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Bootstrapping the Small Sample Critical Values of the Rescaled Range Statistic*

MARWAN IZZELDIN
City University Business School

and

ANTHONY MURPHY
University College Dublin

Abstract: Finite sample critical values of the rescaled range or R/S statistic may be obtained by bootstrapping. The empirical size and power performance of these critical values is good. Using the post blackened, moving block bootstrap helps to replicate the time dependencies in the original data. The Monte Carlo results show that the asymptotic critical values in Lo (1991) should not be used.

I INTRODUCTION

The modified rescaled range or R/S statistic is used to detect long memory in financial, economic, hydrological and other time series (Lo, 1991; Beran, 1994; Baillie, 1996). The R/S statistic for an I(0) time series $\{x_t\}_{t=1}^{T}$, divided by the square root of the sample size T, is just the maximum range of the partial sum of the standardised series $S_T(t) = T^{-1/2}\sum_{s=1}^{t}(x_t - \overline{x}) / \hat{\sigma}_x$:

$$T^{-1/2}R/S = \max_{0 \le t \le T} S_T(t) \; - \; \min_{0 \le t \le T} S_T(t)$$

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where $\bar{x} = \sum_{t=1}^{T} x_t$ is the sample average and $\hat{\sigma}_x$ is a consistent estimate of the standard deviation of x_t. The modified R/S statistic proposed by Lo (1991) uses a consistent estimator of the long run standard deviation, such as the Newey-West (1987) estimator, rather than the ordinary standard deviation to standardise the series. Asymptotically, if the I(0) time series x_t satisfies some weak regularity conditions such as those set out in Lo (1991), the limiting distribution of $T^{-1/2}$ R/S is the maximum range of a standard Brownian bridge or "tied down" Brownian process:

$$T^{-1/2} R/S \Rightarrow \max_r W_b(r) - \min_r W_b(r) \qquad 0 \le r \le 1$$

where $\Rightarrow$ denotes weak convergence, W(r) is a Brownian process and $W_b(r) = W(r) - rW(1)$ is a standard Brownian bridge. The regularity conditions underlying this result are standard in the unit root literature and permit a fair degree of dependence and heterogeneity. ARMA and/or GARCH processes are permitted.

II FINITE SAMPLE CRITICAL VALUES FOR R/S

Unfortunately, as Lo (1991), Harrison and Treacy (1997) and others show, the finite sample distribution of R/S is not well approximated by its asymptotic distribution even when T is large and the time series x_t is i.i.d.. Moreover, very few exact/finite sample results for the distribution of R/S are known. One exception is Anis and Lloyd (1976) who derived the expected value of the R/S statistic when the x_t are i.i.d. normal.

Harrison and Treacy (1997) calculated the small sample distribution of R/S using 100,000 random draws from three i.i.d. distributions — the standard normal, the uniform and the log-normal distributions — for various sample sizes ranging from 25 to 500. They fitted four moment Beta approximations to their results for the two symmetric distributions and derived approximate small sample critical values for R/S. Unfortunately, the Beta approximation for the asymmetric, log-normal distribution is poor. Of course, Harrison and Treacy (1997) may have inadvertently considered a very unfavourable set of parameters for the log-normal distribution. Conniffe and Spencer (1999) suggest two other useful approximations to the small sample distribution of R/S. Their approximations are not sample size specific but the derivations assumed normality so their results may not be robust to departures from normality.

III BOOTSTRAPPING THE R/S STATISTIC

The problem with the various approximations to the finite sample distribution of the R/S statistic is that they may be distribution and, in the Harrison and Treacy case, sample size specific. This problem can often be overcome by using the bootstrap to obtain finite sample critical values etc. for the R/S statistic. The bootstrap approach involves resampling the data which is an option in many econometric packages and, in any case, is easy to programme. The bootstrap may be used to automatically obtain distribution and sample size specific, higher order or better approximations to the small sample distribution of a great many statistics, including the R/S statistic. See Efron and Tibshirami (1993). The reason why the bootstrap works in this case is because $T^{-1/2}$ R/S is asymptotically pivotal, since its distribution does not depend on unknown parameters. This means that the bootstrap distribution will generally provide a better approximation to the finite sample distribution of R/S than the asymptotic distribution tabulated in Lo (1991). Moreover, a large number of bootstrap samples/replications may not be required.

As researchers are predominantly interested in obtaining critical values for the R/S statistic we concentrate on this. The Monte Carlo results presented in the next section show that critical values based on the bootstrapped R/S statistic have good small sample size and power properties for both symmetric and asymmetric distributions, with and without fat tails, when the data are independent and when they are dependent etc. Thus, for example, the bootstrap handles the difficult i.i.d. log-normal case in Harrison and Treacy (1997) without any problems. The Monte Carlo results suggest that as few as 99 bootstrap replications are required to obtain reasonably accurate critical values.

In finite samples, the empirical size of the R/S test statistic based on bootstrapped critical values is generally an order of magnitude closer to the nominal size of the test than the asymptotic critical values generated by Lo (1991). Horowitz (1994) shows that the difference between the true and the bootstrapped critical values is of the order $O(T^{-1})$, as opposed to $O(T^{-1/2})$ in the case of the asymptotic critical values. In particular, this result holds in the presence of short memory processes such as stationary AR, MA, ARMA, ARCH, GARCH etc. processes.

These short memory processes were examined using the post blackened, moving block bootstrap discussed in Davidson and Hinkley (1997) which appears to adequately capture the dependence structure of the data, even using a small number of bootstrap replications. The moving block bootstrap involves resampling possibly overlapping blocks of x_t. There are some problems with the moving block bootstrap (Maddala and Kim, 1998), but the post blackened version works well in practise. Post blackening involves (i) pre-whitening the x_t series by fitting an AR model with a suitably large number of lags to the time series; (ii) resampling blocks of residuals from this estimated model using the moving block bootstrap and then (iii) post blackening these resampled residuals using the estimated parameters of the AR model in order to generate the bootstrapped sample of the x's. Note that the AR model is only used as a device to pre-whiten the data and is not put forward as a model of the DGP.

In finite samples, the empirical size of the R/S test statistic based on these bootstrapped critical values is good with short memory processes. The empirical power of the bootstrapped R/S statistic is also good when fractionally integrated series, the main long memory processes considered in the literature, were examined. This result is at odds with the theoretical result in Wright (1999), who examined the local asymptotic power of the rescaled range and other related tests for fractional integration. He suggested that these tests are poor since they have only trivial asymptotic power against fractionally integrated I(d) alternatives with $d = O(T^{-1/2})$. However, it is easy to show that this limiting result is not a good approximation when dealing with the size of samples which are common in economics, say T in the range 100 to 1000.

IV SOME MONTE CARLO RESULTS

The Monte Carlo results set out in the tables illustrate the good small sample performance of bootstrapped critical values of R/S. Details of the data generation processes etc. used are given in the notes to the tables. Tables 1 and 2 refer to the i.i.d. distributions examined by Harrison and Treacy (1997). Table 1 sets out the empirical size of the R/S test statistic at the nominal 5 and 1 per cent levels using (i) bootstrapped critical values, (ii) the Beta approximation based critical values in Harrison and Treacy (1997, Table 10) and (iii) the asymptotic critical values in Lo (1991). The empirical size of the bootstrapped tests is good. Table 2 shows that a small number of bootstrap replications is sufficient to obtain reasonably accurate critical values.

In Tables 3 to 5, the empirical size of the R/S test statistic for various AR(1), MA(1) and GARCH models using bootstrapped and asymptotic critical values is examined. The bootstrap is performed using the post blackened, moving block resampling method. Results for residual resampling are also available. The nominal and empirical sizes of the test based on bootstrapped critical values are close.

Table 1: *The Empirical Size of the R/S Statistic*
Some Monte Carlo Results for the IID Normal, Uniform and Log-Normal Cases (1000 Replications, 99 Bootstrap Replications)

Distribution	*Sample Size T*	*Bootstrapped Critical Values*		*Harrison & Treacy's Critical Values*		*Lo's Asymptotic Critical Values*	
		5 Per Cent	*1 Per Cent*	*5 Per Cent*	*1 Per Cent*	*5 Per Cent*	*1 Per Cent*
Standard Normal N(0, 1)	100	5.7	0.9	5.2	0.8	1.0	0.3
	200	6.5	1.1	6.6	1.3	3.7	0.7
	500	4.8	0.2	4.8	0.3	3.3	0.9
	1000	4.6	1.1	4.3	1.0	4.3	1.0
Uniform Range [0, 1]	100	3.2	0.4	3.5	0.5	1.5	0.1
	200	5.9	1.3	5.8	1.2	3.3	0.4
	500	5.5	1.1	5.9	1.1	3.9	0.5
	1000	5.2	0.7	4.0	0.6	4.0	0.6
Log-Normal e^z, z~N(0, 1)	100	3.4	1.0	—	—	1.2	0.1
	200	4.5	0.5	—	—	1.4	0.2
	500	4.9	1.0	—	—	2.3	0.4
	1000	4.8	1.2	—	—	3.3	0.4

Table 2: *The Effect of Varying the Number of Bootstrap Replications on the Empirical Size of the R/S Statistic*
Some Monte Carlo Results for the IID Normal, Uniform and Log-Normal Cases (1000 Replications, 100 Observations)

Distribution	*No of Bootstrap Replications B*	*Bootstrapped Critical Values*	
		5 Per Cent Level	*1 Per Cent Level*
Standard Normal N(0, 1)	99	4.6	1.0
	199	5.2	1.4
	499	4.6	0.8
	999	5.0	0.7
Uniform Range [0, 1]	99	5.3	1.4
	199	5.0	1.3
	499	5.9	1.3
	999	3.2	0.4
Log-Normal e^z, z~N(0, 1)	99	4.2	0.7
	199	4.0	0.6
	499	4.3	0.7
	999	4.0	0.6

Table 3: *The Empirical Size of the Modified R/S Statistic in the AR(1) Case Some Monte Carlo Results Using the Post Blackened, Moving Block Bootstrap (1000 Replications)*

DGP	*Sample Size T*	*Value of AR(1) Parameter ρ*	*Bootstrapped Critical Critical Values*		*Lo's Asymptotic Critical Values*	
			5 Per Cent Level	*1 Per Cent Level*	*5 Per Cent Level*	*1 Per Cent Level*
AR(1) Standard Normal Random Errors	200	0.50	6.8	1.3	1.3	0.0
	200	0.75	5.9	1.2	2.0	0.0
	200	0.90	5.0	0.6	5.9	0.0
	200	0.95	7.1	1.2	19.7	0.8
	100	0.75	5.1	0.9	0.0	0.0
	200	0.75	5.9	1.2	2.0	0.0
	500	0.75	4.5	1.0	5.6	1.7
	1000	0.75	4.8	1.1	12.2	2.8
AR(1) Log Normal Random Errors	200	0.50	4.5	0.6	0.5	0.0
	200	0.75	5.0	1.0	1.4	0.0
	200	0.90	5.2	0.6	5.4	0.1
	200	0.95	6.9	1.9	21.0	0.7
	100	0.75	5.9	1.1	0.0	0.0
	200	0.75	4.5	0.9	1.1	0.0
	500	0.75	4.2	1.1	5.3	1.1
	1000	0.75	5.5	1.1	10.2	2.9

Notes: The AR(1) model is $x_t = \rho x_{t-1} + u_t$. The random error u_t is either a standard normal or, in the log normal case, the exponential of a standard normal random variable. The post blackened, moving block bootstrap was used to calculate the bootstrapped critical values. An AR(10) model was used to pre-whiten the data. A block size of 10 was used for the moving block bootstrap. The Newey-West estimator was used to calculate the long run variance. The first 10 estimated autocovariances were used to estimate the long run variance.

Table 4: *The Empirical Size of the Modified R/S Statistic in the MA(1) Case Some Monte Carlo Results using the Post Blackened, Moving Block Bootstrap (1000 Replications, 200/500 Observations)*

DGP	*Sample Size T*	*Value of MA(1) Parameter θ*	*Bootstrapped Critical Values*		*Lo's Asymptotic Critical Values*	
			5 Per Cent Level	*1 Per Cent Level*	*5 Per Cent Level*	*1 Per Cent Level*
		0.50	4.8	1.5	1.2	0.0
MA(1)	200	0.75	4.2	0.7	0.2	0.0
Standard		0.90	4.7	0.6	0.7	0.3
Normal						
Random		0.50	5.6	1.1	3.6	0.3
Errors	500	0.75	3.4	0.5	2.4	0.0
		0.90	4.7	0.8	0.3	0.0

Notes: The MA(1) model is $x_t = u_t + \theta u_{t-1}$. See the Notes to Table 3 for details of the post blackened, moving block bootstrap etc.

Table 5: *The Empirical Size of the Modified R/S Statistic in the ARCH/ GARCH Case Some Monte Carlo Results using the Post Blackened, Moving Block Bootstrap (1000 Replications, 100 Observations)*

DGP	*Bootstrapped Critical Values*		*Lo's Asymptotic Critical Values*	
	5 Per Cent Level	*1 Per Cent Level*	*5 PerCent Level*	*1 Per Cent Level*
ARCH(1)	5.4	0.7	0.3	0.1
GARCH(1,1)	4.1	1.4	0.4	0.0
GARCH(1,1) + MA(1)	5.4	1.7	0.3	0.1

Notes: The GARCH(1,1) + MA(1) model is $x_t = e_t + 0.5e_{t-1}$, $e_t = \sqrt{h_t}u_t, h_t = 1 + 0.3e_{t-1}^2 + 0.4h_{t-1}$, $u_t \sim N(0,1)$. The other two models are special cases. See the Notes to Table 3 for details of the post blackened, moving block bootstrap etc.

Finally, in Table 6 the empirical power of the R/S test statistic is examined when the data are generated as stationary and borderline non-stationary fractionally integrated processes. The empirical power is as good as the power obtained using the asymptotic critical values in Lo (1991). The empirical size and power of the bootstrapped tests compare well with those of the KPSS test (Kwiatkowski *et al.*, 1992) as reported in Lee and Schmidt (1996). It would certainly be interesting to see if bootstrapping improves the finite sample size and power performance of the KPSS test statistic and other test statistics, including those for parameter stability, which involve limiting Brownian bridge processes under the null. In summary, these results strongly suggest that the use of Lo's asymptotic critical values for the R/S statistic be replaced or supplemented by the use of bootstrapped critical values obtained using the post blackened, moving block bootstrap.

Table 6: *The Empirical Power of the R/S Statistic in the Fractionally Integrated Case*
Some Monte Carlo Results using the Post Blackened, Moving Block Bootstrap (1000 Replications, 200 Observations)

Sample Size T	*Fractional Difference Parameter d*	*Bootstrapped Critical Values*		*Lo's Asymptotic Critical Values*	
		5 Per Cent Level	*1 Per Cent Level*	*5 Per Cent Level*	*1 Per Cent Level*
T = 200	d = 0.25	29.6	10.4	10.0	0.1
	d = 0.50	70.9	44.1	57.8	5.5
T = 1000	d = 0.25	76.4	56.9	74.0	54.3
	d = 0.50	88.6	75.2	99.9	99.5

Notes: The model is $x_t = (1 - L)^{-d}u_t$ with $u_t \sim N(0,1)$. The fractionally integrated series were generated using the Choleski decomposition of the variance-covariance matrix. See the Notes to Table 3 for details of the post blackened, moving block bootstrap etc.

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