Difficulties in Practical Application of Game Theory and a Partial Solution

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MUCH effort has been devoted to development of game theory. However, nontrivial practical applications of discrete game theory are few. One reason may be difficulties in evaluating payoff matrices. Another reason may be the very limited extent of situations for which game solutions of a "forcing" nature have been developed. That is, the solutions control what the players can do, according to some reasonable criterion (such as expected payoff received) rather than merely trying to predict what they will do. A third reason may be that very little has been developed for situations where the numbers in a payoff matrix do not satisfy the arithmetic operations. As an example, this occurs for the practically important case where the ranks of the payoff values are the numbers in a payoff matrix. These three restraints on game theory use are discussed here. Then, attention is directed to a new form of game theory, based on median (rather than expected value) considerations, that is much less sensitive to these restraints.

Introduction and Discussion

Game theory, with mixed strategies and consideration of expected payoffs, has been the subject of much research. However, few practical applications that are not of a highly oversimplified nature have occurred. This note identifies and discusses what seem to be the three most important direct restraints on nontrivial practical applications of this expected-value approach to game theory. Then, a new approach with a median (rather than expected value) basis is described. The three restraints are found to be much less important for median game theory. Only the most elementary case, that of two players with finite numbers of

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strategies, is directly considered. However, the discussion about restraints has wide applicability.

The first restraint is concerned with evaluation of the payoff matrices, one for each player. There are three major difficulties. First, even approximate determination of payoff values can experience conceptual difficulties. For example, some losses or gains can involve considerations of an intangible nature (human life, mental anguish, goodwill, morale, etc.). Intangible considerations are often most important in evaluation of some of the largest and/or the smallest values in a payoff matrix. Usually, precise quantification of payoffs involving intangibles is not possible. For the second difficulty, sufficiently accurate evaluation of payoffs is possible but involves large amounts of effort. For example, evaluation of the payoff for a player might require several runs of an elaborate and expensive simulation in which the strategy combination is used. Finally, the number of strategies for each player nearly always is at least moderately large for non-trivial practical applications. Then the number of payoffs to be determined is huge. For example, consider two players that each have 300 strategies. Each payoff matrix contains 90,000 numbers. Determination of a huge number of payoffs is a strong restraint even when each payoff can be evaluated with moderate effort.

The second restraint involves the two types of solutions that have been developed using the expected-value approach to game theory. For one type, and two players, use of an optimum strategy allows each player to simultaneously be as protective as possible for himself and as vindictive as possible toward the other player (in the sense of expected payoffs). This desirable "forcing" type of solution occurs for situations where the payoff matrices satisfy a zero-sum condition (sum of payoffs to players is zero for every strategy combination) or one of some mild modifications of this condition. Unfortunately, this class of situations is very restricted and only a small fraction of the cases arising in practice are of this nature (even for two-player games). Now, consider solutions of the other type. They are applicable to a large fraction of the practical situations but have undesirable characteristics. These solutions have a strong philosophical basis but do not control expected payoffs in a desirable fashion. That is, more emphasis is placed on predicting player actions (from the philosophical basis) than on controlling expected payoffs. Thus, a player can use an "optimum" strategy and do poorly, because the other player was not astute enough to recognise the philosophical basis. Almost all practical applications have been for situations where a "forcing" solution is obtainable, likely because the other type is not of a dependable nature.

The third restraint is concerned with situations where the numbers in a payoff matrix cannot meaningfully be added, subtracted, or multiplied by constants (cardinal numbers). Then, expected payoff, zero-sum condition, etc. are undefined concepts and the expected-value approach of game theory is not usable. In particular, expected-value game theory is not applicable to situations where the numbers in any of the payoff matrices are the ranks of the corresponding (unknown) payoffs to that player. Ranks (ordinal numbers) are often the most that can be obtained in social science fields (psychology, education, etc.). Thus, expected-value game theory is not usable in several practically important fields.

Median Game Theory

Consideration is limited to two players and finite numbers of strategies. When one or both players use a mixed strategy, the payoffs received by the players are random variables. Knowledge of the probability distributions of these random payoffs constitutes the maximum information that is available.

Determination of optimum (mixed) strategies on the basis of probability distributions has many complications. This determination has been simplified, for expected-value game theory, by representing a distribution by its mean value (expected payoff to that player). Another reasonable way is to represent a probability distribution by its median value(s). This is the basis for median game theory, whose fundamental properties are given in (Walsh, 1969).

In general, median game theory has the properties: A largest value P_I (P_{II}) occurs in the payoff matrix for protective player *I* (*II*) such that he can assure himself at least this payoff with probability at least $\frac{1}{2}$. Also, a smallest value P_I^1 (P_{II}^{11}) occurs in the matrix for player *I* (*II*) such that vindictive player *II* (*I*) can assure, with probability at least $\frac{1}{2}$, that player *I* (*II*) receives at most this payoff. Except for P_I, P_I', P_{II}, P_{II}', it is sufficient to know the relative order of the values within each payoff matrix. Moreover, knowledge of this relative order is sufficient to determine the locations of P_I, P_I', P_{II}, P_{II}' in the payoff matrices. It is to be noted that $P_I = P_I'$ and $P_{II} = P_{II}'$ are possibilities.

Analogous to expected-value game theory, situations occur where each player can simultaneously be as protective as possible for himself and as vindictive as possible toward the other player (according to the median criterion). A sufficient condition for such situations is that the game is competitive or generates a competitive game (special case of median competitive games).

A pair of payoffs, one for each player, occurs for every strategy-combination for the players. A game is competitive if the totality of these outcomes can be sequence ordered so that the payoffs for player *I* are nondecreasing and the payoffs to player *II* are nonincreasing.

A game generates a competitive game if there is a sequence ordering for the outcomes such that: First, the payoffs of player *I* (*II*) that are in outcomes above (below) the outcomes corresponding to the matrix location of P_I (P_{II}) are at least (most) equal to P_I (P_{II}), and the payoffs in outcomes below this outcome are at most (least) equal to P_I (P_{II}). Second, also the payoffs of player *I* (*II*) that are in outcomes above the outcome corresponding to the matrix location of P_I' (P_{II}') are at least (most) equal to P_I' (P_{II}'), and the payoffs in outcomes below this outcome are at most (least) equal to P_I' (P_{II}').

Determination of median optimum strategies usually does not require much effort and a method is given in (Walsh, 1969). Pure and mixed median optimum strategies can both occur. However, a pure median optimum strategy is not necessarily a minimax strategy.

Discussion of Restraints—Median Games

First, consider the restraint concerned with evaluation of payoff matrices. Determination of payoff values is, effectively, simplified to the much easier task of determining payoff order in each matrix. Moreover, not even order needs to be determined within some payoff sets that can contain a substantial fraction of the payoffs. As an example, consider a given matrix and a set of largest payoffs (payoffs in this set at least equal to all other payoffs in matrix). Suppose that this set contains a payoff in every column of the matrix and that there is a column that contains the smallest value of the set but no other value in the set. Then, all that needs to be known about this set is the matrix locations of the payoffs equal to the smallest value and the locations of the payoffs exceeding the smallest value. Similar considerations apply to a set of smallest values (in terms of rows and the largest value of the set). Also, it is unneccessary to obtain orderings for payoffs at matrix locations for player I (II) where a payoff either $\geqslant P_I$ ($\geqslant P_{II}$) or $\leqslant P'_I$ ($\leqslant P'_{II}$) does not occur. The matrix locations where these relations do occur can be determined from the method given in (Walsh, 1969) for evaluating P_I, P'_I, P_{II}, P'_{II}.

The difficulty due to intangible considerations is unimportant if only largest and/or smallest payoffs are affected since, ordinarily, these would occur in sets within which not even order needs to be determined. In any case, order may be determinable although actual payoffs cannot be even approximately evaluated. The difficulty due to large effort for evaluation often can be at least partially overcome. Often, many relative orderings of payoffs can be made by a casual examination of the strategies involved. One way is to hold the strategy for one player fixed and consider the relative effects of the strategies of the other player. Also, many relative orderings can be determined with only moderate analysis or simulation effort. Some values, including P_I, P_{II}, P'_I, P'_{II} will have to be accurately evaluated but, overall, the level of effort per evaluation should be a very small fraction of that required for corresponding expected value game theory. Finally, consider the difficulty due to the huge number of payoffs in the matrices. The methods of reducing the level of effort per evaluation are also useful when only a moderate effort is needed for an evaluation. Even here, the average effort per evaluation should be a small fraction of that required for expected-value game theory. This, of course, substantially reduces but does not eliminate the difficulty arising from large numbers of strategies for the players.

Now, consider the second restraint. A desirable solution of a "forcing" nature is obtainable when the situation is of a competitive nature or such that a competitive game is generated. Even when the usual kinds of payoffs are considered (cardinal numbers), the class of situations where this occurs is extremely huge compared to the zero-sum class for expected-value game theory (and includes the zero-sum class). That is, desirable solutions can be obtained for a large class of situations that do not have a "forcing" type of solution for the expected-value approach. Moreover, it seems likely that desirable solutions can be obtained for much broader classes of situations by extension of the median competitive concept.

Finally, consider the third restraint. Median game theory is capable of handling any situation where the numbers within a payoff matrix can be ranked. In particular, this extends the usage of game theory to situations where payoffs in one or more matrices are ordinal numbers.

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