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Scalar Quartic Couplings in Type IIB Supergravity on $AdS_5 \times S^5$

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Abstract

All quartic couplings of scalar fields s^I that are dual to extended chiral primary operators in $\mathcal{N}=4$ SYM_4 are derived by using the covariant equations of motion for type IIB supergravity on $AdS_5 \times S^5$. It is shown that despite some expectations if one keeps the structure of the cubic terms untouched, the quartic action obtained contains terms with *two* and *four* derivatives. It is shown that the quartic action vanishes on shell in the extremal case, e.g. $k_1 = k_2+k_3+k_4$. Consistency of the truncation of the quartic couplings to the massless multiplet of the $\mathcal{N}=8$, $d=5$ supergravity is proven and the explicit values of the couplings are found. It is argued that the consistency of the KK reduction implies non-renormalization of n-point functions of $n-1$ operators dual to the fields from the massless multiplet and one operator dual to a field from a massive multiplet.

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1 Introduction and Summary

The AdS/CFT correspondence [1, 2, 3] provides a powerful method of studing correlation functions in conformal field theories, in particular, in $D = 4$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM_4). According to the proposal by [2, 3], the generating functional of Green functions in SYM_4 at large N and at strong 't Hooft coupling λ coincides with the on-shell value of the type IIB supergravity action on $AdS_5 \times S^5$. Thus, the computation of an n-point Green function requires the knowledge of the supergravity action up to the n-th order. In particular, the quadratic [4] and cubic actions [5, 6, 7] for physical fields of type IIB supergravity determine the normalization constants of two- and three-point Green functions [8]-[29]. In principle, by using the quadratic action [4] and the covariant equations of motion for type IIB supergravity [30, 31, 32] one can easily compute *any* three-point function of gauge invariant operators in SYM_4 in the supergravity approximation, the problem that can be hardly solved in perturbative SYM_4 even at the one-loop approximation.

The problem of computing four-point functions [33]-[45] is obviously much more involved, and consists in general of two independent parts – one first has to derive the relevant part of the supergravity action up to the fourth order, and then to find the minimum of the action that amounts to computing the corresponding exchange and contact Feynman diagrams. However, as was pointed out in [33], in the simplest cases of massless modes of dilaton and axion fields, the relevant part of the supergravity action was known. Computing the corresponding 4-point functions was initiated in [33], and completed in [40]. Unfortunately, these modes correspond to rather complicated operators $\mathrm{tr}\,(F^2 + \cdots)$ and $\mathrm{tr}\,(F\tilde{F} + \cdots)$, and not much seems to be known about their four-point functions in perturbative SYM_4. Nevertheless, the analysis of the four-point functions of these operators performed in [45] allows one to conclude that at strong 't Hooft coupling all operators with large anomalous dimensions, which are dual to massive string states, decouple.

The important and simplest operators in SYM_4 are single-trace [1] chiral primary operators (CPOs) [5] that are of the form $O_k^I = \mathrm{tr}\,(\phi^{(i_1} \cdots \phi^{i_k)})$. It is well known that all other operators in SYM_4 corresponding to type IIB supergravity fields are descendents of CPOs.

Type IIB supergravity on $AdS_5 \times S^5$ contains in its particle spectrum [47, 48] scalar fields s^I that are mixtures of the five form field strength on S^5 and the trace of the graviton on S^5. At the linear approximation to the supergravity, namely these scalars correspond to CPOs, as one can see from their transformation properties with respect to the superconformal group of SYM_4.

[1]Throughout the paper we refer to a single-trace CPO as a CPO. Multi-trace operators may also belong to the same short representation of the supersymmetry algebra as was shown in [46].

Although the correlation functions of CPOs are the simplest ones to compute in SYM_4, the corresponding calculation in the supergravity approximation is nontrivial due to the absence of a relevant action for the scalars s^I. The quadratic and cubic actions for the scalars s^I have been found and used to calculate all three-point functions of normalized CPOs in [5].[2]

To compute four-point functions of CPOs one first has to know all cubic terms that involve two scalar fields s^I, and the s^I-dependent quartic terms, and then to find the on-shell value of the supergravity action.

In the present paper as the first step in this direction we determine all necessary cubic and quartic terms by using the quadratic action [4] and the covariant equations of motion for type IIB supergravity [30, 31, 32]. It is clear that one can consider only the sector of type IIB supergravity that depends on the graviton and the four-form potential. There are four different types of cubic vertices describing interaction of two scalars s^I with (i) symmetric tensor fields coming from the AdS_5 components of the graviton, (ii) with vector fields, (iii) with scalar fields coming from the S^5 components of the graviton, and $(iiii)$ with scalar fields t^I that are mixtures of the trace of the graviton on the sphere and the five form field strength on the sphere. Although all the cubic terms were recently found in [6, 7], we will see that the derivation done in the papers should be reconsidered. The reason is that this time, since we are interested in *cubic* corrections to equations of motion for scalars s^I, dealing with quadratic terms, we have to take into account quadratic corrections to equations of motion for the gravity fields.

Actually, it is straightforward, although cumbersome, to find cubic corrections to equations of motion by decomposing the covariant equations of motion up to the third order, and keeping only relevant terms. The main problem in deriving the quartic couplings comes from the fact that the equations of motion such obtained are $non-Lagrangian$, and one should perform a very complicated and fine analysis to reduce the equations of motion to a Lagrangian form. In particular, although the original equations contain terms with *six* derivatives, we will show that one can remove these terms completely by means of a chain of field redefinitions, and by using nontrivial identities between spherical harmonics of different types. Even after removing the terms with six derivatives, the resulting equations containing terms with four and two derivatives (and without derivatives, of course) are still non-Lagrangian. To make the equations Lagrangian one should again redefine the scalar fields, and use the identities. Since any mistake in the computation would destroy the possibility of obtaining Lagrangian equations of motion, we are

[2] To get rid of higher-derivative terms in the equations of motion for scalars s^I a derivative-dependent field redefinition was made in [5]. By this reason the scalars s^I used in [5] correspond not to CPOs but to extended CPOs involving products of CPOs and their descendants [6]. However, three-point functions of the extended CPOs coincide with the ones of CPOs for generic values of conformal dimensions of CPOs.

pretty sure that the quartic couplings we found are correct.

The fact that the covariant equations of motion are non-Lagrangian, and one has to perform nontrivial field redefinitions to reduce them to a Lagrangian form, explicitly shows that the gravity fields entering the covariant equations of motion *cannot* correspond to *any* operator in SYM_4. The quartic action presented in the next Section is in fact written for the scalar fields s^I corresponding not to CPOs, but to *extended* CPOs, as was discussed in [6]. In principle it seems possible to find an action for the scalars dual to CPOs by performing the field redefinitions reversed to the ones used to reduce the equations of motion to a Lagrangian form [3]. However, the resulting action for the new scalars will be much more complicated and will contain higher-derivative terms with six derivatives. It's worth noting that the equations of motion derived from the new action certainly differ from the original ones despite the fact that one made reversed transformations.

We show that despite some expectations, if we keep the structure of the cubic terms untouched, the action obtained contains quartic terms with *two* and $four$ derivatives, and there is no field redefinition allowing one to remove these terms. Thus, the problem of computing the four-point functions of CPOs will require computing two new types of Feynman diagrams: (i) exchange diagrams involving massive tensor fields of second rank, and (ii) contact diagrams with four-derivative quartic vertices. All other necessary diagrams were computed in [36, 40, 41].

In our previous paper [6] we argued that quartic couplings of the scalars s^I had to vanish in the extremal case when, say, $k_1 = k_2 + k_3 + k_4$. This conjecture was based on the fact that all exchange Feynman diagrams vanished and contact Feynman diagrams had singularity in the extremal case, thus non-vanishing quartic couplings would contradict to the AdS/CFT correspondence. Although the vanishing of the quartic couplings obtained in the present paper is not manifest, we show that this important property does take place after an additional field redefinition. This means that 4-point extremal correlators of extended CPOs vanish, and also implies the non-renormalization theorem [29] for the corresponding extremal correlators of single-trace CPOs. It is clear that since the quartic couplings vanish then there should exist such a representation of the quartic couplings, that makes the vanishing explicit. We, however, have not looked for such a representation yet.

The quartic couplings we found allow us to study the problem of the consistency of the Kaluza-Klein (KK) reduction down to five dimensions.[4] It is customarily believed that the S^5 compactification of type IIB supergravity admits a consistent truncation to the massless

[3] Note that the reversed transformations should be made at the level of the quartic action, but not at the level of equations of motion.

[4] For a recent discussion of the consistency problem see [49, 50], and references therein.

multiplet, which can be identified with the field content of the gauged $\mathcal{N} = 8$, $d = 5$ supergravity [51, 52]. Consistency means that there is no term linear in massive KK modes in the untruncated supergravity action, so that all massive KK fields can be put to zero without any contradiction with equations of motion. From the AdS/CFT correspondence point of view the consistent truncation implies that *any* n-point correlation function of $n-1$ operators dual to the fields from the massless multiplet and one operator dual to a massive KK field vanishes because, as one can easily see there is no exchange Feynman diagram in this case.

It is obvious that the cubic couplings found in [6, 7] obey the consistency condition allowing therefore truncation to the fields from the massless multiplet at the level of the cubic action. In this paper we show that after an additional simple field redefinition the quartic vertices we found indeed vanish when one of the four fields is not from the massless multiplet, proving thereby the consistency of the reduction at the level of the quartic scalar couplings. This in particular provides an additional argument that the scalars s^I (and, in general, any supergravity field) correspond not to CPOs but rather to extended CPOs. Indeed, if we assume that the consistent truncation takes place at all orders in gravity fields, we get that correlators of the form $\langle O_2^{I_1} O_2^{I_2} \cdots O_2^{I_{n-1}} O_k^{I_n} \rangle$ vanish for $k \geq 3$. This is certainly not the case for single-trace CPOs, and we are forced to conclude once more that supergravity fields are in general dual to extended operators which are admixtures of single-trace operators and multi-trace ones.[5] Since an extended operator is uniquely determined by a single-trace one, it is natural to assume that if a correlation function of extended operators vanishes then there exists a kind of a non-renormalization theorem for an analogous correlation function of single-trace operators. If we further assume that type IIB string theory on $AdS_5 \times S^5$ respects the consistent truncation, then the vanishing of n-point correlation functions of $n-1$ extended operators dual to the supergravity modes from the massless multiplet, and one extended operator dual to a massive KK mode seems to imply that

at large N the n-point functions of the corresponding single-trace operators are independent of 't Hooft coupling $\lambda = g_{YM}^2 N$.

If the consistent truncation is valid at quantum level, that seems to be plausible because of a large amount of supersymmetry, then these n-point functions are independent of g_{YM} for any N.

In particular this conjecture is applied to n-point functions of $n-1$ CPOs O_2 and a CPO O_4. It would be interesting to check this in perturbation theory.

[5]Note that the lowest modes s_2 may be dual only to single-trace CPOs. It is possible that any field from the massless supergravity multiplet is dual to a single-trace operator.

We also use the quartic couplings to find quartic action for the scalars s_2^I from the massless multiplet. The 4-derivative terms vanish in this case, and we arrive at an action with 2-derivative and non-derivative quartic couplings. We do not compare the action obtained with the one of the gauged $\mathcal{N}=8$, $d=5$ supergravity on the AdS_5 background. This problem will be considered together with the problem of computing 4-point functions of CPOs O_2 dual to the scalars from the massless multiplet in a latter paper.

The plan of the paper is as follows. In Section 2 we recall equations of motion for the graviton and the four-form potential, introduce notations, and represent the action obtained. In Section 3 we prove that there is the consistent reduction of type IIB supergravity on $AdS_5 \times S^5$ down to five dimensions at the level of the quartic action. In Section 4 we show that the quartic couplings vanish in the extremal case. In Section 5 we discuss the structure of the cubic corrections to the equations of motion due to the contributions of the gravity fields. In Section 6 we explain what steps one should undertake to reduce the equations of motion to a Lagrangian form. In Conclusion we disscuss unsolved problems. In Appendix A we present the values of the quartic couplings obtained, and in the Appendix B we summarize several important identities involving spherical harmonics of different kinds.

2 Quartic Couplings

Quartic couplings of scalars s^I may be derived from cubic corrections to the covariant equations of motion [30, 31, 32] for type IIB supergravity. Only the graviton and the four-form potential give relevant contributions to cubic terms. The equations of motion of the metric and the 4-form potential are

$$F_{M_1\dots M_5} = \frac{1}{5!}\varepsilon_{M_1\dots M_{10}}F^{M_6\dots M_{10}}, \tag{2.1}$$

$$R_{MN} = \frac{1}{3!}F_{MM_1\dots M_4}F_N{}^{M_1\dots M_4}. \tag{2.2}$$

Here $M, N, \dots, = 0, 1, \dots 9$ and we use the following notation

$$F_{M_1\dots M_5} = 5\partial_{[M_1}A_{M_2\dots M_5]} = \partial_{M_1}A_{M_2\dots M_5} + 4 \;\; \text{terms},$$

i.e all antisymmetrizations are with "weight" 1. The dual forms are defined as

$$\varepsilon_{01\dots 9} = \sqrt{-G}, \qquad e^{01\dots 9} = -\frac{1}{\sqrt{-G}},$$
$$\varepsilon^{M_1\dots M_{10}} = G^{M_1N_1}\cdots G^{M_{10}N_{10}}\varepsilon_{N_1\dots N_{10}},$$
$$(F^*)_{M_1\dots M_k} = \frac{1}{k!}\varepsilon_{M_1\dots M_{10}}F^{M_{k+1}\dots M_{10}} = \frac{1}{k!}\varepsilon^{N_1\dots N_{10}}G_{M_1N_1}\cdots G_{M_kN_k}F_{N_{k+1}\dots N_{10}}.$$

In the units in which the radius of S^5 is set to be unity, the $AdS_5 \times S^5$ background solution looks as

$$\begin{aligned}
& ds^2 = \frac{1}{x_0^2}(dx_0^2 + \eta_{ij}dx^i dx^j) + d\Omega_5^2 = g_{MN}dx^M dx^N \\
& R_{abcd} = -g_{ac}g_{bd} + g_{ad}g_{bc}; \quad R_{ab} = -4g_{ab} \\
& R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}; \quad R_{\alpha\beta} = 4g_{\alpha\beta} \\
& \bar{F}_{abcde} = \varepsilon_{abcde}; \quad \bar{F}_{\alpha\beta\gamma\delta\varepsilon} = \varepsilon_{\alpha\beta\gamma\delta\varepsilon},
\end{aligned} \tag{2.3}$$

where $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ are the AdS and the sphere indices respectively and η_{ij} is the 4-dimensional Minkowski metric. We represent the gravitational field and the 4-form potential as

$$G_{MN} = g_{MN} + h_{MN}; \quad A_{MNPQ} = \bar{A}_{MNPQ} + a_{MNPQ}; \quad F = \bar{F} + f.$$

The gauge symmetry of the equations of motion allows one to impose the de Donder gauge:

$$\nabla^\alpha h_{a\alpha} = \nabla^\alpha h_{(\alpha\beta)} = \nabla^\alpha a_{M_1M_2M_3\alpha} = 0; \quad h_{(\alpha\beta)} \equiv h_{\alpha\beta} - \frac{1}{5}g_{\alpha\beta}h^\gamma_\gamma. \tag{2.4}$$

This gauge choice does not remove all the gauge symmetry of the theory, for a detailed discussion of the residual symmetry see [47]. As was shown in [47], the gauge condition (2.4) implies that the components of the 4-form potential of the form $a_{\alpha\beta\gamma\delta}$ and $a_{a\alpha\beta\gamma}$ can be represented as follows:

$$a_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta\varepsilon}\nabla^\varepsilon b; \quad a_{a\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\delta\varepsilon}\nabla^\delta \phi_a^\varepsilon. \tag{2.5}$$

It is also convenient to introduce the dual 1- and 2-forms for a_{abcd} and $a_{abc\alpha}$:

$$a_{abcd} = -\varepsilon_{abcde}Q^e; \quad a_{abc\alpha} = -\varepsilon_{abcde}\phi_\alpha^{de}. \tag{2.6}$$

Then the solution of the first-order self-duality equation can be written as

$$Q^a = \nabla^a b, \quad \phi_\alpha^{ab} = \nabla^{[a}\phi_\alpha^{b]}. \tag{2.7}$$

To write down the action for scalars s^I that can be used to compute 4-point correlation functions of CPOs in SYM_4 we need to expand fields in spherical harmonics[6]

$$\begin{aligned}
& h_\alpha^\alpha(x,y) = \sum \pi^{I_1}(x)Y^{I_1}(y); \quad b(x,y) = \sum b^{I_1}(x)Y^{I_1}(y), \\
& h_{ab}(x,y) = \sum h_{ab}^{I_1}(x)Y^{I_1}(y); \quad \nabla_\beta^2 Y^k = -k(k+4)Y^k = -f(k)Y^k \\
& h_{a\alpha}(x,y) = \sum h_a^{I_5}(x)Y_\alpha^{I_5}(y); \quad \phi_{a\alpha}(x,y) = \sum \phi_a^{I_5}(x)Y_\alpha^{I_5}(y); \\
& (\nabla_\beta^2 - 4)Y_\alpha^k = -(k+1)(k+3)Y_\alpha^k, \\
& h_{(\alpha\beta)}(x,y) = \sum \phi^{I_{14}}(x)Y_{(\alpha\beta)}^{I_{14}}(y); \quad (\nabla_\gamma^2 - 10)Y_{(\alpha\beta)}^k = -(k^2+4k+8)Y_{(\alpha\beta)}^k,
\end{aligned}$$

[6] Here and in what follows we suppose that the spherical harmonics of all types are orthogonal with "weight" 1, i.e. $\int Y^I Y^J = \delta^{IJ}$, $\int Y_\alpha^I Y_\alpha^J = \delta^{IJ}$, $\int Y_{(\alpha\beta)}^I Y_{(\alpha\beta)}^J = \delta^{IJ}$, and summation over α, β is assumed. Namely this normalization was used in [4].

We also need to make a number of fields redefinitions, the simplest ones required to diagonalize the linear equations of motion are [7]

$$\pi_k = 10ks_k + 10(k+4)t_k; \quad b_k = -s_k + t_k \tag{2.8}$$

$$h^k_{ab} = \varphi^k_{ab} + g_{ab}\eta_k + \nabla_a\nabla_b\zeta_k, \tag{2.9}$$

$$\zeta_k = \frac{4}{k+1}s_k + \frac{4}{k+3}t_k. \tag{2.10}$$

$$\eta_k = -\frac{2k(k-1)}{k+1}s_k - \frac{2(k+4)(k+5)}{k+3}t_k \tag{2.11}$$

$$A^k_a = h^k_a - 4(k+3)\phi^k_a; \quad C^k_a = h^k_a + 4(k+1)\phi^k_a \tag{2.12}$$

Note that we use the off-shell shift of h_{ab} (2.9) that was used to find the quadratic action for type IIB supergravity on $AdS_5 \times S^5$ in [4]. It differs from the on-shell shift [47] by higher-order terms.

The field redefinitions that are needed to make the equations of motion Lagrangian and to remove higher-derivative terms from quadratic terms in the equations of motion will be discussed in the next Sections.

Then the action for the scalars s^I may be written in the form

$$S(s) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\ \Big(\mathcal{L}_2(s) + \mathcal{L}_2(t) + \mathcal{L}_2(\phi) + \mathcal{L}_2(\varphi_{ab}) + \mathcal{L}_2(A_a) + \mathcal{L}_2(C_a) + \mathcal{L}_3(s) + \mathcal{L}_3(t) + \mathcal{L}_3(\phi) + \mathcal{L}_3(\varphi_{ab}) + \mathcal{L}_3(A_a) + \mathcal{L}_3(C_a) + \mathcal{L}_4^{(0)} + \mathcal{L}_4^{(2)} + \mathcal{L}_4^{(4)}\Big). \tag{2.13}$$

Here the quadratic terms are given by [4]

$$\mathcal{L}_2(s) = \sum\ \frac{32k(k-1)(k+2)}{k+1}\left(-\frac{1}{2}\nabla_a s_k\nabla^a s_k - \frac{1}{2}m^2 s_k^2\right), \tag{2.14}$$

$$\mathcal{L}_2(t) = \sum\ \frac{32(k+2)(k+4)(k+5)}{k+3}\left(-\frac{1}{2}\nabla_a t_k\nabla^a t_k - \frac{1}{2}m_t^2 t_k^2\right), \tag{2.15}$$

$$\mathcal{L}_2(\phi) = \sum\ \left(-\frac{1}{4}\nabla_a\phi_k\nabla^a\phi_k - \frac{1}{4}f(k)\phi_k^2\right), \tag{2.16}$$

$$\mathcal{L}_2(\varphi_{ab}) = \sum\ \Big(-\frac{1}{4}\nabla_c\varphi^k_{ab}\nabla^c\varphi^{ab}_k + \frac{1}{2}\nabla_a\varphi^{ab}_k\nabla^c\varphi^k_{cb} - \frac{1}{2}\nabla_a\varphi^{ck}_c\nabla_b\varphi^{ba}_k + \frac{1}{4}\nabla_c\varphi^{ak}_a\nabla^c\varphi^{bk}_b + \frac{1}{4}(2-f(k))\varphi^k_{ab}\varphi^{ab}_k + \frac{1}{4}(2+f(k))(\varphi^{ak}_a)^2\Big), \tag{2.17}$$

$$\mathcal{L}_2(A_a) = \sum\ \frac{k+1}{2(k+2)}\left(-\frac{1}{4}(F_{ab}(A^k))^2 - \frac{1}{2}m_A^2(A^k_a)^2\right), \tag{2.18}$$

$$\mathcal{L}_2(C_a) = \sum\ \frac{k+3}{2(k+2)}\left(-\frac{1}{4}(F_{ab}(C^k))^2 - \frac{1}{2}m_C^2(C^k_a)^2\right), \tag{2.19}$$

[7] We often denote π^{I_1} as π_k or as π_1, and a similar notation for other fields.

where the masses of the particles are

$$m^2 = k(k-4), \quad m_t^2 = (k+4)(k+8), \quad m_\phi^2 = m_\varphi^2 = f(k) = k(k+4),$$
$$m_A^2 = k^2 - 1, \quad m_C^2 = (k+3)(k+5)$$

and $F_{ab}(A) = \partial_a A_b - \partial_b A_a$.

The cubic terms were found in [5, 6, 7], and may be written as follows

$$\mathcal{L}_3(s) = S_{I_1I_2I_3}\ s^{I_1}s^{I_2}s^{I_3},$$
$$S_{I_1I_2I_3} = a_{123}\frac{2^7\Sigma((\frac{1}{2}\Sigma)^2-1)((\frac{1}{2}\Sigma)^2-4)\alpha_1\alpha_2\alpha_3}{3(k_1+1)(k_2+1)(k_3+1)}, \tag{2.20}$$
$$\mathcal{L}_3(t) = T_{I_1I_2I_3}\ s^{I_1}s^{I_2}t^{I_3},$$
$$T_{I_1I_2I_3} = a_{123}\frac{2^7(\Sigma+4)(\alpha_1+2)(\alpha_2+2)\alpha_3(\alpha_3-1)(\alpha_3-2)(\alpha_3-3)(\alpha_3-4)}{(k_1+1)(k_2+1)(k_3+3)}, \tag{2.21}$$
$$\mathcal{L}_3(\phi) = \Phi_{I_1I_2I_3}\ s^{I_1}s^{I_2}\phi^{I_3},$$
$$\Phi_{I_1I_2I_3} = \frac{4p_{123}\Sigma(\Sigma+2)}{(k_1+1)(k_2+1)}(\alpha_3-1)(\alpha_3-2), \tag{2.22}$$
$$\mathcal{L}_3(\varphi_{ab}) = G_{I_1I_2I_3}\left(\nabla^a s^{I_1}\nabla^b s^{I_2}\varphi_{ab}^{I_3} - \frac{1}{2}\left(\nabla^a s^{I_1}\nabla_a s^{I_2} + \frac{1}{2}(m_1^2+m_2^2-f_3)s^{I_1}s^{I_2}\right)\varphi_c^{cI_3}\right),$$
$$G_{I_1I_2I_3} = \frac{4(\Sigma+2)(\Sigma+4)\alpha_3(\alpha_3-1)}{(k_1+1)(k_2+1)}a_{123}, \tag{2.23}$$
$$\mathcal{L}_3(A_a) = A_{I_1I_2I_3}\ s^{I_1}\nabla^a s^{I_2}A_a^{I_3},$$
$$A_{I_1I_2I_3} = \frac{2(k_3+1)(\alpha_3-1/2)(\Sigma-1)(\Sigma+1)(\Sigma+3)}{(k_1+1)(k_2+1)(k_3+2)}t_{123}, \tag{2.24}$$
$$\mathcal{L}_3(C_a) = C_{I_1I_2I_3}\ s^{I_1}\nabla^a s^{I_2}C_a^{I_3},$$
$$C_{I_1I_2I_3} = \frac{8(k_3+3)(\alpha_3-1/2)(\alpha_3-3/2)(\alpha_3-5/2)(\Sigma+3)}{(k_1+1)(k_2+1)(k_3+2)}t_{123}. \tag{2.25}$$

Here the summation over $I_1,\ I_2,\ I_3$ is assumed, and we use the following notations

$$\alpha_1 = \frac{1}{2}(k_2+k_3-k_1), \quad \alpha_2 = \frac{1}{2}(k_1+k_3-k_2), \quad \alpha_3 = \frac{1}{2}(k_1+k_2-k_3), \quad \Sigma = k_1+k_2+k_3,$$
$$a_{123} = \int\ Y^{I_1}Y^{I_2}Y^{I_3}, \quad p_{123} = \int\ \nabla^\alpha Y^{I_1}\nabla^\beta Y^{I_2}Y^{I_3}_{(\alpha\beta)}, \quad t_{123} = \int\ \nabla^\alpha Y^{I_1}Y^{I_2}Y^{I_3}_\alpha,$$

and for any function $f_i \equiv f(k_i)$.

The quartic terms represent our main result and are given by

$$\mathcal{L}_4^{(0)} = S^{(0)}_{I_1I_2I_3I_4}\ s^{I_1}s^{I_2}s^{I_3}s^{I_4}; \tag{2.26}$$
$$\mathcal{L}_4^{(2)} = \left(S^{(2)}_{I_1I_2I_3I_4} + A^{(2)}_{I_1I_2I_3I_4}\right)s^{I_1}\nabla_a s^{I_2}s^{I_3}\nabla^a s^{I_4},$$

$$S^{(2)}_{I_1I_2I_3I_4} = S^{(2)}_{I_2I_1I_3I_4} = S^{(2)}_{I_3I_4I_1I_2}, \tag{2.27}$$

$$A^{(2)}_{I_1I_2I_3I_4} = -A^{(2)}_{I_2I_1I_3I_4} = A^{(2)}_{I_3I_4I_1I_2}; \tag{2.28}$$

$$\mathcal{L}^{(4)}_4 = \left(S^{(4)}_{I_1I_2I_3I_4} + A^{(4)}_{I_1I_2I_3I_4}\right) s^{I_1}\nabla_a s^{I_2}\nabla^2_b(s^{I_3}\nabla^a s^{I_4}),$$

$$S^{(4)}_{I_1I_2I_3I_4} = S^{(4)}_{I_2I_1I_3I_4} = S^{(4)}_{I_3I_4I_1I_2}, \tag{2.29}$$

$$A^{(4)}_{I_1I_2I_3I_4} = -A^{(4)}_{I_2I_1I_3I_4} = A^{(4)}_{I_3I_4I_1I_2}. \tag{2.30}$$

The explicit values of the quartic couplings are collected in Appendix A.

3 Reduction to the gauged $\mathcal{N} = 8$ 5-dimensional supergravity

There is much evidence that the S^5 compactification of the IIB supergravity admits a consistent truncation to the massless graviton multiplet, which can be identified with the field content of the gauged $\mathcal{N} = 8$, $d = 5$ supergravity [51, 52]. Consistency means that there is no term linear in massive KK modes in the untruncated action so that all massive KK fields can be put to zero. As was noted in [49, 50] the cubic couplings (2.20)-(2.25) obviously obey this condition allowing therefore the consistent truncation to the massless gravity multiplet.

In this Section we show that after an additional field redefinition the found quartic vertices (2.26)-(2.30) indeed vanish when one of the four fields is not from the massless multiplet, proving thereby the consistency of the reduction at the level of the quartic scalar couplings.

Recall that the gauged $\mathcal{N} = 8$ five-dimensional supergravity has in particular 42 scalars with 20 of them forming the singlet of the global invariance group $SL(2, \mathbf{R})$. These 20 scalars comprise the **20** irrep. of $SO(6)$ and correspond to the IIB supergravity fields[8] s^I_k with $k = 2$. The five-dimensional scalar Lagrangian consists of the kinetic energy and the potential. The maximal number of derivatives appearing in the Lagrangian is two and that is due to the non-linear sigma model type kinetic energy. We have however found the quartic 4-derivative vertices that can not be shifted away by any field redefinition. Thus, a highly non-trivial check of the relation between the compactification of the ten-dimensional theory and the gauged supergravity in five dimensions as well of the results obtained involves showing that the 4-derivative vertices vanish for the modes from the massless multiplet.

We start to analyse the consistency of the truncation with the quartic couplings of 4-derivative vertices and assume the fields $s^{I_2}_2, s^{I_3}_2, s^{I_4}_2$ belong to the massless multiplet. Upon substituting

[8] In this Section we use the explicit notation s^I_k for a scalar transforming in $I = (0, k, 0)$-irrep.

$k_2 = k_3 = k_4 = 2$ the couplings $(A_0)^{(4)}_{I_1I_2,I_3I_4}$ and $(A_{-1})^{(4)}_{I_1I_2,I_3I_4}$ turn to zero. The other couplings are non-zero and, therefore, the only possibility is that their sum should vanish. Moreover, according to the above discussion this vanishing should hold regardless of the fact if the remaing field s_1 is in a massive or in the massless graviton multiplet.

Among the couplings we consider there is a distinguished one, namely, $(A_{t2})^{(4)}_{I_1I_2,I_3I_4}$ since it involves another type of $SO(6)$ tensors. To deal with this coupling we note that for $k_3 = k_4 = 2$ there exists only two values of k_5 for which t_{345} does not vanish, namely, $k_5 = 1$ and $k_5 = 3$. Then one represents

$$(f_5-1)^2 t_{125}t_{345} = \Big((f_5-5)(f_5-21) + 24(f_5-1) - 80\Big) t_{125}t_{345}, \tag{3.1}$$

so that $(f_5-5)(f_5-21)$ vanishes for both $k_5 = 1$ and $k_5 = 3$. By using relations (9.8) and (9.9) the remaing terms in the last formula may be now reduced to involve the same type of the $SO(6)$ tensors as the rest of the 4-derivative quartic couplings.

Finally, we sum up all couplings assuming that three fields are from massless multiplet and the fourth one is s^J_k. The resulting expression Ł$_4$ contains the tensor $(a_{JI_4I_5}a_{I_2I_3I_5} - a_{JI_3I_5}a_{I_2I_4I_5})$ as a multiplier[9], which for given three fields from the massless multiplet restricts a number of possible fields s^I_k to a finite number. Namely, k can be equal only to $2, 4, 6$. The case $k = 6$ is the most simple one, since in this case the only value of k_5 for which the tensor does not vanish is 4. Thus, we can extend the summation index over the whole set and use the fact that

$$\sum_{I_5} a_{JI_2I_5}a_{I_3I_4I_5} = \sum_{I_5} a_{JI_4I_5}a_{I_2I_3I_5} = \sum_{I_5} a_{JI_3I_5}a_{I_2I_4I_5}. \tag{3.2}$$

Hence, for $k = 6$ the sum of the couplings vanishes.

If $k = 4$ then there are two possible values of k_5: $k_5 = 2, 4$. Evaluating Ł$_4$ for these values of k_5 we find

$$Ł_4 = \frac{128}{3}\sum_{k_5}(a_{JI_4I_5}a_{I_2I_3I_5} - a_{JI_3I_5}a_{I_2I_4I_5}),$$

where sum is over $k_5 = 2$ and $k_5 = 4$. Thus, for $k = 4$ the sum Ł$_4$ vanishes by virtue of (3.2).

Finally we have $k = 2$ that allows for k_5 three values: $k_5 = 0, 2, 4$ and corresponds to the case when all fields are from the massless multiplet. Substituting $k = 2$ we find

$$Ł_4 = \frac{1}{324}\sum_{I_5}(a_{JI_4I_5}a_{I_2I_3I_5} - a_{JI_3I_5}a_{I_2I_4I_5})(k_5-2)(k_5-4)f_5(k_5+6)(k_5+8). \tag{3.3}$$

[9]We do not assume here a summation over the index I_5

For these values of k_5 the r.h.s. here vanishes identically. Thus, we have shown that there is no 4-derivative linear couplings of the massive fields with the massless ones and that the 4-derivative vertices are absent for fields from the massless multiplet.

The analysis of the couplings of the 2-derivative and non-derivative vertices proceeds in the same manner. For the $SO(6)$ tensors involving vector spherical harmonics one can use the formula

$$(f_5-1)^3 t_{125}t_{345} = \Big((f_5+23)(f_5-5)(f_5-21)+496(f_5-1)-1920\Big)t_{125}t_{345} \qquad (3.4)$$

to make the nonreducible part vanishing when three of four fields are from the massless multiplet. For the $SO(6)$ tensors involving tensor spherical harmonics the corresponding representations look as

$$f_5^2 p_{125}p_{345} = \Big((f_5-12)^2+24(f_5-12)+144\Big)p_{125}p_{345} \qquad (3.5)$$

$$f_5^3 p_{125}p_{345} = \Big(f_5(f_5-12)^2+24f_5(f_5-12)+144(f_5-12)+1728\Big)p_{125}p_{345} \qquad (3.6)$$

and they are based on the fact that for $k_3=k_4=2$ tensor p_{345} is nonzero only for $k_5=2$.

The relevant part of the quartic Lagrangian involving two derivatives can be written as follows

$$\text{Ł}_2 = \sum_{I_2,I_3,I_4}\Big(S_{JI_2I_3I_4}\nabla_a(s_k^J s_2^{I_2})\nabla^a(s_2^{I_3}s_2^{I_4}) + A_{JI_2I_3I_4}(\nabla_a s_k^J s_2^{I_2} - s_k^J\nabla_a s_2^{I_2})(\nabla^a s_2^{I_3}s_2^{I_4} - s_2^{I_3}\nabla^a s_2^{I_4})\Big)$$

In this Section and throughout the paper we often use the following notations

$$l = a_{125}a_{345}, \quad m = a_{145}a_{235}, \quad n = a_{135}a_{245},$$

where we do not assume summation over the index 5.

Calculating the coefficients in Ł$_2$, we see that they have the structure

$$S = S^l\cdot l + S^m\cdot m + S^n\cdot n, \quad A = A^n\cdot(n-m)$$

Omitting total-derivative terms, taking into account the symmetry of the coefficients in I_3, I_4, and using linear equations of motion, one can rewrite this Lagrangian in the form

$$\begin{aligned}\text{Ł}_2 = &\sum_{I_2,I_3,I_4} a_{JI_2I_5}a_{I_3I_4I_5}s_k^J\Big((-2S^l_{JI_2I_3I_4}+4A^n_{JI_2I_3I_4})s_2^{I_2}\nabla^a s_2^{I_3}\nabla^a s_2^{I_4}\\ &+(-2S^m_{JI_2I_3I_4}-2S^n_{JI_2I_3I_4}-4A^n_{JI_2I_3I_4})\nabla^a s_2^{I_2}s_2^{I_3}\nabla^a s_2^{I_4}\Big)\\ &+\sum_{I_2,I_3,I_4} 8S_{JI_2I_3I_4}s_k^J s_2^{I_2}s_2^{I_3}s_2^{I_4}\end{aligned}$$

To remove the 2-derivative terms, we make the shift

$$s_k^J \to s_k^J + \frac{1}{\kappa_k} J_{JI_2I_3I_4} s_2^{I_2} s_2^{I_3} s_2^{I_4} + \frac{1}{\kappa_k} L_{JI_2I_3I_4} s_2^{I_2} s_2^{I_3} s_2^{I_4},$$

where

$$J = \frac{1}{2}(S^m + S^n + 2A^n) \cdot l, \quad L = (-A^n + \frac{1}{6}(2S^l - S^m - S^n)) \cdot l.$$

Computing L we see that L is completely symmetric in I_2, I_3, I_4 for $k = 4, 6$. After the shift all 2-derivative terms are removed and we get the following Lagrangian

$$Ł_2 = \sum_{I_2,I_3,I_4} \left(8S_{JI_2I_3I_4} + (-12 - m_k^2)(J_{JI_2I_3I_4} + L_{JI_2I_3I_4}) \right) s_k^J s_2^{I_2} s_2^{I_3} s_2^{I_4}.$$

This Lagrangian should be summed up with the non-derivative terms of the quartic Lagrangian whose couplings will be denoted by $S^{(0)}_{JI_2I_3I_4}$. Thus the complete Lagrangian for non-derivative terms is given by

$$Ł_0 = \sum_{I_2,I_3,I_4} \left(4S^{(0)}_{JI_2I_3I_4} + 8S_{JI_2I_3I_4} + (-12 - m_k^2)(J_{JI_2I_3I_4} + L_{JI_2I_3I_4}) \right) s_k^J s_2^{I_2} s_2^{I_3} s_2^{I_4}$$

One can easily check that for $k = 6$ this Lagrangian vanishes, because in this case k_5 can be equal only to 4.

In the case of $k = 4$ there are two possible values of k_5: $k_5 = 2, 4$, and we get at first sight a nonzero result which has the form

$$Ł_0 = \left(\alpha \sum_{k(I_5)=2} a_{JI_2I_5} a_{I_3I_4I_5} + \beta \sum_{k(I_5)=4} a_{JI_2I_5} a_{I_3I_4I_5} \right) s_k^J s_2^{I_2} s_2^{I_3} s_2^{I_4}.$$

However, now we can use identity (9.5) to show that

$$\sum_{k(I_5)=4} a_{JI_2I_5} a_{I_3I_4I_5} s_4^J s_2^{I_2} s_2^{I_3} s_2^{I_4} = \frac{8}{7} \sum_{k(I_5)=2} a_{JI_2I_5} a_{I_3I_4I_5} s_4^J s_2^{I_2} s_2^{I_3} s_2^{I_4}$$

Taking into account this relation we find that non-derivative Lagrangian $Ł_0$ vanishes too.

Thus we have shown that at least at the level of the quartic action for scalars s^I there is a consistent dimensional reduction of the type IIB supergravity to the gauged supergravity on the AdS_5 background. We would like to stress again that this reduction requires non-trivial redefinitions of the fields.

Now we can compute the quartic couplings for the case when all four fields are from the massless multiplet. Summing up the nonvanishing quartic couplings of the two-derivative vertices we find out that the answer contains the terms involving the $SO(6)$ tensors of the form

$a_{I_1I_3I_5}a_{I_2I_4I_5}$ and $a_{I_1I_4I_5}a_{I_2I_3I_5}$. Integrating by parts it is possible to convert the tensor indices to the normal order, namely to $a_{I_1I_2I_5}a_{I_3I_4I_5}$ This also leads to the additional contributions to the quartic couplings of the non-derivative vertices. Recall that $k(I_5)$ runs now the set $0, 2, 4$.

Again it is useful to note that identity (9.5) implies a number of relations between the Lagrangian terms involving tensors $a_{125}a_{345}$. In particular, when all the fields are from the massless multiplet one finds the relation

$$\sum_{k(I_5)=4} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}s_2^{I_2}s_2^{I_3}s_2^{I_4} = \frac{1}{4}\sum_{k(I_5)=2} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}s_2^{I_2}s_2^{I_3}s_2^{I_4} + \sum_{k(I_5)=0} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}s_2^{I_2}s_2^{I_3}s_2^{I_4}.$$

Actually there is no sum over $k(I_5) = 0$ since in this case I_5 is just the trivial representation. Analogously, multiplying both sides of (9.5) by $s_2^{I_1}\nabla_a s_2^{I_2}s_2^{I_3}\nabla^a s_2^{I_4}$ and then integrating by parts and using the previous relation one obtains the following formula:

$$\begin{aligned}\sum_{k(I_5)=0,2,4} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}\nabla_a s_2^{I_2}s_2^{I_3}\nabla^a s_2^{I_4} = {}& \frac{8}{3}\sum_{k(I_5)=0} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}s_2^{I_2}s_2^{I_3}s_2^{I_4} \\ &+ \frac{5}{3}\sum_{k(I_5)=2} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}s_2^{I_2}s_2^{I_3}s_2^{I_4}.\end{aligned}$$

These relations allow one to exclude from the Lagrangian for the scalar fields from the massless multiplet the contributions of the representations I_5 with $k(I_5) = 4$.

In this way we find the following values of the quartic couplings of the 2-derivative vertex

$$\begin{aligned}\mathcal{L}^{(2)}_{AdS5} = {}& \frac{5^2\cdot 2^9}{27}\sum_{k(I_5)=2} a_{I_1I_2I_5}a_{I_3I_4I_5}\nabla_a(s_2^{I_1}s_2^{I_2})\nabla^a(s_2^{I_3}s_2^{I_4}) \\ &+ \frac{2^{13}}{27}\sum_{k(I_5)=0} a_{I_1I_2I_5}a_{I_3I_4I_5}\nabla_a(s_2^{I_1}s_2^{I_2})\nabla^a(s_2^{I_3}s_2^{I_4})\end{aligned} \tag{3.7}$$

and of the non-derivative vertex

$$\mathcal{L}^{(0)}_{AdS5} = -\frac{5^2\cdot 2^{11}}{9}\sum_{k(I_5)=2} a_{I_1I_2I_5}a_{I_3I_4I_5}s_2^{I_1}s_2^{I_2}s_2^{I_3}s_2^{I_4}. \tag{3.8}$$

Note that the contribution of the trivial representation completely dissappears from the non-derivative quartic coupling.

The quartic action can be further simplified by substituting the integrals of spherical harmonics for their explicit value. By using (9.12) one gets

$$\sum_{k(I_5)=2} a_{I_1I_2I_5}a_{I_3I_4I_5} = \frac{2^5\cdot 3}{5^2\pi^3}\sum_{I_5} C^{I_1I_2I_5}C^{I_3I_4I_5},$$

where $C^{I_1I_2I_3} = \langle C^{I_1}C^{I_2}C^{I_3}\rangle$. One can easily establish the following summation formula

$$\sum_I C^I_{ij}C^I_{kl} = \frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{6}\delta_{ij}\delta_{kl}$$

that steams from the fact that the l.h.s. of the expression above is a fourth rank tensor of $SO(6)$, symmetric and traceless both in (ij) and (kl) indices with the normalization condition $C^I_{ij}C^I_{ij} = 20$. Applying this formula one gets

$$\sum_{k(I_5)=2} a_{I_1I_2I_5}a_{I_3I_4I_5} = \frac{2^4\cdot 3}{5^2\pi^3}\left(C^{I_1I_2I_3I_4} + C^{I_1I_2I_4I_3} - \frac{1}{3}\delta^{I_1I_2}\delta^{I_3I_4}\right), \tag{3.9}$$

where the shorthand notation $C^{I_1I_2I_3I_4} = C^{I_1}_{i_1i_2}C^{I_2}_{i_2i_3}C^{I_3}_{i_3i_4}C^{I_4}_{i_4i_1}$ for the trace product of four matrices C^I was introduced. It remains to note that for $k_5 = 0$ the normalization condition for scalar spherical harmonics gives $\sum_{k(I_5)=0} a_{I_1I_2I_5}a_{I_3I_4I_5} = \frac{1}{\pi^3}\delta^{I_1I_2}\delta^{I_3I_4}$. By exploting this formula together with (3.9) the two-derivative Lagrangian may be reduced to the following form:

$$\mathcal{L}^{(2)}_{AdS5} = \frac{2^{14}}{9\pi^3}C_{I_1I_2I_3I_4}\nabla_a(s_2^{I_1}s_2^{I_2})\nabla^a(s_2^{I_3}s_2^{I_4}) \tag{3.10}$$

One can also introduce the fields $s_{ij} \equiv C^I_{ij}s^I$ that provide the natural parametrization of the coset space $SL(6,\mathbf{R})/SO(6)$. Although it is clear that namely this form of the action should be compared to the one of the gauged $\mathcal{N} = 8$, $d = 5$ supergravity, we will not do this here. We only note that the comparison requires to perform the field redefinitions of the type $s^{I_1} \to s^{I_1} + j_{I_1I_2I_3I_4}s^{I_2}s^{I_3}s^{I_4}$ to convert the Lagrangian terms with two derivatives to the form (3.10).

4 Quartic couplings in the extremal case

In our previous paper [6] we conjectured that quartic couplings of scalars s^I vanish in the extremal case when, say, $k_1 = k_2 + k_3 + k_4$. In this Section we show that the quartic couplings we found do satisfy the property after an additional shift of the fields. In principle by using the shift one can find such a representation of the quartic couplings, that makes the vanishing explicit.

The vanishing of the quartic couplings means that correlation functions of extended CPOs vanish in the extremal case [6] and also implies the non-renormalization theorem [29] for the corresponding extremal correlators of single-trace CPOs.

To prove the vanishing we find convenient to use different 4-derivative vertices. Namely, one can easily show that the following relations are valid on-shell

$$A^{(4)}_{1234}\int s_1\nabla_a s_2\nabla^2_b(s_3\nabla^a s_4) = -2A^{(4)}_{1234}\int \nabla_a s_1\nabla_b s_2\nabla^a s_3\nabla^b s_4 \tag{4.1}$$

$$
\begin{aligned}
&- 4A^{(4)}_{1234}\int s_1\nabla_a s_2 s_3\nabla^a s_4 \\
&- \frac{1}{4}A^{(4)}_{1234}(m_1^2-m_2^2)(m_3^2-m_4^2)\int s_1 s_2 s_3 s_4.
\end{aligned}
$$

$$
\begin{aligned}
S^{(4)}_{1234}\int s_1\nabla_a s_2\nabla_b^2(s_3\nabla^a s_4) &= -S^{(4)}_{1234}\int \nabla_a s_1\nabla^a s_2\nabla_b s_3\nabla^b s_4 \\
&+ S^{(4)}_{1234}\left(m_1^2+m_2^2+m_3^2+m_4^2-4\right)\int s_1\nabla_a s_2 s_3\nabla^a s_4 \\
&+ \frac{1}{4}S^{(4)}_{1234}(m_1^2+m_2^2)(m_3^2+m_4^2)\int s_1 s_2 s_3 s_4. \qquad (4.2)
\end{aligned}
$$

Thus we replace all 4-derivative vertices by the ones with only one derivative on each field. This representation has also the advantage that in this case the Hamiltonian reformulation of the quartic action is straightforward, and, therefore, as was shown in [16], there is no need to add boundary terms.

We assume for definiteness that $k_1 = k_2+k_3+k_4$. It is easy to show, by using the description of spherical harmonics as restrictions of functions, vectors and tensors on the $\mathbf{R}^6$ in which the sphere S^5 is embedded [5, 6], that the tensors (we do not assume summation over I_5 here)

$$
t_{125}t_{345}, \quad \text{and} \quad p_{125}p_{345}
$$

vanish in the extremal case, and that the tensors $a_{125}a_{345}$, $a_{135}a_{245}$ and $a_{145}a_{235}$ differ from zero only if $k_5 = k_3+k_4$, $k_5 = k_2+k_4$, $k_5 = k_2+k_3$ respectively. Thus in all vertices we can replace k_5 by a corresponding function of k_2, k_3, k_4, and, then the only dependence on k_5 is in tensors $a_{125}a_{345}$, $a_{135}a_{245}$ and $a_{145}a_{235}$ which are obviously symmetric in 2, 3, 4.

We single out the field s^{I_1} and write the relevant part of the quartic 4-derivative vertices in the form

$$
Ł^{(4)}_{ext} = 4\sum_{I_2,I_3,I_4}\left(-S^{(4)}_{I_1I_2I_3I_4} - A^{(4)}_{I_1I_3I_2I_4} + A^{(4)}_{I_1I_4I_3I_2}\right)\nabla_a s^{I_1}\nabla^a s^{I_2}\nabla_b s^{I_3}\nabla^b s^{I_4},
$$

where we sum over the representations satisfying the extremality condition. Now, we substitute the values of k_5 discussed above, and $k_1 = k_2+k_3+k_4$ in the quartic couplings, and obtain zero.

To analyze 2-derivative terms we represent the 2-derivative Lagrangian as follows

$$
\begin{aligned}
Ł^{(2)}_{ext} = 4\sum_{I_2,I_3,I_4}&\Bigg(\left(-\frac{1}{2}\tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4}\right) s^{I_1}\nabla^a s^{I_2} s^{I_3}\nabla_a s^{I_4} \\
&+ \frac{1}{4}\left(\tilde{A}^{(2)}_{I_1I_2I_3I_4}(m_4^2-m_3^2) - \tilde{S}^{(2)}_{I_1I_2I_3I_4}(m_4^2+m_3^2)\right) s^{I_1}s^{I_2}s^{I_3}s^{I_4}\Bigg),
\end{aligned}
$$

where using (4.2) we define

$$
\begin{aligned}
\tilde{S}^{(2)}_{I_1I_2I_3I_4} &= S^{(2)}_{I_1I_2I_3I_4} + S^{(4)}_{I_1I_2I_3I_4}\left(m_1^2+m_2^2+m_3^2+m_4^2-4\right), \\
\tilde{A}^{(2)}_{I_1I_2I_3I_4} &= A^{(2)}_{I_1I_2I_3I_4} - 4A^{(4)}_{I_1I_2I_3I_4}.
\end{aligned}
$$

This time substituting k_5 and k_1 and symmetrizing the expression obtained in I_2 and I_4, we get a non-zero function which is, however, completely symmetric in I_2, I_3 and I_4. Thus we can remove the 2-derivative term by using the shift

$$s^{I_1} \to s^{I_1} - \frac{2}{3\kappa_1}\left(-\frac{1}{2}\tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4}\right) s^{I_2}s^{I_3}s^{I_4}.$$

This shift also produces an additional contribution to the non-derivative terms which is equal to

$$-\frac{2}{3}\left(-\frac{1}{2}\tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4}\right)(m_2^2 + m_3^2 + m_4^2 - m_1^2)s^{I_1}s^{I_2}s^{I_3}s^{I_4}.$$

After accounting this contribution the non-derivative terms aquire the form

$$\begin{aligned} Ł^{(0)}_{ext} = 4\sum_{I_2,I_3,I_4}\Big(& S^{(0)}_{I_1I_2I_3I_4} - \frac{1}{6}\left(-\frac{1}{2}\tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4}\right)(m_2^2 + m_3^2 + m_4^2 - m_1^2) \\ & + \frac{1}{4}\left(\tilde{A}^{(2)}_{I_1I_2I_3I_4}(m_4^2 - m_3^2) - \tilde{S}^{(2)}_{I_1I_2I_3I_4}(m_4^2 + m_3^2)\right) \\ & + \frac{1}{4}S^{(4)}_{1234}(m_1^2 + m_2^2)(m_3^2 + m_4^2) - \frac{1}{4}A^{(4)}_{1234}(m_1^2 - m_2^2)(m_3^2 - m_4^2)\Big)s^{I_1}s^{I_2}s^{I_3}s^{I_4}. \end{aligned}$$

Substituting k_5 and k_1 and symmetrizing the coefficient obtained in I_2, I_3 and I_4 we end up with zero.

5 Equations of motion

The equations of motion that follow from the action (2.13) have the form

$$\begin{aligned} \kappa_1(\nabla_a^2 - m_1^2)s_1 = & -3S_{125}s_2s_5 - 2T_{125}s_2t_5 - 2\Phi_{125}s_2\phi_5 \\ & + G_{125}\left(2\nabla^a(\nabla^b s_2\varphi^5_{ab}) - \nabla^a(\nabla_a s_2\varphi^c_{c5}) + \frac{1}{2}(m_1^2 + m_2^2 - f_5)s_2\varphi^c_{c5}\right) \\ & - A_{125}(2\nabla_a s_2 A^a_5 + s_2\nabla_a A^a_5) - C_{125}(2\nabla_a s_2 C^a_5 + s_2\nabla_a C^a_5) - \frac{\delta S_4}{\delta s_1}, \end{aligned} \tag{5.1}$$

$$\frac{32(k_5+2)(k_5+4)(k_5+5)}{k_5+3}\left(\nabla_a^2 t_5 - m_t^2 t_5\right) = -T_{125}s_1s_2, \tag{5.2}$$

$$\nabla_a^2\phi_5 - m_\phi^2\phi_5 = -2\Phi_{125}s_1s_2, \tag{5.3}$$

$$Eq_{ab}(\varphi) \equiv -\frac{1}{2}\nabla_c^2\varphi^5_{ab} + \frac{1}{2}\nabla_a\nabla^c\varphi^5_{cb} + \frac{1}{2}\nabla_b\nabla^c\varphi^5_{ca} + (\frac{1}{2}f_5 - 1)\varphi^5_{ab} \tag{5.4}$$

$$\begin{aligned} & -\frac{1}{2}g_{ab}\nabla_c\nabla_d\varphi_5^{cd} - \frac{1}{2}\nabla_a\nabla_b\varphi^c_{c5} + \frac{1}{2}g_{ab}\nabla_c^2\varphi^d_{d5} - g_{ab}(\frac{1}{2}f_5 + 1)\varphi^c_{c5} \\ & = G_{125}\left(\nabla_a s_1\nabla_b s_2 - \frac{1}{2}g_{ab}\nabla_c s_1\nabla^c s_2 - \frac{1}{4}g_{ab}(m_1^2 + m_2^2 - f_5)s_1s_2\right), \end{aligned} \tag{5.5}$$

$$\frac{k_5+1}{2(k_5+2)}\left(\nabla_b^2 A^5_a - \nabla^b\nabla_a A^5_b - m_A^2 A^5_a\right) = -A_{125}s_1\nabla_a s_2, \tag{5.6}$$

$$\frac{k_5+3}{2(k_5+2)}\left(\nabla_b^2 C^5_a - \nabla^b\nabla_a C^5_b - m_C^2 C^5_a\right) = -C_{125}s_1\nabla_a s_2. \tag{5.7}$$

Here $\kappa \equiv \frac{32k(k-1)(k+2)}{k+1}$, the summation over 2 and 5 is assumed, and the masses of all fields except s depend on k_5.

To obtain these equations of motion from the covariant equations (2.1) and (2.2) we first need to decompose them up to the third order, and then to perform a number of fields redefinitions to make the equations Lagrangian. It is convenient to begin by considering quadratic terms in the covariant equations because as we will see they also give contributions to cubic terms. It is also useful to single out contributions coming from different fields.

5.1 Contribution of scalars s

We begin by considering the contribution of the scalars s coming from quadratic terms in the equations of motion for s that are obtained from the covariant equations (2.1) and (2.2). Decomposing these equations up to the second order in fields, and keeping only terms quadratic in s, one can represent the equation for s in the following form[10]

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= D^s_{125}s_2s_5 + E^s_{125}\nabla_a s_2\nabla^a s_5 + F^s_{125}\nabla_a\nabla_b s_2\nabla^a\nabla^b s_5 \\ &+ R^s_{125}s_2(\nabla_b^2 - m_5^2)s_5 + T^s_{125}\nabla_a s_2\nabla^a(\nabla_b^2 - m_5^2)s_5 \end{aligned} \tag{5.8}$$

We see that the r.h.s of (5.8) contains terms proportional to linear part of the equations of motion: $(\nabla_b^2 - m_5^2)s_5$. Although such terms do not give contributions to quadratic terms, and by this reason were neglected in [5], they *do* contribute to cubic terms. To remove the higher-derivative terms on the first line of (5.8) one should make the field redefinition [5]

$$s_1 = {s'}_1 + J^s_{125}s'_2s'_5 + L^s_{125}\nabla_a s'_2\nabla^a s'_5, \tag{5.9}$$

where

$$2L^s_{125} = F^s_{125}, \quad 2J^s_{125} + L^s_{125}(m_2^2 + m_5^2 - m_1^2 - 8) = E^s_{125}.$$

Then (5.8) takes the form

$$(\nabla_a^2 - m_1^2)s'_1 = V^s_{125}s'_2s'_5 + \textit{cubic terms},$$

where

$$V^s_{125} = D^s_{125} - J^s_{125}(m_2^2 + m_5^2 - m_1^2) = -\frac{3}{\kappa_1}S_{125}. \tag{5.10}$$

[10] We do not present the explicit values of the coefficients here and below because they are pretty complicated and not very instructive.

To simplify the form of the cubic terms we make an additional shift[11]

$$\begin{aligned} s_1 \to s_1 \,&+\, F^s_{125}\nabla^a s_2 \nabla_a (J^s_{534}s_3s_4 + L^s_{534}\nabla_a s_3 \nabla^a s_4) \\ &+\, 2J^s_{125}J^s_{534}s_2s_3s_4 + 2J^s_{125}L^s_{534}s_2\nabla^a s_3\nabla_a s_4 \end{aligned}$$

and represent the equation in the form

$$\begin{aligned} (\nabla_a^2 - m_1^2)s_1 \,=\, & V^s_{123}s_2s_3 + (s0)_{1234}s_2s_3s_4 + (s2a)_{1234}\nabla^a s_2 s_3 \nabla_a s_4 + (s2b)_{1234}s_2\nabla^a s_3\nabla_a s_4 \\ &+\, (s4a)_{1234}\nabla^a s_2 \nabla^b s_3 \nabla_a\nabla_b s_4 + (s4b)_{1234}s_2\nabla^b\nabla^b s_3\nabla_a\nabla_b s_4 \\ &+\, (s6a)_{1234}\nabla^a s_2 \nabla^b\nabla^c s_3 \nabla_a\nabla_b\nabla_c s_4, \end{aligned} \tag{5.11}$$

where the coefficients are given by

$$\begin{aligned} (s0)_{1234} &= 2V^s_{125}J^s_{534} + (R^s_{125} - 2J^s_{125})D^s_{534} \\ (s2a)_{1234} &= 2(T^s_{125} - 2L^s_{125})D^s_{534} \\ (s2b)_{1234} &= 2V^s_{125}L^s_{534} + (R^s_{125} - 2J^s_{125})E^s_{534} \\ (s4a)_{1234} &= 2(T^s_{125} - 2L^s_{125})E^s_{534} \\ (s4b)_{1234} &= (R^s_{125} - 2J^s_{125})F^s_{534} \\ (s6a)_{1234} &= 2(T^s_{125} - 2L^s_{125})F^s_{534}. \end{aligned} \tag{5.12}$$

The coefficients s_{1234} in this equation depend on $SO(6)$ tensors of the form

$$\frac{k_5^n}{k_5(k_5-1)(k_5+1)(k_5+2)}a_{125}a_{345}, \quad n \geq 0.$$

5.2 Contribution of scalars t

To obtain the contribution of the scalars t to cubic terms in the equations of motion for s we need to decompose the covariant equations (2.1) and (2.2) up to the second order in fields, and to keep the terms of the form st in the equation for s and terms quadratic in s in the equation for t. We represent the equations for s and t in the following form

$$\begin{aligned} (\nabla_a^2 - m_1^2)s_1 \,=\, & K^t_{125}s_2t_5 + N^t_{125}\nabla_a s_2 \nabla^a t_5 + P^t_{125}\nabla_a\nabla_b s_2 \nabla^a\nabla^b t_5 \\ &+\, R^t_{125}s_2(\nabla_b^2 - m_t^2)t_5 + T^t_{125}\nabla_a s_2 \nabla^a(\nabla_b^2 - m_t^2)t_5, \end{aligned} \tag{5.13}$$

$$(\nabla_a^2 - m_t^2)t_5 \,=\, D^t_{345}s_3s_4 + E^t_{345}\nabla_a s_3 \nabla^a s_4 + F^t_{345}\nabla_a\nabla_b s_3\nabla^a\nabla^b s_4. \tag{5.14}$$

[11]Here and in what follows we omit the primes on redefined fields.

To get rid of the higher-derivative terms in (5.13) we perform the following redefinition of the fields s:

$$s_1 \to s_1 + J^s_{t125} s_2 t_5 + L^s_{t125} \nabla_a s_2 \nabla^a t_5,$$

where

$$2L^s_{t125} = P^t_{125}, \quad 2J^s_{t125} + L^s_{t125}(m_2^2 + m_t^2 - m_1^2 - 8) = N^t_{125}.$$

Then eq.(5.13) takes the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= V^s_{t125} s_2 t_5 + (R^t_{125} - J^s_{t125}) s_2 (\nabla_b^2 - m_t^2) t_5 \\ &+ (T^t_{125} - L^s_{t125}) \nabla_a s_2 \nabla^a (\nabla_b^2 - m_t^2) t_5,\end{aligned} \tag{5.15}$$

where

$$V^s_{t125} = K^t_{125} - J^s_{t125}(m_2^2 + m_t^2 - m_1^2) = -\frac{2}{\kappa_1} T_{125}. \tag{5.16}$$

To take into account the terms proportional to the linear part of the equation for t, one should use eq.(5.14). We also need to reduce eq.(5.14) to the canonical form (5.2). To this end we perform the shift of the field t [6]

$$t_5 \to t_5 + J^t_{345} s_3 s_4 + L^t_{345} \nabla_a s_3 \nabla^a s_4.$$

This shift removes all terms with derivatives, and we end up with eq.(5.2). Finally we make the same shift of t in eq.(5.15), and represent the equation in the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= V^s_{t125} s_2 t_5 + (t0)_{1234} s_2 s_3 s_4 + (t2a)_{1234} \nabla^a s_2 s_3 \nabla_a s_4 + (t2b)_{1234} s_2 \nabla^a s_3 \nabla_a s_4 \\ &+ (t4a)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (t4b)_{1234} s_2 \nabla^b \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &+ (t6a)_{1234} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4,\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}(t0)_{1234} &= V^s_{t125} J^t_{345} + (R^t_{125} - J^s_{t125}) D^t_{345} \\ (t2a)_{1234} &= 2(T^t_{125} - L^s_{t125}) D^t_{345} \\ (t2b)_{1234} &= V^s_{t125} L^t_{345} + (R^t_{125} - J^s_{t125}) E^t_{345} \\ (t4a)_{1234} &= 2(T^t_{125} - L^s_{t125}) E^t_{345} \\ (t4b)_{1234} &= (R^t_{125} - J^s_{t125}) F^t_{345} \\ (t6a)_{1234} &= 2(T^t_{125} - L^s_{t125}) F^t_{345}\end{aligned} \tag{5.18}$$

The coefficients t_{1234} in this equation depend on $SO(6)$ tensors of the form

$$\frac{k_5^n}{(k_5+2)(k_5+3)(k_5+4)(k_5+5)} a_{125} a_{345}, \quad n \geq 0.$$

Summing up the contributions of scalars s^I and t^I we find that the resulting coefficients only depend on the following $SO(6)$ tensors

$$f_5^n a_{125} a_{345}, \quad n \geq -1, \quad \frac{1}{f_5+3} a_{125} a_{345}, \quad \frac{1}{f_5-5} a_{125} a_{345}.$$

5.3 Contribution of scalars ϕ

Decomposing the covariant equations (2.1) and (2.2) up to the second order in fields, and keeping the terms of the form $s\phi$ in the equation for s and terms quadratic in s in the equation for ϕ, we represent the equations for s and ϕ as follows

$$\begin{aligned} (\nabla_a^2 - m_1^2) s_1 &= K^{\phi}_{125} s_2 \phi_5 + N^{\phi}_{125} \nabla_a s_2 \nabla^a \phi_5 + P^{\phi}_{125} \nabla_a \nabla_b s_2 \nabla^a \nabla^b \phi_5 \\ &\quad + R^{\phi}_{125} s_2 (\nabla_a^2 - m_\phi^2) \phi_5, \qquad (5.19) \\ (\nabla_a^2 - m_\phi^2) \phi_5 &= D^{\phi}_{345} s_3 s_4 + E^{\phi}_{345} \nabla_a s_3 \nabla^a s_4 + F^{\phi}_{345} \nabla_a \nabla_b s_3 \nabla^a \nabla^b s_4. \qquad (5.20) \end{aligned}$$

Following the same steps as above, we make the following field redefinitions to remove higher-derivative terms, and to reduce (5.20) to the canonical form (5.3)

$$\begin{aligned} s_1 &\to s_1 + J^s_{\phi 125} s_2 \phi_5 + L^s_{\phi 125} \nabla_a s_2 \nabla^a \phi_5, \\ \phi_5 &\to \phi_5 + J^{\phi}_{345} s_3 s_4 + L^{\phi}_{345} \nabla_a s_3 \nabla^a s_4, \end{aligned} \qquad (5.21)$$

where

$$J^{\phi}_{345} = \frac{2(-4 + k_3^2 + 4k_3k_4 + k_4^2 - f_5)}{(k_3+1)(k_4+1)} p_{345}, \quad L^{\phi}_{345} = -\frac{4p_{345}}{(k_3+1)(k_4+1)}. \qquad (5.22)$$

Then eq.(5.19) takes the form

$$\begin{aligned} (\nabla_a^2 - m_1^2) s_1 &= V^s_{\phi 125} s_2 \phi_3 + (\phi 0)_{1234} s_2 s_3 s_4 + (\phi 2a)_{1234} \nabla^a s_2 s_3 \nabla_a s_4 + (\phi 2b)_{1234} s_2 \nabla^a s_3 \nabla_a s_4 \\ &\quad + (\phi 4a)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (\phi 4b)_{1234} s_2 \nabla^b \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &\quad + (\phi 6a)_{1234} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4, \end{aligned} \qquad (5.23)$$

where

$$(\phi 0)_{1234} = V^s_{\phi 125} J^{\phi}_{345} + (R^{\phi}_{125} - J^s_{\phi 125}) D^{\phi}_{345}$$

$$\begin{aligned}
(\phi 2a)_{1234} &= -2L^s_{\phi 125} D^\phi_{345} \\
(\phi 2b)_{1234} &= V^s_{\phi 125} L^\phi_{345} + (R^\phi_{125} - J^s_{\phi 125}) E^\phi_{345} \\
(\phi 4a)_{1234} &= -2L^s_{\phi 125} E^\phi_{345} \\
(\phi 4b)_{1234} &= (R^\phi_{125} - J^s_{\phi 125}) F^\phi_{345} \\
(\phi 6a)_{1234} &= -2L^s_{\phi 125} F^\phi_{345}.
\end{aligned} \tag{5.24}$$

The coefficients ϕ_{1234} in this equation depend on $SO(6)$ tensors of the form

$$f_5^n p_{125} p_{345}, \quad n = 0, 1, 2, 3.$$

The surprizing fact is that this equation is Lagrangian. Namely, it can be derived from the following Lagrangian:

$$\begin{aligned}
\mathcal{L}_\phi &= \mathcal{L}_2(s) + \frac{1}{4} \nabla_a (J^\phi_{125} s_1 s_2 + L^\phi_{125} \nabla_b s_1 \nabla^b s_2) \nabla^a (J^\phi_{345} s_3 s_4 + L^\phi_{345} \nabla_c s_3 \nabla^c s_4) \\
&+ \frac{1}{4} f_5 (J^\phi_{125} s_1 s_2 + L^\phi_{125} \nabla_b s_1 \nabla^b s_2)(J^\phi_{345} s_3 s_4 + L^\phi_{345} \nabla_c s_3 \nabla^c s_4) \\
&+ \Phi_{125} s_1 s_2 (J_{345} s_3 s_4 + L^\phi_{345} \nabla_b s_3 \nabla^b s_4),
\end{aligned} \tag{5.25}$$

where J^ϕ_{125} and L^ϕ_{125} are given by (5.22). One can easily see that if one makes a redefinition of ϕ inverse to (5.21): $\phi_5 \to \phi_5 - J^\phi_{125} s_1 s_2 - L^\phi_{125} \nabla_a s_1 \nabla^a s_2$, then the quartic terms in (5.25) will be removed from the action, but we obtain additional cubic higher-derivative terms.

5.4 Contribution of massive gravitons

We loosely refer to symmetric tensor fields coming from the AdS_5 components of the graviton as massive gravitons. To account for the massive graviton contribution we first need to derive equations of motion for the massive gravitons. In principle to obtain these equations one should consider the Einstein equations (2.2) not only with the indices (a, b), but also with the indices (a, α) and (α, β). The reason is that the equations for $\nabla^a \varphi_{ab}$ and φ^a_a that are constraints and, therefore should be a consequence of a true equation, do not follow from (2.2) if one restricts oneself by considering only indices (a, b). To find the true equation for the massive gravitons, it is convenient to replace (2.2) by the following equivalent equation

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3!} \left(F_{MM_2 \dots M_4} F_N^{M_2 \dots M_4} - \frac{1}{5} g_{ab} F_{M_1 \dots M_5} F^{M_1 \dots M_5} \right). \tag{5.26}$$

Namely this equation one would derive from the usual Lagrangian for the metric and the nonchiral five-form in ten dimensions. An important property of the equation is that after performing the off-shell shift of h_{ab} (2.9), its linear part coincides with the linear part of (5.5)

$$\left(R_{ab} - \frac{1}{2}g_{ab}R - \frac{1}{3!}\left(F_aF_b - \frac{1}{5}g_{ab}F_{M_1\dots M_5}F^{M_1\dots M_5}\right)\right)^{(1)} = Eq_{ab}(\varphi). \tag{5.27}$$

Decomposing eq. (5.26) up to the second order in fields, and keeping the terms quadratic in s, we can represent the equations for φ_{ab} in the following form

$$\begin{aligned}
Eq_{ab}(\varphi) &+ \alpha_{123}\nabla_a s_1\nabla_b s_2 + \beta_{123}\nabla_c\nabla_a s_1\nabla^c\nabla_b s_2 + \gamma_{123}\nabla_c\nabla^d\nabla_a s_1\nabla^c\nabla_d\nabla_b s_2 \\
&+ \mu_{123}(\nabla_a(\nabla_b s_1 s_2) + \nabla_b(\nabla_a s_1 s_2)) + \nu_{123}(\nabla_a(\nabla^c s_1\nabla_b\nabla_c s_2) + \nabla_b(\nabla^c s_1\nabla_a\nabla_c s_2)) \\
&+ \rho_{123}\nabla_a\nabla_b(s_1 s_2) + \sigma_{123}\nabla_a\nabla_b(\nabla_c\nabla_d s_1\nabla^c\nabla^d s_2) \\
&+ \delta_{123}(\nabla_a(\nabla^c\nabla^d s_1\nabla_c\nabla_d\nabla_b s_2) + \nabla_b(\nabla^c\nabla^d s_1\nabla_c\nabla_d\nabla_a s_2)) + g_{ab}C = 0.
\end{aligned} \tag{5.28}$$

Here C denotes the following contribution:

$$C = T^1_{123}s_1s_2 + T^2_{123}\nabla_a s_1\nabla^a s_2 + T^3_{123}\nabla_a\nabla_b s_1\nabla^a\nabla^b s_2 - \frac{1}{2}L_{123}\nabla_a\nabla_b\nabla_c s_1\nabla^a\nabla^b\nabla^c s_2. \tag{5.29}$$

To remove the higher-derivative and total-derivative terms we perform the following shift of the massive gravitons:

$$\varphi^3_{ab} = \varphi'^3_{ab} + \nabla_a\xi^3_b + \nabla_b\xi^3_a + g_{ab}\eta^3 + J_{123}\nabla_a s_1\nabla_b s_2 + L_{123}\nabla_c\nabla_a s_1\nabla^c\nabla_b s_2. \tag{5.30}$$

Here J_{123} and L_{123} depend on the coefficients α, β, γ as follows

$$L_{123} = \gamma_{123}, \quad J_{123} = \beta_{123} - \frac{1}{2}L_{123}(m_1^2 + m_2^2 - f_3 - 18)$$

and the cubic vertex G_{123} is expressed through them as

$$G_{123} = -\alpha_{123} + \frac{1}{2}J_{123}(m_1^2 + m_2^2 - f_3 - 6) - L_{123}(m_1^2 + m_2^2).$$

To get rid of the total derivative terms with the coefficients $\mu, \nu, \rho, \sigma, \delta$ one also has to impose the following relation

$$\frac{1}{2}f_3\xi^3_a - \frac{3}{4}\nabla_a\eta^3 + U_{123}s_2\nabla_a s_1 + H_{123}\nabla_c s_1\nabla^c\nabla_a s_2 = 0, \tag{5.31}$$

where

$$\begin{aligned}
U_{123} &= \frac{1}{2}(m_2^2 J_{123} - 3m_2^2 L_{123} + 2\rho_{123} + 2\mu_{123} + 2m_2^2\sigma_{123}), \\
H_{123} &= \frac{1}{2}(m_1^2 - 3)L_{123} + \nu_{123} - \sigma_{123}.
\end{aligned}$$

Thus only the coefficient η has not been fixed yet. Actually, a change of the coefficient (with the simultaneous change of ξ according to (5.31)) results only in a change of the interaction of the trace φ_a^a of the massive graviton with the scalars s. In particular, one can choose η in such a way that only traceless part of a massive graviton interacts with the scalars s. However, this choice leads to the appearance of quartic couplings with 6 derivatives. Terms with 6 derivatives are absent only if we choose the cubic vertex as in eq.(2.23). This vertex is a natural generalization of the interaction vertex of a massless graviton with scalar fields. To determine η we take the trace of eq.(5.28) and represent the resulting equation as

$$-\frac{3}{2}\nabla_a(\nabla_b\varphi^{ab} - \nabla^a\varphi_c^c) - 2(f+3)\varphi_c^c + \tilde{\alpha}_{123}\nabla_a s_1\nabla^a s_2$$
$$+\tilde{\beta}_{123}\nabla_a\nabla_b s_1\nabla^a\nabla^b s_2 + \tilde{\gamma}_{123}\nabla_a\nabla_b\nabla_c s_1\nabla^a\nabla^b\nabla^c s_2 + \tilde{\delta}_{123}s_1 s_2 = 0. \quad (5.32)$$

Then, by requiring that after the shift (5.30) eq.(5.32) coincides with the trace of (5.5) and assuming that η has the form

$$\eta_3 = A_{123}s_1 s_2 + B_{123}\nabla_a s_1\nabla^a s_2 + C_{123}\nabla_a\nabla_b s_1\nabla^a\nabla^b s_2,$$

we find the following relations

$$8H_{123} + \left(\frac{3}{2}J_{123} - (\frac{33}{2} + 2f_3)K_{123}\right) + \tilde{\beta}_{123} - 10(f_3+3)C_{123} = 0, \quad (5.33)$$
$$8H_{123}(m_2^2 - 4) + 8U_{123} - 2(f_3+6)J_{123} + (6 - \frac{3}{2}(m_1^2-4)(m_2^2-4)K_{123})$$
$$+\tilde{\alpha}_{123} - 10(f_3+3)B_{123} = \frac{3}{2}G_{123},$$
$$8U_{123}m_1^2 + \frac{3}{2}m_1^2 m_2^2(3K_{123} - J_{123}) + \tilde{\delta}_{123} - 10(f_3+3)A_{123} = \frac{3}{4}(m_1^2 + m_2^2 - f_3)G_{123}.$$

In particular, one can show that $C_{123} = 0$.

To find the massive graviton contribution to the equations for s we also need to know equations of motion for $\nabla^b\varphi_{ab}$ ($\nabla^b\varphi'_{ab}$) and φ_a^a (φ'^a_a). The simplest way to derive the equations is first to take into account that φ'_{ab} satisfies eq.(5.5), and then to use the graviton redefinition (5.30) to find $\nabla^a\varphi_{ab}$ and φ_a^a. Differentiating and taking the trace of (5.5) one can easily obtain

$$\varphi'^a_{a3} = -\frac{3G_{123}}{2f_3(f_3+3)}\left(-\frac{1}{4}(m_1^2 - m_2^2)^2 - \frac{1}{6}f_3(m_1^2+m_2^2) + \frac{5}{12}f_3^2\right)s_1 s_2,$$
$$\nabla^b\varphi'^3_{ab} = \frac{G_{123}}{f_3}(m_2^2 - m_1^2 + f_3)\nabla_a s_1 s_2 + \nabla_a\varphi'^b_{b3}.$$

These equations also explain why we took the interaction vertex of massive gravitons in the form (2.23), namely, there are no terms with derivatives in the r.h.s. of the equation for φ'^a_a under this choice.

Now we can proceed with the massive graviton contribution to the equations for s. To find the equations we should decompose the covariant equations of motion up to the second order, make the shift (2.9), and keep only the terms of the form $s\varphi$. To simplify the consideration we also find convenient not to shift the trace of the original gravitons h_a^a (however we do not decompose h_{ab} in the sum of a traceless tensor and a trace) but to take into account the contribution of h_a^a later. This can be easily done because the equation of motion for h_a^a follows from the Einstein equation (2.2) with indices (α, β). Then the equations for s take the form

$$(\nabla_a^2 - m_1^2)s_1 = V_{123}^g \nabla^a \nabla^b s_2 \varphi_{ab}^3 + R_{123}^g \nabla^a s_2 \nabla^b \varphi_{ab}^3 + T_{123}^g s_2 \varphi_a^{a3}, \tag{5.34}$$

where $V_{123}^g = \frac{2}{\kappa_1} G_{123}$.

Finally, substituting the massive graviton redefinition (5.30) in (5.34), and performing the following redefinition of s to simplify the equation

$$s_1 \to s_1 + V_{123}^g \nabla^b s_2 \xi_b^3,$$

we represent the contribution of the massive gravitons in the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 = {} & V_{125}^g \left(\nabla^a (\nabla^b s_2 \varphi'^5_{ab}) - \frac{1}{2} \nabla^a (\nabla_a s_2 \varphi'^c_{c5}) + \frac{1}{4}(m_1^2 + m_2^2 - f_5) s_2 \varphi'^c_{c5}) \right) \\ & + (g0)_{1234} s_2 s_3 s_4 + (g2a)_{1234} \nabla^a s_2 s_3 \nabla_a s_4 + (g2b)_{1234} s_2 \nabla_a s_3 \nabla^a s_4 \\ & + (g4a)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (g4b)_{1234} s_2 \nabla_a \nabla_b s_3 \nabla^a \nabla^b s_4 \\ & + (g4c)_{1234} \nabla^a \nabla^b s_2 \nabla_a s_3 \nabla_b s_4 \\ & + (g6a)_{1234} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4 \\ & + (g6c)_{1234} \nabla^a \nabla^b s_2 \nabla_c \nabla_a s_3 \nabla^c \nabla_b s_4. \end{aligned} \tag{5.35}$$

The coefficients g_{1234} in this equation depend on $SO(6)$ tensors of the form

$$f_5^n a_{125} a_{345}, \quad n \geq -1, \quad \frac{1}{f_5 + 3} a_{125} a_{345}.$$

5.5 Contribution of the trace of massive gravitons

It is known [47] that at linear order the graviton trace h_a^a is equal to $-\frac{3}{5}\pi$. By using the Einstein equation (2.2) with indices (α, β) one can easily find that the combination

$$\bar{\varphi}_a^a = h_a^a + \frac{3}{5}\pi$$

is equal to

$$\bar{\varphi}_a^a = \Omega_{123}^1 s_1 s_2 + \Omega_{123}^2 \nabla_a s_1 \nabla^a s_2 + \Omega_{123}^2 \nabla_a \nabla_b s_1 \nabla^a \nabla^b s_2.$$

Taking into account that the terms of the form $s\bar{\varphi}_a^a$ enter the equation for s as follows

$$(\nabla_a^2 - m_a^2)s_1 = \mu_{123}s_2\bar{\varphi}_{a3}^a + \nu_{123}\nabla_a s_2 \nabla^a \bar{\varphi}_{c3}^c$$

we obtain the contribution of $\bar{\varphi}_a^a$

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 =& (tr0)_{1234}s_2s_3s_4 + (tr2a)_{1234}\nabla^a s_2 s_3 \nabla_a s_4 + (tr2b)_{1234}s_2\nabla^a s_3 \nabla_a s_4 \\ &+ (tr4a)_{1234}\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (tr4b)_{1234}s_2 \nabla^b \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &+ (tr6a)_{1234}\nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4. \end{aligned} \tag{5.36}$$

The coefficients tr_{1234} in this equation depend on $SO(6)$ tensors of the form

$$f_5^n a_{125}a_{345}, \quad n \geq -1, \quad \frac{1}{f_5 - 5}a_{125}a_{345}.$$

5.6 Contribution of vector fields

In this subsection V_a denotes either the vector field A_a or C_a. The contribution of the vector fields to the equations of motion for the scalars s, and to the equations of motion for the vector fields may be written in the form

$$\begin{aligned}(\nabla_a^2 - m^2)s^1 =& K_{125}^V \nabla^a s^2 V_a^5 + N_{125}^V \nabla^a \nabla^b s^2 \nabla_a V_b^5 \\ &+ R_{125}^V s^2 \nabla^a V_a^5 + T_{125}^V \nabla^a s^2 (\nabla_b^2 V_a^5 - \nabla^b \nabla_a V_b^5 - m_V^2 V_b^5), \end{aligned} \tag{5.37}$$

$$\begin{aligned}\nabla_a^2 V_b^5 - \nabla^a \nabla_b V_a^5 - m_V^2 V_a^5 =& \nabla_a V + D_{125}^V s_1 \nabla_a s_2 + E_{125}^V \nabla^b s_1 \nabla_a \nabla_b s_2 \\ &+ F_{125}^V \nabla^b \nabla^c s_1 \nabla_a \nabla_b \nabla_c s_2, \end{aligned} \tag{5.38}$$

where the constants D, E, F are antisymmetric in 1, 2, and V has the following dependence on s

$$V = Q_{123}^V s_1 s_2 + H_{123}^V \nabla^a s_1 \nabla_a s_2.$$

To get rid of higher-derivative terms in eq.(5.37) we perform the following shift

$$s_1 \to s_1 + \frac{1}{2}N_{123}^V \nabla^a s_2 V_a^3.$$

Then the equation acquires the form

$$\begin{aligned}(\nabla_a^2 - m^2)s_1 =& V_{123}\nabla^a s_2 V_a^3 + R_{123}s_2 \nabla^a V_a^3 - \frac{1}{2}N_{123}\nabla^b s_2 \nabla_b(\nabla^a V_a^3) \\ &+ (T_{123} - \frac{1}{2}N_{123})\nabla^a s_2 (\nabla_b^2 V_a^3 - \nabla^b \nabla_a V_b^3 - m_V^2 V_b^3). \end{aligned}$$

To remove higher-derivative and total-derivative term from eq.(5.38), and to reduce it to the canonical form we make the following fields redefinition [6]

$$V_a^3 = V'^3_a - \frac{1}{m_V^2}\nabla_a \bar{V} + J^V_{453} s_4 \nabla_a s_5 + L^V_{453} \nabla^b s_4 \nabla_a \nabla_b s_5.$$

Finally we should substitute the redefinition in eq.(5.39), and represent it in the form

$$\begin{aligned}(\nabla_a^2 - m^2)s_1 = & \; V_{125}\left(\nabla^a s_2 V'^5_a + \frac{1}{2} s_2 \nabla^a V'^5_a\right) + (V0)_{1234} s_2 s_3 s_4 \\ & + (V2a)_{1234} \nabla^a s_2 s_3 \nabla_a s_4 + (V2b)_{1234} s_2 \nabla^a s_3 \nabla_a s_4 \\ & + (V4a)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (V4b)_{1234} s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & + (V6a)_{1234} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4.\end{aligned}$$

Summing up the contributions of the vectors A and C we see that the coefficients V_{1234} depend on the $SO(6)$ tensors of the form

$$f_5^n t_{125} t_{345}, \quad n \geq 0, \quad \frac{1}{f_5 - 5} t_{125} t_{345}.$$

5.7 Contribution of contact terms

Finally we have to take into account the contribution of contact terms that appear when we decompose the covariant equations of motion (2.1) and (2.2) up to the third order in the fields, and keep only terms cubic in the scalars s. This contribution has the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 = & \; (c0)_{1234} s_2 s_3 s_4 + (c2a)_{1234} \nabla^a s_2 s_3 \nabla_a s_4 + (c2b)_{1234} s_2 \nabla_a s_3 \nabla^a s_4 \\ & + (c4a)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (c4b)_{1234} s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & + (c4c)_{1234} \nabla_a \nabla_b s_2 \nabla^a s_3 \nabla^b s_4 \\ & + (c6a)_{1234} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4 \\ & + (c6c)_{1234} \nabla^a \nabla^b s_2 \nabla_b \nabla_c s_3 \nabla_a \nabla^c s_4. \end{aligned} \tag{5.39}$$

The coefficients c_{1234} in this equation depend on $SO(6)$ tensors of the form

$$f_5^n a_{125} a_{345}, \quad n \geq -1.$$

6 Analysis of the equations

In this Section we explain what steps one should undertake to obtain Lagrangian equations of motion from the original ones. Looking at the contributions derived in the previous section we see that the cubic corrections to the equations of motion for the scalars s^I have the form

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 &= (w0)_{1234}s_2s_3s_4 + (w2a)_{1234}\nabla^a s_2 s_3 \nabla_a s_4 \\
&+ (w4a)_{1234}\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (w4b)_{1234} s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&+ (w6a)_{1234}\nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4 \\
&+ (w6c)_{1234}\nabla^a \nabla^b s_2 \nabla_b \nabla_c s_3 \nabla_a \nabla^c s_4
\end{aligned} \tag{6.1}
$$

The coefficients w_{1234} in this equation may in general depend on $SO(6)$ tensors of the form

$$
f_5^n a_{125}a_{345}, \quad f_5^n t_{125}t_{345}, \quad f_5^n p_{125}p_{345}, \quad n \geq 0
$$

$$
f_5^{-1}a_{125}a_{345}, \quad \frac{1}{f_5+3}a_{125}a_{345}, \quad \frac{1}{f_5-5}a_{125}a_{345}, \quad \frac{1}{f_5-5}t_{125}t_{345},
$$

and tensors obtained from them by permutation of the indices 1, 2, 3, 4.

However, by using the identities (9.8), (9.9), (9.10) and (9.11) from the Appendix, we may reduce the tensors $t_{125}t_{345}$ and $f_5t_{125}t_{345}$ to the tensors $f_5^n a_{125}a_{345},\ n \geq -1$, and $p_{125}p_{345}$ and $f_5p_{125}p_{345}$ to the tensors $f_5^n a_{125}a_{345},\ n \geq -1$, $f_5^2 t_{125}t_{345}$, $\frac{1}{f_5-5}a_{125}a_{345}$, and $\frac{1}{f_5-5}t_{125}t_{345}$, and tensors obtained from them by permutation of the indices 1, 2, 3, 4. Then we find that (after the additional shift) the tensors of the form $\frac{1}{f_5+3}a_{125}a_{345}$, $\frac{1}{f_5-5}t_{125}t_{345}$ completely disappear from the total contribution, and the tensor $\frac{1}{f_5-5}a_{125}a_{345}$ occures only in the terms without derivatives, and with two derivatives.

6.1 6-derivative terms

We begin our analysis of (6.1) with the six-derivative terms. We see that the equation contains in particular the following term coming from the vectors contribution after the use of the identities (9.8) and (9.9)

$$
(\nabla_a^2 - m_1^2)s_1 = w_{1234}f_5(a_{135}a_{245} - a_{145}a_{235})\nabla^a s_2 \nabla_b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4. \tag{6.2}
$$

All other terms in the coefficients $w6a$ and $w6c$ only depend on tensors $f_5^n a_{125}a_{345},\ n = 0, 1, 2$. To compare (6.2) with the other contributions we perform the shift

$$
s_1 \to s_1 + J_{1234}\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4, \tag{6.3}
$$

and choose

$$
J_{1234} = j_{1234}f_5(a_{135}a_{245} - a_{145}a_{235}),
$$

where

$$j_{1234} = \frac{1}{2}\left(\frac{1}{3}(w_{1234} + w_{1324}) + w_{1234} - w_{1324}\right) = \frac{2}{3}w_{1234} - \frac{1}{3}w_{1324}.$$

This results in the following change of eq.(6.2)

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= -2(j_{1324} - j_{1432})f_5 a_{125}a_{345}\nabla^a\nabla_b s_2\nabla^b\nabla^c s_3\nabla_a\nabla_c s_4 \\ &+ j_{1324}(f_1 + f_2 + f_3 + f_4 - 3f_5)a_{125}a_{345}\nabla^a s_2\nabla^b\nabla_c s_3\nabla_a\nabla_b\nabla_c s_4 \\ &+ (dw4a)_{1234}\nabla^a s_2\nabla^b s_3\nabla_a\nabla_b s_4 + (dw2a)_{1234}\nabla^a s_2 s_3\nabla_a s_4,\end{aligned}$$

where

$$\begin{aligned}(dw4a)_{1234} &= -f_5(a_{135}a_{245} - a_{145}a_{235})(j_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 18) - 2j_{1243}) \\ &- 2j_{1423}f_5(a_{125}a_{245} - a_{135}a_{245}), \\ (dw2a)_{1234} &= 2((j_{1234} + j_{1243})(a_{135}a_{245} - a_{145}a_{235}) + j_{1324}(a_{125}a_{345} - a_{145}a_{235}))f_5 m_3^2\end{aligned}$$

are the additional contributions to the coefficients $w4a$ and $w2a$. Now we can use the symmetry of the 4-derivative term and the 2-derivative term under the permutation of the indices 2, 3, and 2, 4 respectively, and identity (9.5) from the Appendix to express $f_5 a_{145}a_{235}$ and the whole right hand side of the equation only through $a_{125}a_{345}$. Then the coefficients $dw4a$ and $dw2a$ acquire the form

$$\begin{aligned}(dw4a)_{1234} &= -\Big[f_5\Big((2j_{1324} + j_{1234})(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 18) \\ &- 2(2j_{1342} - j_{1423} + j_{1432} + j_{1243})\Big) \\ &- (f_1 + f_2 + f_3 + f_4)(j_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 18) - 2j_{1243})\Big]\, a_{125}a_{345} \\ (dw2a)_{1234} &= 2\left[(j_{1234} + j_{1243})(f_1 + f_2 + f_3 + f_4 - f_5)\right. \\ &+ \left.(j_{1324} - j_{1342} - 2j_{1432} - 2j_{1423})f_5\right] m_3^2 a_{125}a_{345}\end{aligned}$$

Thus we reduced all 6-derivative terms to terms which depend only on the tensors $f_5^n a_{125}a_{345}$.

Now summing up the vector fields contribution with the contributions of all the other fields we get that the term $w6c$ has the following structure

$$(w6c)_{1234} = (w6c0)_{1234}a_{125}a_{345} + (w6c1)_{1234}f_5 a_{125}a_{345} - \frac{f_5^2}{4\rho}a_{125}a_{345}, \tag{6.4}$$

where $(w6c1)_{1234}$ is a function symmetric under permutation of 2, 3, and 4, and we denote $\rho = (k_1 - 1)k_1(k_1 + 2)(k_2 + 1)(k_3 + 1)(k_4 + 1)$. Thus we may use the identity

$$w_{1234}f_5 a_{125}a_{345} = \frac{1}{3}w_{1234}(f_1 + f_2 + f_3 + f_4)a_{125}a_{345},$$

valid for any function symmetric in 2, 3, 4. So, we see that $w6c$ does not depend on the tensor $f_5 a_{125} a_{345}$, and, moreover $wc60$ should be symmetrized in 2, 3, 4 because it is multiplied by a symmetric tensor $\nabla_a \nabla^b s_2 \nabla_b \nabla^c s_3 \nabla^a \nabla_c s_4$.

Looking at the term $w6a$ we see that this term has the same form (6.4), and moreover, the coefficient $w6a1$ is also symmetric in 2, 3, 4, and, therefore, can be reduced to $w6a0$. The term $wa62$ proportional to $f_5^2 a_{125} a_{345}$ comes from massive gravitons, vectors and scalars ϕ and is given explicitly

$$(\nabla_a^2 - m_1^2) s_1 = -\frac{f_5^2}{2\rho} a_{145} a_{235} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4. \tag{6.5}$$

We may reduce the term to the structure $\nabla_a \nabla^b s_2 \nabla_b \nabla^c s_3 \nabla^a \nabla_c s_4$ by performing the shift

$$s_1 \to s_1 - \frac{f_5^2}{8\rho} a_{145} a_{235} \nabla^b s_2 \nabla^c s_3 \nabla_b \nabla_c s_4$$

that results in

$$\begin{aligned}(\nabla_a^2 - m_1^2) s_1 = & \frac{f_5^2}{4\rho} a_{125} a_{345} \nabla_a \nabla^b s_2 \nabla_b \nabla^c s_3 \nabla^a \nabla_c s_4 \\ & + (\frac{f_5^2}{2\rho} a_{125} a_{345} + \frac{f_5^2}{8\rho} a_{145} a_{235} (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 18)) \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & - m_3^2 \frac{f_5^2}{2\rho} a_{125} a_{345} \nabla^a s_2 s_3 \nabla_a s_4 + m_2^2 \frac{f_5^2}{4\rho} a_{125} a_{345} s_2 \nabla^a s_3 \nabla_a s_4.\end{aligned} \tag{6.6}$$

Summing up the coefficient on the first line of (6.6) with $w6c$ we obtain that the final contribution does not depend on the tensor $f_5^2 a_{125} a_{345}$. So the new coefficient $w6c$ depends only on the tensor $a_{125} a_{345}$, and we can easily reduce it to $w6a$ by means of the shift

$$s_1 \to s_1 + \frac{1}{2} (w6c)_{1234} \nabla^b s_2 \nabla^c s_3 \nabla_b \nabla_c s_4$$

This results in

$$\begin{aligned}(\nabla_a^2 - m_1^2) s_1 = & -2(w6c)_{1234} \nabla^a s_2 \nabla^b \nabla^c s_3 \nabla_a \nabla_b \nabla_c s_4 \\ & - \left(2(w6c)_{1234} + \frac{1}{2}(w6c)_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 18)\right) \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & + (w6c)_{1234} m_3^2 \nabla^a s_2 s_3 \nabla_a s_4.\end{aligned}$$

Adding the coefficient $-2(w6c)$ from the first line of the equation to $w6a$ we obtain a new coefficient that is symmetric in 2, 3, and, therefore, the first term on the r.h.s. of the equation can be transformed to the structure $\nabla_a \nabla^b s_2 \nabla_b \nabla^c s_3 \nabla^a \nabla_c s_4$ by using (6.3). Symmetrizing the coefficient in front of the 6-derivative term we obtain zero. This shift also produces additional contributions to the coefficients $w4a$ and $w2a$.

Thus we have shown that all 6-derivative terms could be shifted away.

6.2 4-derivative terms

We proceed with 4-derivative terms for which, we take into account all the additional contributions appeared in the previous subsection due to our way of working with 6-derivative terms. The coefficient $w4a$ contains the term $\frac{16(f_5-1)^2}{\rho} t_{125} t_{345}$ that gives Lagrangian contribution to the equations of motion. Other contributions are nonlagrangian, and we analyze them by decomposing the coefficients $w4a$ and $w4b$ in Laurent series in f_5.

I. 4-derivative terms with $1/f_5$.

These terms give the following contribution to the equations of motion

$$(\nabla_a^2 - m_1^2)s_1 = (w4ad)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 + (w4bd)_{1234} s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4.$$

We decompose $w4ad$ into parts symmetric and antisymmetric in 3, 4, and shift its symmetric part to the 4b-structure by using the field redefinition

$$s_1 \to s_1 + J_{1234} s_2 \nabla^a s_3 \nabla_a s_4. \tag{6.7}$$

The resulting 4-derivative vertices can be written in the form:

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 = {} & \frac{2(f_1-f_2)(f_3-f_4)}{\rho f_5} a_{125} a_{345} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & - \frac{(f_1-f_2)(f_3-f_4)}{\rho f_5} a_{125} a_{345} \nabla^a \nabla^b s_2 s_3 \nabla_a \nabla_b s_4.\end{aligned}$$

Finally performing the shift

$$s_1 \to s_1 + J_{1234} \nabla^b s_2 s_3 \nabla_b s_4 \tag{6.8}$$

and using the symmetry of the vertex w.r.t. 3, 4, we represent the final result for the 4-derivative vertex as follows

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 = {} & \frac{3(f_1-f_2)(f_3-f_4)}{\rho f_5} a_{125} a_{345} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & + \frac{(f_1-f_2)(f_3-f_4)}{2\rho f_5} (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) \nabla^a s_2 s_3 \nabla_a s_4.\end{aligned}$$

The 4-derivative term represents a Lagrangian contribution to the equation of motion, which can be derived from the Lagrangian of the form

$$\mathcal{L} = A^{(4)}_{1234} \int s_1 \nabla_a s_2 \nabla_b^2 (s_3 \nabla^a s_4), \tag{6.9}$$

where the quartic coupling A^4_{1234} is *antisymmetric* in 1, 2 and 3, 4, and *symmetric* under the interchange (1,2) and (3,4), and is given by

$$A^{(4)}_{1234} = -\kappa \frac{3}{8\rho f_5}(f_1 - f_2)(f_3 - f_4).$$

The equations of motion that follow from the Lagrangian are

$$\begin{aligned}\kappa(\nabla_a^2 - m_1^2)s_1 = & -8A^{(4)}_{1234}\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & - 4(m_3^2 + m_4^2 - 4)A^{(4)}_{1234}\nabla^a s_2 s_3 \nabla_a s_4 \\ & - 2(m_4^2 - m_3^2)A^{(4)}_{1234} s_2 \nabla^a s_3 \nabla_a s_4 \\ & - (m_3^2 + m_4^2 - 4)(m_4^2 - m_3^2)A^{(4)}_{1234} s_2 s_3 s_4.\end{aligned}$$

It is clear that the 4-derivative term cannot be removed by any field redefinition.

II. 4-derivative terms with f_5^3.

The contribution of the terms with f_5^3 is given by

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 = & -\frac{f_5^3}{64\rho}(3l + 8m)\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & - \frac{f_5^3}{128\rho}(3l + 4m + 4n) s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4.\end{aligned}$$

Performing the shift

$$s_1 \to s_1 + J_{1234} s_2 \nabla^b s_3 \nabla_b s_4, \tag{6.10}$$

where $2J_{1234} = -\frac{f_5^3}{128\rho}(3l + 4m + 4n)$ is symmetric in $3, 4$, we obtain the Lagrangian 4-derivative term

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 = & \frac{f_5^3}{16\rho}(n - m)\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ & + \frac{f_5^3}{256\rho}(3l + 4n + 4m)(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) s_2 \nabla^a s_3 \nabla_a s_4,\end{aligned}$$

that can be again derived from a vertex of the form (6.9).

III. 4-derivative terms with f_5^2.

Here we first consider the term of the 4a-type. The term of the 4b-type is also nonzero and we consider it later.

$$(\nabla_a^2 - m_1^2)s_1 = (h^n_{1234} n + h^m_{1234} m)\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4.$$

Here h^n, h^m denote the coefficients of the corresponding structures n, m. We can rewrite this equation as follows

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 &= (h^n_{1234}n + h^n_{1243}m)\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&+ \omega_{1432} a_{125} a_{345} \nabla^a \nabla^b s_2 \nabla_a s_3 \nabla_b s_4,
\end{aligned}
$$

where

$$
\omega_{1234} = (h^m_{1234} - h^n_{1243}).
$$

To convert the equation to the one containing only the 4b-type structure $s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4$ we make the field redefinition

$$
s_1 \to s_1 + \frac{1}{2}\omega_{1432} a_{125} a_{345} \nabla^b s_2 s_3 \nabla_b s_4 + j_{1234} s_2 \nabla^b s_3 \nabla_b s_4,
$$

where

$$
j_{1234} = \frac{1}{4}(h^n_{1234}n + h^n_{1243}m - \omega_{1432}l).
$$

Then the equation transforms as follows

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 &= -\frac{1}{2}(\omega_{1423}n + \omega_{1324}m + h^n_{1234}n + h^n_{1243}m - \omega_{1432}l)s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&- j_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)s_2 \nabla^a s_3 \nabla_a s_4 \\
&- \frac{1}{2}\omega_{1432} l (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)\nabla^a s_2 s_3 \nabla_a s_4.
\end{aligned}
$$

Now we sum the r.h.s. of the equation with the contribution of 4b-type and get a Lagrangian 4-derivative term

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 &= \frac{(2l - n - m)f_5^2}{32\rho}(-28 + 3f_1 + 3f_2 + 3f_3 + 3f_4)s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&- j_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)s_2 \nabla^a s_3 \nabla_a s_4 \\
&- \frac{1}{2}\omega_{1432} l (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)\nabla^a s_2 s_3 \nabla_a s_4.
\end{aligned}
$$

It is convenient, however, to reduce the 4-derivative term to the term of 4a-type by means of a field redefinition of the form,

$$
s_1 \to s_1 + J_{1234} s_2 \nabla^a s_3 \nabla_a s_4
$$

and by using the symmetry of the 4a-type term under the permutation of the indices 2, 3. The resulting equation looks as

$$
\begin{aligned}
&(\nabla_a^2 - m_1^2)s_1 = -\frac{f_5^2(n - m)}{16\rho}(3(f_1 + f_2 + f_3 + f_4) - 28)\nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&-\frac{f_5^2(2l - n - m)}{64\rho}(3(f_1 + f_2 + f_3 + f_4) - 28)(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)s_2 \nabla^a s_3 \nabla_a s_4 \\
&-j_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)s_2 \nabla^a s_3 \nabla_a s_4 \\
&-\frac{1}{2}\omega_{1432} l (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8)\nabla^a s_2 s_3 \nabla_a s_4,
\end{aligned}
$$

and the 4-derivative term can be obtained from the vertex of the form (6.9).

IV. 4-derivative terms with f_5.

We can reduce the 4b-type term to the 4a-type one by means of the shift

$$s_1 \to s_1 + j_{1234} s_2 \nabla^b s_3 \nabla_b s_4, \qquad j_{1234} = \frac{1}{2}(w4b)_{1234} f_5.$$

This results in

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= (w4a - 2\cdot w4b)_{1234} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &- j_{1234}(m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) s_2 \nabla_a s_3 \nabla^a s_4.\end{aligned}$$

Representing

$$(w4a - 2\cdot w4b)_{1234} = P^l_{1234} l + P^n_{1234} n + P^m_{1234} m,$$

using the identity (9.5), and changing the summation indices 2 and 3 we rewrite the equation in the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= (P^l_{1234} + P^n_{1324} - P^m_{1324} - P^m_{1234}) f_5 a_{125} a_{345} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &+ P^m_{1324}(f_1 + f_2 + f_3 + f_4) \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &- j_{1234} f_5 (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) s_2 \nabla_a s_3 \nabla^a s_4.\end{aligned}$$

The 4-derivative term represents a Lagrangian contribution as can be seen by decomposing the coefficients in front of $f_5 a_{125} a_{345}$ in parts antisymmetric and symmetric in 3, 4:

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= (Y^a + Y^s)_{1234}\, f_5 a_{125} a_{345} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &+ P^m_{1324}(f_1 + f_2 + f_3 + f_4) \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\ &- j_{1234} f_5 (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) s_2 \nabla_a s_3 \nabla^a s_4,\end{aligned}$$

where

$$\begin{aligned}Y^a_{1234} &= \frac{3(f_1 - f_2)(f_3 - f_4)}{16\rho} \\ Y^s_{1234} &= \frac{3(f_1 + f_2 + f_3 + f_4 - 2)(f_1 + f_2 + f_3 + f_4 - 12)}{8\rho}.\end{aligned}$$

It is convenient to get rid of the symmetric in 3, 4 4-derivative contribution by using the following identity

$$w_{1234} f_5 a_{125} a_{345} = \frac{1}{3} w_{1234}((f_1 + f_2 + f_3 + f_4) a_{125} a_{345} + f_5(a_{135} a_{245} - a_{145} a_{235})),$$

valid for any function w_{1234} symmetric in $2,3$. The final contribution is given by

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 &= \left(Y^a_{1234} f_5 l + \frac{1}{3} Y^s_{1234} f_5 (n-m)\right) \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&+ \left(\frac{1}{3} Y^s_{1234} + P^m_{1324}(f_1+f_2+f_3+f_4)\right) \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&- j_{1234} f_5 (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) s_2 \nabla_a s_3 \nabla^a s_4 .
\end{aligned}
$$

Thus, we are left only with the antisymmetric 4-derivative Lagrangian contribution and the additional 4a-type terms without f_5.

V. 4-derivative terms without f_5.

Just as above we use the shift

$$
s_1 \to s_1 + j_{1234} s_2 \nabla^b s_3 \nabla_b s_4, \qquad j_{1234} = \frac{1}{2}(w4b0)_{1234},
$$

to get rid of the $4b$ structure, and take into account the additional contribution coming from the terms with f_5. Then we symmetrize the resulting coefficient $w4a0$ in 2 and 3, and decompose it into parts symmetric and antisymmetric in 3 and 4. Then we shift the symmetric part back to the 4b structure and get the equation

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 &= (L4a0)_{1234} a_{125} a_{345} \nabla^a s_2 \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&+ (L4b0)_{1234} a_{125} a_{345} s_2 \nabla^a \nabla^b s_3 \nabla_a \nabla_b s_4 \\
&+ 2(L4b0)_{1234} (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) a_{125} a_{345} s_2 \nabla_a s_3 \nabla^a s_4 \\
&- j_{1234} (m_2^2 + m_3^2 + m_4^2 - m_1^2 - 8) s_2 \nabla_a s_3 \nabla^a s_4 ,
\end{aligned} \tag{6.11}
$$

where

$$
\begin{aligned}
(L4a0)_{1234} &= -\frac{21(f_1-f_2)(f_3-f_4)}{16\rho} \\
(L4b0)_{1234} &= -\frac{7\left(2f_1f_2 + 2f_3f_4 - (f_1+f_2)(f_3+f_4)\right)}{8\rho} .
\end{aligned}
$$

Both the 4-derivative terms are Lagrangian. The antisymmetric term can be derived from a Lagrangian of the form (6.9), and the symmetric term can be obtain from the following Lagrangian

$$
\mathcal{L} = S^{(4)}_{1234} \int s_1 \nabla_a s_2 \nabla_b^2 (s_3 \nabla^a s_4), \tag{6.12}
$$

where the quartic coupling S^4_{1234} is *symmetric* in 1, 2 and 3, 4, and *symmetric* under the interchange (1,2) and (3,4), and is given by

$$
S^{(4)}_{1234} = \frac{\kappa}{4}(L4b0)_{1234} .
$$

Equations of motion that follow from the Lagrangian are

$$\begin{aligned}\kappa(\nabla_a^2 - m_1^2)s_1 &= 4S^{(4)}_{1234}s_2\nabla^a\nabla^b s_3\nabla_a\nabla_b s_4\\&+ 4(m_3^2+m_4^2-6)S^{(4)}_{1234}s_2\nabla^a s_3\nabla_a s_4\\&+ (m_3^2+m_4^2-4)(m_3^2+m_4^2)S^{(4)}_{1234}s_2s_3s_4.\end{aligned}$$

This completes considering 4-derivative terms.

6.3 2-derivative terms

We proceed with 2-derivative terms for which, we should take into account all the additional contributions appeared because of the shifts used in the previous subsections, and contributions which appear when one represents the 4-derivative terms as variations of the vertices of the types (6.9) and (6.12).

The coefficient $w2a$ contains four Lagrangian terms proportional to

$$f_5^2 p_{125}p_{345}, \quad (f_5-1)^3 t_{125}t_{345}, \quad (f_5-1)^2 t_{125}t_{345} \quad \text{and} \quad \frac{1}{f_5-5}a_{125}a_{345}$$

that can be found in Section 2.

We find convenient to represent the contribution of the other 2-derivative terms in the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= (w2a)_{1234}a_{125}a_{345}\nabla^a s_2 s_3\nabla_a s_4\\&+ (w2b)_{1234}a_{125}a_{345}s_2\nabla^b s_3\nabla_a s_4,\end{aligned}$$

where the coefficients $w2a$ and $w2b$ may depend on f_5.

This equation is non-Lagrangian, and we again analyze it by decomposing the coefficients $w2a$ and $w2b$ in Laurent series in f_5.

I. 2-derivative terms with $1/f_5$.

Taking into account all additional contributions we represent the 2-derivative contribution in the form

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= (A2ad)_{1234}\frac{1}{f_5}a_{125}a_{345}\nabla^a s_2 s_3\nabla_a s_4\\&+ (S2ad)_{1234}\frac{1}{f_5}a_{125}a_{345}\nabla^a s_2 s_3\nabla_a s_4 + (L2bd)_{1234}\frac{1}{f_5}a_{125}a_{345}s_2\nabla^a s_3\nabla_a s_4\end{aligned}$$

Here we decompose the coefficient of the 2a type on the antisymmetric $A2ad$ and symmetric $S2ad$ parts with respect to permutation of the indices 3, 4.

The antisymmetric part is Lagrangian and the coefficient is given by

$$\begin{aligned}(A2ad)_{1234} = &\frac{1}{2\rho}(f_1-f_2)(f_3-f_4)\\ &\times\,(36+f_1+f_2+f_3+f_4-20(k_1+k_2+k_3+k_4)+10(k_1+k_2)(k_3+k_4)).\end{aligned}$$

The corresponding 2-derivative term can be derived from the following Lagrangian

$$\mathcal{L} = A^{(2)}_{1234}\int s_1\nabla_a s_2 s_3\nabla^a s_4, \tag{6.13}$$

where the quartic coupling A^2_{1234} is *antisymmetric* in 1, 2 and 3, 4, and *symmetric* under the interchange (1,2) and (3,4). The equations of motion that follow from the Lagrangian are

$$\kappa(\nabla_a^2-m_1^2)s_1 = -4A^{(2)}_{1234}\nabla^a s_2 s_3\nabla_a s_4-(m_4^2-m_3^2)A^{(2)}_{1234}s_2s_3s_4,$$

and, therefore, we have

$$A^{(2)}_{1234} = -\frac{\kappa}{4}(A2ad)_{1234}\frac{1}{f_5}a_{125}a_{345}.$$

Now we shift the remaining type 2a structure to the type 2b one and get

$$\begin{aligned}(\nabla_a^2-m_1^2)s_1 = &\,(A2ad)_{1234}\frac{1}{f_5}a_{125}a_{345}\nabla^a s_2 s_3\nabla_a s_4 &(6.14)\\ &+\,(L2bd-\frac{1}{2}S2ad)_{1234}\frac{1}{f_5}a_{125}a_{345}s_2\nabla^a s_3\nabla_a s_4\\ &-\,\frac{1}{4}(S2ad)_{1234}(m_2^2+m_3^2+m_4^2-m_1^2)\frac{1}{f_5}a_{125}a_{345}s_2s_3s_4. &(6.15)\end{aligned}$$

Now we see that the 2b structure turns out to be Lagrangian with

$$(S2bd)_{1234} = -\frac{1}{2\rho}(k_1-k_2)(k_3-k_4)(f_1-f_2)(f_3-f_4). \tag{6.16}$$

The corresponding 2-derivative term can be derived from the Lagrangian

$$\mathcal{L} = S^{(2)}_{1234}\int s_1\nabla_a s_2 s_3\nabla^a s_4, \tag{6.17}$$

where the quartic coupling S^2_{1234} is *symmetric* in 1, 2 and 3, 4, and *symmetric* under the interchange (1,2) and (3,4). The equations of motion that follow from the Lagrangian are

$$\kappa(\nabla_a^2-m_1^2)s_1 = 2S^{(2)}_{1234}s_2\nabla^a s_3\nabla_a s_4+(m_4^2+m_3^2)S^{(2)}_{1234}s_2s_3s_4,$$

and, therefore, we have

$$S^{(2)}_{1234} = \frac{\kappa}{2}(S2bd)_{1234}\frac{1}{f_5}a_{125}a_{345}.$$

We omit considering terms with f_5^4, f_5^3 and f_5^2, because their analisys goes the same line as before. We just remark that we used the shift

$$s_1 \to s_1 + J_{1234}s_2s_3s_4$$

to remove the terms completely symmetric with respect to permutation of indices 2, 3 and 4 from the equations of motion.

II. 2-derivative terms with f_5.

By using a field redefinition, we shift the type 2b term to the type 2a one, and represent the equation in the form

$$(\nabla_a^2 - m_1^2)s_1 = ((w2)^l_{1234}l + (w2)^n_{1234}n + (w2)^m_{1432}m)f_5\nabla^a s_2s_3\nabla_a s_4.$$

This equation can be further rewritten as follows

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= ((w2)^l_{1234} - (w2)^n_{1234} - (w2)^n_{1432} + (w2)^m_{1432})f_5a_{125}a_{345}\nabla^a s_2s_3\nabla_a s_4\\ &+ (w2)^n_{1234}(f_1+f_2+f_3+f_4)\nabla^a s_2s_3\nabla_a s_4,\end{aligned}$$

where we use identity (9.5) and the symmetry of the tensor $\nabla^a s_2s_3\nabla_a s_4$ in 2 and 4. Introducing the notation

$$F_{1234} = (w2)^l_{1234} - (w2)^n_{1234} - (w2)^n_{1432} + (w2)^m_{1432},$$

and by using again (9.5) we rewrite the equation as follows

$$\begin{aligned}(\nabla_a^2 - m_1^2)s_1 &= (F^a + \frac{1}{3}F^s)_{1234}f_5a_{125}a_{345}\nabla^a s_2s_3\nabla_a s_4\\ &- \frac{1}{3}F^s_{1324}f_5a_{125}a_{345}s_2\nabla^a s_3\nabla_a s_4\\ &+ \left((w2)^n_{1234} + \frac{1}{3}F^s_{1234}\right)(f_1+f_2+f_3+f_4)a_{125}a_{345}\nabla^a s_2s_3\nabla_a s_4,\end{aligned}$$

where F^a and F^s denote the parts of F antisymmetric and symmetric in 2 and 4 respectively. Finally we use a field redefinition to shift the 2b structure to the 2a one, decompose the resulting 2a coefficient into parts symmetric and antisymmetric in 3 and 4

$$(F^a + \frac{1}{3}F^s)_{1234} + \frac{2}{3}F^s_{1324} = S_{1234} + A_{1234},$$

and shift the symmetric part to the 2b structure. The resulting equation with Lagrangian 2-derivative terms A_{1234} and S_{1234} looks as follows

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 = &\; A_{1234} f_5 a_{125} a_{345} \nabla^a s_2 s_3 \nabla_a s_4 \\
&- \frac{1}{2} S_{1234} f_5 a_{125} a_{345} s_2 \nabla^a s_3 \nabla_a s_4 \\
&+ \left((w2)^n + \frac{1}{3} F^s \right)_{1234} (f_1 + f_2 + f_3 + f_4) a_{125} a_{345} \nabla^a s_2 s_3 \nabla_a s_4 \\
&- \frac{1}{4} S_{1234} (m_2^2 + m_3^2 + m_4^2 - m_1^2) f_5 a_{125} a_{345} s_2 s_3 s_4 \\
&+ \frac{1}{6} F^s_{1234} (m_2^2 + m_3^2 + m_4^2 - m_1^2) f_5 a_{125} a_{345} s_2 s_3 s_4.
\end{aligned} \tag{6.18}
$$

The consideration of the terms without f_5 is simple. We sum up all the additional contributions, shift the 2b structure to the 2a one, and finally we use a shift to remove the part symmetric in 2, 3 and 4. After these steps we obtain a Lagrangian term. As before we shift the symmetric part to the 2b structure to have a simple Lagrangian.

6.4 Non-derivative terms

The consideration of non-derivative terms is the simplest one. Summing up all contributions we immediately obtain Lagrangian terms for all cases except the case with f_5 and without f_5. The equation of motion for the term with f_5 has the form

$$
(\nabla_a^2 - m_1^2)s_1 = Q_{1234} f_5 a_{125} a_{345} s_2 s_3 s_4.
$$

Now we write the equation as

$$
(\nabla_a^2 - m_1^2)s_1 = \frac{f_5}{3} \left(2Q_{1234} a_{125} a_{345} + Q_{1324} a_{135} a_{245} \right) s_2 s_3 s_4 \tag{6.19}
$$

and then apply the identity (9.5). We get

$$
\begin{aligned}
(\nabla_a^2 - m_1^2)s_1 = &\; \frac{f_5}{3} \left(2Q_{1234} - Q_{1324} - Q_{1342} \right) a_{125} a_{345} s_2 s_3 s_4 \\
&+ \frac{1}{3} Q_{1324} (f_1 + f_2 + f_3 + f_4) s_2 s_3 s_4.
\end{aligned} \tag{6.20}
$$

Here the term:

$$
\frac{f_5}{3} \left(2Q_{1234} - Q_{1324} - Q_{1342} \right) a_{125} a_{345}
$$

appears to be Lagrangian, and the additional term without f_5 makes the total contribution to the term without f_5 Lagrangian as well.

Thus we showed that the equations of motion for the scalars s^I can be reduced to the Lagrangian form by means of a number of field redefinitions.

7 Conclusion

In this paper we derived all quartic couplings of the scalars dual to extended chiral primary operators in $\mathcal{N} = 4$ SYM$_4$ by using the covariant equations of motion for type IIB supergravity. The quartic terms appeared to contain vertices with two and four derivatives. The appearance of 2-derivative vertices was of course expected. Some of the 4-derivative vertices may be removed by such a field redefinition that changes the structure of cubic terms, namely, one gets scalar cubic terms with two derivatives, and cubic terms describing non-minimal interaction of two scalars with vector fields of the form $V_{IJK}\nabla^a s^I \nabla^b s^J F_{ab}^K$. However, we do not know if all of the 4-derivative terms can be removed in such a way. It would be interesting to clarify this point because the derivation of the Callan-Symanzik equations in the AdS/CFT framework performed in [53] was based on a gravity action which does not contain terms with four or more derivatives.

Since we know the gravity action for the scalars s^I up to the fourth order, we can start computing 4-point functions of CPOs. In general this will require calculating two new types of Feynman diagrams: (i) contact diagrams with 4-derivative vertices, and (ii) exchange diagrams involving massive gravitons. It is not difficult to show that all contact diagrams with 4-derivative vertices can be reduced to a sum of terms corresponding to simple non-derivative quartic couplings, just as this was done in [40] for the case of contact diagrams with 2-derivative vertices. Thus the only real problem is to compute the exchange diagrams involving massive gravitons. However, the 4-point functions of CPOs O_2 can be easily found because all necessary diagrams have been already calculated. This problem is now under consideration.

We proved that, as was conjectured in [6], the quartic couplings obtained vanish in the extremal case, for which $k_1 = k_2 + k_3 + k_4$. This also implies the non-renormalization of extremal 4-point functions of single-trace CPOs. The vanishing of the quartic couplings is not manifest, and requires an additional field redefinition. Although the quartic couplings can be easily used for computing any 4-point function of CPOs, it would be useful to find such a representation for the quartic couplings that makes the vanishing in the extremal case explicit.

We showed that the quartic couplings admit the consistent KK truncation, and argued that the consistency of the KK reduction implies a non-renormalization theorem of n-point functions of $n-1$ single-trace operators dual to the fields from the massless multiplet and one single-trace operator dual to a field from a massive multiplet. It would be interesting to check the non-renormalization of the 5-point function of four CPOs O_2 and one CPO O_4 in perturbation theory.

The simplest example of the 4-point function of three CPOs O_2 and a CPO O_4 belongs, actually, to the class of so-called "next-to-extremal" 4-point functions, for which $k_1 = k_2 +$

$k_3 + k_4 - 2$. The non-renormalization of such correlation functions was proven in [54], and very recently checked to first order in perturbation theory in [55]. The non-renormalization theorem also implies the vanishing of the corresponding functions of extended CPOs and, since it is not difficult to show that there is no exchange diagram in this case, the corresponding "next-to-extremal" quartic couplings of scalars s^I have to vanish too. It would be interesting to check this.

Note added.

We have recently shown that the relevant part of the gauged $\mathcal{N} = 8$ 5-dimensional supergravity action coincides with the action for the scalar s_2 we found in the paper.

8 Appendix A

Here we collect the quartic couplings of the scalars s^I representing our main result. The couplings are given by sums of terms depending on various independent $SO(6)$ tensors. To simplify the presentation we sometimes use the following notations

$$x \equiv k_1, \quad y \equiv k_2, \quad t \equiv k_3, \quad w \equiv k_4, \quad z \equiv k_5,$$

$$\delta = (x+1)(y+1)(t+1)(w+1).$$

All the $SO(6)$ tensors are given by tensors of the form $F(f_5) a_{I_1 I_2 I_5} a_{I_3 I_4 I_5}$, $(f_5 - 1)^n t_{I_1 I_2 I_5} t_{I_3 I_4 I_5}$ and $f_5^n p_{I_1 I_2 I_5} p_{I_3 I_4 I_5}$, where $F(f_5)$ is a function of f_5, and summation over the index I_5 is assumed. To distinguish the couplings with different functions F we use an additional subscript in notation of a coupling.

Quartic couplings of 4-derivative vertices

$$\begin{aligned}
(A_3)^{(4)}_{I_1 I_2 I_3 I_4} &= \frac{1}{4\delta} f_5^3 \left(a_{145} a_{235} - a_{135} a_{245}\right). \\
(A_2)^{(4)}_{I_1 I_2 I_3 I_4} &= -\frac{1}{4\delta}(3(f_1 + f_2 + f_3 + f_4) - 28) f_5^2 \left(a_{145} a_{235} - a_{135} a_{245}\right). \\
(A_1)^{(4)}_{I_1 I_2 I_3 I_4} &= -\frac{3}{4\delta}(f_1 - f_2)(f_3 - f_4) f_5 a_{125} a_{345} \\
&\quad - \frac{1}{\delta}(f_1 + f_2 + f_3 + f_4 - 2)(f_1 + f_2 + f_3 + f_4 - 12) f_5 \left(a_{145} a_{235} - a_{135} a_{245}\right). \\
(A_0)^{(4)}_{I_1 I_2 I_3 I_4} &= \frac{21}{4\delta}(f_1 - f_2)(f_3 - f_4) a_{125} a_{345}.
\end{aligned}$$

$$(S_0)^{(4)}_{I_1I_2I_3I_4} = \frac{7}{4\delta}\left(2f_1f_2 + 2f_3f_4 - (f_1+f_2)(f_3+f_4)\right)a_{125}a_{345}.$$

$$(A_{-1})^{(4)}_{I_1I_2I_3I_4} = -\frac{12}{\delta}(f_1-f_2)(f_3-f_4)f_5^{-1}a_{125}a_{345}.$$

$$(A_{t2})^{(4)}_{I_1I_2I_3I_4} = -\frac{3}{\delta}(f_5-1)^2 t_{125}t_{345}.$$

Quartic couplings of 2-derivative vertices

$$(A_4)^{(2)}_{I_1I_2I_3I_4} = \frac{5}{48\delta}f_5^4\left(a_{145}a_{235} - a_{135}a_{245}\right).$$

$$(A_3)^{(2)}_{I_1I_2I_3I_4} = -\frac{1}{2\delta}(k_1-k_2)(k_3-k_4)f_5^3 a_{125}a_{345}.$$

$$\begin{aligned}(S_3)^{(2)}_{I_1I_2I_3I_4} = \frac{1}{16\delta}\Big(&137 - 80(k_1+k_2+k_3+k_4) + 2(f_1+f_2+f_3+f_4)\\ &+ 32(k_1k_2+k_3k_4) + 24(k_1+k_2)(k_3+k_4)\Big)f_5^3 a_{125}a_{345}.\end{aligned}$$

$$\begin{aligned}(A_2)^{(2)}_{I_1I_2I_3I_4} = \frac{(k_1-k_2)(k_3-k_4)}{4\delta}\Big(&40 - 12(k_1+k_2+k_3+k_4) + 2(f_1+f_2+f_3+f_4)\\ &+ 16(k_1k_2+k_3k_4) + (k_1+k_2)(k_3+k_4)\Big)f_5^2 a_{125}a_{345}.\end{aligned}$$

$$\begin{aligned}(S_2)^{(2)}_{I_1I_2I_3I_4} = -\frac{1}{16\delta}\Big(&-3741 + 2984t - 342t^2 - 56t^3 + 31t^4 + 2984w - 2272tw + 376t^2w\\
&+ 128t^3w - 342w^2 + 376tw^2 + 42t^2w^2 - 56w^3 + 128tw^3 + 31w^4 + 2984x\\
&- 1760tx + 144t^2x + 88t^3x - 1760wx + 832twx + 88t^2wx + 144w^2x + 88tw^2x\\
&+ 88w^3x - 342x^2 + 144tx^2 + 40t^2x^2 + 144wx^2 + 192twx^2 + 40w^2x^2 - 56x^3\\
&+ 88tx^3 + 88wx^3 + 31x^4 + 2984y - 1760ty + 144t^2y + 88t^3y - 1760wy\\
&+ 832twy + 88t^2wy + 144w^2y + 88tw^2y + 88w^3y - 2272xy + 832txy + 192t^2xy\\
&+ 832wxy - 128twxy + 192w^2xy + 376x^2y + 88tx^2y + 88wx^2y + 128x^3y - 342y^2\\
&+ 144ty^2 + 40t^2y^2 + 144wy^2 + 192twy^2 + 40w^2y^2 + 376xy^2 + 88txy^2 + 88wxy^2\\
&+ 42x^2y^2 - 56y^3 + 88ty^3 + 88wy^3 + 128xy^3 + 31y^4\Big)f_5^2 a_{125}a_{345}.\end{aligned}$$

$$\begin{aligned}(A_1)^{(2)}_{I_1I_2I_3I_4} = \frac{(t-w)(x-y)}{48\delta}\Big(&-1840 - 1964t + 160t^2 + 156t^3 + 16t^4 - 1964w\\
&+ 1312tw - 388t^2w - 128t^3w + 160w^2 - 388tw^2 - 120t^2w^2 + 156w^3 - 128tw^3\\
&+ 16w^4 - 1964x + 645tx - 48t^2x - 25t^3x + 645wx - 952twx - 73t^2wx\\
&- 48w^2x - 73tw^2x - 25w^3x + 160x^2 - 48tx^2 - 56t^2x^2 - 48wx^2 - 328twx^2\\
&- 56w^2x^2 + 156x^3 - 25tx^3 - 25wx^3 + 16x^4 - 1964y + 645ty - 48t^2y\\
&- 25t^3y + 645wy - 952twy - 73t^2wy - 48w^2y - 73tw^2y - 25w^3y + 1312xy\\
&- 952txy - 328t^2xy - 952wxy - 656twxy - 328w^2xy - 388x^2y - 73tx^2y\\
&- 73wx^2y - 128x^3y + 160y^2 - 48ty^2 - 56t^2y^2 - 48wy^2 - 328twy^2 - 56w^2y^2\\
&- 388xy^2 - 73txy^2 - 73wxy^2 - 120x^2y^2 + 156y^3 - 25ty^3 - 25wy^3\\
&- 128xy^3 + 16y^4\Big)f_5 a_{125}a_{345}\end{aligned}$$

$$
\begin{aligned}
(S_1)^{(2)}_{I_1I_2I_3I_4} = \; & \frac{1}{48\delta}\Big(20979 - 53784t + 18666t^2 + 4056t^3 - 1197t^4 + 192t^5 + 72t^6 \\
& - 53784w + 59648tw - 17792t^2w - 2816t^3w + 1896t^4w + 256t^5w + 18666w^2 \\
& - 17792tw^2 + 2736t^2w^2 + 1344t^3w^2 + 98t^4w^2 + 4056w^3 - 2816tw^3 + 1344t^2w^3 \\
& + 256t^3w^3 - 1197w^4 + 1896tw^4 + 98t^2w^4 + 192w^5 + 256tw^5 + 72w^6 \\
& - 53784x + 65168tx - 11900t^2x - 3296t^3x + 1428t^4x + 208t^5x + 65168wx \\
& - 53760twx + 7296t^2wx + 4000t^3wx + 144t^4wx - 11900w^2x + 7296tw^2x \\
& + 1760t^2w^2x + 104t^3w^2x - 3296w^3x + 4000tw^3x + 104t^2w^3x + 1428w^4x \\
& + 144tw^4x + 208w^5x + 18666x^2 - 11900tx^2 + 801t^2x^2 + 1488t^3x^2 \\
& + 173t^4x^2 - 11900wx^2 + 3840twx^2 + 4472t^2wx^2 + 704t^3wx^2 + 801w^2x^2 \\
& + 4472tw^2x^2 + 252t^2w^2x^2 + 1488w^3x^2 + 704tw^3x^2 + 173w^4x^2 + 4056x^3 \\
& - 3296tx^3 + 1488t^2x^3 + 424t^3x^3 - 3296wx^3 + 5632twx^3 + 464t^2wx^3 \\
& + 1488w^2x^3 + 464tw^2x^3 + 424w^3x^3 - 1197x^4 + 1428tx^4 + 173t^2x^4 \\
& + 1428wx^4 + 576twx^4 + 173w^2x^4 + 192x^5 + 208tx^5 + 208wx^5 + 72x^6 \\
& - 53784y + 65168ty - 11900t^2y - 3296t^3y + 1428t^4y + 208t^5y + 65168wy \\
& - 53760twy + 7296t^2wy + 4000t^3wy + 144t^4wy - 11900w^2y + 7296tw^2y \\
& + 1760t^2w^2y + 104t^3w^2y - 3296w^3y + 4000tw^3y + 104t^2w^3y + 1428w^4y \\
& + 144tw^4y + 208w^5y + 59648xy - 53760txy + 3840t^2xy + 5632t^3xy \\
& + 576t^4xy - 53760wxy + 23040twxy + 3264t^2wxy - 384t^3wxy + 3840w^2xy \\
& + 3264tw^2xy - 128t^2w^2xy + 5632w^3xy - 384tw^3xy + 576w^4xy - 17792x^2y \\
& + 7296tx^2y + 4472t^2x^2y + 464t^3x^2y + 7296wx^2y + 3264twx^2y + 40t^2wx^2y \\
& + 4472w^2x^2y + 40tw^2x^2y + 464w^3x^2y - 2816x^3y + 4000tx^3y + 704t^2x^3y \\
& + 4000wx^3y - 384twx^3y + 704w^2x^3y + 1896x^4y + 144tx^4y + 144wx^4y + 256x^5y \\
& + 18666y^2 - 11900ty^2 + 801t^2y^2 + 1488t^3y^2 + 173t^4y^2 - 11900wy^2 + 3840twy^2 \\
& + 4472t^2wy^2 + 704t^3wy^2 + 801w^2y^2 + 4472tw^2y^2 + 252t^2w^2y^2 + 1488w^3y^2 \\
& + 704tw^3y^2 + 173w^4y^2 - 17792xy^2 + 7296txy^2 + 4472t^2xy^2 + 464t^3xy^2 \\
& + 7296wxy^2 + 3264twxy^2 + 40t^2wxy^2 + 4472w^2xy^2 + 40tw^2xy^2 + 464w^3xy^2 \\
& + 2736x^2y^2 + 1760wx^2y^2 - 128twx^2y^2 + 252w^2x^2y^2 + 1344x^3y^2 + 104tx^3y^2 \\
& + 104wx^3y^2 + 98x^4y^2 + 4056y^3 - 3296ty^3 + 1488t^2y^3 + 424t^3y^3 \\
& - 3296wy^3 + 5632twy^3 + 464t^2wy^3 + 1488w^2y^3 + 464tw^2y^3 + 424w^3y^3 \\
& - 2816xy^3 + 4000txy^3 + 704t^2xy^3 + 4000wxy^3 - 384twxy^3 + 704w^2xy^3 \\
& + 1344x^2y^3 + 104tx^2y^3 + 104wx^2y^3 + 256x^3y^3 - 1197y^4 + 1428ty^4 + 173t^2y^4 \\
& + 1428wy^4 + 576twy^4 + 173w^2y^4 + 1896xy^4 + 144txy^4 + 144wxy^4 + 98x^2y^4 \\
& + 192y^5 + 208ty^5 + 208wy^5 + 256xy^5 + 72y^6\Big) f_5 a_{125} a_{345}.
\end{aligned}
$$

$$
\begin{aligned}
(A_0)^{(2)}_{I_1I_2I_3I_4} = \; & -\frac{(x-y)(t-w)}{192\delta}\Big(-144288 + 74776t + 10752t^2 - 5264t^3 \\
& + 992t^4 + 440t^5 + 32t^6 + 74776w + 37504tw - 11664t^2w
\end{aligned}
$$

$$
\begin{aligned}
&+ 2016t^3w + 1400t^4w + 128t^5w + 10752w^2 - 11664tw^2 + 4512t^2w^2 \\
&+ 2088t^3w^2 + 176t^4w^2 - 5264w^3 + 2016tw^3 + 2088t^2w^3 + 256t^3w^3 \\
&+ 992w^4 + 1400tw^4 + 176t^2w^4 + 440w^5 + 128tw^5 + 32w^6 \\
&+ 74776x - 26042tx - 5888t^2x + 3948t^3x + 888t^4x + 46t^5x \\
&- 26042wx - 7648twx + 6380t^2wx + 1784t^3wx + 142t^4wx - 5888w^2x \\
&+ 6380tw^2x + 1880t^2w^2x + 170t^3w^2x + 3948w^3x + 1784tw^3x + 170t^2w^3x \\
&+ 888w^4x + 142tw^4x + 46w^5x + 10752x^2 - 5888tx^2 + 832t^2x^2 \\
&+ 1272t^3x^2 + 160t^4x^2 - 5888wx^2 + 768twx^2 + 1784t^2wx^2 + 144t^3wx^2 \\
&+ 832w^2x^2 + 1784tw^2x^2 + 16t^2w^2x^2 + 1272w^3x^2 + 144tw^3x^2 + 160w^4x^2 \\
&- 5264x^3 + 3948tx^3 + 1272t^2x^3 + 5t^3x^3 + 3948wx^3 + 1480twx^3 \\
&- 91t^2wx^3 + 1272w^2x^3 - 91tw^2x^3 + 5w^3x^3 + 992x^4 + 888tx^4 \\
&+ 160t^2x^4 + 888wx^4 + 208twx^4 + 160w^2x^4 + 440x^5 + 46tx^5 \\
&+ 46wx^5 + 32x^6 + 74776y - 26042ty - 5888t^2y + 3948t^3y \\
&+ 888t^4y + 46t^5y - 26042wy - 7648twy + 6380t^2wy + 1784t^3wy \\
&+ 142t^4wy - 5888w^2y + 6380tw^2y + 1880t^2w^2y + 170t^3w^2y + 3948w^3y \\
&+ 1784tw^3y + 170t^2w^3y + 888w^4y + 142tw^4y + 46w^5y + 37504xy \\
&- 7648txy + 768t^2xy + 1480t^3xy + 208t^4xy - 7648wxy + 448twxy \\
&+ 1608t^2wxy + 32t^3wxy + 768w^2xy + 1608tw^2xy - 352t^2w^2xy + 1480w^3xy \\
&+ 32tw^3xy + 208w^4xy - 11664x^2y + 6380tx^2y + 1784t^2x^2y - 91t^3x^2y \\
&+ 6380wx^2y + 1608twx^2y - 571t^2wx^2y + 1784w^2x^2y - 571tw^2x^2y - 91w^3x^2y \\
&+ 2016x^3y + 1784tx^3y + 144t^2x^3y + 1784wx^3y + 32twx^3y + 144w^2x^3y \\
&+ 1400x^4y + 142tx^4y + 142wx^4y + 128x^5y + 10752y^2 - 5888ty^2 \\
&+ 832t^2y^2 + 1272t^3y^2 + 160t^4y^2 - 5888wy^2 + 768twy^2 + 1784t^2wy^2 \\
&+ 144t^3wy^2 + 832w^2y^2 + 1784tw^2y^2 + 16t^2w^2y^2 + 1272w^3y^2 + 144tw^3y^2 \\
&+ 160w^4y^2 - 11664xy^2 + 6380txy^2 + 1784t^2xy^2 - 91t^3xy^2 + 6380wxy^2 \\
&+ 1608twxy^2 - 571t^2wxy^2 + 1784w^2xy^2 - 571tw^2xy^2 - 91w^3xy^2 + 4512x^2y^2 \\
&+ 1880tx^2y^2 + 16t^2x^2y^2 + 1880wx^2y^2 - 352twx^2y^2 + 16w^2x^2y^2 + 2088x^3y^2 \\
&+ 170tx^3y^2 + 170wx^3y^2 + 176x^4y^2 - 5264y^3 + 3948ty^3 + 1272t^2y^3 \\
&+ 5t^3y^3 + 3948wy^3 + 1480twy^3 - 91t^2wy^3 + 1272w^2y^3 - 91tw^2y^3 \\
&+ 5w^3y^3 + 2016xy^3 + 1784txy^3 + 144t^2xy^3 + 1784wxy^3 + 32twxy^3 \\
&+ 144w^2xy^3 + 2088x^2y^3 + 170tx^2y^3 + 170wx^2y^3 + 256x^3y^3 + 992y^4 \\
&+ 888ty^4 + 160t^2y^4 + 888wy^4 + 208twy^4 + 160w^2y^4 + 1400xy^4 \\
&+ 142txy^4 + 142wxy^4 + 176x^2y^4 + 440y^5 + 46ty^5 + 46wy^5 \\
&+ 128xy^5 + 32y^6\Big) a_{125}a_{345}.
\end{aligned}
$$

$$
(S_0)^{(2)}_{I_1I_2I_3I_4} = \frac{1}{576\delta}\Big(-288576tw + 149552t^2w + 21504t^3w - 10528t^4w
$$

$$
\begin{aligned}
&+ 1984t^5w + 880t^6w + 64t^7w + 149552tw^2 - 52084t^2w^2 - 11776t^3w^2\\
&+ 7896t^4w^2 + 1776t^5w^2 + 92t^6w^2 + 21504tw^3 - 11776t^2w^3 + 1664t^3w^3\\
&+ 2544t^4w^3 + 320t^5w^3 - 10528tw^4 + 7896t^2w^4 + 2544t^3w^4 + 10t^4w^4\\
&+ 1984tw^5 + 1776t^2w^5 + 320t^3w^5 + 880tw^6 + 92t^2w^6 + 64tw^7\\
&+ 144288tx - 74776t^2x - 10752t^3x + 5264t^4x - 992t^5x - 440t^6x\\
&- 32t^7x + 144288wx + 26752t^2wx - 6400t^3wx + 1024t^4wx + 960t^5wx\\
&+ 96t^6wx - 74776w^2x + 26752tw^2x - 3520t^2w^2x + 2368t^3w^2x + 1104t^4w^2x\\
&+ 144t^5w^2x - 10752w^3x - 6400tw^3x + 2368t^2w^3x + 1024t^3w^3x - 112t^4w^3x\\
&+ 5264w^4x + 1024tw^4x + 1104t^2w^4x - 112t^3w^4x - 992w^5x + 960tw^5x\\
&+ 144t^2w^5x - 440w^6x + 96tw^6x - 32w^7x - 74776tx^2 + 26042t^2x^2\\
&+ 5888t^3x^2 - 3948t^4x^2 - 888t^5x^2 - 46t^6x^2 - 74776wx^2 - 53504twx^2\\
&+ 1760t^2wx^2 + 2560t^3wx^2 + 480t^4wx^2 + 26042w^2x^2 + 1760tw^2x^2 + 96t^3w^2x^2\\
&+ 28t^4w^2x^2 + 5888w^3x^2 + 2560tw^3x^2 + 96t^2w^3x^2 - 256t^3w^3x^2 - 3948w^4x^2\\
&+ 480tw^4x^2 + 28t^2w^4x^2 - 888w^5x^2 - 46w^6x^2 - 10752tx^3 + 5888t^2x^3\\
&- 832t^3x^3 - 1272t^4x^3 - 160t^5x^3 - 10752wx^3 + 12800twx^3 - 4928t^2wx^3\\
&- 512t^3wx^3 + 176t^4wx^3 + 5888w^2x^3 - 4928tw^2x^3 - 192t^2w^2x^3 + 128t^3w^2x^3\\
&- 832w^3x^3 - 512tw^3x^3 + 128t^2w^3x^3 - 1272w^4x^3 + 176tw^4x^3 - 160w^5x^3\\
&+ 5264tx^4 - 3948t^2x^4 - 1272t^3x^4 - 5t^4x^4 + 5264wx^4 - 2048twx^4\\
&- 1584t^2wx^4 - 64t^3wx^4 - 3948w^2x^4 - 1584tw^2x^4 - 56t^2w^2x^4 - 1272w^3x^4\\
&- 64tw^3x^4 - 5w^4x^4 - 992tx^5 - 888t^2x^5 - 160t^3x^5 - 992wx^5\\
&- 1920twx^5 - 144t^2wx^5 - 888w^2x^5 - 144tw^2x^5 - 160w^3x^5 - 440tx^6\\
&- 46t^2x^6 - 440wx^6 - 192twx^6 - 46w^2x^6 - 32tx^7 - 32wx^7\\
&+ 144288ty - 74776t^2y - 10752t^3y + 5264t^4y - 992t^5y - 440t^6y\\
&- 32t^7y + 144288wy + 26752t^2wy - 6400t^3wy + 1024t^4wy + 960t^5wy\\
&+ 96t^6wy - 74776w^2y + 26752tw^2y - 3520t^2w^2y + 2368t^3w^2y + 1104t^4w^2y\\
&+ 144t^5w^2y - 10752w^3y - 6400tw^3y + 2368t^2w^3y + 1024t^3w^3y - 112t^4w^3y\\
&+ 5264w^4y + 1024tw^4y + 1104t^2w^4y - 112t^3w^4y - 992w^5y + 960tw^5y\\
&+ 144t^2w^5y - 440w^6y + 96tw^6y - 32w^7y - 288576xy - 53504t^2xy\\
&+ 12800t^3xy - 2048t^4xy - 1920t^5xy - 192t^6xy - 53504w^2xy - 512t^2w^2xy\\
&- 768t^3w^2xy - 320t^4w^2xy + 12800w^3xy - 768t^2w^3xy - 768t^3w^3xy - 2048w^4xy\\
&- 320t^2w^4xy - 1920w^5xy - 192w^6xy + 149552x^2y + 26752tx^2y + 1760t^2x^2y\\
&- 4928t^3x^2y - 1584t^4x^2y - 144t^5x^2y + 26752wx^2y + 256t^2wx^2y + 384t^3wx^2y\\
&+ 160t^4wx^2y + 1760w^2x^2y + 256tw^2x^2y - 256t^3w^2x^2y - 4928w^3x^2y + 384tw^3x^2y\\
&- 256t^2w^3x^2y - 1584w^4x^2y + 160tw^4x^2y - 144w^5x^2y + 21504x^3y - 6400tx^3y\\
&+ 2560t^2x^3y - 512t^3x^3y - 64t^4x^3y - 6400wx^3y + 384t^2wx^3y + 384t^3wx^3y\\
&+ 2560w^2x^3y + 384tw^2x^3y + 512t^2w^2x^3y - 512w^3x^3y + 384tw^3x^3y - 64w^4x^3y
\end{aligned}
$$

$$
\begin{aligned}
&- 10528x^4y + 1024tx^4y + 480t^2x^4y + 176t^3x^4y + 1024wx^4y + 160t^2wx^4y \\
&+ 480w^2x^4y + 160tw^2x^4y + 176w^3x^4y + 1984x^5y + 960tx^5y + 960wx^5y \\
&+ 880x^6y + 96tx^6y + 96wx^6y + 64x^7y - 74776ty^2 + 26042t^2y^2 \\
&+ 5888t^3y^2 - 3948t^4y^2 - 888t^5y^2 - 46t^6y^2 - 74776wy^2 - 53504twy^2 \\
&+ 1760t^2wy^2 + 2560t^3wy^2 + 480t^4wy^2 + 26042w^2y^2 + 1760tw^2y^2 + 96t^3w^2y^2 \\
&+ 28t^4w^2y^2 + 5888w^3y^2 + 2560tw^3y^2 + 96t^2w^3y^2 - 256t^3w^3y^2 - 3948w^4y^2 \\
&+ 480tw^4y^2 + 28t^2w^4y^2 - 888w^5y^2 - 46w^6y^2 + 149552xy^2 + 26752txy^2 \\
&+ 1760t^2xy^2 - 4928t^3xy^2 - 1584t^4xy^2 - 144t^5xy^2 + 26752wxy^2 + 256t^2wxy^2 \\
&+ 384t^3wxy^2 + 160t^4wxy^2 + 1760w^2xy^2 + 256tw^2xy^2 - 256t^3w^2xy^2 - 4928w^3xy^2 \\
&+ 384tw^3xy^2 - 256t^2w^3xy^2 - 1584w^4xy^2 + 160tw^4xy^2 - 144w^5xy^2 - 52084x^2y^2 \\
&- 3520tx^2y^2 - 192t^3x^2y^2 - 56t^4x^2y^2 - 3520wx^2y^2 - 512twx^2y^2 + 512t^3wx^2y^2 \\
&- 192w^3x^2y^2 + 512tw^3x^2y^2 - 56w^4x^2y^2 - 11776x^3y^2 + 2368tx^3y^2 + 96t^2x^3y^2 \\
&+ 128t^3x^3y^2 + 2368wx^3y^2 - 768twx^3y^2 - 256t^2wx^3y^2 + 96w^2x^3y^2 - 256tw^2x^3y^2 \\
&+ 128w^3x^3y^2 + 7896x^4y^2 + 1104tx^4y^2 + 28t^2x^4y^2 + 1104wx^4y^2 - 320twx^4y^2 \\
&+ 28w^2x^4y^2 + 1776x^5y^2 + 144tx^5y^2 + 144wx^5y^2 + 92x^6y^2 \\
&- 10752ty^3 + 5888t^2y^3 - 832t^3y^3 - 1272t^4y^3 - 160t^5y^3 - 10752wy^3 \\
&+ 12800twy^3 - 4928t^2wy^3 - 512t^3wy^3 + 176t^4wy^3 + 5888w^2y^3 - 4928tw^2y^3 \\
&- 192t^2w^2y^3 + 128t^3w^2y^3 - 832w^3y^3 - 512tw^3y^3 + 128t^2w^3y^3 - 1272w^4y^3 \\
&+ 176tw^4y^3 - 160w^5y^3 + 21504xy^3 - 6400txy^3 + 2560t^2xy^3 - 512t^3xy^3 \\
&- 64t^4xy^3 - 6400wxy^3 + 384t^2wxy^3 + 384t^3wxy^3 + 2560w^2xy^3 + 384tw^2xy^3 \\
&+ 512t^2w^2xy^3 - 512w^3xy^3 + 384tw^3xy^3 - 64w^4xy^3 - 11776x^2y^3 + 2368tx^2y^3 \\
&+ 96t^2x^2y^3 + 128t^3x^2y^3 + 2368wx^2y^3 - 768twx^2y^3 - 256t^2wx^2y^3 + 96w^2x^2y^3 \\
&- 256tw^2x^2y^3 + 128w^3x^2y^3 + 1664x^3y^3 + 1024tx^3y^3 - 256t^2x^3y^3 + 1024wx^3y^3 \\
&- 768twx^3y^3 - 256w^2x^3y^3 + 2544x^4y^3 - 112tx^4y^3 - 112wx^4y^3 + 320x^5y^3 \\
&+ 5264ty^4 - 3948t^2y^4 - 1272t^3y^4 - 5t^4y^4 + 5264wy^4 - 2048twy^4 \\
&- 1584t^2wy^4 - 64t^3wy^4 - 3948w^2y^4 - 1584tw^2y^4 - 56t^2w^2y^4 - 1272w^3y^4 \\
&- 64tw^3y^4 - 5w^4y^4 - 10528xy^4 + 1024txy^4 + 480t^2xy^4 + 176t^3xy^4 \\
&+ 1024wxy^4 + 160t^2wxy^4 + 480w^2xy^4 + 160tw^2xy^4 + 176w^3xy^4 + 7896x^2y^4 \\
&+ 1104tx^2y^4 + 28t^2x^2y^4 + 1104wx^2y^4 - 320twx^2y^4 + 28w^2x^2y^4 + 2544x^3y^4 \\
&- 112tx^3y^4 - 112wx^3y^4 + 10x^4y^4 - 992ty^5 - 888t^2y^5 - 160t^3y^5 \\
&- 992wy^5 - 1920twy^5 - 144t^2wy^5 - 888w^2y^5 - 144tw^2y^5 - 160w^3y^5 \\
&+ 1984xy^5 + 960txy^5 + 960wxy^5 + 1776x^2y^5 + 144tx^2y^5 + 144wx^2y^5 \\
&+ 320x^3y^5 - 440ty^6 - 46t^2y^6 - 440wy^6 - 192twy^6 - 46w^2y^6 \\
&+ 880xy^6 + 96txy^6 + 96wxy^6 + 92x^2y^6 - 32ty^7 - 32wy^7 \\
&+ 64xy^7\Big) a_{125}a_{345}.
\end{aligned}
$$

$$(A_{-1})^{(2)}_{I_1I_2I_3I_4} = -\frac{4}{\delta}(f_1-f_2)(f_3-f_4)\Big(36-20(k_1+k_2+k_3+k_4)+f_1+f_2+f_3+f_4 + 10(k_1+k_2)(k_3+k_4)\Big)f_5^{-1}a_{125}a_{345}.$$

$$(S_{-1})^{(2)}_{I_1I_2I_3I_4} = -\frac{8}{\delta}(f_1-f_2)(f_3-f_4)(k_1-k_2)(k_3-k_4)f_5^{-1}a_{125}a_{345}.$$

$$(A_{t3})^{(2)}_{I_1I_2I_3I_4} = -\frac{2}{\delta}(f_5-1)^3 t_{125}t_{345}.$$

$$(A_{t2})^{(2)}_{I_1I_2I_3I_4} = -\frac{1}{\delta}(f_5-1)^2 t_{125}t_{345}(k_1^2+k_2^2+k_3^2+k_4^2 - 16(k_1+k_2+k_3+k_4)+10(k_1+k_2)(k_3+k_4)+44).$$

$$(S_{t2})^{(2)}_{I_1I_2I_3I_4} = -\frac{2}{\delta}(k_1-k_2)(k_3-k_4)(f_5-1)^2 t_{125}t_{345}.$$

$$(S_{p2})^{(2)}_{I_1I_2I_3I_4} = \frac{4}{\delta}f_5^2 p_{125}p_{345}.$$

$$(S_d)^{(2)}_{I_1I_2I_3I_4} = \frac{9a_{125}a_{345}}{16\delta(f_5-5)}(-1+k_1-k_2)(1+k_1-k_2)(3+k_1+k_2)(5+k_1+k_2) \times (-1+k_3-k_4)(1+k_3-k_4)(3+k_3+k_4)(5+k_3+k_4).$$

Quartic couplings of non-derivative vertices

$$(S_5)^{(0)}_{I_1I_2I_3I_4} = \frac{3}{64\delta}f_5^5 a_{125}a_{345}.$$

$$(S_4)^{(0)}_{I_1I_2I_3I_4} = -\frac{1}{192\delta}f_5^4 a_{125}a_{345}\Big(747-368(k_1+k_2+k_3+k_4)+65(k_1^2+k_2^2+k_3^2+k_4^2) + 132(k_1k_2+k_3k_4)-96(k_1+k_2)(k_3+k_4)\Big).$$

$$\begin{aligned}(S_3)^{(0)}_{I_1I_2I_3I_4} = \frac{1}{64\delta}f_5^3 a_{125}a_{345}\Big(&-3293+4036t-1012t^2-96t^3\\ &+35t^4+4036w-2428tw+48t^2w+88t^3w-1012w^2+48tw^2\\ &+122t^2w^2-96w^3+88tw^3+35w^4+4036x-2976tx+360t^2x\\ &+40t^3x-2976wx+1056twx-8t^2wx+360w^2x-8tw^2x+40w^3x\\ &-1012x^2+360tx^2+44t^2x^2+360wx^2-8twx^2+44w^2x^2-96x^3\\ &+40tx^3+40wx^3+35x^4+4036y-2976ty+360t^2y+40t^3y\\ &-2976wy+1056twy-8t^2wy+360w^2y-8tw^2y+40w^3y-2428xy\\ &+1056txy-8t^2xy+1056wxy-8w^2xy+48x^2y-8tx^2y-8wx^2y\\ &+88x^3y-1012y^2+360ty^2+44t^2y^2+360wy^2-8twy^2+44w^2y^2\\ &+48xy^2-8txy^2-8wxy^2+122x^2y^2-96y^3+40ty^3\\ &+40wy^3+88xy^3+35y^4\Big).\end{aligned}$$

$$\begin{aligned}(S_2)^{(0)}_{I_1I_2I_3I_4} = -\frac{1}{64\delta}f_5^2 a_{125}a_{345}\Big(&8273-20116t+9396t^2+1008t^3\\ &-1227t^4+36t^5+26t^6-20116w+25644tw-2688t^2w\\ &-3544t^3w+356t^4w+76t^5w+9396w^2-2688tw^2-2778t^2w^2\end{aligned}$$

$$
\begin{aligned}
&+ 664t^3w^2 + 46t^4w^2 + 1008w^3 - 3544tw^3 + 664t^2w^3 + 104t^3w^3 \\
&- 1227w^4 + 356tw^4 + 46t^2w^4 + 36w^5 + 76tw^5 + 26w^6 - 20116x \\
&+ 32384tx - 9032t^2x - 1696t^3x + 492t^4x + 8t^5x + 32384wx \\
&- 23776twx + 224t^2wx + 1152t^3wx - 104t^4wx - 9032w^2x + 224tw^2x \\
&+ 1096t^2w^2x - 224t^3w^2x - 1696w^3x + 1152tw^3x - 224t^2w^3x + 492w^4x \\
&- 104tw^4x + 8w^5x + 9396x^2 - 9032tx^2 + 332t^2x^2 + 288t^3x^2 \\
&+ 60t^4x^2 - 9032wx^2 + 2152twx^2 - 96t^2wx^2 + 144t^3wx^2 + 332w^2x^2 \\
&- 96tw^2x^2 + 96t^2w^2x^2 + 288w^3x^2 + 144tw^3x^2 + 60w^4x^2 + 1008x^3 \\
&- 1696tx^3 + 288t^2x^3 + 80t^3x^3 - 1696wx^3 + 608twx^3 + 32t^2wx^3 \\
&+ 288w^2x^3 + 32tw^2x^3 + 80w^3x^3 - 1227x^4 + 492tx^4 + 60t^2x^4 \\
&+ 492wx^4 - 20twx^4 + 60w^2x^4 + 36x^5 + 8tx^5 + 8wx^5 + 26x^6 \\
&- 20116y + 32384ty - 9032t^2y - 1696t^3y + 492t^4y + 8t^5y \\
&+ 32384wy - 23776twy + 224t^2wy + 1152t^3wy - 104t^4wy - 9032w^2y \\
&+ 224tw^2y + 1096t^2w^2y - 224t^3w^2y - 1696w^3y + 1152tw^3y - 224t^2w^3y \\
&+ 492w^4y - 104tw^4y + 8w^5y + 25644xy - 23776txy + 2152t^2xy \\
&+ 608t^3xy - 20t^4xy - 23776wxy + 10496twxy + 32t^2wxy - 128t^3wxy \\
&+ 2152w^2xy + 32tw^2xy + 168t^2w^2xy + 608w^3xy - 128tw^3xy - 20w^4xy \\
&- 2688x^2y + 224tx^2y - 96t^2x^2y + 32t^3x^2y + 224wx^2y + 32twx^2y - 16t^2wx^2y \\
&- 96w^2x^2y - 16tw^2x^2y + 32w^3x^2y - 3544x^3y + 1152tx^3y + 144t^2x^3y \\
&+ 1152wx^3y - 128twx^3y + 144w^2x^3y + 356x^4y - 104tx^4y - 104wx^4y \\
&+ 76x^5y + 9396y^2 - 9032ty^2 + 332t^2y^2 + 288t^3y^2 + 60t^4y^2 \\
&- 9032wy^2 + 2152twy^2 - 96t^2wy^2 + 144t^3wy^2 + 332w^2y^2 - 96tw^2y^2 \\
&+ 96t^2w^2y^2 + 288w^3y^2 + 144tw^3y^2 + 60w^4y^2 - 2688xy^2 + 224txy^2 \\
&- 96t^2xy^2 + 32t^3xy^2 + 224wxy^2 + 32twxy^2 - 16t^2wxy^2 - 96w^2xy^2 - 16tw^2xy^2 \\
&+ 32w^3xy^2 - 2778x^2y^2 + 1096tx^2y^2 + 96t^2x^2y^2 + 1096wx^2y^2 + 168twx^2y^2 \\
&+ 96w^2x^2y^2 + 664x^3y^2 - 224tx^3y^2 - 224wx^3y^2 + 46x^4y^2 + 1008y^3 \\
&- 1696ty^3 + 288t^2y^3 + 80t^3y^3 - 1696wy^3 + 608twy^3 + 32t^2wy^3 \\
&+ 288w^2y^3 + 32tw^2y^3 + 80w^3y^3 - 3544xy^3 + 1152txy^3 + 144t^2xy^3 \\
&+ 1152wxy^3 - 128twxy^3 + 144w^2xy^3 + 664x^2y^3 - 224tx^2y^3 - 224wx^2y^3 \\
&+ 104x^3y^3 - 1227y^4 + 492ty^4 + 60t^2y^4 + 492wy^4 - 20twy^4 \\
&+ 60w^2y^4 + 356xy^4 - 104txy^4 - 104wxy^4 + 46x^2y^4 + 36y^5 \\
&+ 8ty^5 + 8wy^5 + 76xy^5 + 26y^6\Big).
\end{aligned}
$$

$$
\begin{aligned}
(S_1)^{(0)}_{I_1I_2I_3I_4} &= \frac{(w-x)(t-y)}{288\delta} f_5 a_{125} a_{345}\Big(163692 - 128440t + 28616t^2 + 2052t^3 \\
&- 3460t^4 + 484t^5 + 80t^6 - 128440w + 72314tw - 3096t^2w \\
&+ 393t^3w + 468t^4w - 119t^5w + 28616w^2 - 3096tw^2 - 1208t^2w^2
\end{aligned}
$$

$$
\begin{aligned}
&- 864t^3w^2 + 184t^4w^2 + 2052w^3 + 393tw^3 - 864t^2w^3 + 41t^3w^3 \\
&- 3460w^4 + 468tw^4 + 184t^2w^4 + 484w^5 - 119tw^5 + 80w^6 - 128440x \\
&+ 72314tx - 3096t^2x + 393t^3x + 468t^4x - 119t^5x + 88352wx \\
&- 31064twx + 13912t^2wx - 2360t^3wx - 392t^4wx - 18716w^2x + 15049tw^2x \\
&- 5472t^2w^2x + 641t^3w^2x - 416w^3x - 1416tw^3x + 312t^2w^3x + 1380w^4x \\
&- 335tw^4x - 256w^5x + 28616x^2 - 3096tx^2 - 1208t^2x^2 - 864t^3x^2 \\
&+ 184t^4x^2 - 18716wx^2 + 15049twx^2 - 5472t^2wx^2 + 641t^3wx^2 - 880w^2x^2 \\
&- 4496tw^2x^2 + 2392t^2w^2x^2 + 992w^3x^2 + 524tw^3x^2 + 104w^4x^2 + 2052x^3 + 393tx^3 \\
&- 864t^2x^3 + 41t^3x^3 - 416wx^3 - 1416twx^3 + 312t^2wx^3 + 992w^2x^3 \\
&+ 524tw^2x^3 - 368w^3x^3 - 3460x^4 + 468tx^4 + 184t^2x^4 + 1380wx^4 \\
&- 335twx^4 + 104w^2x^4 + 484x^5 - 119tx^5 - 256wx^5 + 80x^6 \\
&- 128440y + 88352ty - 18716t^2y - 416t^3y + 1380t^4y - 256t^5y \\
&+ 72314wy - 31064twy + 15049t^2wy - 1416t^3wy - 335t^4wy - 3096w^2y \\
&+ 13912tw^2y - 5472t^2w^2y + 312t^3w^2y + 393w^3y - 2360tw^3y + 641t^2w^3y \\
&+ 468w^4y - 392tw^4y - 119w^5y + 72314xy - 31064txy + 15049t^2xy \\
&- 1416t^3xy - 335t^4xy - 31064wxy + 34928twxy - 15928t^2wxy + 1664t^3wxy \\
&+ 15049w^2xy - 15928tw^2xy + 4313t^2w^2xy - 1416w^3xy + 1664tw^3xy - 335w^4xy \\
&- 3096x^2y + 13912tx^2y - 5472t^2x^2y + 312t^3x^2y + 15049wx^2y - 15928twx^2y \\
&+ 4313t^2wx^2y - 4496w^2x^2y + 3488tw^2x^2y + 524w^3x^2y + 393x^3y - 2360tx^3y \\
&+ 641t^2x^3y - 1416wx^3y + 1664twx^3y + 524w^2x^3y + 468x^4y - 392tx^4y \\
&- 335wx^4y - 119x^5y + 28616y^2 - 18716ty^2 - 880t^2y^2 + 992t^3y^2 \\
&+ 104t^4y^2 - 3096wy^2 + 15049twy^2 - 4496t^2wy^2 + 524t^3wy^2 - 1208w^2y^2 \\
&- 5472tw^2y^2 + 2392t^2w^2y^2 - 864w^3y^2 + 641tw^3y^2 + 184w^4y^2 - 3096xy^2 \\
&+ 15049txy^2 - 4496t^2xy^2 + 524t^3xy^2 + 13912wxy^2 - 15928twxy^2 + 3488t^2wxy^2 \\
&- 5472w^2xy^2 + 4313tw^2xy^2 + 312w^3xy^2 - 1208x^2y^2 - 5472tx^2y^2 + 2392t^2x^2y^2 \\
&- 5472wx^2y^2 + 4313twx^2y^2 + 2392w^2x^2y^2 - 864x^3y^2 + 641tx^3y^2 + 312wx^3y^2 \\
&+ 184x^4y^2 + 2052y^3 - 416ty^3 + 992t^2y^3 - 368t^3y^3 + 393wy^3 \\
&- 1416twy^3 + 524t^2wy^3 - 864w^2y^3 + 312tw^2y^3 + 41w^3y^3 + 393xy^3 \\
&- 1416txy^3 + 524t^2xy^3 - 2360wxy^3 + 1664twxy^3 + 641w^2xy^3 - 864x^2y^3 \\
&+ 312tx^2y^3 + 641wx^2y^3 + 41x^3y^3 - 3460y^4 + 1380ty^4 + 104t^2y^4 \\
&+ 468wy^4 - 335twy^4 + 184w^2y^4 + 468xy^4 - 335txy^4 - 392wxy^4 \\
&+ 184x^2y^4 + 484y^5 - 256ty^5 - 119wy^5 - 119xy^5 + 80y^6\Big).
\end{aligned}
$$

$$
\begin{aligned}
(S_0)^{(0)}_{I_1I_2I_3I_4} &= \frac{1}{576\delta}a_{125}a_{345}\Big(-18225 - 24300t + 292830t^2 - 71028t^3 \\
&- 111795t^4 + 33444t^5 + 8640t^6 - 5124t^7 - 618t^8 + 192t^9 \\
&+ 24t^{10} - 24300w - 256068tw - 66756t^2w - 120628t^3w + 177012t^4w
\end{aligned}
$$

$$
\begin{aligned}
&- 17836t^5w - 26900t^6w + 132t^7w + 800t^8w + 32t^9w + 292830w^2\\
&- 66756tw^2 + 177178t^2w^2 + 160520t^3w^2 - 94964t^4w^2 - 31788t^5w^2 + 1416t^6w^2\\
&+ 1136t^7w^2 + 112t^8w^2 - 71028w^3 - 120628tw^3 + 160520t^2w^3 - 11080t^3w^3\\
&- 46764t^4w^3 - 3076t^5w^3 + 2640t^6w^3 + 336t^7w^3 - 111795w^4 + 177012tw^4\\
&- 94964t^2w^4 - 46764t^3w^4 + 6638t^4w^4 + 2920t^5w^4 + 224t^6w^4 + 33444w^5\\
&- 17836tw^5 - 31788t^2w^5 - 3076t^3w^5 + 2920t^4w^5 + 416t^5w^5 + 8640w^6\\
&- 26900tw^6 + 1416t^2w^6 + 2640t^3w^6 + 224t^4w^6 - 5124w^7 + 132tw^7\\
&+ 1136t^2w^7 + 336t^3w^7 - 618w^8 + 800tw^8 + 112t^2w^8 + 192w^9\\
&+ 32tw^9 + 24w^{10} - 24300x - 256068tx - 66756t^2x - 120628t^3x\\
&+ 177012t^4x - 17836t^5x - 26900t^6x + 132t^7x + 800t^8x + 32t^9x\\
&- 256068wx + 12816twx + 319756t^2wx + 370976t^3wx\\
&- 220300t^4wx - 57456t^5wx + 5828t^6wx + 1024t^7wx - 96t^8wx\\
&- 66756w^2x + 319756tw^2x + 184632t^2w^2x - 247264t^3w^2x\\
&- 68020t^4w^2x - 2724t^5w^2x + 1792t^6w^2x + 272t^7w^2x - 120628w^3x\\
&+ 370976tw^3x - 247264t^2w^3x - 124960t^3w^3x + 6940t^4w^3x + 4000t^5w^3x\\
&+ 352t^6w^3x + 177012w^4x - 220300tw^4x - 68020t^2w^4x + 6940t^3w^4x\\
&+ 1672t^4w^4x + 268t^5w^4x - 17836w^5x - 57456tw^5x - 2724t^2w^5x + 4000t^3w^5x\\
&+ 268t^4w^5x - 26900w^6x + 5828tw^6x + 1792t^2w^6x + 352t^3w^6x + 132w^7x\\
&+ 1024tw^7x + 272t^2w^7x + 800w^8x - 96tw^8x + 32w^9x + 292830x^2\\
&- 66756tx^2 + 177178t^2x^2 + 160520t^3x^2 - 94964t^4x^2 - 31788t^5x^2\\
&+ 1416t^6x^2 + 1136t^7x^2 + 112t^8x^2 - 66756wx^2 + 319756twx^2 + 184632t^2wx^2\\
&- 247264t^3wx^2 - 68020t^4wx^2 - 2724t^5wx^2 + 1792t^6wx^2\\
&+ 272t^7wx^2 + 177178w^2x^2 + 184632tw^2x^2 - 335076t^2w^2x^2\\
&- 134456t^3w^2x^2 - 1458t^4w^2x^2 + 4296t^5w^2x^2\\
&+ 848t^6w^2x^2 + 160520w^3x^2 - 247264tw^3x^2 - 134456t^2w^3x^2 + 7688t^3w^3x^2\\
&+ 4056t^4w^3x^2 + 592t^5w^3x^2 - 94964w^4x^2 - 68020tw^4x^2 - 1458t^2w^4x^2\\
&+ 4056t^3w^4x^2 - 56t^4w^4x^2 - 31788w^5x^2 - 2724tw^5x^2 + 4296t^2w^5x^2 + 592t^3w^5x^2\\
&+ 1416w^6x^2 + 1792tw^6x^2 + 848t^2w^6x^2 + 1136w^7x^2 + 272tw^7x^2 + 112w^8x^2\\
&- 71028x^3 - 120628tx^3 + 160520t^2x^3 - 11080t^3x^3 - 46764t^4x^3\\
&- 3076t^5x^3 + 2640t^6x^3 + 336t^7x^3 - 120628wx^3 + 370976twx^3\\
&- 247264t^2wx^3 - 124960t^3wx^3 + 6940t^4wx^3 + 4000t^5wx^3 + 352t^6wx^3\\
&+ 160520w^2x^3 - 247264tw^2x^3 - 134456t^2w^2x^3 + 7688t^3w^2x^3 + 4056t^4w^2x^3\\
&+ 592t^5w^2x^3 - 11080w^3x^3 - 124960tw^3x^3 + 7688t^2w^3x^3 + 6720t^3w^3x^3\\
&- 1096t^4w^3x^3 - 46764w^4x^3 + 6940tw^4x^3 + 4056t^2w^4x^3\\
&- 1096t^3w^4x^3 - 3076w^5x^3 + 4000tw^5x^3 + 592t^2w^5x^3\\
&+ 2640w^6x^3 + 352tw^6x^3 + 336w^7x^3 - 111795x^4 + 177012tx^4 - 94964t^2x^4
\end{aligned}
$$

$$
\begin{aligned}
&- 46764t^3x^4 + 6638t^4x^4 + 2920t^5x^4 + 224t^6x^4 + 177012wx^4 - 220300twx^4 \\
&- 68020t^2wx^4 + 6940t^3wx^4 + 1672t^4wx^4 + 268t^5wx^4 - 94964w^2x^4 - 68020tw^2x^4 \\
&- 1458t^2w^2x^4 + 4056t^3w^2x^4 - 56t^4w^2x^4 - 46764w^3x^4 + 6940tw^3x^4 + 4056t^2w^3x^4 \\
&- 1096t^3w^3x^4 + 6638w^4x^4 + 1672tw^4x^4 - 56t^2w^4x^4 + 2920w^5x^4 + 268tw^5x^4 \\
&+ 224w^6x^4 + 33444x^5 - 17836tx^5 - 31788t^2x^5 - 3076t^3x^5 + 2920t^4x^5 \\
&+ 416t^5x^5 - 17836wx^5 - 57456twx^5 - 2724t^2wx^5 + 4000t^3wx^5 + 268t^4wx^5 \\
&- 31788w^2x^5 - 2724tw^2x^5 + 4296t^2w^2x^5 + 592t^3w^2x^5 - 3076w^3x^5 + 4000tw^3x^5 \\
&+ 592t^2w^3x^5 + 2920w^4x^5 + 268tw^4x^5 + 416w^5x^5 + 8640x^6 - 26900tx^6 \\
&+ 1416t^2x^6 + 2640t^3x^6 + 224t^4x^6 - 26900wx^6 + 5828twx^6 + 1792t^2wx^6 \\
&+ 352t^3wx^6 + 1416w^2x^6 + 1792tw^2x^6 + 848t^2w^2x^6 + 2640w^3x^6 + 352tw^3x^6 \\
&+ 224w^4x^6 - 5124x^7 + 132tx^7 + 1136t^2x^7 + 336t^3x^7 + 132wx^7 \\
&+ 1024twx^7 + 272t^2wx^7 + 1136w^2x^7 + 272tw^2x^7 + 336w^3x^7 - 618x^8 \\
&+ 800tx^8 + 112t^2x^8 + 800wx^8 - 96twx^8 + 112w^2x^8 + 192x^9 + 32tx^9 \\
&+ 32wx^9 + 24x^{10} - 24300y - 256068ty - 66756t^2y - 120628t^3y \\
&+ 177012t^4y - 17836t^5y - 26900t^6y + 132t^7y + 800t^8y + 32t^9y \\
&- 256068wy + 12816twy + 319756t^2wy + 370976t^3wy - 220300t^4wy \\
&- 57456t^5wy + 5828t^6wy + 1024t^7wy - 96t^8wy - 66756w^2y \\
&+ 319756tw^2y + 184632t^2w^2y - 247264t^3w^2y - 68020t^4w^2y - 2724t^5w^2y \\
&+ 1792t^6w^2y + 272t^7w^2y - 120628w^3y + 370976tw^3y - 247264t^2w^3y \\
&- 124960t^3w^3y + 6940t^4w^3y + 4000t^5w^3y + 352t^6w^3y + 177012w^4y - 220300tw^4y \\
&- 68020t^2w^4y + 6940t^3w^4y + 1672t^4w^4y + 268t^5w^4y - 17836w^5y \\
&- 57456tw^5y - 2724t^2w^5y + 4000t^3w^5y + 268t^4w^5y - 26900w^6y \\
&+ 5828tw^6y + 1792t^2w^6y + 352t^3w^6y + 132w^7y \\
&+ 1024tw^7y + 272t^2w^7y + 800w^8y - 96tw^8y \\
&+ 32w^9y - 256068xy + 12816txy + 319756t^2xy + 370976t^3xy - 220300t^4xy \\
&- 57456t^5xy + 5828t^6xy + 1024t^7xy - 96t^8xy + 12816wxy + 2404608twxy \\
&- 48096t^2wxy - 749568t^3wxy - 41136t^4wxy \\
&+ 4992t^5wxy - 2112t^6wxy - 384t^7wxy \\
&+ 319756w^2xy - 48096tw^2xy - 403336t^2w^2xy - 128736t^3w^2xy - 11588t^4w^2xy \\
&+ 2432t^5w^2xy + 688t^6w^2xy + 370976w^3xy - 749568tw^3xy - 128736t^2w^3xy \\
&+ 39680t^3w^3xy + 896t^4w^3xy + 384t^5w^3xy - 220300w^4xy - 41136tw^4xy \\
&- 11588t^2w^4xy + 896t^3w^4xy + 608t^4w^4xy \\
&- 57456w^5xy + 4992tw^5xy + 2432t^2w^5xy \\
&+ 384t^3w^5xy + 5828w^6xy - 2112tw^6xy + 688t^2w^6xy + 1024w^7xy - 384tw^7xy \\
&- 96w^8xy - 66756x^2y + 319756tx^2y + 184632t^2x^2y - 247264t^3x^2y - 68020t^4x^2y \\
&- 2724t^5x^2y + 1792t^6x^2y + 272t^7x^2y + 319756wx^2y - 48096twx^2y
\end{aligned}
$$

$$
\begin{aligned}
&- 403336t^2wx^2y - 128736t^3wx^2y - 11588t^4wx^2y + 2432t^5wx^2y \\
&+ 688t^6wx^2y + 184632w^2x^2y - 403336tw^2x^2y - 169560t^2w^2x^2y \\
&+ 17288t^3w^2x^2y + 4136t^4w^2x^2y + 1112t^5w^2x^2y - 247264w^3x^2y \\
&- 128736tw^3x^2y + 17288t^2w^3x^2y + 2496t^3w^3x^2y \\
&- 1856t^4w^3x^2y - 68020w^4x^2y - 11588tw^4x^2y + 4136t^2w^4x^2y \\
&- 1856t^3w^4x^2y - 2724w^5x^2y + 2432tw^5x^2y \\
&+ 1112t^2w^5x^2y + 1792w^6x^2y + 688tw^6x^2y + 272w^7x^2y - 120628x^3y \\
&+ 370976tx^3y - 247264t^2x^3y - 124960t^3x^3y + 6940t^4x^3y + 4000t^5x^3y \\
&+ 352t^6x^3y + 370976wx^3y - 749568twx^3y - 128736t^2wx^3y + 39680t^3wx^3y \\
&+ 896t^4wx^3y + 384t^5wx^3y - 247264w^2x^3y - 128736tw^2x^3y + 17288t^2w^2x^3y \\
&+ 2496t^3w^2x^3y - 1856t^4w^2x^3y - 124960w^3x^3y + 39680tw^3x^3y + 2496t^2w^3x^3y \\
&- 6912t^3w^3x^3y + 6940w^4x^3y + 896tw^4x^3y \\
&- 1856t^2w^4x^3y + 4000w^5x^3y + 384tw^5x^3y \\
&+ 352w^6x^3y + 177012x^4y - 220300tx^4y - 68020t^2x^4y + 6940t^3x^4y + 1672t^4x^4y \\
&+ 268t^5x^4y - 220300wx^4y - 41136twx^4y \\
&- 11588t^2wx^4y + 896t^3wx^4y + 608t^4wx^4y \\
&- 68020w^2x^4y - 11588tw^2x^4y + 4136t^2w^2x^4y - 1856t^3w^2x^4y + 6940w^3x^4y \\
&+ 896tw^3x^4y - 1856t^2w^3x^4y + 1672w^4x^4y + 608tw^4x^4y + 268w^5x^4y - 17836x^5y \\
&- 57456tx^5y - 2724t^2x^5y + 4000t^3x^5y + 268t^4x^5y - 57456wx^5y + 4992twx^5y \\
&+ 2432t^2wx^5y + 384t^3wx^5y - 2724w^2x^5y \\
&+ 2432tw^2x^5y + 1112t^2w^2x^5y + 4000w^3x^5y \\
&+ 384tw^3x^5y + 268w^4x^5y - 26900x^6y + 5828tx^6y + 1792t^2x^6y + 352t^3x^6y \\
&+ 5828wx^6y - 2112twx^6y + 688t^2wx^6y + 1792w^2x^6y + 688tw^2x^6y + 352w^3x^6y \\
&+ 132x^7y + 1024tx^7y + 272t^2x^7y + 1024wx^7y - 384twx^7y + 272w^2x^7y \\
&+ 800x^8y - 96tx^8y - 96wx^8y + 32x^9y + 292830y^2 - 66756ty^2 \\
&+ 177178t^2y^2 + 160520t^3y^2 - 94964t^4y^2 - 31788t^5y^2 + 1416t^6y^2 \\
&+ 1136t^7y^2 + 112t^8y^2 - 66756wy^2 + 319756twy^2 + 184632t^2wy^2 \\
&- 247264t^3wy^2 - 68020t^4wy^2 - 2724t^5wy^2 + 1792t^6wy^2 + 272t^7wy^2 \\
&+ 177178w^2y^2 + 184632tw^2y^2 - 335076t^2w^2y^2 - 134456t^3w^2y^2 - 1458t^4w^2y^2 \\
&+ 4296t^5w^2y^2 + 848t^6w^2y^2 + 160520w^3y^2 - 247264tw^3y^2 - 134456t^2w^3y^2 \\
&+ 7688t^3w^3y^2 + 4056t^4w^3y^2 + 592t^5w^3y^2 - 94964w^4y^2 \\
&- 68020tw^4y^2 - 1458t^2w^4y^2 + 4056t^3w^4y^2 - 56t^4w^4y^2 \\
&- 31788w^5y^2 - 2724tw^5y^2 + 4296t^2w^5y^2 + 592t^3w^5y^2 \\
&+ 1416w^6y^2 + 1792tw^6y^2 + 848t^2w^6y^2 + 1136w^7y^2 + 272tw^7y^2 + 112w^8y^2 \\
&- 66756xy^2 + 319756txy^2 + 184632t^2xy^2 - 247264t^3xy^2 - 68020t^4xy^2 \\
&- 2724t^5xy^2 + 1792t^6xy^2 + 272t^7xy^2 + 319756wxy^2 - 48096twxy^2
\end{aligned}
$$

$- 403336t^2wxy^2 - 128736t^3wxy^2 - 11588t^4wxy^2 + 2432t^5wxy^2$
$+ 688t^6wxy^2 + 184632w^2xy^2 - 403336tw^2xy^2 - 169560t^2w^2xy^2$
$+ 17288t^3w^2xy^2 + 4136t^4w^2xy^2 + 1112t^5w^2xy^2 - 247264w^3xy^2$
$- 128736tw^3xy^2 + 17288t^2w^3xy^2 + 2496t^3w^3xy^2$
$- 1856t^4w^3xy^2 - 68020w^4xy^2 - 11588tw^4xy^2 + 4136t^2w^4xy^2 - 1856t^3w^4xy^2$
$- 2724w^5xy^2 + 2432tw^5xy^2 + 1112t^2w^5xy^2 + 1792w^6xy^2 + 688tw^6xy^2$
$+ 272w^7xy^2 + 177178x^2y^2 + 184632tx^2y^2 - 335076t^2x^2y^2 - 134456t^3x^2y^2$
$- 1458t^4x^2y^2 + 4296t^5x^2y^2 + 848t^6x^2y^2 + 184632wx^2y^2 - 403336twx^2y^2$
$- 169560t^2wx^2y^2 + 17288t^3wx^2y^2 + 4136t^4wx^2y^2 + 1112t^5wx^2y^2$
$- 335076w^2x^2y^2 - 169560tw^2x^2y^2 + 62988t^2w^2x^2y^2 + 10224t^3w^2x^2y^2$
$- 3408t^4w^2x^2y^2 - 134456w^3x^2y^2 + 17288tw^3x^2y^2 + 10224t^2w^3x^2y^2$
$- 9680t^3w^3x^2y^2 - 1458w^4x^2y^2 + 4136tw^4x^2y^2$
$- 3408t^2w^4x^2y^2 + 4296w^5x^2y^2 + 1112tw^5x^2y^2 + 848w^6x^2y^2 + 160520x^3y^2$
$- 247264tx^3y^2 - 134456t^2x^3y^2 + 7688t^3x^3y^2 + 4056t^4x^3y^2 + 592t^5x^3y^2$
$- 247264wx^3y^2 - 128736twx^3y^2 + 17288t^2wx^3y^2 + 2496t^3wx^3y^2$
$- 1856t^4wx^3y^2 - 134456w^2x^3y^2 + 17288tw^2x^3y^2 + 10224t^2w^2x^3y^2$
$- 9680t^3w^2x^3y^2 + 7688w^3x^3y^2 + 2496tw^3x^3y^2 - 9680t^2w^3x^3y^2$
$+ 4056w^4x^3y^2 - 1856tw^4x^3y^2 + 592w^5x^3y^2$
$- 94964x^4y^2 - 68020tx^4y^2 - 1458t^2x^4y^2 + 4056t^3x^4y^2 - 56t^4x^4y^2$
$- 68020wx^4y^2 - 11588twx^4y^2 + 4136t^2wx^4y^2 - 1856t^3wx^4y^2 - 1458w^2x^4y^2$
$+ 4136tw^2x^4y^2 - 3408t^2w^2x^4y^2 + 4056w^3x^4y^2 - 1856tw^3x^4y^2 - 56w^4x^4y^2$
$- 31788x^5y^2 - 2724tx^5y^2 + 4296t^2x^5y^2 + 592t^3x^5y^2 - 2724wx^5y^2$
$+ 2432twx^5y^2 + 1112t^2wx^5y^2 + 4296w^2x^5y^2 + 1112tw^2x^5y^2 + 592w^3x^5y^2$
$+ 1416x^6y^2 + 1792tx^6y^2 + 848t^2x^6y^2 + 1792wx^6y^2 + 688twx^6y^2$
$+ 848w^2x^6y^2 + 1136x^7y^2 + 272tx^7y^2 + 272wx^7y^2 + 112x^8y^2$
$- 71028y^3 - 120628ty^3 + 160520t^2y^3 - 11080t^3y^3 - 46764t^4y^3$
$- 3076t^5y^3 + 2640t^6y^3 + 336t^7y^3 - 120628wy^3 + 370976twy^3$
$- 247264t^2wy^3 - 124960t^3wy^3 + 6940t^4wy^3 + 4000t^5wy^3 + 352t^6wy^3$
$+ 160520w^2y^3 - 247264tw^2y^3 - 134456t^2w^2y^3 + 7688t^3w^2y^3 + 4056t^4w^2y^3$
$+ 592t^5w^2y^3 - 11080w^3y^3 - 124960tw^3y^3 + 7688t^2w^3y^3 + 6720t^3w^3y^3$
$- 1096t^4w^3y^3 - 46764w^4y^3 + 6940tw^4y^3 + 4056t^2w^4y^3 - 1096t^3w^4y^3$
$- 3076w^5y^3 + 4000tw^5y^3 + 592t^2w^5y^3 + 2640w^6y^3 + 352tw^6y^3$
$+ 336w^7y^3 - 120628xy^3 + 370976txy^3 - 247264t^2xy^3 - 124960t^3xy^3$
$+ 6940t^4xy^3 + 4000t^5xy^3 + 352t^6xy^3 + 370976wxy^3 - 749568twxy^3$
$- 128736t^2wxy^3 + 39680t^3wxy^3 + 896t^4wxy^3 + 384t^5wxy^3 - 247264w^2xy^3$
$- 128736tw^2xy^3 + 17288t^2w^2xy^3 + 2496t^3w^2xy^3 - 1856t^4w^2xy^3 - 124960w^3xy^3$

$$
\begin{aligned}
&+ 39680tw^3xy^3 + 2496t^2w^3xy^3 - 6912t^3w^3xy^3 + 6940w^4xy^3 + 896tw^4xy^3\\
&- 1856t^2w^4xy^3 + 4000w^5xy^3 + 384tw^5xy^3 + 352w^6xy^3 + 160520x^2y^3\\
&- 247264tx^2y^3 - 134456t^2x^2y^3 + 7688t^3x^2y^3 + 4056t^4x^2y^3 + 592t^5x^2y^3\\
&- 247264wx^2y^3 - 128736twx^2y^3 + 17288t^2wx^2y^3 + 2496t^3wx^2y^3\\
&- 1856t^4wx^2y^3 - 134456w^2x^2y^3 + 17288tw^2x^2y^3 + 10224t^2w^2x^2y^3\\
&- 9680t^3w^2x^2y^3 + 7688w^3x^2y^3 + 2496tw^3x^2y^3 - 9680t^2w^3x^2y^3\\
&+ 4056w^4x^2y^3 - 1856tw^4x^2y^3 + 592w^5x^2y^3\\
&- 11080x^3y^3 - 124960tx^3y^3 + 7688t^2x^3y^3 + 6720t^3x^3y^3 - 1096t^4x^3y^3\\
&- 124960wx^3y^3 + 39680twx^3y^3 + 2496t^2wx^3y^3 - 6912t^3wx^3y^3 + 7688w^2x^3y^3\\
&+ 2496tw^2x^3y^3 - 9680t^2w^2x^3y^3 + 6720w^3x^3y^3 - 6912tw^3x^3y^3 - 1096w^4x^3y^3\\
&- 46764x^4y^3 + 6940tx^4y^3 + 4056t^2x^4y^3 - 1096t^3x^4y^3 + 6940wx^4y^3\\
&+ 896twx^4y^3 - 1856t^2wx^4y^3 + 4056w^2x^4y^3 - 1856tw^2x^4y^3 - 1096w^3x^4y^3\\
&- 3076x^5y^3 + 4000tx^5y^3 + 592t^2x^5y^3 + 4000wx^5y^3 + 384twx^5y^3\\
&+ 592w^2x^5y^3 + 2640x^6y^3 + 352tx^6y^3 + 352wx^6y^3 + 336x^7y^3\\
&- 111795y^4 + 177012ty^4 - 94964t^2y^4 - 46764t^3y^4 + 6638t^4y^4\\
&+ 2920t^5y^4 + 224t^6y^4 + 177012wy^4 - 220300twy^4 - 68020t^2wy^4\\
&+ 6940t^3wy^4 + 1672t^4wy^4 + 268t^5wy^4 - 94964w^2y^4 - 68020tw^2y^4\\
&- 1458t^2w^2y^4 + 4056t^3w^2y^4 - 56t^4w^2y^4 - 46764w^3y^4 + 6940tw^3y^4\\
&+ 4056t^2w^3y^4 - 1096t^3w^3y^4 + 6638w^4y^4 + 1672tw^4y^4 - 56t^2w^4y^4 + 2920w^5y^4\\
&+ 268tw^5y^4 + 224w^6y^4 + 177012xy^4 - 220300txy^4 - 68020t^2xy^4\\
&+ 6940t^3xy^4 + 1672t^4xy^4 + 268t^5xy^4 - 220300wxy^4 - 41136twxy^4\\
&- 11588t^2wxy^4 + 896t^3wxy^4 + 608t^4wxy^4 - 68020w^2xy^4 - 11588tw^2xy^4\\
&+ 4136t^2w^2xy^4 - 1856t^3w^2xy^4 + 6940w^3xy^4 + 896tw^3xy^4 - 1856t^2w^3xy^4\\
&+ 1672w^4xy^4 + 608tw^4xy^4 + 268w^5xy^4 - 94964x^2y^4 - 68020tx^2y^4\\
&- 1458t^2x^2y^4 + 4056t^3x^2y^4 - 56t^4x^2y^4 - 68020wx^2y^4 - 11588twx^2y^4\\
&+ 4136t^2wx^2y^4 - 1856t^3wx^2y^4 - 1458w^2x^2y^4 + 4136tw^2x^2y^4 - 3408t^2w^2x^2y^4\\
&+ 4056w^3x^2y^4 - 1856tw^3x^2y^4 - 56w^4x^2y^4 - 46764x^3y^4 + 6940tx^3y^4\\
&+ 4056t^2x^3y^4 - 1096t^3x^3y^4 + 6940wx^3y^4 + 896twx^3y^4 - 1856t^2wx^3y^4\\
&+ 4056w^2x^3y^4 - 1856tw^2x^3y^4 - 1096w^3x^3y^4 + 6638x^4y^4 + 1672tx^4y^4\\
&- 56t^2x^4y^4 + 1672wx^4y^4 + 608twx^4y^4 - 56w^2x^4y^4 + 2920x^5y^4\\
&+ 268tx^5y^4 + 268wx^5y^4 + 224x^6y^4 + 33444y^5 - 17836ty^5\\
&- 31788t^2y^5 - 3076t^3y^5 + 2920t^4y^5 + 416t^5y^5 - 17836wy^5\\
&- 57456twy^5 - 2724t^2wy^5 + 4000t^3wy^5 + 268t^4wy^5 - 31788w^2y^5\\
&- 2724tw^2y^5 + 4296t^2w^2y^5 + 592t^3w^2y^5 - 3076w^3y^5 + 4000tw^3y^5\\
&+ 592t^2w^3y^5 + 2920w^4y^5 + 268tw^4y^5 + 416w^5y^5 - 17836xy^5\\
&- 57456txy^5 - 2724t^2xy^5 + 4000t^3xy^5 + 268t^4xy^5 - 57456wxy^5
\end{aligned}
$$

$$
\begin{aligned}
&+ 4992twxy^5 + 2432t^2wxy^5 + 384t^3wxy^5 - 2724w^2xy^5 + 2432tw^2xy^5\\
&+ 1112t^2w^2xy^5 + 4000w^3xy^5 + 384tw^3xy^5 + 268w^4xy^5 - 31788x^2y^5\\
&- 2724tx^2y^5 + 4296t^2x^2y^5 + 592t^3x^2y^5 - 2724wx^2y^5 + 2432twx^2y^5\\
&+ 1112t^2wx^2y^5 + 4296w^2x^2y^5 + 1112tw^2x^2y^5 + 592w^3x^2y^5 - 3076x^3y^5\\
&+ 4000tx^3y^5 + 592t^2x^3y^5 + 4000wx^3y^5 + 384twx^3y^5 + 592w^2x^3y^5\\
&+ 2920x^4y^5 + 268tx^4y^5 + 268wx^4y^5 + 416x^5y^5 + 8640y^6\\
&- 26900ty^6 + 1416t^2y^6 + 2640t^3y^6 + 224t^4y^6 - 26900wy^6\\
&+ 5828twy^6 + 1792t^2wy^6 + 352t^3wy^6 + 1416w^2y^6 + 1792tw^2y^6\\
&+ 848t^2w^2y^6 + 2640w^3y^6 + 352tw^3y^6 + 224w^4y^6 - 26900xy^6\\
&+ 5828txy^6 + 1792t^2xy^6 + 352t^3xy^6 + 5828wxy^6 - 2112twxy^6\\
&+ 688t^2wxy^6 + 1792w^2xy^6 + 688tw^2xy^6 + 352w^3xy^6 + 1416x^2y^6\\
&+ 1792tx^2y^6 + 848t^2x^2y^6 + 1792wx^2y^6 + 688twx^2y^6 + 848w^2x^2y^6\\
&+ 2640x^3y^6 + 352tx^3y^6 + 352wx^3y^6 + 224x^4y^6 - 5124y^7 + 132ty^7\\
&+ 1136t^2y^7 + 336t^3y^7 + 132wy^7 + 1024twy^7 + 272t^2wy^7\\
&+ 1136w^2y^7 + 272tw^2y^7 + 336w^3y^7 + 132xy^7 + 1024txy^7 + 272t^2xy^7\\
&+ 1024wxy^7 - 384twxy^7 + 272w^2xy^7 + 1136x^2y^7 + 272tx^2y^7\\
&+ 272wx^2y^7 + 336x^3y^7 - 618y^8 + 800ty^8 + 112t^2y^8 + 800wy^8\\
&- 96twy^8 + 112w^2y^8 + 800xy^8 - 96txy^8 - 96wxy^8 + 112x^2y^8\\
&+ 192y^9 + 32ty^9 + 32wy^9 + 32xy^9 + 24y^{10}\Big).
\end{aligned}
$$

$$
\begin{aligned}
(S_{-1})^{(0)}_{I_1I_2I_3I_4} &= \frac{2}{\delta} f_5^{-1} a_{125}a_{345}(k_1-k_2)(k_3-k_4)(f_1-f_2)(f_3-f_4)\\
&\times \Big(-36 + 2(k_1+k_2+k_3+k_4) + f_1+f_2+f_3+f_4 - 2k_1k_2 - 2k_3k_4\Big).\\
(S_{t2})^{(0)}_{I_1I_2I_3I_4} &= \frac{(f_5-1)^2 t_{125}t_{345}}{2\delta}(k_1-k_2)(k_3-k_4)(f_1+f_2+f_3+f_4\\
&+ 2(k_1+k_2+k_3+k_4) - 2(k_1k_2+k_3k_4) - 36).\\
(S_{p3})^{(0)}_{I_1I_2I_3I_4} &= -\frac{1}{\delta} f_5^3 p_{125}p_{345}.\\
(S_{p2})^{(0)}_{I_1I_2I_3I_4} &= \frac{2}{\delta} f_5^2 p_{125}p_{345}(k_1^2+k_2^2+k_3^2+k_4^2 - 2(k_1+k_2+k_3+k_4) + 2(k_1k_2+k_3k_4) - 4).\\
(S_d)^{(0)}_{I_1I_2I_3I_4} &= \frac{9a_{125}a_{345}}{64\delta(f_5-5)}(-1+k_1-k_2)(1+k_1-k_2)(3+k_1+k_2)(5+k_1+k_2)\\
&\times (-1+k_3-k_4)(1+k_3-k_4)(3+k_3+k_4)(5+k_3+k_4)\\
&\times (2k_1^2+2k_2^2+2k_3^2+2k_4^2+4k_1k_2+4k_3k_4-4(k_1+k_2+k_3+k_4)-5).
\end{aligned}
$$

9 Appendix B

We need a number of relations between the structure constants a_{123}, t_{123} and p_{123}, defined as[12]

$$a_{123} = \int Y^{I_1} Y^{I_2} Y^{I_3} \tag{9.1}$$

$$t_{123} = \int \nabla^\alpha Y^{I_1} Y^{I_2} Y^{I_3}_\alpha \tag{9.2}$$

$$p_{123} = \int \nabla^\alpha Y^{I_1} \nabla^\beta Y^{I_2} Y^{I_3}_{(\alpha\beta)} \tag{9.3}$$

In deriving the equations of motions for scalar fields s_k and t_k and for tensor φ^k_{ab} one comes across a number of integrals of scalar spherical harmonics, all of them can be reduced to a_{123}. Introducing the concise notation $f_i = f(k_i) = k_i(k_i+4)$ we present below the corresponding formulae:

$$\begin{aligned} b_{123} &= \int \nabla^\alpha Y^{I_1} \nabla_\alpha Y^{I_2} Y^{I_3} = \frac{1}{2}(f_1 + f_2 - f_3) a_{123}, \\ c_{123} &= \int \nabla^\alpha \nabla^\beta Y^{I_1} \nabla_\alpha \nabla_\beta Y^{I_2} Y^{I_3} = \frac{1}{2}\Big(-f_1 f_3 - f_2 f_3 + \frac{3}{5} f_1 f_2 + \frac{1}{2} f_1^2 \\ &+ \frac{1}{2} f_2^2 + \frac{1}{2} f_3^2 - 4(f_1 + f_2 - f_3)\Big) a_{123}. \end{aligned}$$

Since any scalar function on a sphere can be decomposed in scalar spherical harmonics, we have the following relations

$$Y^1 Y^2 = a_{123} Y^3, \quad \nabla^\alpha Y^1 \nabla_\alpha Y^2 = b_{123} Y^3, \quad \nabla^\alpha \nabla^\beta Y^1 \nabla_\alpha \nabla_\beta Y^2 = c_{123} Y^3. \tag{9.4}$$

These relations allow one to calculate some integrals involving 4 scalar spherical harmonics. In particular we have

$$\begin{aligned} a_{1234} &= \int Y^{I_1} Y^{I_2} Y^{I_3} Y^{I_4} = a_{125} a_{345} = a_{135} a_{245} = a_{145} a_{235}, \\ b_{1234} &= \int Y^{I_1} Y^{I_2} \nabla^\alpha Y^{I_3} \nabla_\alpha Y^{I_4} = a_{125} b_{345}, \\ c_{1234} &= \int Y^{I_1} Y^{I_2} \nabla^\alpha \nabla^\beta Y^{I_3} \nabla_\alpha \nabla_\beta Y^{I_4} = a_{125} c_{345}, \end{aligned}$$

where the summation over the index 5 is assumed.

There is the following important relation

$$f_5(a_{125} a_{345} + a_{135} a_{245} + a_{235} a_{145}) = (f_1 + f_2 + f_3 + f_4) a_{125} a_{345}, \tag{9.5}$$

[12] For a detailed description of spherical harmonics on S^5 see [5, 6].

that shows that among the three tensors $f_5 a_{125}a_{345}$, $f_5 a_{135}a_{245}$ and $f_5 a_{235}a_{145}$ only the following two tensors are independent

$$f_5 a_{125}a_{345} \quad \text{and} \quad f_5(a_{135}a_{245} - a_{235}a_{145}).$$

We also encounter tensors of the form $f_5^n t_{125}t_{345}$ and $f_5^n p_{125}p_{345}$. Some of them can be reduced to sums of tensors of the form $f_5^n a_{125}a_{345}$. To see this we note that any vector and any traceless symmetric tensor on a sphere can be decomposed as follows

$$\nabla_\alpha Y^1 Y^2 = t_{125} Y^5_\alpha + \frac{b_{152}}{f_5}\nabla_\alpha Y^5, \tag{9.6}$$

$$\nabla_{(\alpha} Y^1 \nabla_{\beta)} Y^2 = p_{125} Y^5_{(\alpha\beta)} + \mu_{125}\nabla_{(\alpha} Y^5_{\beta)} + \nu_{125}\nabla_{(\alpha}\nabla_{\beta)} Y^5, \tag{9.7}$$

where

$$\mu_{125} = \frac{f_2 - f_1}{f_5 - 5} t_{125}, \quad \nu_{125} = \frac{1}{q_5} d_{125},$$
$$q_5 = \frac{4}{5} f_5(f_5 - 5), \quad d_{125} = c_{125} + (\frac{1}{5} f_5 - f_2 + 4) b_{125} + f_1 b_{251}.$$

Note that in the paper we use the following normalizations

$$\int Y^I_{(\alpha\beta)} Y_J^{(\alpha\beta)} = \delta^I_J, \quad \int Y^I_\alpha Y^\alpha_J = \delta^I_J,$$

where summation over α, β is assumed.

By using the decompositions one can find the following relations

$$t_{125}t_{345} = -\frac{(f_1 - f_2)(f_3 - f_4)}{4 f_5} a_{125}a_{345} + \frac{1}{4} f_5 (a_{145}a_{235} - a_{245}a_{135}), \tag{9.8}$$

$$(1 - f_5) t_{125}t_{345} = \frac{1}{4}(f_5^2 - f_5(f_1 + f_2 + f_3 + f_4 - 4))(a_{145}a_{235} - a_{135}a_{245}) \tag{9.9}$$
$$- \frac{4 - f_5}{4 f_5}(f_1 - f_2)(f_3 - f_4) a_{125}a_{345}$$

$$p_{125}p_{345} = -\frac{(f_1 - f_2)(f_3 - f_4)}{2(f_5 - 5)} t_{125}t_{345} - \frac{1}{q_5} d_{125}d_{345}$$
$$- \frac{1}{20}(f_1 + f_2 - f_5)(f_3 + f_4 - f_5) a_{125}a_{345} + \frac{1}{8}(f_1 + f_3 - f_5)(f_2 + f_4 - f_5) a_{135}a_{245}$$
$$+ \frac{1}{8}(f_1 + f_4 - f_5)(f_2 + f_3 - f_5) a_{145}a_{235}, \tag{9.10}$$

$$p_{125}(2 - f_5) p_{345} = \int \nabla^2_\gamma (\nabla^\alpha Y^1 \nabla^\beta Y^2) \nabla^{(\alpha} Y^3 \nabla^{\beta)} Y^4 - \frac{10 - f_5}{q_5} d_{125}d_{345}$$
$$- \frac{(f_1 - f_2)(f_3 - f_4)}{(f_5 - 5)} t_{125}t_{345} + \frac{1}{2}(f_1 - f_2)(f_3 - f_4) t_{125}t_{345}, \tag{9.11}$$

where

$$\int \nabla^2_\gamma(\nabla^\alpha Y^1 \nabla^\beta Y^2)\nabla^{(\alpha}Y^3\nabla^{\beta)}Y^4 = \frac{1}{8}(8-f_1-f_2)(f_1+f_3-f_5)(f_2+f_4-f_5)a_{135}a_{245}$$
$$+\frac{1}{8}(8-f_1-f_2)(f_1+f_4-f_5)(f_2+f_3-f_5)a_{145}a_{235}$$
$$+\frac{1}{20}(f_1+f_2-f_5)(f_3+f_4-f_5)f_5a_{125}a_{345}+g_{1324}+g_{1423}$$

and

$$g_{1234} = \frac{1}{16f_5}(f_1+f_5-f_2)(f_1+f_2-f_5)(f_3+f_5-f_4)(f_3+f_4-f_5)a_{125}a_{345}$$
$$+\frac{1}{4}(f_1+f_2-4)(f_3+f_4-4)t_{125}t_{345}+\frac{1}{4}(f_1+f_2+f_3+f_4-8)t_{125}(1-f_5)t_{345}$$
$$+\frac{1}{4}t_{125}(1-f_5)^2t_{345}.$$

The following integrals are also used in deriving the equations of motion for scalars s_k:

$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}Y^{I_2}\nabla_\alpha Y^{I_3}_\beta = \frac{1}{2}\left(f_3+f_1-f_2-5\right)t_{123},$$
$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}\nabla_\alpha Y^{I_2}Y^{I_3}_\beta = \frac{1}{2}\left(f_2+\frac{3}{5}f_1-f_3-3\right)t_{123},$$
$$\int \nabla^{(\alpha}\nabla^{\gamma)}Y^{I_1}\nabla^\beta\nabla_\gamma Y^{I_2}Y^{I_3}_{(\alpha\beta)} = \frac{1}{10}(3f_1+5f_2-5f_3-30)p_{123},$$
$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}\nabla_\gamma Y^{I_2}\nabla^\gamma Y^{I_3}_{(\alpha\beta)} = \frac{1}{2}(f_1-f_2-f_3-8)p_{123}.$$

Considering the action for the fields from the massive graviton multiplet we need the following explicit expression for the integral a_{123} of scalar spherical harmonics [5, 6]:

$$a_{123} = (z(k_1)z(k_2)z(k_3))^{-1/2}\frac{\pi^3}{(\frac{1}{2}\Sigma+2)!2^{\frac{1}{2}(\Sigma-2)}}\frac{k_1!k_2!k_3!}{\alpha_1!\alpha_2!\alpha_3!}\langle C^{I_1}C^{I_2}C^{I_3}\rangle \tag{9.12}$$

Here

$$z(k) = \frac{\pi^3}{2^{k-1}(k+1)(k+2)}$$

and $\alpha_i = \frac{1}{2}(k_j+k_l-k_i)$, $j\neq l\neq i$. Notation $\langle C^{I_1}C^{I_2}C^{I_3}\rangle$ is used to denote the unique $SO(6)$ invariant obtained by contracting α_1 indices between C^{I_2} and C^{I_3}, α_2 indices between C^{I_3} and C^{I_1}, and α_3 indices between C^{I_2} and C^{I_1}.

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