Stability of QED

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It is shown for a class of random, time-independent, square-integrable, three-dimensional magnetic fields that the one-loop effective fermion action of four-dimensional QED increases faster than a quadratic in **B** in the strong coupling limit. The limit is universal. The result relies on the paramagnetism of charged spin-1/2 fermions and the diamagnetism of charged scalar bosons.

I. INTRODUCTION

Integrating out the fermion fields in four-dimensional QED continued to the Euclidean metric results in the measure for the gauge field integration

$$d\mu(A) = Z^{-1} e^{-\int d^4 x(1/4 F_{\mu\nu} F_{\mu\nu} + \text{gauge fixing})}$$
$$\times \det_{\text{ren}}(1 - eS\slashed{A}) \prod_{x,\mu} dA_\mu(x), \qquad (1.1)$$

where $\det_{\text{ren}}$ is the renormalized fermion determinant defined in Sec. II; S is the free fermion propagator, and Z is chosen so that $\int d\mu(A) = 1$. In the limit $e = 0$, the Gaussian measure for the potential A_μ is chosen to have mean zero and covariance

$$\int d\mu(A) A_\mu(x) A_\nu(y) = D_{\mu\nu}(x - y), \qquad (1.2)$$

where $D_{\mu\nu}$ is the free photon propagator in some fixed gauge. Naively, integration over the fermion fields produces the ratio of determinants $\det(\slashed{P} - e\slashed{A} + m)/\det(\slashed{P} + m)$, which is not well defined; $\det_{\text{ren}}$ makes sense of this ratio. It is gauge invariant and depends only on the field strength $F_{\mu\nu}$ and invariants formed from it.

We have chosen to introduce this paper with an abrupt intrusion of definitions in order to emphasize the central role of $\det_{\text{ren}}$ in QED: it is everywhere. It is the origin of all fermion loops in QED. If there are multiple charged fermions, then $\det_{\text{ren}}$ is replaced by a product of renormalized determinants, one for each species. For our purpose here, it is sufficient to consider one fermion.

The nonperturbative calculation of $\det_{\text{ren}}$ reduces to finding the eigenvalues of $S\slashed{A}$,

$$\int d^4 y S(x - y) \slashed{A}(y) \psi_n(y) = \frac{1}{e_n} \psi_n(x). \qquad (1.3)$$

There are at least two complications. First, $S\slashed{A}$ is not a self-adjoint operator, and so many powerful theorems from analysis do not apply. Second, since A_μ is part of a functional measure, it is a random field, making the task of calculating the e_n for all admissible fields impossible. What can be done is to expand $\ln \det_{\text{ren}}$, the one-loop effective action, in a power series in e. Then the functional

integration can be done term-by-term to obtain textbook QED.

The first nonperturbative calculation of $\det_{\text{ren}}$ was done by Heisenberg and Euler [1] 75 years ago for the special case of constant electric and magnetic fields. Their paper gave rise to a vast subfield known as quantum field theory under the influence of external conditions. A comprehensive review of this body of work relevant to $\det_{\text{ren}}$ is given by Dunne [2].

An outstanding problem is the strong field behavior of $\det_{\text{ren}}$ that goes beyond constant fields or slowly varying fields or special fields rapidly varying in one variable [2,3].[1] That is, what is the strong field behavior of $\det_{\text{ren}}$ for a class of random fields $F_{\mu\nu}$ on $\mathbb{R}^4$? What if $\ln \det_{\text{ren}}$ increases faster than a quadratic in $F_{\mu\nu}$ for such fields? Is $\det_{\text{ren}}$ integrable for any Gaussian measure in this case? This is a question with profound implications for the stability of QED in isolation. Of course, QED is part of the standard model, thereby making the overall stability question a much more intricate one. Nevertheless, the stability of QED in isolation remains unknown and deserves an answer.

In this paper, we consider the case of square-integrable, time-independent magnetic fields $\mathbf{B}(x)$ defined on $\mathbb{R}^3$. There are additional technical conditions on $\mathbf{B}$ introduced later. The magnetic field lines are typically twisted, tangled loops. We find that

$$\lim_{e \to \infty} \frac{\ln \det_{\text{ren}}}{e^2 \ln e} = \frac{\|\mathbf{B}\|^2 T}{24\pi^2}, \qquad (1.4)$$

where $\|\mathbf{B}\|^2 = \int d^3 x \mathbf{B} \cdot \mathbf{B}(x)$, and T is the size of the time box. Since e always multiplies $\mathbf{B}$, this means that $\ln \det_{\text{ren}}$ is growing faster than a quadratic in $\mathbf{B}$. In the constant field case, this result is formally equivalent to the

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[1]We note here progress in scalar QED$_4$ since the review [2] in going beyond these fields. Using the multidimensional worldline instanton technique, the vacuum pair production rate has been calculated from the one-loop effective action of a charged scalar particle in selected two- and three-dimensional electric fields [4]. These fields have to be sufficiently regular in order to define a formal functional semiclassical expansion of the quantum mechanical path integral representation of the effective action. The extension of this technique to spinor QED has not been done yet.

Heisenberg-Euler result [1] and to calculations relating the effective Lagrangian to the short-distance behavior of QED via its perturbative β-function [2]. What is notable here is that the strong coupling limit of $\ln \det_{\mathrm{ren}}$ is universal.

To achieve universality, the derivation of (1.4) must rely on general principles. One of these is the conjectured "diamagnetic" inequality for Euclidean three-dimensional QED, namely

$$|\det_{\mathrm{QED}_3}(1 - eS\!\!\!/A)| \leq 1. \qquad (1.5)$$

The fermion determinant in (1.5) is defined in Sec. II. The diamagnetic inequality is known to be true for lattice formulations of QED_3 obeying reflection positivity and using Wilson fermions [5–7]. Since Wilson fermions are *CP* invariant, there is no Chern-Simons term to interfere with the uniqueness of $\det_{\mathrm{QED}_3}$ [8]. And since $\det_{\mathrm{QED}_3}$ is gauge invariant, there are no divergences when the lattice spacing for the fermions is sent to zero. As stated by Seiler [7], (1.5) is more an obvious truth than a conjecture.

Since $\det_{\mathrm{QED}_3}|_{e=0} = 1$ and $\det_{\mathrm{QED}_3}$ has no zeros in e for real values of e when $m \neq 0$ [9], (1.5) can be rewritten as

$$0 < \det_{\mathrm{QED}_3} \leq 1. \qquad (1.6)$$

An inspection of Eq. (2.4) below indicates that (1.6) is a reflection of the tendency of an external magnetic field to lower the energy of a charged fermion. Therefore,

the historic heading of (1.5) and (1.6) as diamagnetic inequalities is a misnomer; paramagnetic inequalities would be a more accurate designation. The detailed justification for going from (1.5) to (1.6) is given in Sec. II.

The second general principle underlying (1.4) is the diamagnetism of charged spin-0 bosons in an external magnetic field. This is encapsulated in one of the versions of Kato's inequality discussed in Sec. III.

The final essential input to (1.4) is a restriction on the class of fields needed to obtain the limit. These restrictions are summarized in Sec. IV. As the foregoing remarks indicate, QED_3 is central to the derivation of (1.4), and it is to the connection between QED_3 and QED_4 that we now turn.

II. QED₃ AND QED₄

A. The connection

The connection has been dealt with previously [10]. In order to make this paper reasonably self-contained, we will review the relevant definitions and results. The upper bound on $\det_{\mathrm{ren}}$ obtained in [10] is not optimal; it will be optimized here.

The renormalized and regularized fermion determinant in Wick-rotated Euclidean QED_4 with on-shell renormalization, $\det_{\mathrm{ren}}$, may be defined by Schwinger's proper time representation [11]

$$\det{}_{\mathrm{ren}}(1 - eS\!\!\!/A) = \frac{1}{2}\int_0^\infty \frac{dt}{t}\left(\mathrm{Tr}\left\{e^{-P^2 t} - \exp\left[-(D^2 + \frac{e}{2}\sigma_{\mu\nu}F_{\mu\nu})t\right]\right\} + \frac{e^2\|F\|^2}{24\pi^2}\right)e^{-tm^2}, \qquad (2.1)$$

where $D_\mu = P_\mu - eA_\mu$, $\sigma_{\mu\nu} = (1/2i)[\gamma_\mu\gamma_\nu]$, $\gamma_\mu^\dagger = -\gamma_\mu$, $\|F\|^2 = \int d^4x F_{\mu\nu}^2(x)$, and e is assumed to be real. We choose the chiral representation of the γ-matrices so that

$$\sigma_{ij} = \begin{pmatrix} -\sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix},$$

$i, j, k = 1, 2, 3$ in cyclic order. Since we will consider time-independent magnetic fields, we set $A_\mu = (0, \mathbf{A}(x))$ with x in $\mathbb{R}^3$. Then (2.1) reduces to

$$\ln \det{}_{\mathrm{ren}} = \frac{T}{2}\int_0^\infty \frac{dt}{t}\left[\frac{2}{(4\pi t)^{1/2}}\mathrm{Tr}(e^{-P^2 t} - \exp\{-[(\mathbf{P} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}]t\}) + \frac{e^2\|\mathbf{B}\|^2}{12\pi^2}\right]e^{-tm^2}, \qquad (2.2)$$

where T is the dimension of the time box, and the factor 2 is from the partial spin trace. Clearly, we must have $\mathbf{B} \in L^2(\mathbb{R}^3)$. If $\mathbf{A}$ is assumed to be in the Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, then by the Sobolev-Talenti-Aubin inequality [12]

$$\int d^3x \mathbf{B}(x) \cdot \mathbf{B}(x) \geq \left(\frac{27\pi^4}{16}\right)^{1/3}\sum_{i=1}^3\left(\int d^3x |A_i(x)|^6\right)^{1/3}. \qquad (2.3)$$

So we must also have $\mathbf{A} \in L^6(\mathbb{R}^3)$.

In analogy with $\det_{\mathrm{ren}}$ in (2.1), without the charge renormalization subtraction, $\det_{\mathrm{QED}_3}$ may be defined by

$$\ln\det{}_{\mathrm{QED}_3}(m^2)$$
$$= \frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}(e^{-P^2 t} - \exp\{-[(\mathbf{P} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}]t\})e^{-tm^2}. \qquad (2.4)$$

This definition and regularization of $\det_{\mathrm{QED}_3}$ is parity conserving and gives no Chern-Simons term. Substituting (2.4) in (2.2) and, noting that $\pi^{-1}\int_0^\infty dE e^{-tE^2} = (4\pi t)^{-1/2}$, we obtain [10]

$$\ln \det_{\text{ren}} = \frac{2T}{\pi} \int_0^\infty dE \Big(\ln \det_{\text{QED}_3}(E^2 + m^2)$$
$$+ \frac{e^2 \|\mathbf{B}\|^2}{24 \pi^{3/2}} \int_0^\infty \frac{dt}{t^{1/2}} e^{-(E^2+m^2)t} \Big)$$
$$= \frac{T}{\pi} \int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \Big(\ln \det_{\text{QED}_3}(M^2) + \frac{e^2 \|\mathbf{B}\|^2}{24 \pi \sqrt{M^2}} \Big).$$
(2.5)

Result (2.5) will be referred to repeatedly in what follows.

B. Justification of (1.6)

Continuing our review of previous work, we turn to the derivation of the upper bound on $\ln \det_{\text{ren}}$ in (1.4). Since the degrees of divergence of the first-, second-, and third-order contributions to $\ln \det_{\text{QED}_3}$ are 2, 1, and 0, respectively, these must be dealt with separately. Their definition is obtained from the expansion of (2.4) through $O(e^3)$, resulting in

$$\ln \det_{\text{QED}_3}(1 - eS\!\!\!/A)$$
$$= -\frac{e^2}{4\pi} \int \frac{d^3k}{(2\pi)^3} |\hat{\mathbf{B}}(k)|^2 \int_0^1 dz \frac{z(1-z)}{[z(1-z)k^2 + m^2]^{1/2}}$$
$$+ \ln \det_4(1 - eS\!\!\!/A),$$
(2.6)

where $\ln \det_4$ defines the remainder and $\hat{\mathbf{B}}$ is the Fourier transform of $\mathbf{B}$. Definition (2.4) assigns the value of zero to the terms of order e and e^3. The argument of $\det_{\text{QED}_3}$ has been changed to indicate its origin as the formal ratio of QED$_3$ determinants $\det(\not{p} - e\not{A} + m)/\det(\not{p} + m)$. Note the minus sign in (2.6) pointing to paramagnetism.

The following theorems are essential for what follows:

Theorem 1 [6,13,14]. Let the operator $S\!\!\!/A$ in $\det_4$ be transformed by a similarity transformation to $K = (p^2 + m^2)^{1/4} S\!\!\!/A (p^2 + m^2)^{-1/4}$. This leaves the eigenvalues of $S\!\!\!/A$ invariant. Then K is a bounded operator on $L^2(\mathbb{R}^3, d^3x; \mathbb{C}^2)$ for $\mathbf{A} \in L^p(\mathbb{R}^3)$ for $p > 3$. Moreover, K is a compact operator belonging to the trace ideal $\mathcal{I}_p$, $p > 3$.

The trace ideal $\mathcal{I}_p (1 \le p < \infty)$ is defined as those compact operators A with $\|A\|_p^p = \text{Tr}((A^\dagger A)^{p/2}) < \infty$. From this it follows that the eigenvalues $1/e_n$ of $S\!\!\!/A$ obtained from (1.3) specialized to three dimensions are of finite multiplicity and satisfy $\sum_{n=1}^\infty |e_n|^{-p} < \infty$ for $p > 3$. The eigenfunctions ψ_n belong to the Sobolev space $L^2(\mathbb{R}^3, \sqrt{k^2 + m^2} d^3k; \mathbb{C})$. None of the e_n are real for $m \neq 0$ [9].

Theorem 2 [15–17]. Define the regularized determinant

$$\det_n(1 + A) = \det\Big[(1 + A) \exp\Big(\sum_{k=1}^{n-1} (-1)^k A^k / k \Big) \Big]. \quad (2.7)$$

Then $\det_n$ can be expressed in terms of the eigenvalues of $A \in \mathcal{I}_p$ for $n \ge p$.

Accordingly, $\det_4$ in (2.6) is defined and can be represented as [17]

$$\det_4(1 - eS\!\!\!/A) = \prod_{n=1}^\infty \Big[\Big(1 - \frac{e}{e_n} \Big) \exp\Big(\sum_{k=1}^3 \Big(\frac{e}{e_n} \Big)^k / k \Big) \Big].$$
(2.8)

The reality of $\det_4$ for real e and C-invariance require that the eigenvalues e_n appear in the complex plane as quartets $\pm e_n$, $\pm e_n^*$, or as imaginary pairs when $m \neq 0$. As expected, the expansion of $\ln \det_4$ in powers of e begins in fourth order.

We have established that $\det_4|_{e=0} = 1$ and that $\det_4$ has no zeros for real values of e. Therefore, by (2.6) $\det_{\text{QED}_3} > 0$ for all real e, thereby allowing one to go from (1.5) and (1.6). It might be objected that this is obvious, but we will need the detailed information introduced about $\det_4$ in the sequel.

The determinant $\det_4$ is an entire function of e considered as a complex variable, meaning that it is holomorphic in the entire complex e plane. Since $\sum_{n=1}^\infty |e_n|^{-3-\epsilon} < \infty$ for $\epsilon > 0$, its order is at most 3 [16,18]. This means that for any complex value of e, and positive constants A, K, $|\det_4| < A(\epsilon) \exp(K(\epsilon)|e|^{3+\epsilon})$ for any $\epsilon > 0$ From (1.6) and (2.6), and for real values of e

$$\ln \det_4 \le \frac{e^2}{4\pi} \int \frac{d^3k}{(2\pi)^3} |\hat{\mathbf{B}}(k)|^2$$
$$\times \int_0^1 dz \frac{z(1-z)}{[z(1-z)k^2 + m^2]^{1/2}}. \quad (2.9)$$

This is a truly remarkable inequality. Referring to (2.9), $\det_4$'s growth is slower on the real e axis than its potential growth in other directions. We also note that $\det_4$ is largely unknown. Even the reduction of the fourth-order term in its expansion to an explicitly gauge invariant form involving only $\mathbf{B}$ fields requires a huge effort when the fields are not constant [19]. The sixth-order reduction has not been completed as far as the author knows.

C. Upper bound on $\det_{\text{ren}}$

Insert (2.6) in (2.5) and get

$$\ln \det_{\text{ren}} = \frac{e^2 T}{4\pi^2} \int \frac{d^3k}{(2\pi)^3} |\hat{\mathbf{B}}(k)|^2 \int_0^\infty dz z(1-z)$$
$$\times \ln\Big[\frac{z(1-z)k^2 + m^2}{m^2} \Big]$$
$$+ \frac{T}{\pi} \int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \ln \det_4(M^2). \quad (2.10)$$

The objective here is to obtain the behavior of $\ln \det_{\text{ren}}$ when the coupling e is large, real, and positive. Since e always multiplies $\mathbf{B}$, we introduce the scale parameter $\mathcal{B} = \max_x |\mathbf{B}|$, which has the dimension of M^2. Why $\mathcal{B}$

is finite will be explained in Sec. III B. Then the integral in (2.10) is broken up into $\int_{m^2}^{e\mathcal{B}}$ and $\int_{e\mathcal{B}}^{\infty}$.

Substitution of (2.9) into the lower range integral gives

$$
\begin{aligned}
\ln\det_{\mathrm{ren}} \leq{}& \frac{e^2 T}{4\pi^2} \int \frac{d^3 k}{(2\pi)^3} |\hat{\mathbf{B}}(k)|^2 \int_0^1 dz z(1-z) \\
&\times \ln\left(\frac{4e\mathcal{B} + 2z(1-z)k^2 - 2m^2}{m^2}\right) \\
&+ \frac{T}{\pi} \int_{e\mathcal{B}}^{\infty} \frac{dM^2}{\sqrt{M^2 - m^2}} \ln\det_4(e\mathbf{B}, M^2). \quad (2.11)
\end{aligned}
$$

We have simplified the argument of the logarithm using $2\sqrt{xy} \leq x + y$ for $x, y \geq 0$. Then for $e\mathcal{B} \gg m^2$,

$$
\begin{aligned}
\ln\det_{\mathrm{ren}} \leq{}& \frac{e^2 T \|\mathbf{B}\|^2}{24\pi^2} \ln\left(\frac{4e\mathcal{B}}{m^2}\right) \\
&+ \frac{T}{\pi} \int_{e\mathcal{B}}^{\infty} \frac{dM^2}{\sqrt{M^2 - m^2}} \ln\det_4(e\mathbf{B}, M^2) \\
&+ O\left(\frac{eT \int d^3 x \mathbf{B} \cdot \nabla^2 \mathbf{B}}{\mathcal{B}}\right). \quad (2.12)
\end{aligned}
$$

The integral in (2.12) can be estimated by making a large mass expansion of $\ln\det_4$. This is facilitated by inserting (2.6) in (2.4) and examining the small t region of $\ln\det_4$'s resulting proper time representation. The details of this expansion are in Sec. 3B of [10], and give the result

$$
\begin{aligned}
\ln\det_4(e\mathbf{B}, M^2) \underset{M\to\infty}{=}{}& \frac{1}{2} \int_0^{\infty} \frac{dt}{t} (4\pi t)^{-3/2} e^{-tM^2} \int d^3 x \left[\frac{2}{45} e^4 t^4 (\mathbf{B} \cdot \mathbf{B})^2 + O(e^4 t^5 \mathbf{B} \cdot \mathbf{BB} \cdot \nabla^2 \mathbf{B})\right] \\
={}& \frac{e^4 \int (\mathbf{B} \cdot \mathbf{B})^2}{480\pi M^5} + O\left(\frac{e^4 \int \mathbf{B} \cdot \mathbf{BB} \cdot \nabla^2 \mathbf{B}}{M^7}\right). \quad (2.13)
\end{aligned}
$$

In the first line of (2.13), it is assumed that the heat kernel expansion is an asymptotic expansion in t in the strict sense of its definition, namely [20]

$$
\begin{aligned}
&< x|e^{-t[(\mathbf{P} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}]}|x > -(4\pi t)^{-3/2} \sum_{n=0}^{N} a_n(x) t^n \\
&\underset{t\to 0^+}{\sim} (4\pi t)^{-3/2} a_{N+1}(x) t^{N+1}. \quad (2.14)
\end{aligned}
$$

This must hold for every N. A necessary condition for (2.14) is that $\mathbf{B}$ be infinitely differentiable to ensure that each coefficient a_n is finite. As far as the author knows, it is not known yet if this is a sufficient condition. So (2.14) is an assumption that may require additional conditions on $\mathbf{B}$. Only coefficients a_n of $O(e^{2n})$, $n \geq 2$ are present in $\ln\det_4$'s expansion.

The t integration in (2.13), although extending to infinity, is limited to small t since $M \to \infty$ due to the parameter $e\mathcal{B}$ in (2.12). Substituting (2.13) in (2.12) results in

$$
\begin{aligned}
\ln\det_{\mathrm{ren}} \leq{}& \frac{e^2 \|\mathbf{B}\|^2 T}{24\pi^2} \ln\left(\frac{4e\mathcal{B}}{m^2}\right) + O\left(\frac{e^2 T \int (\mathbf{B} \cdot \mathbf{B})^2}{\mathcal{B}^2}\right) \\
&+ O\left(\frac{eT \int \mathbf{B} \cdot \nabla^2 \mathbf{B}}{\mathcal{B}}\right), \quad (2.15)
\end{aligned}
$$

or

$$
\lim_{e\to\infty} \frac{\ln\det_{\mathrm{ren}}}{e^2 \ln e} \leq \frac{\|\mathbf{B}\|^2 T}{24\pi^2}, \quad (2.16)
$$

consistent with (1.4). This bound is independent of the charge renormalization subtraction point. If the subtraction

were made at photon momentum $k^2 = \mu^2$ instead of $k^2 = 0$, then the $\ln m^2$ terms in (2.10) and (2.11) would be replaced with $\ln[z(1-z)\mu^2 + m^2]$, which has nothing to do with strong coupling.

The scaling procedure used here is designed to obtain the least upper bound on $\ln\det_{\mathrm{ren}}$. In [10], we chose to break up the M integral as $\int_{m^2}^{e^4\|\mathbf{B}\|^4}$ and $\int_{e^4\|\mathbf{B}\|^4}^{\infty}$. This resulted in a fast $1/e^4$ falloff of the $\ln\det_4$ terms compared to e^2 here, but gave a weaker upper bound on $\ln\det_{\mathrm{ren}}$, namely

$$
\lim_{e\to\infty} \frac{\ln\det_{\mathrm{ren}}}{e^2 \ln e} \leq \frac{\|\mathbf{B}\|^2 T}{6\pi^2}. \quad (2.17)
$$

We mention that the coefficient $1/960$ in (3.16) in [10] should be $1/360$.

Here we might have chosen a more general scaling, such as $e^{\alpha}(\ln e)^{\beta}\mathcal{B}$ or $e^{\alpha}(\ln \ln e)^{\beta}\mathcal{B}$, etc., with $\alpha \geq 1$, $\beta > 0$. Then the right-hand side of (2.16) would have been replaced with $\alpha\|\mathbf{B}\|^2 T/24\pi^2$. The case $\alpha < 1$ causes the first remainder term in (2.15) to be no longer subdominant. Therefore, our scaling $e\mathcal{B}$ is an optimal one.

III. LOWER BOUND ON $\det_{\mathrm{ren}}$

A. Fundamentals

On referring to (2.5) the lower bound on $\det_{\mathrm{ren}}$ will come from operations on $\ln\det_{\mathrm{QED}_3}$. We begin with the operator identity (A2) in Appendix A applied to $\ln\det_{\mathrm{QED}_3}$ in (2.4). Letting $X = (\mathbf{P} - e\mathbf{A})^2$ and $Y = -e\boldsymbol{\sigma} \cdot \mathbf{B}$, we obtain

$$\ln\det{}_{\text{QED}_3} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\,\text{Tr}\Big(e\int_0^t ds\,e^{-(t-s)(\mathbf{P}-e\mathbf{A})^2}\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s(\mathbf{P}-e\mathbf{A})^2} + e^2\int_0^t ds_1\int_0^{t-s_1} ds_2 e^{-(t-s_1-s_2)[(\mathbf{P}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{B}]}$$

$$\times\,\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s_2(\mathbf{P}-e\mathbf{A})^2}\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s_1(\mathbf{P}-e\mathbf{A})^2}\Big)e^{-m^2 t} + \frac{1}{2}\int_0^\infty\frac{dt}{t}\,\text{Tr}(e^{-\mathbf{P}^2 t} - e^{-(\mathbf{P}-e\mathbf{A})^2 t})e^{-tm^2}. \tag{3.1}$$

The spin trace in the first term is zero, and the last term, after tracing over spin, is the one-loop effective action of scalar QED_3,

$$\ln\det{}_{\text{SQED}_3} = \int_0^\infty\frac{dt}{t}e^{-tm^2}\,\text{Tr}(e^{-\mathbf{P}^2 t} - e^{-(\mathbf{P}-e\mathbf{A})^2 t}). \tag{3.2}$$

Thus,

$$\ln\det{}_{\text{QED}_3} = \ln\det{}_{\text{SQED}_3} - \frac{e^2}{2}\int_0^\infty\frac{dt}{t}e^{-tm^2}\,\text{Tr}\Big(\int_0^t ds_1\int_0^{t-s_1} ds_2 e^{-(t-s_1-s_2)[(\mathbf{P}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{B}]}\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s_2(\mathbf{P}-e\mathbf{A})^2}\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s_1(\mathbf{P}-e\mathbf{A})^2}\Big), \tag{3.3}$$

remembering that the factor $1/2$ in the last term of (3.1) is canceled by the spin trace.

Let $\Delta_A = [(\mathbf{P}-e\mathbf{A})^2 + m^2]^{-1}$. In Appendix B, it is shown that $\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}\in\mathcal{I}_2$; that is, it is a Hilbert-Schmidt operator provided $\mathbf{B}\in L^2$ and $m\neq 0$. Then (2.7) gives

$$\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}) = \ln\det[(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})e^{e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}}]$$

$$= \text{Tr}\,\ln[(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})e^{e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}}]$$

$$= \text{Tr}\Big[\int_0^\infty\frac{dt}{t}e^{-tm^2}(e^{-(\mathbf{P}-e\mathbf{A})^2 t} - e^{-[(\mathbf{P}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{B}]t} + e\Delta_A\boldsymbol{\sigma}\cdot\mathbf{B}\Big]$$

$$= -e^2\int_0^\infty\frac{dt}{t}e^{-tm^2}\int_0^t ds_1\int_0^{t-s_1} ds_2\,\text{Tr}(e^{-(t-s_1-s_2)[(\mathbf{P}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{B}]}$$

$$\times\,\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s_2(\mathbf{P}-e\mathbf{A})^2}\boldsymbol{\sigma}\cdot\mathbf{B}e^{-s_1(\mathbf{P}-e\mathbf{A})^2}). \tag{3.4}$$

In going from the penultimate to the last line in (3.4), use was again made of the identity (A2). Substituting (3.4) in (3.3) gives

$$\ln\det{}_{\text{QED}_3} = \frac{1}{2}\,\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})$$

$$+ \ln\det{}_{\text{SQED}_3}. \tag{3.5}$$

As $\ln\det_{\text{QED}_3}$ and $\ln\det_2$ are well-defined by our choice of fields, so is $\ln\det_{\text{SQED}_3}$ in (3.5). What has been accomplished here is to isolate the Zeeman term $\boldsymbol{\sigma}\cdot\mathbf{B}$ in $\ln\det_2$. Since $\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}$ is Hilbert-Schmidt and self-adjoint, $\ln\det_2$ is susceptible to extensive analytic analysis.

Substitute (3.5) in (2.5):

$$\ln\det{}_{\text{ren}} = \frac{T}{\pi}\int_{m^2}^\infty\frac{dM^2}{\sqrt{M^2-m^2}}\Big(\frac{1}{2}\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})$$

$$+ \ln\det{}_{\text{SQED}_3} + \frac{e^2\|\mathbf{B}\|^2}{24\pi\sqrt{M^2}}\Big). \tag{3.6}$$

We now introduce two central inequalities. The first relies on the diamagnetism of charged scalar bosons as expressed by Kato's inequality in the form [21,22]

$$\text{Tr}\,(e^{-(\mathbf{P}-e\mathbf{A})^2 t}) \leq \text{Tr}\,e^{-\mathbf{P}^2 t}. \tag{3.7}$$

This implies that on average the energy eigenvalues of such bosons rise in a magnetic field and hence by (3.2) that [22]

$$\ln\det{}_{\text{SQED}_3} \geq 0. \tag{3.8}$$

The second inequality is introduced beginning with the penultimate line of (3.4). Noting that the spin trace of the $\Delta_A\boldsymbol{\sigma}\cdot\mathbf{B}$ term is zero, then

$$\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})$$

$$= \int_0^\infty\frac{dt}{t}e^{-tm^2}\,\text{Tr}(e^{-t(\mathbf{P}-e\mathbf{A})^2} - e^{-[(\mathbf{P}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{B}]t}). \tag{3.9}$$

By the Bogoliubov-Peierls inequality [23,24]and Sec. 2.1, 8 of [25]

$$\text{Tr}\,e^{-[(\mathbf{P}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{B}]t} \geq \text{Tr}\,e^{-t(\mathbf{P}-e\mathbf{A})^2 t}e^{-te\langle\boldsymbol{\sigma}\cdot\mathbf{B}\rangle}, \tag{3.10}$$

where

$$\langle\boldsymbol{\sigma}\cdot\mathbf{B}\rangle = \frac{\text{Tr}(\boldsymbol{\sigma}\cdot\mathbf{B}e^{-t(\mathbf{P}-e\mathbf{A})^2 t})}{\text{Tr}\,e^{-(\mathbf{P}-e\mathbf{A})^2 t}} = 0. \tag{3.11}$$

Hence,

$$\ln \det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}) \le 0, \qquad (3.12)$$

which is consistent with (3.5) when combined with (1.6) and (3.8). There is another reason why (3.12) holds. Let $C = e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}$ Since C is Hilbert-Schmidt,

$$\begin{aligned}
\ln \det{}_2(1 - C) &= \ln \det[(1 - C)e^C]\\
&= \mathrm{Tr}[\ln(1 - C) + C]\\
&= \frac{1}{2}\,\mathrm{Tr}\,\ln(1 - C^2)\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\ln(1 - \lambda_n^2). \qquad (3.13)
\end{aligned}$$

The third line of (3.13) follows from the second since the trace over spin eliminates all odd powers of C. In the last

line, we introduced the real eigenvalues λ_n of $e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}$. Since $\ln \det_2$ is real and finite, then $|\lambda_n| < 1$ for all n, giving (3.12). Because $\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}\epsilon \mathcal{I}_2$, it is a compact operator, and so the λ_n are countable and of finite multiplicity.

Now consider

$$\begin{aligned}
&\frac{\partial}{\partial m^2}\ln\det{}_2(m^2)\\
&= \int_0^{\infty} dt\, e^{-tm^2}\,\mathrm{Tr}(e^{-[\mathbf{P}-e\mathbf{A}]^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}]t} - e^{(\mathbf{P}-e\mathbf{A})^2 t}) \ge 0,
\end{aligned}$$
$$(3.14)$$

by (3.9), (3.10), and (3.11). Therefore, $\det_2$ is a monotonically increasing function of m^2. Next, break up the M integral in (3.6) as in Sec. II C:

$$\begin{aligned}
\ln\det{}_{\mathrm{ren}} &= \frac{T}{\pi}\int_{m^2}^{e\mathcal{B}}\frac{dM^2}{\sqrt{M^2 - m^2}}\left(\frac{1}{2}\,\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}) + \ln\det{}_{\mathrm{SQED}_3} + \frac{e^2\|\mathbf{B}\|^2}{24\pi\sqrt{M^2}}\right)\\
&\quad + \frac{T}{\pi}\int_{e\mathcal{B}}^{\infty}\frac{dM^2}{\sqrt{M^2 - m^2}}\left(\ln\det{}_{\mathrm{QED}_3} + \frac{e^2\|\mathbf{B}\|^2}{24\pi\sqrt{M^2}}\right),
\end{aligned}$$
$$(3.15)$$

where we reinserted (3.5) into the upper-range M integral. By (3.14)

$$\int_{m^2}^{e\mathcal{B}}\frac{dM^2}{\sqrt{M^2 - m^2}}\ln\det{}_2(M^2) \ge \ln\det{}_2|_{M^2 = m^2}\int_{m^2}^{e\mathcal{B}}\frac{dM^2}{\sqrt{M^2 - m^2}} = 2\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})|_{M^2 = m^2}\sqrt{e\mathcal{B} - m^2}.$$
$$(3.16)$$

Hence, (3.8) and (3.16) result in (3.15) becoming

$$\begin{aligned}
\ln\det{}_{\mathrm{ren}} &\ge \frac{T}{\pi}\sqrt{e\mathcal{B} - m^2}\ln\det{}_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2})|_{M^2 = m^2} + \frac{e^2 T\|\mathbf{B}\|^2}{24\pi^2}\,\ln\left(\frac{e\mathcal{B}}{m^2}\right)\\
&\quad + \frac{e^2 T}{12\pi^2}\|\mathbf{B}\|^2\ln\left(1 + \sqrt{1 - \frac{m^2}{e\mathcal{B}}}\right) + \frac{T}{\pi}\int_{e\mathcal{B}}^{\infty}\frac{dM^2}{\sqrt{M^2 - m^2}}\left(\ln\det{}_{\mathrm{QED}_3} + \frac{e^2\|\mathbf{B}\|^2}{24\pi\sqrt{M^2}}\right). \qquad (3.17)
\end{aligned}$$

We now turn to the strong coupling behavior of $\ln\det_2$.

B. Strong coupling behavior of $\ln\det_2$

The eigenvalues λ_n in (3.13) are obtained from

$$e\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}\varphi_n = \lambda_n\varphi_n, \qquad (3.18)$$

for $\varphi_n \in L^2$ following the remark under (B3) in Appendix B. Letting $\Delta_A^{1/2}\varphi_n = \psi_n$ gives

$$\left[(\mathbf{P} - e\mathbf{A})^2 - \frac{e\boldsymbol{\sigma}\cdot\mathbf{B}}{\lambda_n}\right]\psi_n = -m^2\psi_n, \qquad (3.19)$$

where $\psi_n \in L^2$ provided $m \ne 0$. This follows from (B5) and Young's inequality (B7). The requirement that $m \ne 0$ follows from the role of the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ as adjustable coupling constants whose discrete values result in bound states with energy $-m^2$ for a fixed value of e.

Since the operator $(\mathbf{P} - e\mathbf{A})^2 - e\boldsymbol{\sigma}\cdot\mathbf{B} \ge 0$, such bound states are impossible unless $|\lambda_n| < 1$ for all n, which is the physical reason why (3.12) is true. Inspection of (3.19) suggests that as e increases $|\lambda_n|$ likewise increases for fixed n to maintain the bound state energy at $-m^2$. This is illustrated by the constant field case that is excluded from our analysis:

$$|\lambda_n| = \frac{|eB|}{(2n + 1)|eB| + m^2}, \qquad n = 0, 1, \dots. \quad (3.20)$$

Because the operator $\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}$ is Hilbert-Schmidt, the eigenfunction φ_n has finite multiplicity, and the λ_n in (3.13) are counted up to this multiplicity. To estimate the multiplicity, note that the eigenfunctions φ_n and ψ_n are in one-to-one correspondence. Next, note that for $\psi \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ and a generic λ with $|\lambda| < 1$,

$$\left(\psi, \left[(\mathbf{P} - e\mathbf{A})^2 - \frac{e}{\lambda}\boldsymbol{\sigma} \cdot \mathbf{B}\right]\psi\right)$$

$$\geq \left(\psi, \left[(\mathbf{P} - e\mathbf{A})^2 - \left|\frac{e}{\lambda}\right||\mathbf{B}|\right]\psi\right). \qquad (3.21)$$

Thus, the Hamiltonian on the left, H_+, dominates that on the right, H_-. Let $N_{-m^2}(H)$ denote the dimension of the spectral projection onto the eigenstates of Hamiltonian H with eigenvalues less than or equal to $-m^2$. Because $H_+ \geq H_-$, then $N_{-m^2}(H_+) \leq N_{-m^2}(H_-)$. $N_{-m^2}(H_+)$ is an overestimate of the number of the bound states of H_+ at $-m^2$ for a fixed value of λ but satisfactory for our purpose here.

By the Cwinkel-Lieb-Rozenblum bound in the form [26]

$$N_{-m^2}(H_-) \leq C \int d^3x \left[\left|\frac{e}{\lambda}\right||\mathbf{B}(x)| - m^2\right]_+^{3/2}, \quad (3.22)$$

where $[a]_+ = \max(a, 0)$ and $C = 2 \times 0.1156$. The factor 2 accounts for the additional spin degrees of freedom in the present estimate. Since $|\lambda_n| = O(1)$, we are confident that the degeneracy/multiplicity associated with each λ_n in (3.13) does not exceed $c|e|^{3/2}\int d^3x|\mathbf{B}|^{3/2}$, where $c \geq 0.2312$ is another finite constant. This estimate has to be modified for values of $n > N$ beyond which λ_n assumes its asymptotic form as discussed below. Therefore, for $n \leq N$ we will estimate the sum in (3.13) by factoring out the common maximal degeneracy $c|e|^{3/2}\int d^3x|\mathbf{B}|^{3/2}$ and treat each λ_n in the factored sum as having multiplicity equal to one. Those λ_n, if any, that vanish as $e \to \infty$ give a subdominant contribution to $\ln\det_2$ in (3.13) since by inspection their contribution grows at most as $\lambda_n^2|e/\lambda_n|^{3/2}$.

We now turn to the large e dependence of λ_n. From here on we assume that ψ_n is normalized to one. By C-invariance we may assume $e > 0$. Now consider the expectation value of (3.19):

$$\langle n|(\mathbf{P} - e\mathbf{A})^2|n\rangle - \frac{e}{\lambda_n}\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle = -m^2. \qquad (3.23)$$

From (3.23) if $\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle > 0$ then $\lambda_n > 0$ and *vice versa*. Therefore, we need only consider $\lambda_n > 0$ and write

$$\lambda_n = \left[\frac{\langle n|(\mathbf{P} - e\mathbf{A})^2|n\rangle}{e\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle} + \frac{m^2}{e\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle}\right]^{-1}, \qquad (3.24)$$

where $\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle \neq 0$ as (3.23) must be satisfied. The case $\lambda_n = 0$ for some n corresponding to $\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle = 0$ can be ignored as $\lambda_n = 0$ contributes nothing to $\ln\det_2$ in (3.13). An easy estimate gives

$$|(\psi_n, \boldsymbol{\sigma} \cdot \mathbf{B}\psi_n)| \leq (\psi_n, |\mathbf{B}|\psi_n) \leq \max_x|\mathbf{B}(x)|. \qquad (3.25)$$

Because $\mathbf{B} \in L^2$ and is assumed infinitely differentiable, then $\max_x|\mathbf{B}|$ is finite. Hence, $\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle$ is a bounded function of e and n.

Now consider the ratio $R_n = \langle n|(\mathbf{P} - e\mathbf{A})^2|n\rangle/e\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle$ in (3.24). The case $R_n \to_{e\gg1} 0$ is ruled out

since this implies $\lambda_n \to \infty$. The case $R_n \to_{e\gg1} \infty$ implies $\lambda_n \to 0$, which gives a subdominant contribution to (3.13) as discussed above. The final possibility is $1 \leq R_n < \infty$ for $e \to \infty$. The case $R_n \to 1$ for $e \to \infty$ happens if $\langle n|(\mathbf{P} - e\mathbf{A})^2|n\rangle \sim e\langle n|\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle$. Since $\psi_n \in L^2$, $\langle n|(\mathbf{P} - e\mathbf{A})^2 - e\boldsymbol{\sigma} \cdot \mathbf{B}|n\rangle = 0$ implies $\boldsymbol{\sigma} \cdot (\mathbf{P} - e\mathbf{A})\psi_n = 0$. Now this may happen for the $\mathbf{B}$ fields considered so far. But if we exclude zero-mode supporting $\mathbf{B}$ fields [27] from our analysis it cannot. By so doing, we can exclude the case $\lambda_n = 1 - \delta_n(e)$, $\delta_n(\infty) = 0$. We will see below why this is necessary.

We proceed to estimate the strong coupling limit of $\ln\det_2$ in (3.13). First, consider the sum for $n \leq N$. We need only consider $0 < |\lambda_n| < 1$ for all e, including $e = \infty$ as concluded above. Hence, on factoring out the common maximal multiplicity of the λ_n we get

$$\lim_{e\gg1}\left|\sum_{n=1}^N \ln(1 - \lambda_n^2)\right| \leq c_1 e^{3/2}\int d^3x|\mathbf{B}|^{3/2}, \qquad (3.26)$$

where c_1 is a constant and noting again that the eigenvalues $\lambda_n \to 0$ are subdominant.

Since $\lambda_n \to 0$ for $n \to \infty$ and $1/2 \leq |\ln(1 - \lambda_n^2)/\lambda_n^2| \leq 3/2$ for $\lambda_n^2 < 1/2$, the absolute convergence of the series in (3.13) requires $\sum_{n=1}^\infty \lambda_n^2 < \infty$. Consider this sum for $n > N$ and indicate the degeneracy factors μ_n explicitly:

$$S \equiv \sum_{n>N}^\infty \mu_n(e)\lambda_n^2(e). \qquad (3.27)$$

We estimated from (3.22) that $\mu_n \leq c|e/\lambda_n|^{3/2}\int d^3x|\mathbf{B}|^{3/2}$. So

$$S \leq ce^{3/2}\int d^3x|\mathbf{B}|^{3/2}\underset{\substack{n>N \\ \text{no degeneracy}}}{\overset{\infty}{\sum}}|\lambda_n|^{1/2} < \infty. \qquad (3.28)$$

This implies that for $n > N$

$$|\lambda_n(e)| = \frac{C_n(e)}{n^{2+\epsilon}}, \qquad (3.29)$$

where $\epsilon > 0$ and C_n is a bounded function of n and e with $\lim_{e\to\infty}C_n(e) < \infty$. Otherwise, $|\lambda_n| < 1$ for any n cannot be satisfied. Accordingly, the series in (3.28) is uniformly convergent in e by the Weierstrass M test and so

$$\lim_{e\to\infty}\left|\sum_{n>N}^\infty \ln(1 - \lambda_n^2)\right|/e^{3/2} \leq c_2\int d^3x|\mathbf{B}|^{3/2}, \qquad (3.30)$$

where c_2 is a constant. From (3.13), (3.26), and (3.30), we conclude

$$\lim_{e\to\infty}|\ln\det_2(1 - e\Delta_A^{1/2}\boldsymbol{\sigma} \cdot \mathbf{B}\Delta_A^{1/2})|/e^{3/2}$$

$$\leq c_3\int d^3x|\mathbf{B}|^{3/2}, \qquad (3.31)$$

where c_3 is another constant.

As a check on (3.31), refer to (3.5). For $\mathbf{B} \in L^{3/2}(\mathbb{R}^2)$, we found [10]

$$\ln \det_{\mathrm{QED}_3} \geq -\frac{Ze^{3/2}}{6\pi} \int d^2x |B(x)|^{3/2}, \qquad (3.32)$$

for $B(x) \geq 0$ or $B(x) \leq 0$, $x \in \mathbb{R}^2$. Z is the dimension of the remaining space box. We know that $\ln \det_2 \leq 0$ and $\ln \det_{\mathrm{SQED}_3} \geq 0$ in (3.5). Specializing (3.31) to these $\mathbf{B}$ fields, it is seen that the strong coupling growth of $\ln \det_2$ is consistent with (3.32).

Finally, if zero-mode supporting $\mathbf{B}$ fields were allowed, we would have obtained $\ln \det_2 =_{e \gg 1} O(e^{3/2} \ln \rho(e))$, $\rho(e) \to_{e \gg 1} 0$, since when $|\lambda_n| \sim_{e \gg 1} 1 - \delta_n(e)$, $\delta_n(\infty) = 0$, the logarithm in (3.13) gives an additional factor $\ln \delta_n$. As will be seen below the limit (1.4) requires $\lim_{e \to \infty} \ln \det_2 / e^{3/2} = $ finite (or zero).

C. Strong coupling limit of (3.17)

It remains to estimate the large coupling limit of the last term in (3.17),

$$
\begin{aligned}
I &\equiv \frac{T}{\pi} \int_{e\mathcal{B}}^{\infty} \frac{dM^2}{\sqrt{M^2 - m^2}} \left(\ln \det_{\mathrm{QED}_3} + \frac{e^2 \|\mathbf{B}\|^2}{24\pi \sqrt{M^2}} \right) \\
&= \frac{T}{\pi} \int_{e\mathcal{B}}^{\infty} \frac{dM^2}{\sqrt{M^2 - m^2}} \left(-\frac{e^2}{4\pi} \int \frac{d^3k}{(2\pi)^2} |\hat{\mathbf{B}}(k)|^2 \right. \\
&\quad \times \int_0^1 dz \frac{z(1-z)}{[z(1-z)k^2 + M^2]^{1/2}} + \frac{e^2 \|\mathbf{B}\|^2}{24\pi \sqrt{M^2}} \\
&\quad \left. + \ln \det_4 (1 - eS\mathcal{A}) \right),
\end{aligned}
\qquad (3.33)
$$

where we substituted (2.6) for $\ln \det_{\mathrm{QED}_3}$. Calculation of the first two terms in (3.33) is straightforward. The last term has already been estimated in Sec. II and is given by the second term in (2.15). Hence,

$$I = O\left(\frac{e^2 T \int (\mathbf{B} \cdot \mathbf{B})^2}{\mathcal{B}^2} \right) + O\left(\frac{eT \int \mathbf{B} \cdot \nabla^2 \mathbf{B}}{\mathcal{B}} \right). \qquad (3.34)$$

Taking into account (3.31) and (3.34), we obtain from (3.17)

$$\lim_{e \to \infty} \frac{\ln \det_{\mathrm{ren}}}{e^2 \ln e} \geq \frac{\|\mathbf{B}\|^2 T}{24\pi^2}. \qquad (3.35)$$

Equations. (2.16) and (3.35) therefore establish (1.4).

IV. SUMMARY

The two assumptions underlying (1.4) are first that the continuum limit of the lattice diamagnetic inequality coincides with (1.5), and second that the heat kernel expansion of the Pauli operator in (2.14) is an asymptotic series. These assumptions can and should be proven or falsified.

In addition, the result (1.4) assumes that the vector potential and magnetic field satisfy the following conditions:

$\mathbf{B} \in L^2(\mathbb{R}^3)$ to define $\ln \det_{\mathrm{ren}}$ in (2.5) and to ensure that $\Delta_A^{1/2} \boldsymbol{\sigma} \cdot \mathbf{B} \Delta_A^{1/2} \in \mathcal{I}_2$ following Appendix B. In addition $\mathbf{B} \in L^{3/2}(\mathbb{R}^3)$ in order that the degeneracy estimate in (3.22) is defined. To ensure that the bound in (3.31) holds, zero-mode supporting $\mathbf{B}$ fields are excluded. Also, $\mathbf{B}$ must be infinitely differentiable (C^∞) to ensure that the expansion coefficients in (2.14) are finite.

If $\mathbf{A}$ is assumed to be in the Coulomb gauge, then by (2.3) $\mathbf{A} \in L^6(\mathbb{R}^3)$. If $\mathbf{B} \in L^{3/2}(\mathbb{R}^3)$, then $\mathbf{A} \in L^3(\mathbb{R}^3)$ by the Sobolev-Talenti-Aubin inequality [12]. In order to define $\det_{\mathrm{QED}_3}$, it is necessary to assume $\mathbf{A} \in L^r(\mathbb{R}^3)$, $r > 3$, following the discussion under (2.6). If $\mathbf{A} \in L^3(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$, then $\mathbf{A} \in L^r(\mathbb{R}^3)$, $3 < r < 6$ also. This follows from Hölder's inequality [28]

$$\|fg\|_r \leq \|f\|_p \|g\|_q, \qquad (4.1)$$

with $p^{-1} + q^{-1} = r^{-1}$, $p, q, r \geq 1$. Since $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{B} \in C^\infty$, then $\mathbf{A} \in C^\infty$.

We note that the sample functions $A_\mu(x)$ supporting the Gaussian measure in (1.2) with probability one are not C^∞. It is generally accepted that they belong to $\mathcal{S}'(\mathbb{R}^4)$, the space of tempered distributions. Therefore, we point out here that the C^∞ functions we introduced can be related to $A_\mu \in \mathcal{S}'(\mathbb{R}^4)$ by the convoluted field $A_\mu^\Lambda(x) = \int d^4y f_\Lambda(x - y) A_\mu(y) \in C^\infty$, provided $f_\Lambda \in \mathcal{S}(\mathbb{R}^4)$, the functions of rapid decrease. Then the Fourier transform of the covariance $\int d\mu(A) A_\mu^\Lambda(x) A_\nu^\Lambda(y)$ derived from (1.2) is $\hat{D}_{\mu\nu}(k) |\hat{f}_\Lambda(k)|^2$, where $\hat{f}_\Lambda \in C^\infty$. Since QED_4 must be ultraviolet regulated before renormalizing, $\hat{f}_\Lambda$ can serve as the regulator by choosing, for example, $\hat{f}_\Lambda = 1$, $k^2 \leq \Lambda^2$, and $\hat{f}_\Lambda = 0$, $k^2 \geq 2\Lambda^2$. So the need to regulate can serve as a natural way to introduce C^∞ background fields A_μ^Λ into $\det_{\mathrm{ren}}$—but not the rest of $d\mu$ in (1.1)—and into whatever else one is calculating. This procedure is a generalization of that used in the two-dimensional Yukawa model [29].

Finally, the obvious generalization of (1.4) for an admissible class of fields on $\mathbb{R}^4$ is

$$\lim_{e \to \infty} \frac{\ln \det_{\mathrm{ren}}}{e^2 \ln e} = \frac{1}{48\pi^2} \int d^4x F_{\mu\nu}^2(x). \qquad (4.2)$$

There is no chiral anomaly term since $F_{\mu\nu}$ falls off faster than $1/|x|^2$ and $\int d^4x \tilde{F}_{\mu\nu} F_{\mu\nu} = \int d^4x \partial_\alpha (\epsilon_{\alpha\beta\mu\nu} A_\beta F_{\mu\nu}) = 0$, where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$. Equation (4.2) remains to be verified.

If (1.4) and (4.2) do indeed indicate instability, then they are yet another reason why QED should not be considered in isolation.

APPENDIX A

The operator identity on which (3.1) is based is obtained as follows. Let [30]

$$F_t = e^{-t(X+Y)}e^{tX}.$$

Then

$$\frac{dF_t}{dt} = -e^{-t(X+Y)}Ye^{tX}.$$

Integrating gives

$$e^{-t(X+Y)} - e^{-tX} = -\int_0^t ds\, e^{-(t-s)(X+Y)}Ye^{-sX}, \quad \text{(A1)}$$

known as Duhamel's formula. Iterating once gives the required identity:

$$e^{-t(X+Y)} - e^{-tX}$$
$$= -\int_0^t ds\, e^{-(t-s)X}Ye^{-sX} + \int_0^t ds_1$$
$$\times \int_0^{t-s_1} ds_2\, e^{-(t-s_1-s_2)(X+Y)}Ye^{-s_2X}Ye^{-s_1X}. \quad \text{(A2)}$$

APPENDIX B

Here we show that the operator $K = \Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2} \in I_2$ and hence that K is Hilbert-Schmidt. This follows [14,15,28] if and only if K is a bounded operator on $L^2(\mathbb{R}^3, d^3x; \mathbb{C}^2)$ having a representation of the form

$$(Kf)(x) = \int \mathcal{K}(x, y)f(y)d^3y, \quad f \in L^2, \quad \text{(B1)}$$

where

$$\mathcal{K}(x, y) = \langle x|\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2}|y\rangle, \quad \text{(B2)}$$

and where $\mathcal{K} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3; d^3x \times d^3y)$. Moreover,

$$\|K\|_2^2 = \int |\mathcal{K}(x, y)|^2 d^3x\, d^3y. \quad \text{(B3)}$$

If it can be shown that $\mathcal{K} \in L^2$, then it trivially follows that K maps L^2 into itself. So consider

$$\|\mathcal{K}\|_{L^2} = 2\sum_i \int d^3x\, d^3y\, B_i(x)\Delta_A(x, y)B_i(y)\Delta_A(y, x)$$
$$\leq 2\sum_i \int d^3x\, d^3y\, |B_i(x)|\|\Delta_A(x, y)\||B_i(y)|\|\Delta_A(y, x)|. \quad \text{(B4)}$$

A form of Kato's inequality [5,21,31] asserts that the interacting scalar propagator is bounded by the free propagator

$$|\Delta_A(x, y)| \leq \Delta(x - y), \quad \text{(B5)}$$

where $\Delta(x) = (4\pi|x|)^{-1}e^{-m|x|}$ in three dimensions. Then

$$\|\mathcal{K}\|_{L^2} \leq \frac{1}{8\pi^2}\sum_i \int d^3x\, d^3y\, |B_i(x)|$$
$$\times \left|\frac{1}{|x-y|^2}e^{-2m|x-y|}\right||B_i(y)|. \quad \text{(B6)}$$

By Young's inequality in the form [23]

$$\left|\int d^3x\, d^3y\, f(x)g(x-y)h(y)\right| \leq \|f\|_p\|g\|_q\|h\|_r, \quad \text{(B7)}$$

where $p^{-1} + q^{-1} + r^{-1} = 2$, p, q, $r \geq 1$, and $\|f\|_p = (\int d^3x|f(x)|^p)^{1/p}$ etc., obtain from (B6) with $p = r = 2$, $q = 1$

$$\|\mathcal{K}\|_{L^2} \leq \frac{1}{8\pi^2}\sum_i \|B_i\|^2 \int d^3x\, e^{-2m|x|}/x^2$$
$$= \|\mathbf{B}\|^2(4\pi m)^{-1}. \quad \text{(B8)}$$

Therefore, by the theorem that began this appendix, $\Delta_A^{1/2}\boldsymbol{\sigma}\cdot\mathbf{B}\Delta_A^{1/2} \in I_2$ when $m \neq 0$ and $\mathbf{B} \in L^2$. We mention that this can be proved even when $m = 0$ provided $\mathbf{B} \in L^{3/2}$.

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