an expected-value basis. An optimum strategy for player i in this game is an optimum α_i -vindictive strategy for him.

Overall Optimum Solutions

The results for $OP\alpha_i C$ games are stated first. Then, competitive games, extensions of the concept of competitive games, and generation of $OP\alpha_i C$ games are considered.

A pair of payoffs, one to each player, occurs for every possible combination of a (pure) strategy for each player. These pairs of payoffs are the possible outcomes for the game.

Let $s_i(\alpha_i)$ consist of those outcomes where, as ranked by player i , the desirability level of the payoff to player i is at least $P_i(\alpha_i)$ and also the desirability level of the payoff to player j is at most $P'_j(\alpha_i)$. A game is $OP\alpha_i C$ for player i if, and only if, he can assure an outcome of $s_i(\alpha_i)$ with probability at least α_i .

To determine whether a game is $OP\alpha_i C$ for player i , first mark the positions in his matrix of the outcomes in $s_i(\alpha_i)$. Then replace the marked positions by unity and the unmarked positions by zero. Consider the resulting matrix of ones and zeros to be for player i in a zero-sum game with an expected-value basis. The game is $OP\alpha_i C$ for player i if, and only if, the value of this game (to him) is at least α_i .

When the value of the zero-sum game is at least α_i , an optimum strategy for player i in this game is an α_i -optimum strategy for player i .

A value of α_i is achievable for a game that is $OP\alpha_i C$ for player i if, and only if, player i cannot assure an outcome of $s_i(\alpha_i)$ with probability greater than α_i . That is, player i can assure an outcome of $s_i(\alpha_i)$ with probability at least α_i but not exceeding α_i .

Now, consider competitive games and some extensions. A game is one-player-competitive for player i if, and only if, the possible outcomes can be arranged in a sequence so that, according to the rankings by player i , the desirability level of the payoffs to player 1 is nondecreasing and also the desirability level of the payoffs to player 2 is nonincreasing. A game is competitive when it is one-player-competitive for both players.

Next, consider a one-player extension for player i that depends on the value used for α_i . A game is α_i -one-player-competitive ($\alpha_i OPC$) for player i if, and only if, according to the rankings by player i , all outcomes whose payoffs to player i have desirability levels at least $P_i(\alpha_i)$ contain payoffs to player j whose desirability levels are at most $P'_j(\alpha_i)$; also, these outcomes can be arranged in sequence so that the desirability level of the payoffs to player i is nondecreasing and simultaneously the desirability level of the payoffs to player j (as ranked by player i) is nonincreasing. A game can be $\alpha_i OPC$ for one value of α_i but not for another value.

Also, for $\alpha_i = \alpha_j$, a game can be α_i OPC for player i but not α_j OPC for player j . If a game is one-player-competitive for player i , however, it is α_i OPC for him with any permissible value for α_i .

A game is said to be α_1 - α_2 -competitive when it is α_1 OPC for player 1 and also α_2 OPC for player 2. A competitive game is α_1 - α_2 -competitive for all permissible values of α_1 and α_2 .

Finally, consider the implications of competitive, one-player-competitive, α_i OPC, and α_1 - α_2 -competitive games with respect to occurrence of $OP\alpha_iC$ games. A competitive game is $OP\alpha_iC$ for player i for $i = 1, 2$ and all permissible values of α_i . A one-player-competitive game for player i is $OP\alpha_iC$ for player i for all permissible values of α_i . An α_i OPC game for player i is $OP\alpha_iC$ for player i with this value of α_i , and an α_1 - α_2 -competitive game is $OP\alpha_1C$ for player 1 and also $OP\alpha_2C$ for player 2 (with the stated values for α_1 and α_2).

Basis for Results

First, consider the method that was employed in marking positions of payoff matrices. This method requires that the positions of all payoffs of equal desirability be marked at the same time. Use of this method tends to minimise the application effort and also to maximise the probabilities for marked sets. Other methods, such as establishment of preferred sequence orders for the outcomes (see [1]), could be developed in a straightforward manner. However, only the stated method is considered here.

Next, consider the basis for the statements of protective, vindictive, and $OP\alpha_iC$ results. Motivation for the first steps in identifying $P_i(\alpha_i)$ and $P'_j(\alpha_i)$ follows from

THEOREM 1. *Let the number of strategies for player i be denoted by $r(i)$, ($i = 1, 2$). When the marked payoffs in the matrix for player i are such that marks in all columns can be obtained from $r(i) - t(i)$ rows, with $t(i) \geq 0$, player i can assure a marked payoff with probability at least $[r(i) - t(i)]^{-1}$, or at least α_i when $r(i) - t(i) \leq 1/\alpha_i$.*

COROLLARY. *When the unmarked payoffs in the matrix for player j are such that unmarked payoffs in all columns can be obtained from $r(j) - t(j)$ rows, player j can assure an unmarked payoff of his matrix with probability at least $[r(j) - t(j)]^{-1}$. Thus, player i can assure a marked value in this matrix with probability at most $1 - [r(j) - t(j)]^{-1}$, or at most α_i when $r(j) - t(j) \leq (1 - \alpha_i)^{-1}$.*

PROOF OF THEOREM 1: When $r(i) - t(i) = 1$, so that a row is completely marked, some one of the marked outcomes can be assured with unit probability by player i .

Suppose that $r(i) - t(i) \geq 2$. Let $p_1^{(u)}, \dots, p_r^{(u)}$ be the mixed strategy used by player u , ($u = 1, 2$), where a unit probability (pure strategy) is possible. The probability that a marked value occurs is

$$\sum_{v=1}^{r(i)} p_v^{(i)} Q_v^{(i)}$$

where $Qv^{(j)}$ is the sum of those of $p_1^{(j)}, \dots, p_r^{(j)}$ for the columns that have marks in the v -th row. The largest value of this probability that player i can assure, through choice of $p_1^{(i)}, \dots, p_r^{(i)}$, is

$$G^{(i)} = p_1^{(i)}, \min, p_r^{(i)} \left(\max_v Qv^{(j)} \right)$$

Let $v[1], \dots, v[r(i)-t(i)]$ identify $r(i)-t(i)$ rows that together contain marked payoffs in all columns. For any minimising choice of the values for $p_1^{(j)}, \dots, p_r^{(j)}$, all of $Qv_{[1]}^{(j)}, \dots, Qv_{[r(i)-t(i)]}^{(j)}$ are at most $G^{(i)}$. Thus,

$$[r(i)-t(i)]G^{(i)} \geq Qv_{[1]}^{(j)} + \dots + Qv_{[r(i)-t(i)]}^{(j)} \geq 1$$

and a probability of at least $[r(i)-t(i)]^{-1}$, for obtaining a marked payoff, can be assured by player i .

The remaining results, including determination of when a game is $OP\alpha_i C$ for player i and determination of optimum strategies, can be verified by appropriate use of

THEOREM 2. *A sharp lower bound on the probability that player i can assure some payoff of a set that is marked in his matrix, and one or more corresponding optimum strategies for him in accomplishing this, are determined by the solution of a zero-sum game with an expected value basis. The value of this game (to player i) is the sharp lower bound and an optimum strategy for player i in this game is optimum with respect to assuring a payoff of the marked set. The payoff matrix, for player i , in this zero-sum game has value unity at all marked positions and zero at the unmarked positions.*

COROLLARY. *A sharp upper bound on the probability that player i can assure some payoff of a set that is marked in the matrix for player j , and one or more corresponding optimum strategies for player i , are determined by solution of a zero-sum game with an expected-value basis. Unity minus the value of this game (to player j) is the sharp upper bound and an optimum strategy for player i in this game is optimum with respect to assuring a payoff of the marked set. The payoff matrix, for player j , in this zero-sum game has zero value at all marked positions and unit value at all unmarked positions.*

PROOF OF THEOREM 2: Let each player use an arbitrary but specified mixed strategy (with pure strategies possible) for the zero-sum game. The expression for the expected payoff to player i with these strategies is also the expression for the probability of occurrence of some one of the payoffs that are marked in the actual payoff matrix for player i .

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