One-loop correction to the energy of spinning strings in $S^5$

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We revisit the computation of the 1-loop AdS$_5 \times S^5$ superstring sigma model correction to the energy of a closed circular string rotating in $S^5$. The string is spinning around its center of mass with two equal angular momenta $J_1 = J_2$ and its center of mass angular momentum is $J$. We revise the argument that the 1-loop correction is suppressed by $\frac{1}{\sqrt{\lambda}}$ factor $[J = J_1 + 2J_2]$ is the total SO(6) spin relative to the classical term in the energy and use numerical methods to compute the leading 1-loop coefficient. The corresponding gauge-theory result is known only in the $J_1 \to 0$ limit when the string solution becomes unstable and thus the 1-loop shift of the energy formally contains an imaginary part. While the comparison with gauge-theory may not be well defined in this case, our numerical string-theory value of the 1-loop coefficient seems to disagree with the gauge-theory one. A plausible explanation should be in the different order of limits taken on the gauge-theory and the string-theory sides of the anti-de Sitter/conformal field theory duality.

I. INTRODUCTION

Recently, there was an interesting progress in understanding anti-de Sitter (AdS)/conformal field theories (CFT) duality by extending the Berenstein-Maldacena-Nastase (BMN) approach [1] to other sectors of semiclassical [2] string states (see [3] for reviews and references).

In general, for a classical rotating closed string solution in $S^5$ its energy has a regular expansion [4–8] $E_0 = J + c_1 \frac{\lambda}{\sqrt{\lambda}} + c_2 \frac{\lambda^2}{\sqrt{\lambda}} + \ldots = J(1 + c_1 \lambda + c_2 \lambda^2 + \ldots)$, where $J$ is the total SO(6) spin $J = \sum_{i=1}^{3} J_i$ and $\frac{\lambda}{\sqrt{\lambda}}$ is an effective semiclassical expansion parameter. $c_n = c_a(\frac{\lambda}{J})^n$ are functions of ratios of the spins which are finite in the semiclassical string-theory limit $J_i \gg 1$, $\frac{\lambda}{\sqrt{\lambda}}$ is fixed. Generic 3-spin solutions are described by an integrable Neumann model [7,8] and the coefficients $c_a$ are expressed in terms of genus-two hyperelliptic functions.

Formally, string $\alpha'$ corrections are suppressed in the limit $J \to \infty$, $\frac{\lambda}{\sqrt{\lambda}}$ is fixed since $\alpha' = \frac{\alpha^2}{\sqrt{\lambda}} \sim \frac{1}{J \sqrt{\lambda}}$. However, to expect [4] to be able to compare these classical energies to the super Yang-Mills (SYM) anomalous dimensions [9–11] one should check that the $\frac{1}{\sqrt{\lambda}}$ corrections are again analytic in $\lambda$ (as they are in the BMN case [12–14]), i.e., the expansion in large $J$ and small $\lambda$ is well defined on the string side,

\[ E = J \left[ 1 + \lambda \left( c_1 + \frac{d_1}{J} + \ldots \right) + \lambda^2 \left( c_2 + \frac{d_2}{J} + \ldots \right) + \ldots \right], \]

\[ \lambda = \frac{\lambda}{J^2}. \]

with the classical energy being the $J \to \infty$ limit of the exact expression.

This question was first addressed in [5] on the example of the simplest stable 3-spin solution of [4]: a circular string orbiting in $S^5$ with center of mass angular momentum $J$ and two equal SO(6) angular momenta $J_1 = J_2$. The nonzero SO(6) spin components are

\[ \sigma_{\mu
u
\alpha}\sigma_{\mu
u
\beta}\sigma_{\mu
u
\gamma}\sigma_{\mu
u
\delta}\sigma_{\mu
u
\epsilon} \] $'$

\[ \bigg( d\gamma^1 + \cos^2 \gamma d\varphi_1^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_2^2 + \sin^2 \psi d\varphi_3^2) \bigg) \]$'$

the solution is [4,5] (see also Appendix A): $t = \kappa \tau, \, \gamma = \gamma_0, \, \varphi_1 = \nu \tau, \, \varphi_2 = \varphi_3 = \mu \tau, \, \psi = k \tau$, where $k, \, \gamma_0, \nu \, w$ are constants, $k$ is an integer and $w^2 = \nu^2 + k^2, \, \nu^2 = \kappa^2 - 2k^2q, \, q = \sin^2 \gamma_0$. The three independent parameters are $k, \, \gamma_0$, and $q$. The nonzero SO(6) spin components are $J_1 = \sqrt{\lambda} \nu (1 - q), \, J_2 = J_3 = \frac{1}{2} \sqrt{\lambda} \sqrt{\nu^2 + k^2q}$. The classical energy $E = \sqrt{\lambda} \kappa$ can then be represented as a function of the spins $E = E(J_2, J, k; \lambda)$.

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1Here we change the notation compared to [5]: there we had $J_1 = J, \, J_2 = J_3 = J'$. Below $J$ will stand for the total angular momentum $J = J_1 + 2J_2$. Written in terms of the AdS$_5$ time coordinate $t$ and the angles of $S^5$ (with the metric (ds$^5$)$_{S^5} = d\gamma^2 + \cos^2 \gamma d\varphi_1^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_2^2 + \sin^2 \psi d\varphi_3^2)$) the solution is [4,5] (see also Appendix A): $t = \kappa \tau, \, \gamma = \gamma_0, \, \varphi_1 = \nu \tau, \, \varphi_2 = \varphi_3 = \mu \tau, \, \psi = k \tau$, where $k, \, \gamma_0, \nu \, w$ are constants, $k$ is an integer and $w^2 = \nu^2 + k^2, \, \nu^2 = \kappa^2 - 2k^2q, \, q = \sin^2 \gamma_0$. The three independent parameters are $k, \, \gamma_0$, and $q$. The nonzero SO(6) spin components are $J_1 = \sqrt{\lambda} \nu (1 - q), \, J_2 = J_3 = \frac{1}{2} \sqrt{\lambda} \sqrt{\nu^2 + k^2q}$. The classical energy $E = \sqrt{\lambda} \kappa$ can then be represented as a function of the spins $E = E(J_2, J, k; \lambda)$.

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form implying that the corresponding quadratic fluctuation action has constant coefficients (as in the BMN case corresponding to the limit \(J_2 = J_3 = 0\)). As a result, the fluctuation frequencies which determine the 1-loop correction to the energy (conjugate to the AdS\(_3\) time \(t = \kappa \tau\))

\[ E_1 = \frac{1}{2\kappa} \left( \sum_{n \in \mathbb{Z}} \omega_n^B - \sum_{r \in \mathbb{Z} + 1/2} \omega_r^F \right) \tag{3} \]

can be readily found. Still, \(\omega\)'s are given [5] by the roots of 4th-order polynomials (see Appendix A) and thus are rather involved functions of \(J_2, J\), and \(k\), making it difficult to compute the infinite sums in (3). Attempting to evaluate (3) analytically, in [5] the sums were converted into integrals, but it turns out that this direct procedure fails due to a singularity of the functions involved.

In this paper we shall first improve the general argument in [5] about the form (1) of the expansion of \(E_1\) at large \(J\) and small \(~\lambda\) and then use numerical methods to evaluate the first subleading coefficient \(d_1\).

A motivation behind this work is to compare the 1-loop string correction to the corresponding \(1/2\) correction in the anomalous dimensions of the SYM operators \(\text{tr}(\Phi_1^J \Phi_2^J \Phi_3^J) + \ldots\). On the gauge-theory side, one first expands in \(\lambda\) and then expands in \(1/J\), so that the anomalous dimensions should have the structure

\[ \Delta = J + \lambda \left( a_1 \frac{d_1}{J} + a_2 \frac{d_2}{J^2} + \ldots \right) + \lambda^2 \left( b_1 \frac{d_1}{J^2} + b_2 \frac{d_2}{J^3} + \ldots \right) + \ldots \tag{4} \]

The form of this expansion in the 2-spin [SU(2)] sector was indeed verifed to first few leading orders in [10,15]. Moreover, it was checked on specific examples [10,16–18] and also in general [19–21] that the expressions for \(a_1\) and \(a_2\) match the coefficients \(c_1, c_2\) in (1). Similar conclusion \((a_1 = c_1)\) was reached in the SU(3) sector [11,22–24] (fluctuations near the circular 3-spin solution of [4] also match [25]).

However, it was observed in [18,26] that disagreements start at \(\lambda^3\) order, \(a_1 \neq c_1\), with a plausible (“order-of-limits”) explanation suggested in [18,27,28]. For that reason, it would be interesting to see if the \(1/2\) subleading coefficient \(b_1\) in (4) agrees with the 1-loop coefficient \(d_1\) in (1). So far, \(b_1\) was computed [15] only for a specific 2-spin Bethe ansatz state corresponding to an unstable state on the string-theory side for which \(d_1\) formally has an imaginary part. In that case, a priori the comparison does not seem to be well defined. Apart from clarifying this issue, it remains to compute \(b_1\) for the 3-spin state with \(J_1 \neq 0\), extending the Bethe ansatz analysis of [11] where \(a_1\) was found. Once this is done, one will be in position to compare to the results for \(d_1\) on the string side presented below.

An attempt of comparison of our numerical result for \(d_1\) in (1) for the 2-spin (unstable) case with the gauge-theory result of [15] for \(b_1\) indicates a disagreement (see Sec. III). We suspect that the disagreement may remain also in the stable 3-spin case. This seems also to suggest that a similar “1-loop” (order \(\lambda\)) disagreement may be present for the \(1/2\) correction to scaling dimensions of BMN operators. An explanation of these disagreements may be again related to the noncommutativity [18,27] of the “string-theory” (large \(J\), then small \(~\lambda\)) and the “gauge-theory” (small \(\lambda\), then large \(J\)) limits.

II. STRUCTURE OF ONE-LOOP CORRECTION

In [5] it was attempted to find the one-loop correction \(E_1\) in (3) as an expansion in \(1/\kappa = \bar{\lambda} + \ldots\), i.e.,

\[ E_1 = \frac{1}{\kappa} e_1(q, k) + \frac{1}{\kappa} e_2(q, k) + \ldots \]

\[ = \bar{\lambda} d_1 \left( \frac{J}{J} \right) + \bar{\lambda}^2 d_2 \left( \frac{J}{J} \right) k + \ldots, \tag{5} \]

and the expression for the leading-order coefficient \(e_1\) was presented. We used that [5]

\[ \frac{1}{\kappa} = \bar{\lambda} - \bar{\lambda}^2 2k^2 J \frac{J}{J} + \ldots, \quad q = \sin^2 \gamma_0 = \frac{2k}{} + \ldots, \]

implying

\[ d_1 = e_1, \quad d_2 = e_2 - k^2 q e_1. \tag{7} \]

However, later analysis revealed that the functions that appear at higher orders have unexpected irregularities, so that the method of [5] needs a modification. Here we shall briefly discuss the nature of this modification (which turns out to be rather involved, prohibiting a direct analytic computation) and then turn to numerical methods to evaluate \(E_1\).

It was noticed in [5] that the bosonic and fermionic frequencies (see Appendix A below) admit the following large \(\kappa\) expansion (with \(\frac{n}{\kappa}\) and \(\frac{r}{\kappa}\) kept fixed)

\[ \omega_n^B = \kappa \alpha_1^B \left( \frac{n}{\kappa} \right) + \frac{1}{\kappa} \alpha_1^B \left( \frac{n}{\kappa} \right) + \frac{1}{\kappa} \alpha_3^B \left( \frac{n}{\kappa} \right) + \cdots, \tag{8} \]

\[ \omega_r^F = \kappa \alpha_1^F \left( \frac{r}{\kappa} \right) + \frac{1}{\kappa} \alpha_1^F \left( \frac{r}{\kappa} \right) + \frac{1}{\kappa} \alpha_3^F \left( \frac{r}{\kappa} \right) + \cdots. \tag{9} \]

One can think of \(\alpha_a(x)\) as the values of functions \(\alpha_a(x)\) at points \(x_m = \frac{m}{\kappa}\). It was implicitly assumed in [5] that all \(\alpha_a(x)\)'s are regular. In that case one could replace the bosonic and fermionic series in (3) by integrals, and then \(\alpha_{2a+1}(\frac{x}{\kappa})\) with \(\alpha \geq 1\) would not contribute to the order \(\frac{1}{\kappa}\) in the large \(\kappa\) expansion. However, it turns out that \(\alpha_a(x)\) with \(\alpha \geq 3\) in general have singularities (see (B6) and comments below it in Appendix B). A more careful analysis of \(\alpha_{2a+1}(\frac{x}{\kappa})\) shows that at small values of \(x\) they behave as \(\frac{1}{x^3} + O(\frac{1}{x^5})\). For this reason the analysis of the large \(\kappa\)-expansion in [5] needs to be modified. One must first subtract from \(\alpha_a(x)\) the singular contributions and after that convert the series into integrals. The singular contri-
suitable for taking the
This can be further rewritten in a form which is more
should regularize them first. Let us use the standard "su-
contributions cannot be represented by integrals and have to be
computed directly. However, then all the terms of the order
in $\alpha_{n+1}^2$ with $a \geq 1$ contribute to $e_i(q, k)$ in (5). For this
reason, obtaining the complete answer for the coefficient $e_i(q, k)$ (and, in general, for higher order coefficients $e_p$) along these lines would be hard in practice.

In the $\kappa \to \infty$ limit the one-loop energy correction must
go to zero because this strict limit is essentially like a BPS
limit—the bosonic and fermionic contributions should then cancel against each other due to supersymmetry.\footnote{A heuristic reason is that in the strict limit $\kappa \to \infty$ the world
surface of the string becomes a collection of BMN geodesics [29] with contribution of tension between different string bits effectively suppressed (see also [12] for a related argument).}

This implies that only negative powers of $\kappa$ can appear in
the large $\kappa$-expansion of $E_1$. Indeed, examining the
functions $\alpha_a(x)$ with $a > 1$ one can show that the one-
loop correction does have the large $\kappa$ expansion as given in
(5).

To prepare the ground for a numerical evaluation of $e_1$ and
$e_2$ in (5) let us first discuss the convergence of the 1-
loop correction (3) (which is expected due to the conformal
invariance of the underlying $AdS_5 \times S^5$ string sigma
model [30] and can be demonstrated for a generic string
solution, see [5]). Each of the two sums—over the bosonic
and the fermionic frequencies—is divergent, and so one
should regularize them first. Let us use the standard "su-

\begin{equation}
E_1 = \frac{1}{2\kappa} \left( \sum_{n \in \mathbb{Z}} e^{-\epsilon|n|} \omega_n^B - \sum_{r \in \mathbb{Z}+1/2} e^{-\epsilon|r|} \omega_r^F \right).
\end{equation}

Here $\omega_n^B$ ($\omega_r^F$) is the sum of eight bosonic (fermionic)

\begin{equation}
E_1 = \frac{1}{2\kappa} \left[ \sum_{n \in \mathbb{Z}} e^{-\epsilon|n|} \omega_n^B - \frac{1}{2} e^{-\epsilon|n+1/2|} \omega_{n+1/2}^F + \frac{1}{2} e^{-\epsilon|n+1/2|} \omega_{n+1/2}^F \right].
\end{equation}

This can be further rewritten in a form which is more
suitable for taking the $\epsilon \to 0$ limit

\begin{equation}
E_1 = \frac{1}{2\kappa} \left[ \sum_{n \in \mathbb{Z}} e^{-\epsilon|n|} \left( \omega_n^B - \frac{1}{2} \omega_{n+1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \right) + \frac{1}{2} \sum_{n \in \mathbb{Z}} (e^{-\epsilon|n|} - e^{-\epsilon|n+1/2|}) \omega_{n+1/2}^F \right].
\end{equation}

A nice feature of (12) is that the series on the first line is
convergent even for $\kappa = 0$ because at large $|n|$ (see
Appendices A and B of [5])

\begin{equation}
\omega_n^B = \frac{1}{2} \omega_{n-1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \sim \frac{1}{|n|^3}.
\end{equation}

On the other hand, the series in the second line of (12) can
be easily computed in the limit $\kappa \to 0$. First, we note that the fermionic frequencies $\omega_n^F$ are even under $r \to -r$, and rewrite the second line of (12) as

\begin{equation}
\frac{1}{2\kappa} (e^{-\epsilon/2} + e^{\epsilon/2} - 2) \sum_{r=0} e^{-\epsilon r} \omega_r^F.
\end{equation}

Using the large $r$ expansion in [5]

\begin{equation}
\omega_r^F = 8r + 4k^2 - k^2 q \frac{1}{r} + O \left( \frac{1}{r^3} \right)
\end{equation}

we find that only the first term, $8r$, contributes in the limit
$\epsilon \to 0$:

\begin{equation}
\lim_{\epsilon \to 0} (e^{-\epsilon/2} + e^{\epsilon/2} - 2) \sum_{r=0} e^{-\epsilon r} \omega_r^F = 2.
\end{equation}

Thus, the one-loop sigma model correction to the classical
energy can be represented by the following convergent
sum:\footnote{Let us stress again that one cannot formally rearrange the sum
without loosing the convergence.}

\begin{equation}
E_1 = \frac{1}{2\kappa} \left[ 2 + \sum_{n \in \mathbb{Z}} \left( \omega_n^B - \frac{1}{2} \omega_{n+1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \right) \right].
\end{equation}

It is useful also to single out the contribution of the $n = 0$ term.
Then

\begin{equation}
E_1 = \frac{1}{2\kappa} \left[ 1 + \frac{1}{2} \omega_0^B - \frac{1}{2} \omega_0^F + \sum_{n=1}^{N} \left( \omega_n^B - \frac{1}{2} \omega_n^F - \frac{1}{2} \omega_{n+1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \right) \right].
\end{equation}

where we have used the symmetry of the summand under
$n \to -n$. Having in mind a numerical computation of $E_1$
we have also introduced the upper limit $N \to \infty$. The
results discussed below were obtained for $N = 400.00$.

\section{III. RESULTS OF NUMERICAL EVALUATION}

Fixing the values of the parameters ($\kappa, q, k$) one can,
using Mathematica or Maple, numerically solve the
characteristic Eqs. (A7) and (A8) and (17) for $\omega_n^B$ and $\omega_n^F$. The solutions are then substituted into (17) to yield numerical values of

\begin{equation}
E_1 = E_1(\kappa, q, k).
\end{equation}

Since, in general, $E_1$ has a large $\kappa$-expansion as given in
(5), it is more convenient to compute not $E_1$ but $\kappa^2 E_1$. For
large enough $\kappa$, the value of $\kappa^2 E_1$ is very close to $d_1 = e_1$, assuming $e_2/\kappa^2$ is much smaller than $d_1$.

We consider $\kappa = 50, 100, 200$ for fixed values of $k = 1, 2, 4, 8$ and $q = \frac{k}{17}, h = 0, 1, 2, \ldots, 12$ and for $k = 1$ we
set $q = \frac{h}{17}$. Evaluating $\kappa^2 E_1$ numerically it is possible then
to estimate $d_2$ to be sure it is small enough, and plot the functions $d_1$. For example, we find: $50^2 E_1(50, \frac{1}{12}, 2) = 0.2626$, $100^2 E_1(100, \frac{1}{12}, 2) = 0.2627$, $200^2 E_1(200, \frac{1}{12}, 2) = 0.2648$. Using the Mathematica Fit function yields the following $\kappa$-dependence,

$$\kappa^2 E_1\left(\frac{1}{12}, 2\right) = 0.2641 - 4.1752 \frac{1}{\kappa^2}. \quad (19)$$

We see that $d_2$ is of order 1, and, therefore, $d_2/\kappa^2$ is much smaller than the value of $d_1 = e_i$ which for this case is $d_1 = 0.26$. The values of $\text{N}$ and $\kappa$ that we have used for the computation do not allow us to find $d_2$ reliably because we neglected the “tail” contribution $\left(\sum_{m=N+1}^{\infty} \right)$ in (16). It is shown in Appendix B that the tail contribution to $\kappa^2 E_1$ is of order $\kappa^3/\text{N}^2$. Because of that one cannot make $\kappa$ too large. For $\text{N} = 400[00]$ and $\kappa = 200$ one has $\kappa^3/\text{N}^2 = 0.005$, and, therefore, our computation is accurate at least to 0.01. This procedure can be repeated for the other values of $q = \frac{2k}{\text{N}}$ and $\kappa$. The resulting data is shown in Table I and is used to plot the $q$-dependence of $d_1$ in Fig. 1–8.

It is important to note that the circular string solution in question is stable, i.e., the frequencies $\omega_b^r$ and thus $E_1$ are real, only in the following range of values of $q$ (for fixed $k$):

$$q \leq q_s, \quad \text{where} \quad q_s = 1 - (1 - \frac{1}{\kappa} \frac{1}{2})^2 \quad (\text{see} \quad [4,5] \quad \text{and} \quad \text{Appendix A}).$$

While the plots are valid only in the “stable” regions of $q$, we have interpolated them to all values of $q \leq 1$ by simply dropping the imaginary parts.

Let us first discuss the $q$-dependence of $d_1$ for $k = 1$. The plot of $d_1$ is shown in Fig. 1. As discussed above the solution is stable for $q \leq 0.75$. One can see from the plot that the curve has a corner at $q = 0.75$. This is a general property for all values of $k$—the energy is not a differentiable function of $q$ at the edge of the stability region. An interesting feature of the graph is that it crosses the $q$-axis twice, at $q = 0.31$ and $q = 0.72$, i.e., for these values of $q$ the coefficient $d_1$ vanishes.

The plot of $d_1$ for $k = 2$ is shown in Fig. 2. In this case the solution is stable for $q \leq 0.4375$. One can see that the curve crosses the $q$-axis only once in the stability region. To see that the energy is not differentiable in this case either, it is useful to plot $d_2$, see Fig. 3. Even though, as discussed above, the values of $d_2$ are not reliable, one can clearly see that $d_2$ is not smooth at the edge of the stability region.

The plots of $d_1$ for $k = 4$ and $k = 8$ are shown in Fig. 4 and 5. The solution is stable for $q \leq 0.2344$ and $q \leq 0.1211$, respectively. The plots of $d_1$ for all values of $k$ have similar shapes. In particular, $d_1$ vanishes for at least one value $q = q_s$ in the stability region. This value $q_s$ depends on $k$, suggesting that one should not expect to find a simple dependence of $d_1$ on $k$ for fixed $q$.

Indeed, while the leading correction in the classical energy $E_0$ (2) scales with $k$ as $k^2$, there is no a priori reason why the leading coefficient in $E_1$ should also have a simple dependence on $k$. The dependence of frequencies $\omega_b^r$ and $\omega_r^r$ on $k$ is such that it can be eliminated by rescaling $\omega$'s, $\kappa$, and $n$, $r$ by $k$ [5] (see also Appendix A), but since this transformation involves a rescaling of summation indices, the resulting $E_1 (3)$ should, in general, be a nontrivial function of $k$. The dependence of $d_1$ on $k$ becomes nontrivial already for $q = \frac{1}{12}$ (see Fig. 6). In the formal case of $q = 0$ it is linear ($d_1(0, k) = -\frac{1}{2} k$) as expected.

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4From numerical data in Table I we find, in agreement with the analytic expression $q_s = 1 - (1 - \frac{1}{\kappa} \frac{1}{2})^2$, that $E_1$ becomes complex for $q > q_s$, where $18/24 < q_s < 19/24$ for $k = 1$, $5/12 < q_s < 6/12$ for $k = 2, 2/12 < q_s < 3/12$ for $k = 4$, and $1/12 < q_s < 2/12$ for $k = 8$.

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5Even though the coefficient $d_2$ does not vanish at these points, there exists a curve $F(\lambda, q) = 0$ on which the first (one-loop) sigma model correction, $E_1$, vanishes. Thus, for the corresponding values of $\lambda$ and $q$ the energy of the classical circular string coincides with the exact energy of the quantum string. This may be considered as a kind of nonrenormalization theorem valid only for special $\lambda$ and $q$ (for which we do not have any obvious explanation).
Table I shows.) Since \( d_1(\frac{1}{12}, 1) = -0.34 \) is negative and all other values of \( d_1(\frac{1}{12}, k) \) are positive, the \( k \)-dependence cannot be given by a power function \( k^n \). Rather the curve should be approximated by a polynomial of \( k \). This example shows that in general \( d_1 \) has a complicated dependence on \( k \). As stated in the introduction, the one-loop gauge-theory computation of the corresponding coefficient \( b_1 \) in (4) was carried out [15] only in the (unstable) \( q = 1 \) case with the result \( b_1 = \frac{1}{2}k^2 \). If that gauge-theory prediction \( b_1 \sim k^2 \) applies also to the stable \( q < 1 \) cases, then our results would indicate a disagreement between the string-theory (\( d_1 \)) and the perturbative (1-loop) gauge-theory (\( b_1 \)) coefficients.

Let us now consider the special case of \( q = 1 \), i.e., \( J_1 = 0, J_2 = J_3 = \frac{1}{2}J \) in more detail. Here the frequencies can be found in a simple analytic form [5] and the computation of \( E_1 \) becomes more explicit (see Appendix B). The corresponding gauge-theory (4) result [15] written in the form (1) reads

\[
\Delta = J \left[ 1 + \frac{1}{2} \hbar k^2 \left( 1 + \frac{1}{J} + \cdots \right) + \cdots \right]. \tag{20}
\]

Here the leading-order term agrees [10] with the classical string energy (2); in order for the \( \frac{1}{2} \) term in (20) to be in agreement with the one-loop string correction in (5) one should find that \( d_1(q = 1) = \frac{1}{2}k^2 \). As already mentioned, an apparent problem for checking this is that the \( q = 1 \) solution is unstable for any \( k \geq 1 \) [4,5]. In the simplest case of \( k = 1 \) there is one imaginary bosonic frequency (for larger \( k \) there are several unstable modes). As a result, the definition and interpretation of the 1-loop correction to the energy becomes nontrivial (formally, the 1-loop correction then contains an imaginary part determining the rate of decay of the unstable state, see, e.g., [31]). In order to see if string-theory result may be put into an agreement with the gauge-theory result (20) we may try to use one of the following definitions of \( E_1 \) (for definiteness, we shall consider the case of \( k = 1 \)):\(^6\)

(i) compute \( E_1 \) as a sum over all frequencies as in (3) and (17), and omit the imaginary part; this amounts to ignoring the contribution of the one unstable bosonic mode with \( n = 1 \).

(ii) analytically continue the value of the mass of the “tachyonic” \( n = 1 \) mode; i.e., include the contribution of its frequency to \( E_1 \) with the modulus sign.\(^7\)

In the case (i) we find by the numerical evaluation of the sum that \( d_1 \approx -0.446 \). In the case (ii) we get instead \( d_1 \approx 0.42 \) (the additional contribution of the modulus of the frequency of the unstable \( n = 1 \) mode is

\(^6\)Formally, the unstable mode is not “seen” on the gauge-theory side [10]. More precisely, an unstable mode of the 3-spin circular solution corresponds to the configuration when a Bethe root moves off the real axis [25,26]; this case does not correspond to a true eigenstate of the Hermitian Hamiltonian of the Heisenberg ferromagnet [9]. Still, this suggests that some analytic continuation may apply. Related aspect of this problem is that the spin chain states found using the Bethe ansatz [10] are exact quantum states, while on the string-theory side we are considering semiclassical states dual to coherent states of the spin chain [19,21,32]. One can show (see also [23]) that the corresponding unstable mode is present also in the “Landau-Lifshits” sigma model which is the coherent state effective action following in the low-energy approximation from the Heisenberg spin chain Hamiltonian and which agrees [19] with a large spin limit of the string sigma model action.

\(^7\)A possible way to support the second prescription is to view the \( J_2 = J_3, J_1 = 0 \) case as an analytic continuation of the stable solution with \( J_2 = J_3, J_1 > \frac{1}{2}J_2 \). For this stable solution the spectrum of fluctuations (and thus \( E_1 \)) is real and matches with Bethe ansatz \( J = \infty \) spectrum [11]. Then we may analytically continue all relations in the angular momentum plane and try to define the \( J_1 \rightarrow 0 \) limit.
\[ \delta d_1 = \frac{\sqrt{2}}{q} = 0.866. \] This may look close to the 0.5 value in (20), but our estimate of the numerical error is much smaller than 0.08, so we are inclined to conclude that there is a disagreement between the gauge-theory and string-theory values for \( d_1 \), with a plausible explanation being the same order-of-limits problem as in [27] (see also Sec. IV below).

Finally, let us comment on the \( k \)-dependence of \( d_1 \) in this \( q = 1 \) case. Following the first prescription (i), i.e., keeping only the real part of \( d_1(k) \) we get the plot in Fig. 7. It is interesting to note that the curve can be well-approximated by the power function \( -0.446k^p \) with \( p = 1.46 \).

With the second prescription (ii), i.e., taking the sum of the real part and the imaginary part\(^8 \) of \( d_1 \) we get the plot in Fig. 7. In this case it cannot be approximated by a power function. Comparing to (20) suggests again that there is a disagreement between the \( 1/J \) string-theory and gauge-theory results.

### IV. CONCLUDING REMARKS

In this paper we have used numerical methods to analyze the leading 1-loop sigma model correction to the energy of the classical circular spinning string [4]. We have confirmed the expected large \( J \) expansion of the energy, and studied the dependence of the first subleading coefficient \( d_1 \) in (1) on the two parameters — \( q = \frac{2k}{J} \) and “winding number” \( k \). Comparing our results with the known gauge-theory result [15] for the corresponding spin chain state (dual to unstable 2-spin circular string with \( q = 1 \)), we have found a discrepancy not only in the numerical value but also in the \( k \)-dependence of the leading 1-loop coefficient \( d_1 \). Even though our computation is unambiguous and reliable only for the stable 3-spin string states with \( q \leq q_c < 1 \), the different \( k \)-dependence of \( d_1 \) may be viewed as an indication that there is a string/gauge-theory disagreement for the \( 1/J \) correction starting already at one-loop order on the gauge-theory side. We should add a reservation that it is still possible that the disagreement we find is due to the fact that the semiclassical quantization and its comparison to the gauge-theory side are not directly applicable in the case of the unstable 2-spin solution, and there is still a chance that one may find a one-loop agreement for the stable 3-spin string states once one computes the corresponding \( 1/J \) gauge-theory corrections using the SU(3) Bethe ansatz of [11].

Assuming this \( 1/J \) disagreement persists, it should have the same origin as the previously found mismatch between the string and gauge-theory results at 3-loop order in \( \lambda \) and the leading-order in \( J \) [18]. As was suggested in [18,27], the latter can be explained by adding “wrapping” contributions to the dilatation operator (and thus to the Bethe ansatz relations) on the gauge-theory side. For example, in the \( q = 1 \) case one may use the function like \( \lambda^J/(1 + \lambda)^{\ell} \), which is one in the string-theory limit (\( J \to \infty \) with fixed \( \lambda/J^2 \equiv \lambda \)) but zero in the perturbative gauge-theory limit to interpolate between the different \( \frac{1}{J} \) results as follows:

\[
\Delta = J + \frac{\lambda}{2J} - \frac{\lambda^2}{8J^3} + \frac{\lambda^3}{16J^5} \left( 1 + \lambda \right)^{-3} + \ldots.
\]

This expression agrees with both the string (\( E = \sqrt{J^2 + \lambda} \) [4]) and the perturbative gauge-theory (\( \Delta_{\text{pert}} = J + \frac{\lambda}{2J} - \frac{\lambda^2}{2J} + 0 \times \lambda^3 + \ldots \) [18]) results. The same idea may be applied to explain the discrepancy at order \( \frac{1}{J^2} \), for ex-

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\(^8\)All unstable modes happen to have purely-imaginary frequencies, so their modulus is equal to the absolute value of their imaginary part.
ample, if we assume that the interpolation formula contains also a term

$$\Delta = \ldots + \frac{\lambda}{2J^2} \left[ 1 + a \frac{\lambda^{J-1}}{(1 + \lambda)^{J-1}} \right] + \ldots,$$

then the gauge-theory limit result for the coefficient \(d_1\) will be \(\frac{1}{2}(1 + a)\), explaining the apparent disagreement of our result with that of [15].

A related observation is that this apparent \(1/J\) disagreement can be easily accommodated and thus explained within the generalized Bethe ansatz for quantum string spectrum recently proposed in [28]. To this end all one should do is to assume a definite large \(L \equiv J\) expansion of the functions \(c_r(g, L) (g \equiv \sqrt{8\pi^2\lambda})\) appearing in the Bethe ansatz of [28]. In particular, one can see that if \(c_r\) have an expansion of the form \(c_r(g, L) \sim \frac{1}{\lambda} L^{2r+4} + \frac{1}{\lambda} L^{2r+3}\), then at order \(1/L = 1/J\) there is a string/gauge-theory mismatch already for the coefficient of the one-loop \((\sim \frac{1}{\lambda})\) term. The Bethe ansatz of [28] also implies that if there is a string/gauge-theory (dis)agreement for spinning string states at order \(1/J^n\) then a similar (dis)agreement should exist also for the BMN states at order \(1/J^{n+1}\). In view of the above discussion, this suggests that for the BMN states the disagreement at order \(1/J^2\) should start not for 3-loop \((\lambda^3)\) terms as at \(1/J\) order but already for the 1-loop \((\lambda)\) terms.

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**APPENDIX A: CLASSICAL SOLUTION AND QUADRATIC FLUCTUATIONS**

The solution we discussed above was found in [4], and the characteristic equations for the quadratic fluctuations near it were obtained in [5]. Here we briefly review the derivation of the bosonic characteristic equation while in the fermionic case we only quote the final result referring to [5] for more details. The bosonic part of the string action in the conformal gauge is

$$I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{dx}{2\pi} \times (L_{\text{AdS}} + L_S),$$


(A1)

where

$$L_S = -\frac{1}{2} \partial_a X_M \partial^a X_M - \frac{1}{2} \Lambda (X_M X_M - 1),$$

(A2)

$$L_{\text{AdS}} = -\frac{1}{2} \eta_{\rho\sigma} \partial_\rho Y_\rho \partial_\sigma Y_Q + \frac{1}{2} \Lambda (\eta_{\rho\sigma} Y_\rho Y_Q + 1).$$

Here \(X^M, M = 1, \ldots, 6\) and \(Y^\rho, P = 0, \ldots, 5\) are the embedding coordinates with a flat Euclidean metric for \(S^8\) and with \(\eta_{MN} = (-1, +1, +1, +1, +1, -1)\) for the AdS$_5$ respectively. We consider the configuration where the string is located in the center of AdS$_5$ while rotating in $S_8$. The AdS$_5$ part of the solution is trivial \((Y_5 + iY_0 = e^{\rho t}, Y_1, \ldots, Y_4 = 0)\) with the global AdS$_5$ time being set to \(t = \kappa\tau\), while the $S^8$ part is

$$X_1 + iX_2 = \sqrt{q} \cos k\sigma e^{i\omega \tau},$$

$$X_3 + iX_4 = \sqrt{q} \sin k\sigma e^{i\omega \tau},$$

$$X_5 + iX_6 = \sqrt{1 - q} e^{i\nu \tau},$$

(A3)

with

$$w^2 = \nu^2 + k^2, \quad \Lambda = \nu^2,$$

$$\nu^2 = \kappa^2 - 2k^2 q, \quad q = \sin^2 \gamma_0.$$

It was shown in [5] that the quadratic fluctuation...
Lagrangian around this solution can be written as

\[
L_2 = (\partial_\tau \bar{X}_3)^2 - (\partial_\sigma \bar{X}_3)^2 + 4\sqrt{q}\bar{X}_5 \partial_\tau \bar{X}_6 \\
- 4w\left(\sqrt{1 - q}\bar{X}_5 \partial_\tau \bar{X}_2 - \bar{X}_3 \partial_\tau \bar{X}_4\right) \\
+ 4k\left(\sqrt{1 - q}\bar{X}_5 \partial_\sigma \bar{X}_3 - \bar{X}_2 \partial_\sigma \bar{X}_4\right).
\] (A5)

The corresponding fluctuation spectrum is found by using

\[
B_8(\Omega) = \Omega^4 + \Omega^3(-8k^2 - 4n^2 + 20k^2q - 8k^2) + \Omega^2(16k^4 + 32k^2\kappa^2 + 16\kappa^4 + 8k^2n^2 + 16\kappa^2n^2 + 6n^4 - 80k^2\kappa^2q \\
- 80k^4q - 36k^2n^2q + 96k^4q^2) + \Omega(-32k^4n^2 - 32k^2\kappa^2n^2 + 8k^2n^4 - 8\kappa^4n^4 - 4n^6 + 96\kappa^2n^2q + 48k^2\kappa^2n^2q \\
+ 12k^2n^4q - 96k^4n^2q^2) + 16k^4n^4 - 8k^2n^6 + n^8 - 16\kappa^4n^4q + 4k^2n^6q).
\] (A7)

The 4 + 4 $S^5$-frequencies are obtained as $\omega^R_{n,\kappa^2} = \pm \sqrt{\Omega}$ where $\Omega$ is one of the four roots of $B_8 = 0$. In addition, there are 4 + 4 AdS$_5$ frequencies $\omega_n = \pm \sqrt{n^2 + \kappa^2}$. By following the analogous steps one can show [5] that the fermionic characteristic frequencies $\omega^F_n$ are determined by $F_8(\Omega) = 0$ where

\[
F_8 = 2\Omega^4 + \Omega^3(-8k^2 - 12k^2 - 8r^2 + 20k^2q) + \Omega^2(12k^4 + 28k^2\kappa^2 + 18\kappa^4 + 8k^2r^2 + 28\kappa^2r^2 + 12r^4 - 52\kappa^4q \\
- 64k^2\kappa^2q - 36k^2r^2q + 59k^4q^2) + \Omega(-8k^6 - 20k^4\kappa^2 - 20k^2\kappa^4 - 8\kappa^6 + 8k^4r^2 + 8k^2\kappa^2r^2 - 20k^4r^2 + 8k^2r^4 \\
- 20k^2r^4 - 8r^6 + 44k^6q + 80k^4\kappa^2q + 44k^4\kappa^2q + 24k^2\kappa^2r^2q + 32k^4\kappa^2r^2q + 12k^4r^2q - 78k^6q^2 - 79k^4\kappa^2q^2 \\
+ 2k^2r^2q^2 + 5k^6q^4) + 2k^8 + 4k^6\kappa^2 + 4k^4\kappa^2 - 8k^2r^2q - 4k^4\kappa^2r^2 - 4k^4\kappa^4r^2 + 12k^4r^4 - 4k^4\kappa^2r^4 + 4k^4r^4 \\
- 8k^2r^6 + 4k^4r^6 + 2r^8 - 12k^8 - 16k^6\kappa^2q - 4k^4\kappa^4q + 28k^6r^2q + 16k^4\kappa^2r^2q + 4k^2\kappa^4r^2q - 20k^4r^4q \\
+ 4k^6r^6q + 27k^8q^2 + 21k^6\kappa^2q^2 + 2k^4\kappa^4q^2 - 30k^2\kappa^2r^2q^2 - 11k^4\kappa^2r^2q^2 + 11k^4r^4q^2 - 27k^8q^3 - 9k^6\kappa^2q^3 \\
+ 9k^6r^2q^3 + \frac{81k^8q^4}{8}.
\] (A8)

Unlike the bosonic case, the AdS$_5$ and $S^5$ parts are not decoupled in the fermionic case. The eight fermionic frequencies are obtained by solving $F_8 = 0$ and taking $\omega^F = \sqrt{\Omega}$ with double degeneracy.

When solving $B_8 = 0$ one may set $k = 1$; then the $k$-dependence can be restored by the following rescaling,

\[
\omega_n \rightarrow \omega_n \frac{n}{k}, \quad n \rightarrow n \frac{k}{\kappa}, \quad \kappa \rightarrow \frac{k}{\kappa}.
\] (A9)

Similar rescaling can be done in the fermionic case [5].

Let us now consider the large $\kappa$-expansion of the bosonic frequencies to analyze the stability condition in that limit

\[
\omega_n^2 \rightarrow \frac{h_0}{4\kappa^2} + \frac{h_1}{\kappa^4} + \cdots.
\] (A10)

Here

\[
h_0 = n^2 \left[ 2k^2(2 - 3q) + n^2 \\
\pm 2k\sqrt{4n^2(1 - q) + k^2q(9q - 8)} \right]
\] (A11)

The stability condition that follows from positivity of $h_0$ is [5]

\[
q \leq q^*, \quad q^* = 1 - \left(1 - \frac{1}{2k^2}\right)^2.
\] (A12)

**APPENDIX B: $q = 1$ CASE: $J_1 = 0, J_2 = J_3$**

In the case of $q = 1$ the string is stretched around the big circle of $S^5$ and rotates about its center of mass with two equal angular momenta. Here the characteristic Eqs. (A7) and (A8) can be solved explicitly [5] and one finds that the bosonic $S^5$ frequencies are (up to an overall sign change)

\[
\omega_n^B = \left[n^2 + 2k^2 - 2k^2 \pm 2\sqrt{(k^2 - k^2)^2 + n^2k^2}\right]^{1/2} \omega_n^B \\
= \sqrt{n^2 + k^2 - 2k^2} \pm \sqrt{k^2 - 2k^2}.
\] (B1)

This may be compared to the AdS$_5$ fluctuation frequencies $\omega_n^B = \sqrt{n^2 + \kappa^2}$. The fermionic frequencies are (with double degeneracy)
ONE-LOOP CORRECTION TO THE ENERGY OF...

\[ a_{\ell} = \frac{1}{2} \left( 2\sqrt{\kappa^2 + k^2} - k^2 \pm \sqrt{\kappa^2 - k^2} \pm \sqrt{\kappa^2 - 2k^2} \right) \]  
(B2)

Using (B1) and (B2) we get the explicit form of (17) is

\[ E_1 = \frac{1}{\kappa} \left[ 1 + \left( \sqrt{\kappa^2 - 2k^2} + \sqrt{\kappa^2 - k^2} + 2\kappa \right) \right. \\
- 4 \left( \frac{1}{4} + \kappa^2 - k^2 \right) + \sum_{n=1}^{\infty} S(n, \kappa, k) \]  
(B3)

where

\[ S = \sqrt{\left( n + \sqrt{n^2 - 4k^2} \right)^2 + 4k^2 + 2\sqrt{n^2 - 2k^2 + k^2} \right. \\
+ 4\sqrt{n^2 + k^2} - 4\sqrt{\left( n - 1/2 \right)^2 - k^2 + k^2} \\
- 4\sqrt{\left( n + 1/2 \right)^2 - k^2 + k^2} \]  
(B4)

We have used that

\[ \left[ n^2 + 2\kappa^2 - 2k^2 + 2\sqrt{(\kappa^2 - k^2)^2 + n^2 \kappa^2} \right]^{1/2} \]  
(B5)

Note also that \( S_{\kappa \to \infty} \to \frac{1}{2\kappa} \left[ 2k^2 - 2 - n^2 + n\sqrt{n^2 - 4k^2} \right] + O(\frac{1}{\kappa^3}) \) and \( S_{n \to \infty} \to \frac{\kappa^2}{n^2} - \kappa + O(\frac{1}{n^3}) \), in agreement with (13).

One may be tempted to evaluate the sum in (B3) by converting it into an integral. This conversion is not possible, however, due to singularities of the summand. To see this let us set \( k = 1 \) and follow the procedure of [5] to perform the large-\( \kappa \)-expansion of \( S \) for fixed \( \frac{2}{\kappa} \equiv x \),

\[ S(x) \equiv S(\kappa x, \kappa, 1) \to \frac{1}{\kappa} \left( \frac{1}{1 + x^2} \right)^{3/2} \]  
(B6)

Note that the first term in the coefficient of \( \frac{1}{\kappa^2} \) has a singularity at \( x = 0 \). This is an example of the singularities that we discussed below (9). Even though we singled out the zero mode contribution in (17), it is this divergent behavior near \( x = 0 \) that is responsible for the failure of the conversion of the sum to an integral.

To estimate the accuracy of the numerical method that we used in Sec. III we approximately evaluate the tail of the sum in (B3). It is possible to convert the tail part of the sum in (B3), i.e., \( \frac{1}{\kappa} \sum_{n=1}^{\infty} S \), to an integral over \( x \equiv \frac{2}{\kappa} \) since it does not contain \( x = 0 \) or its neighborhood. Let us follow [5] and use that (we will not distinguish between \( N \) and \( N + 1 \) since \( N \gg 1 \))

\[ \frac{1}{\kappa} \sum_{n=1}^{\infty} S \approx \int_{x}^{\infty} dx g(x) + O\left( \frac{1}{\kappa^6} \right) \]  
(B7)

where

\[ g(x) = -\frac{2}{15} S\left( x - \frac{1}{2\kappa} \right) + \frac{6}{5} S\left( x - \frac{1}{2k} \right) + \frac{1}{30} S(x) \]  
(B8)

and \( S(x) \equiv S(\kappa x, \kappa, 1) \). To evaluate the integral in (B7) consider the following large-\( x \) expansion of \( g \),

\[ g = -\frac{1 + \frac{2}{x^2} - \frac{1}{x^4} - \frac{3}{x^4} + \frac{3}{x^4} + \frac{2}{x^4}}{4\kappa^5} \]  
(B9)

Up to the order given in (B9) the integral yields

\[ \int_{x}^{\infty} dx g(x) = -\frac{1}{2} \left( 1 + \frac{1}{\kappa^2} \frac{k}{N^2} - \frac{1}{2} \left( 1 + \frac{1}{\kappa^2} \right) \frac{k^3}{N^4} \right) \]  
(B10)

For \( N = 400 \) and \( \kappa = 50, 100, 200 \) we find that the correction is small compared to the numerically found value of \( E_1 \) (which is of order \( 10^{-3} \)). However, the correction grows if we increase \( \kappa \) for fixed \( N \) (e.g., consider \( \kappa = 1000 \)).