Four graviton scattering amplitude from $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model

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Abstract

In the IR limit the Matrix string theory is expected to be described by the $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model. Recently Dijkgraaf, Verlinde and Verlinde proposed a vertex that may describe the type IIA string interaction. In this paper using this interaction vertex we derive the four graviton scattering amplitude from the orbifold model in the large N limit.

Keywords: Matrix string; orbifold conformal field theory.

1 Introduction

According to the Matrix theory conjecture [1] the quantum mechanics of N D-particles [2] of type IIA string theory in the large N limit describes the eleven-dimensional dynamics of M-theory [3, 4]. In particular, some subspace of the Hilbert space of the quantum mechanics model admits an interpretation in terms of the second-quantized Fock space of the M-theory states, and the S-matrix of the model is directly related to the scattering amplitudes of the M-theory particles. The consistency of this conjecture was already examined in many ways (for recent review see [5]).

Compactifying Matrix theory on a circle one arrives at the $\mathcal{N}=8$ two-dimensional supersymmetric $SU(N)$ Yang-Mills model [6]. It was recently argued in [7, 8, 9] that in the large N limit the Yang-Mills theory describes non-perturbative dynamics of type IIA string theory, and the Yang-Mills and string coupling constants were shown to be inverse to each other. The argumentation was based on the observation that in the IR limit the gauge theory is strongly coupled and the IR fixed point may be described by the $\mathcal{N}=8$ supersymmetric conformal field theory on the orbifold target space $S^N\mathbf{R}^8$. In particular, it is known [10] that the Hilbert space of the orbifold model coincides (to be precise, contains) in the large N limit with the Fock space of the free second-quantized type IIA string theory.

Basing on the string interpretation of the Hilbert space of the orbifold model, Dijkgraaf, Verlinde and Verlinde (DVV) [9] suggested that perturbative string dynamics in the first order in the string coupling constant can be described by the $S^N\mathbf{R}^8$ supersymmetric orbifold conformal model perturbed by an irrelevant operator of conformal dimension $(3/2, 3/2)$. Moreover, they determined an explicit form of this operator and showed that it preserved the space-time supersymmetry and nicely fitted the conventional formalism of the light-cone string theory.

The described sigma-model approach to the perturbative second-quantized string theory is not limited only to the type IIA strings. On the same grounds one may easily define the DVV interaction vertices for the sigma-model description of bosonic, heterotic [11], and type IIB strings.

An important problem posed by the above-described stringy interpretation of the S^N orbifold sigma models is to obtain the usual string scattering amplitudes directly from the models. The positive result would obviously provide a strong evidence that Yang-Mills models indeed capture nonperturbative string dynamics. This problem seems to be nontrivial due to the nonabelian nature of the S^N orbifold models. In our previous paper [12] we

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obtained the four tachyon scattering amplitude from the $S^N\mathbf{R}^{24}$ orbifold conformal field theory perturbed by the bosonic analog of the DVV interaction vertex.

The aim of the present paper is to derive the four graviton scattering amplitude for type II strings from the $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model, the closed string interaction being described by the DVV interaction vertex. We treat in detail only the more complicated case of type IIA strings since for type IIB strings the left- and right-moving sectors are identical.

Our consideration of the scattering amplitudes starts with defining incoming and outgoing asymptotic states $|i\rangle$ and $|f\rangle$ that should be identified with some states in the Hilbert space of the orbifold conformal field theory. In CFT any state is created by some conformal field and, therefore, the first step consists in finding the conformal fields of the orbifold CFT corresponding to the asymptotic states. Recall that the Hilbert space of the orbifold sigma model is decomposed into the direct sum of Hilbert spaces of twisted sectors. Each twisted sector describes asymptotic states of several strings. The vacuum state of a twisted sector corresponds to a ground state twist operator. If the orbifold sigma model originates from the IR limit of the Yang-Mills theory, then the energy of all vacuum states should be the same and, therefore, the conformal dimensions of the ground state twist operators must be equal. We will show that this is indeed the case for the supersymmetric orbifold sigma model. In contrast, in the bosonic case the conformal dimensions were found [12] to be different and, therefore, the bosonic sigma model does not describe the IR limit of the Yang-Mills theory with 24 matter fields in the adjoint representation of the $U(N)$ gauge group.

Then, by the conventional quantum field theory, the g_s^n-order scattering amplitude A can be extracted from the S-matrix element described as a correlation function of n interaction vertices $V(z_i)$ with the subsequent integration over the insertion points z_i:

$$\langle f|S|i\rangle \sim \int \prod_i d^2 z_i \langle f|V(z_1)\dots V(z_n)|i\rangle.$$

The paper is organized as follows. In the second section we remind the description of the Hilbert space of the orbifold model. In the third section the vertex operators that create the states of the Hilbert space are introduced and their conformal dimensions are calculated. We pay special attention to the fact that in nonabelian orbifold models there is no decomposition of vertex operators into the tensor product of bosonic and fermionic (holomorphic and antiholomorphic) twist fields. We also recall the construction of the DVV interaction vertex. In the fourth section we describe the S-matrix element corresponding to the scattering of four gravitons and reduce the problem of it's calculation to the one of computing special correlation functions in the orbifold CFT. In Section 5 we compute the bosonic correlation functions up to normalization constants by using the stress-energy tensor method. To compute the fermionic contribution, in the next section we describe a bosonization procedure for fermions of the orbifold model in the $SU(4)\times U(1)$ formalism. In Section 7 we find normalization constants for the correlation functions. Finally, in Section 8 we combine the results obtained in the previous sections and derive the well-known four graviton scattering amplitude that appears to be automatically Lorentz-invariant. In Conclusion we discuss unsolved problems.

2 $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model

The target space of the supersymmetric orbifold sigma model is the symmetric product space $S^N\mathbf{R}^8 = (\mathbf{R}^8)^N/S_N$, where S_N is the permutation group of N objects. The model on a cylinder with coordinates (σ,τ) is described by the following action

$$S = \frac{1}{2\pi}\int d\tau d\sigma(\partial_\tau X_I^i\partial_\tau X_I^i - \partial_\sigma X_I^i\partial_\sigma X_I^i + \frac{i}{2}\theta_I^a(\partial_\tau+\partial_\sigma)\theta_I^a + \frac{i}{2}\theta_I^{\dot a}(\partial_\tau-\partial_\sigma)\theta_I^{\dot a}), \tag{2.1}$$

Here $0\le\sigma<2\pi$, $I=1,2,\dots,N$. The real bosonic fields X^i, $i=1,2,\dots,8$ transform in the $\mathbf{8_V}$ representation of the $SO(8)$ group, while the components $\theta^a,\theta^{\dot a}$, $a,\dot a=1,\dots,8$ of the 16-component Majorana-Weyl spinor θ^α transform in the $\mathbf{8_S}$ and $\mathbf{8_C}$ representations respectively. One has also to identify all configurations (X,θ) related by arbitrary S_N transformations:

$$X\sim hX,\quad \theta\sim h\theta,\qquad h\in S_N. \tag{2.2}$$

As usual in orbifold models [13, 14], the fields X^i, θ^α can have twisted boundary conditions

$$X^i(\sigma+2\pi) = gX^i(\sigma), \qquad \theta^\alpha(\sigma+2\pi) = g\theta^\alpha(\sigma), \qquad g \in S_N. \tag{2.3}$$

Note that the untwisted sector corresponds to the Ramond boundary condition.

Multiplying (2.3) by some element $h \in S_N$ and taking into account the identification (2.2), one gets that all possible boundary conditions are in one-to-one correspondence with the conjugacy classes of the symmetric group. Therefore, the Hilbert space of the orbifold model is decomposed into the direct sum of Hilbert spaces of the twisted sectors corresponding to the conjugacy classes $[g]$ of S_N [10]

$$\mathcal{H}(S^N\mathbf{R}^D) = \bigoplus_{[g]} \mathcal{H}_{[g]}.$$

It is well-known that the conjugacy classes of S_N are described by partitions $\{N_n\}$ of N

$$N = \sum_{n=1}^{s} nN_n$$

and can be represented as

$$[g] = (1)^{N_1}(2)^{N_2}\cdots(s)^{N_s}. \tag{2.4}$$

Here N_n is the multiplicity of the cyclic permutation (n) of n elements.

In any conjugacy class $[g]$ there is the only element g_c that has the canonical block-diagonal form

$$g_c = diag(\underbrace{\omega_1,...,\omega_1}_{N_1\ times}, \underbrace{\omega_2,...,\omega_2}_{N_2\ times}, ..., \underbrace{\omega_s,...,\omega_s}_{N_s\ times}),$$

where ω_n is an $n\times n$ matrix that generates the cyclic permutation (n) of n elements

$$\omega_n = \sum_{i=1}^{n-1} E_{i,i+1} + E_{n1}$$

and E_{ij} are matrix unities.
It is not difficult to show that ω_n generates the $\mathbf{Z}_n$ group, since $\omega_n^n = 1$, and that only the matrices ω_n^k from $\mathbf{Z}_n$ commute with ω_n. Since the centralizer subgroup C_g of any element $g \in [g]$ is isomorphic to C_{g_c} one concludes that

$$C_g = \prod_{n=1}^{s} S_{N_n} \times \mathbf{Z}_n^{N_n},$$

where the symmetric group S_{N_n} permutes the N_n cycles (n). It is obvious that the centralizer C_g contains $\prod_{n=1}^{s} N_n! n^{N_n}$ elements.

Due to the factorization (2.4) of $[g]$, the Hilbert space $\mathcal{H}_{[g]} \equiv \mathcal{H}_{\{N_n\}}$ of each twisted sector can be decomposed into the graded N_n-fold symmetric tensor products of the Hilbert spaces $\mathcal{H}_{(n)}$ which correspond to the cycles of length n

$$\mathcal{H}_{\{N_n\}} = \bigotimes_{n=1}^{s} S^{N_n}\mathcal{H}_{(n)} = \bigotimes_{n=1}^{s} \Bigg(\underbrace{\mathcal{H}_{(n)}\otimes\cdots\otimes\mathcal{H}_{(n)}}_{N_n\ times}\Bigg)^{S_{N_n}}.$$

The space $\mathcal{H}_{(n)}$ is $\mathbf{Z}_n$ invariant subspace of the Hilbert space of a sigma model of $8n$ bosonic fields X_I^i and $16n$ fermionic fields θ^α with the cyclic boundary condition

$$X_I^i(\sigma+2\pi) = X_{I+1}^i(\sigma), \qquad \theta_I^\alpha(\sigma+2\pi) = \theta_{I+1}^\alpha(\sigma), \quad I = 1,2,...,n. \tag{2.5}$$

The fields $X_I(\sigma)$ $(\theta_I(\sigma))$ can be glued together into one field $X(\sigma)$ $(\theta(\sigma))$ that is identified with a long string of the length n. The states of the space $\mathcal{H}_{(n)}$ are obtained by acting by the creation operators of the string on eigenvectors of the momentum operator. These eigenvectors have the standard normalization

$$\langle \mathbf{q}|\mathbf{k}\rangle = \delta^D(\mathbf{q}+\mathbf{k})$$

and can be regarded as states obtained by acting by the operator $\mathrm{e}^{\,i\mathbf{k}x}$ on the vacuum state [1] (that is not normalizable): $|\mathbf{k}\rangle = \mathrm{e}^{\,i\mathbf{k}x}|0\rangle$, $\langle \mathbf{q}| = \langle 0|\,\mathrm{e}^{\,i\mathbf{q}x}$.

The $\mathbf{Z}_n$ invariant subspace is singled out by imposing the condition

$$(L_0 - \bar{L}_0)|\Psi\rangle = nm|\Psi\rangle,$$

where m is an integer and L_0 is the canonically normalized L_0 operator of the single string.

The Fock space of the second-quantized IIA type string is recovered in the limit $N\to\infty$, $\frac{n_i}{N}\to p_i^+$ [10], where the finite ratio $\frac{n_i}{N}$ is identified with the p_i^+ momentum of a long string. The $\mathbf{Z}_n$ projection reduces in this limit to the usual level-matching condition $L_0^{(i)} - \bar{L}_0^{(i)} = 0$. The individual p_i^- light-cone momentum is defined by means of the standard mass-shell condition $p_i^+ p_i^- = L_0^{(i)}$.

3 Vertex operators

Let us consider conformal field theory of free fields described by the action (2.1). It is convinient to perform the Wick rotation $\tau\to -i\tau$ and to map the cylinder onto the sphere: $z = \mathrm{e}^{\,\tau+i\sigma}$, $\bar{z} = \mathrm{e}^{\,\tau-i\sigma}$.

The NS vacuum state $|0\rangle$ of the CFT is annihilated by the momentum operators and by annihilation operators, and has to be normalizable. To be able to identify this vacuum state with the vacuum state of the untwisted sector of the orbifold sigma model we choose the following normalization of $|0\rangle$

$$\langle 0|0\rangle = R^{8N}.$$

Here R should be regarded as a regularization parameter of the sigma model. We regularize the sigma model by compactifying the coordinates x_I^i on circles of radius R. Then the norm of the eigenvectors of the momentum operators in the untwisted sector is given by

$$\langle \mathbf{q}|\mathbf{k}\rangle = (2\pi)^{-8N}\int_0^{2\pi R} d^{8N}x\,\mathrm{e}^{\,i(\mathbf{q}+\mathbf{k})x} = \prod_{I=1}^{N}\delta_R^8(\mathbf{q}_I+\mathbf{k}_I),$$

where $k_I^i = \frac{m_I^i}{R}$ and $q_I^i = \frac{n_I^i}{R}$ are momenta of the states, m_I^i and n_I^i are integers since we compactified the coordinates, and $\delta_R^8(\mathbf{k}) = R^8\prod_{i=1}^8 \delta_{m^i 0}$ is the regularized δ-function. In the limit $R\to\infty$ one recovers the usual normalization of the eigenvectors.

The asymptotic states of the orbifold CFT model should be created by some vertex operators applied to the NS vacuum $|0\rangle$. Evidently, the vertex operators creating the ground states of twisted sectors are in one-to-one correspondence with the conjugacy classes of S_N. For nonabelian groups a conjugacy class $[g]$ of an element g contains many group elements. It enforces us to define a vertex operator $V_{[g]}$ in two steps. First one introduces vertex operators V_g corresponding to elements $g\in S_N$. The fundamental fields obey the twisted boundary condition (2.3) around an insertion point of an operator V_g. Under the group action, V_g transforms into $V_{h^{-1}gh}$ and, therefore, to define an invariant operator $V_{[g]}$ one should sum up all vertex operators from a given conjugacy class:

$$V_{[g]}(z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} V_{h^{-1}gh}(z,\bar{z}).$$

Vertex operators creating excited states of the twisted sectors can be defined in an analogous way. The main requirement imposed on all invariant vertex operators is that they should form a closed operator algebra.

[1] As is discussed below, the vacuum state carries a representation of the Clifford algebra, since the long string is in the Ramond sector.

Schematically the OPE of any noninvariant vertex operators is of the form

$$V_{g_1}(z,\bar{z})V_{g_2}(0)=\frac{1}{z^{\Delta}\bar{z}^{\bar{\Delta}}}\left(C^{g_1g_2}_{g_1,g_2}V_{g_1g_2}(0)+C^{g_2g_1}_{g_1,g_2}V_{g_2g_1}(0)\right)+\cdots, \tag{3.1}$$

where $\Delta,\bar{\Delta}$ are defined by the conformal symmetry. Here the two leading terms appear because there are two different ways to go around the points z and 0. It is not difficult to see that g_1g_2 and g_2g_1 belong to the same conjugacy class and, hence, $\Delta_{g_1g_2}=\Delta_{g_2g_1}$. If one requires the operator algebra of invariant operators to be closed, then one faces hard restrictions on the structure constants in (3.1). In particular, the structure constants occuring in the OPE of operators V_g creating ground states should be invariant with respect to the global action of S_N, e.g.:

$$C^{h^{-1}g_1g_2h}_{h^{-1}g_1h,h^{-1}g_2h}=C^{g_1g_2}_{g_1,g_2}.$$

Then OPE (3.1) leads to the following OPE for invariant ground state operators:

$$V_{[g_1]}(z,\bar{z})V_{[g_2]}(0)=\frac{1}{N!}\sum_{h\in S_N}\frac{1}{z^{\Delta_h}\bar{z}^{\bar{\Delta}_h}}\left(C^{g_1h^{-1}g_2h}_{g_1,h^{-1}g_2h}+C^{h^{-1}g_2hg_1}_{g_1,h^{-1}g_2h}\right)V_{[g_1h^{-1}g_2h]}(0)+\cdots,$$

i.e., for leading terms the operator algebra is closed.

Naively, one can think that vertex operators V_g receiving contributions from bosons and fermions can be decomposed into tensor product of bosonic and fermionic twist fields: $V_g=\sigma_g\otimes\Sigma_g$. Obviously, the OPE for the fields σ_g and Σ_g should be of the same type as for V_g (3.1). Then one can easily see that the tensor product structure of V_g leads to the appearance of unwanted terms in the OPE:

$$\begin{aligned}
V_{g_1}V_{g_2} &= (\sigma_{g_1}\otimes\Sigma_{g_1})(\sigma_{g_2}\otimes\Sigma_{g_2})=\sigma_{g_1}\sigma_{g_2}\otimes\Sigma_{g_1}\Sigma_{g_2}\\
&\sim \left(B^{g_1g_2}_{g_1,g_2}\sigma_{g_1g_2}+B^{g_2g_1}_{g_1,g_2}\sigma_{g_2g_1}\right)\otimes\left(F^{g_1g_2}_{g_1,g_2}\Sigma_{g_1g_2}+F^{g_2g_1}_{g_1,g_2}\Sigma_{g_2g_1}\right)+\cdots\\
&= B^{g_1g_2}_{g_1,g_2}F^{g_1g_2}_{g_1,g_2}V_{g_1g_2}+B^{g_2g_1}_{g_1,g_2}F^{g_2g_1}_{g_1,g_2}V_{g_2g_1}+\\
&\quad B^{g_1g_2}_{g_1,g_2}F^{g_2g_1}_{g_1,g_2}\sigma_{g_1g_2}\otimes\Sigma_{g_2g_1}+B^{g_2g_1}_{g_1,g_2}F^{g_1g_2}_{g_1,g_2}\sigma_{g_2g_1}\otimes\Sigma_{g_1g_2}+\cdots.
\end{aligned}$$

The same arguments also reveal the absence of decomposition of V_g into the tensor product of holomorphic and antiholomorphic parts. A reason for the absence of tensor product structure lies, of course, in the nonabelian nature of the S_N orbifold CFT. However, in what follows to simplify the notation we represent V_g as a product of bosonic and fermionic (holomorphic and antiholomorphic) twist fields: $V_g(z,\bar{z})=\sigma_g(z)\Sigma_g(z)\bar{\sigma}_g(\bar{z})\bar{\Sigma}_g(\bar{z})$.

It is known that any $g\in S_N$ has the decomposition

$$(n_1)(n_2)\cdots(n_{N_{str}}), \tag{3.2}$$

where each cycle of length n has a definite set of indices ordered up to a cyclic permutation and generates the action of the subgroup $\mathbf{Z}_n$. Due to this decomposition the vertex operator V_g can be represented as the following product

$$V_g=\prod_{\alpha=1}^{N_{str}}V_{(n_\alpha)},$$

where $V_{(n)}$ is a vertex operator that creates the vacuum state of the space $\mathcal{H}_{(n)}$ of the sigma model of fundamental fields with cyclic boundary condition (2.5).

It is obvious that the conformal dimensions of the vertex operators corresponding to cycles of the same length coincide[2] and therefore Δ_g depends only on $[g]$ and is given by the equation

$$\Delta_g=\sum_{\alpha=1}^{N_{str}}\Delta_{n_\alpha}=\sum_{n=1}^{s}N_n\Delta_n, \tag{3.3}$$

where Δ_n denotes the conformal dimension of the vertex operator $V_{(n)}$. Thus, it is enough to consider the operator $V_{(n)}$, which we again represent as a product of bosonic and fermionic twist fields.

[2]It explains why we do not specify a set of indices occured in a given cycle.

We begin with describing the bosonic twist operator. As usual, the field $X(z,\bar{z})$ can be decomposed into the left- and right-moving components

$$2X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}). \tag{3.4}$$

In what follows we shall mainly concentrate our attention on the left-moving sector.

Let $\sigma_{(n)}(z,\bar{z})$ be a primary field [15] that creates a bosonic vacuum of the twisted sector at the point z, i.e. the fields $X^i(z)$ satisfy the following monodromy conditions

$$X^i(z\,\mathrm{e}^{2\pi i}, \bar{z}\,\mathrm{e}^{-2\pi i})\sigma_{(n)}(0) = \omega_n X^i(z,\bar{z})\sigma_{(n)}(0), \tag{3.5}$$

where ω_n generates the cyclic permutation of n elements.

It is also convinient to regard the twist field $\sigma_{(n)}(z,\bar{z})$ as a product $\sigma_{(n)}(z,\bar{z}) = \sigma_{(n)}(z)\bar{\sigma}_{(n)}(\bar{z})$. To simplify calculations, we require that under the world-sheet parity transformation $z \to \bar{z}$, and $X(z) \to \bar{X}(\bar{z})$ the field $\sigma_{(n)}(z)$ transforms into $\bar{\sigma}_{(-n)}(\bar{z})$, where $(-n)$ denotes the cycle with the reversed orientation corresponding to the element ω_n^{-1}.

Note that (3.5) does not completely specify the field σ_g but it contains enough information to derive it's conformal dimension. For later use we consider the general case of Dn bosonic fields.

Let the twist field $\sigma_{(n)}$ be located at $z = 0$ and let us denote the vacuum state [3] as $|(n)\rangle = \sigma_{(n)}(0,0)|0\rangle$. Since the twist field $\sigma_{(n)}$ creates one long string we normalize the vacuum state $|(n)\rangle$ as

$$\langle (n)|(n)\rangle = R^D. \tag{3.6}$$

The fields $X(z)$ have the following decomposition in the vicinity of $z = 0$

$$\partial X^i_I(z) = -\frac{i}{n}\sum_m \alpha^i_m\,\mathrm{e}^{-\frac{2\pi i}{n}Im} z^{-\frac{m}{n}-1}, \tag{3.7}$$

where α^i_m $(m \neq 0)$ are the usual creation and annihilation operators with the commutation relations

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m+n,0}, \tag{3.8}$$

and α^i_0 is proportional to the momentum operator [4].
The vacuum state $|(n)\rangle$ is annihilated by the operators α^i_m for $m \geq 0$.

Since $\sigma_{(n)}$ is a primary field, the conformal dimension Δ^b_n can be found from the equation

$$\langle (n)|T(z)|(n)\rangle = \frac{\Delta^b_n}{z^2}\langle (n)|(n)\rangle,$$

where $T(z)$ is the stress-energy tensor.
By using eqs. (3.7) and (3.8), one calculates the correlation function

$$\langle (n)|\partial X^i_I(z)\partial X^j_I(w)|(n)\rangle = -\delta^{ij}\frac{(zw)^{\frac{1}{n}-1}}{n^2(z^{\frac{1}{n}} - w^{\frac{1}{n}})^2}\langle (n)|(n)\rangle.$$

Taking into account that the stress-energy tensor is defined as

$$T(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{n}\left(\partial X^i_I(z)\partial X^i_I(w) + \frac{1}{(z-w)^2}\right),$$

one gets

$$\Delta^b_n = \frac{D}{24}(n - \frac{1}{n}). \tag{3.9}$$

[3] This vacuum state is a primary state of the CFT.
[4] $\alpha^i_0 = \frac{1}{2}p^i$ in string units $\alpha' = \frac{1}{2}$.

The excited states of this sigma model are obtained by acting on $|(n)\rangle$ by some vertex operators. In particular the state corresponding to a scalar particle with momentum $\mathbf{k}$ is given by

$$\sigma_{(n)}[\mathbf{k}](0,0)|0\rangle =: \mathrm{e}^{ik_I^i X_I^i(0,0)} : |(n)\rangle, \tag{3.10}$$

where the summation over i and I is assumed, $k_I^i = \frac{m_I^i}{R}$ is a momentum carried by the field $X_I^i(z,\bar{z})$ and $k^i = \sum_{I=1}^n k_I^i$ is a total momentum of the long string.
By using the definition of the vacuum state $|(n)\rangle$, one can rewrite eq.(3.10) in the form

$$\sigma_{(n)}[\mathbf{k}](0,0)|0\rangle =: \mathrm{e}^{i\frac{k^i}{\sqrt{n}}Y^i(0,0)} : |(n)\rangle, \tag{3.11}$$

where

$$Y^i(z,\bar{z}) = \frac{1}{\sqrt{n}}\sum_{I=1}^n X_I^i(z,\bar{z}). \tag{3.12}$$

The field $Y(z)$ is canonically normalized, i.e. the part of the stress-energy tensor depending on Y is $-\frac{1}{2} : \partial Y(z)\partial Y(z) :$, and has the trivial monodromy around $z=0$.

It is obvious from eq.(3.11) that the conformal dimension of the primary field

$$\sigma_{(n)}[\mathbf{k}](z,\bar{z}) =: \mathrm{e}^{i\frac{k^i}{\sqrt{n}}Y^i(z,\bar{z})} : \sigma_{(n)}(z,\bar{z})$$

is equal to

$$\Delta_n^b[\mathbf{k}] = \Delta_n^b + \frac{\mathbf{k}^2}{8n} = \frac{D}{24}(n-\frac{1}{n}) + \frac{\mathbf{k}^2}{8n},$$

where the decomposition (3.4) was taken into account.

To simplify calculations it will be convinient to treat $\sigma_{(n)}[\mathbf{k}](z,\bar{z})$ as a product of holomorphic and antiholomorphic parts $\sigma_{(n)}[\mathbf{k}/2](z)\sigma_{(n)}[\mathbf{k}/2](\bar{z})$.

Other excited states of the model can be produced by considering the OPE of the fields ∂X with the twist fields. By using (3.7) and the definition of the vacuum state $|(n)\rangle$ one can see that the most singular term of the OPE looks as

$$\partial X_I^i(z)\sigma_{(n)}(w,\bar{w}) = (z-w)^{-\left(1-\frac{1}{n}\right)} e^{\frac{2\pi i}{n}I}\tau_{(n)}^i(w,\bar{w}) + \dots, \tag{3.13}$$

where $\tau_{(n)}^i(0,0) = -\frac{i}{n}\alpha_{-1}^i|(n)\rangle$ is the first excited state in the twisted sector. According to our conventions, the field $\tau_{(n)}^i(z,\bar{z})$ can be represented as a product: $\tau_{(n)}^i(z,\bar{z}) = \tau_{(n)}^i(z)\bar{\sigma}_{(n)}(\bar{z})$. In particular, since the element $g_{IJ} = 1 - E_{II} - E_{JJ} + E_{IJ} + E_{JI}$ transposing the fields X_I and X_J has just one cycle of length 2, one can define the field $\tau_{IJ} \equiv \tau_{(2)}$. This twist field will be used to define the DVV interaction vertex.

Similarly to the vertex operator V_g, the twist field σ_g can be represented as

$$\sigma_g = \prod_{\alpha=1}^{N_{str}} \sigma_{(n_\alpha)}.$$

Due to eqs. (3.3) and (3.9), the conformal dimension of σ_g is given by

$$\Delta_g = \sum_{n=1}^s N_n \frac{D}{24}(n-\frac{1}{n}) = \frac{D}{24}(N - \sum_{n=1}^s \frac{N_n}{n}).$$

One can also introduce a primary field that creates scalar particles with momenta k_α^i, $\alpha = 1,2,...,N_1+N_2+\dots+N_s \equiv N_{str}$

$$\sigma_g[\{\mathbf{k}_\alpha\}](z,\bar{z}) =: \mathrm{e}^{i\frac{k_\alpha^i}{\sqrt{n_\alpha}}Y_\alpha^i(z,\bar{z})} : \sigma_g(z,\bar{z}) = \prod_{\alpha=1}^{N_{str}} \sigma_{(n_\alpha)}[\mathbf{k}_\alpha],$$

where $n_1 = n_2 = \dots = n_{N_1} = 1$, $n_{N_1+1} = n_{N_1+2} = \dots = n_{N_1+N_2} = 2$ and so on, Y_α^i corresponds to the cycle (n_α) and is defined by eq.(3.12), and the summation over i and α is assumed.

The conformal dimension of the field $\sigma_g[\{\mathbf{k}_\alpha\}]$ is equal to

$$\Delta_g[\{\mathbf{k}_\alpha\}] = \frac{D}{24}(N - \sum_{n=1}^{s} \frac{N_n}{n}) + \sum_\alpha \frac{\mathbf{k}_\alpha^2}{8n_\alpha}. \tag{3.14}$$

It is obvious that the two-point correlation function of the twist fields σ_{g_1} and σ_{g_2} is not equal to zero if and only if $g_1 g_2 = 1$. Taking into account the normalization (3.6), we find [5]

$$\langle \sigma_{g^{-1}}(\infty)\sigma_g(0)\rangle = R^{DN_{str}}.$$

It means that the fields $\sigma_{g^{-1}}$ and σ_g have the following OPE

$$\sigma_{g^{-1}}(z,\bar{z})\sigma_g(0,0) = \frac{R^{D(N_{str}-N)}}{|z|^{4\Delta_g}} + \cdots .$$

Here we assume that $\langle 0|0\rangle = R^{DN}$.

The two-point correlation function of $\sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}]$ and $\sigma_g[\{\mathbf{k}_\alpha\}]$ is respectively equal to

$$\langle \sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}](\infty)\sigma_g[\{\mathbf{k}_\alpha\}](0)\rangle = \prod_\alpha \delta_R^D(\mathbf{q}_\alpha + \mathbf{k}_\alpha). \tag{3.15}$$

Now we proceed with describing the fermion twist fields that create the vacuum states corresponding to long strings. We begin with the case of one long string of length n.

Equations of motion corresponding to the cyclic boundary condition (2.5) imply that θ^a and $\theta^{\dot{a}}$ are the following holomorphic and antiholomorphic functions on the z-plain:

$$\theta_I^a(z) = \frac{1}{\sqrt{n}} \sum_m \theta_m^a e^{-\frac{2\pi i}{n}Im} z^{-\frac{m}{n}-\frac{1}{2}}, \tag{3.16}$$
$$\theta_I^{\dot{a}}(\bar{z}) = \frac{1}{\sqrt{n}} \sum_m \theta_m^{\dot{a}} e^{\frac{2\pi i}{n}Im} \bar{z}^{-\frac{m}{n}-\frac{1}{2}},$$

where we have taken into account that under conformal mappings fermion fields have the scaling dimension 1/2. Thus, on the z-plain the fermions satisfy $\theta_I^a(ze^{2\pi i}) = -\theta_{I+1}^a(z)$ and analogously for $\theta_I^{\dot{a}}$. In what follows we mainly concentrate on the left-moving sector.

In eq.(3.16) the creation and annihilation operators satisfy the standard commutation relations:

$$\{\theta_m^a, \theta_n^b\} = \delta^{ab}\delta_{m+n}, \tag{3.17}$$

i.e. zero modes θ_0^a form the Clifford algebra. Therefore, the vacuum state anihilated by θ_m^a for $m > 0$ carry an irreducible representation of the Clifford algebra. By the triality the representation space can be choosen as the direct sum $\mathbf{8_V} + \mathbf{8_C}$. It means that the vacuum state is a 16-component vector with components $|i\rangle$ and $|\dot{a}\rangle$ normalized in the standard fashion $\langle i|j\rangle = \delta^{ij}$, $\langle \dot{a}|\dot{b}\rangle = \delta^{\dot{a}\dot{b}}$ and transforming under the action of θ_0^a as follows

$$\theta_0^a|i\rangle = \frac{1}{\sqrt{2}}\gamma_{a\dot{a}}^i|\dot{a}\rangle, \qquad \theta_0^a|\dot{a}\rangle = \frac{1}{\sqrt{2}}\gamma_{a\dot{a}}^i|i\rangle. \tag{3.18}$$

In the CFT the vacuum states $|i\rangle$ and $|\dot{a}\rangle$ are created by the primary (spin) fields $\Sigma_{(n)}^i$ and $\Sigma_{(n)}^{\dot{a}}$. Their conformal dimension can be found similarly to the bosonic case. Denoting by $\Sigma_{(n)}^{\dot{\mu}}$ one of the fields $\Sigma_{(n)}^i$, $\Sigma_{(n)}^{\dot{a}}$ and using eq.(3.17) we obtain

$$\langle \Sigma_{(n)}^{\dot{\mu}}|\theta_I^a(z)\partial\theta_I^b(w)|\Sigma_{(n)}^{\dot{\mu}}\rangle = -\frac{1}{2}\frac{\langle \Sigma_{(n)}^{\dot{\mu}}|\theta_0^a\theta_0^b|\Sigma_{(n)}^{\dot{\mu}}\rangle}{nz^{1/2}w^{3/2}} + \frac{\delta^{ab}}{nz^{1/2}}\left(\frac{\left(\frac{1}{n}-\frac{1}{2}\right)w^{1/n-3/2}}{z^{1/n}-w^{1/n}} + \frac{\frac{1}{n}w^{2/n-3/2}}{(z^{1/n}-w^{1/n})^2}\right).$$

[5] it is clear that $[g^{-1}] = [g]$ and therefore $\Delta_{g^{-1}} = \Delta_g$

Taking into account the definition of the stress-energy tensor T^F for fermion fields

$$T^F(z) = -\frac{1}{2}\lim_{w\to z}\sum_{a=1}^{8}\sum_{I=1}^{n}\left(\theta_I^a(z)\partial\theta_I^b(w) - \frac{1}{(z-w)^2}\right),$$

one finds

$$\Delta_n^f = \frac{n}{6} + \frac{1}{3n}. \tag{3.19}$$

The transformation properties (3.18) are encoded in the following OPE:

$$\theta_I^a(z)\Sigma_{(n)}^i(0) = \frac{1}{\sqrt{n}z^{1/2}}\frac{\gamma_{a\dot a}^i}{\sqrt{2}}\Sigma_{(n)}^{\dot a}(0) + \dots, \qquad \theta_I^a(z)\Sigma_{(n)}^{\dot a}(0) = \frac{1}{\sqrt{n}z^{1/2}}\frac{\gamma_{a\dot a}^i}{\sqrt{2}}\Sigma_{(n)}^i(0) + \dots \tag{3.20}$$

Note that the r.h.s. of this OPE contains other (less) singular terms that correspond to the excited states of the twisted sector.

The twist fields $\bar\Sigma_{(n)}^{\mu}(\bar z)$ for the right-moving sector are introduced in the same manner, they realize the representation space $\mathbf{8_V} + \mathbf{8_S}$, and have the same conformal dimensions. The fields $\theta^{\dot a}(\bar z)$ have the following OPE with the twist fields

$$\theta_I^{\dot a}(\bar z)\bar\Sigma_{(n)}^i(0) = -\frac{1}{\sqrt{n}\bar z^{1/2}}\frac{\gamma_{a\dot a}^i}{\sqrt{2}}\bar\Sigma_{(n)}^{a}(0) + \dots, \qquad \theta_I^{\dot a}(\bar z)\bar\Sigma_{(n)}^{a}(0) = -\frac{1}{\sqrt{n}\bar z^{1/2}}\frac{\gamma_{a\dot a}^i}{\sqrt{2}}\bar\Sigma_{(n)}^i(0) + \dots \tag{3.21}$$

Comparing eqs.(3.20) and (3.21) one can see that under the world-sheet parity transformation $z \to \bar z$ and the space reflection $X^3 \to -X^3$ the fermions and twist fields transform as follows:

$$\begin{aligned}
&\theta^a(z) \leftrightarrow \theta^{\dot a}(\bar z); \qquad \Sigma_{(n)}^{\dot a}(z) \leftrightarrow \bar\Sigma_{(-n)}^{a}(\bar z);\\
&\Sigma_{(n)}^{i}(z) \leftrightarrow \bar\Sigma_{(-n)}^{i}(\bar z), \qquad i \neq 3; \qquad \Sigma_{(n)}^{3}(z) \leftrightarrow -\bar\Sigma_{(-n)}^{3}(\bar z).
\end{aligned} \tag{3.22}$$

The third direction is singled out since in our conventions $\gamma^3 = 1$ (see Appendix A).

At last combining the fermionic vacuum states of the holomorphic and the antiholomorphic sectors with the vacuum state of the bosonic sector we obtain 256 states that describes the spectrum of the IIA supergravity. The IIA supergravity states with the momentum $\mathbf{k}$ are

$$|V(\mathbf{k},\dot\mu,\nu)\rangle = |\mathbf{k}\rangle \otimes |\dot\mu\rangle \otimes |\nu\rangle,$$

where $\dot\mu = (i,\dot a)$ and $\mu = (i,a)$. Clearly, these states can be generated from the NS vacuum $|0\rangle$ by the following vertex operators:

$$V_{(n)}[\mathbf{k},\dot\mu,\nu](z,\bar z) = \sigma_{(n)}[\mathbf{k}](z,\bar z)\Sigma_{(n)}^{\dot\mu}(z)\bar\Sigma_{(n)}^{\nu}(\bar z).$$

The conformal dimension of the vertex operator is equal to

$$\Delta_n = \frac{n}{2} + \frac{\mathbf{k}^2}{8n}. \tag{3.23}$$

In particular, a graviton with a momentum $\mathbf{k}$ and a polarization ζ is created by

$$V_{(n)}[\mathbf{k},\zeta](z,\bar z) = \zeta_{ij}\sigma_{(n)}[\mathbf{k}](z,\bar z)\Sigma_{(n)}^{i}(z)\bar\Sigma_{(n)}^{j}(\bar z),$$

where ζ_{ij} is a symmetric tensor.

It is worth noting that due to eq.(3.22) under the world-sheet parity transformation $z \to \bar z$ and the space reflection $X^3 \to -X^3$ the graviton vertex operator $V_{(n)}[\mathbf{k},\zeta]$ transforms into $V_{(-n)}[\tilde{\mathbf{k}},\tilde\zeta]$, where $\tilde k^i, \tilde\zeta_{ij}$ are the space reflected momenta and polarizations respectively ($\tilde k^3 = -k^3$).

Due to the factorization (3.2) the vertex operator corresponding to any element $g \in S_N$ has the following decomposition into the tensor product of $V_{(n)}[\mathbf{k},\dot\mu,\nu]$:

$$V_g[\{\mathbf{k}_\alpha,\dot\mu_\alpha,\nu_\alpha\}] = \prod_{\alpha=1}^{N_{str}} V_{(n_\alpha)}[\mathbf{k}_\alpha,\dot\mu_\alpha,\nu_\alpha].$$

According to (3.23) the conformal dimension of V_g is given by

$$\Delta_g = \frac{N}{2} + \sum_{\alpha=1}^{N_{str}} \frac{\mathbf{k}_\alpha^2}{8n}. \tag{3.24}$$

Thus, we see that the operators creating ground states ($\mathbf{k}_\alpha = 0$) have the same conformal dimension that does not depend on a particular group element.

As was discussed before an invariant vertex operator is defined by summing up all the twist fields from one conjugacy class:

$$V_{[g]}[\{\mathbf{k}_\alpha, \dot{\mu}_\alpha, \nu_\alpha\}] = \frac{1}{N!} \sum_{h\in S_N} \prod_{\alpha=1}^{N_{str}} V_{h^{-1}(n_\alpha)h}[\mathbf{k}_\alpha, \dot{\mu}_\alpha, \nu_\alpha]. \tag{3.25}$$

One can easily check that the vertex operators are invariant with respect to the simultaneous permutation of $\mathbf{k}_\alpha$, $\dot{\mu}_\alpha$ and ν_α which correspond to cycles (n_α) of the same length.

By using this definition, one can easily calculate the two-point correlation function

$$\langle V_{[g]}[\{\mathbf{k}_\alpha, \dot{\mu}_\alpha, \rho_\alpha\}](\infty) V_{[g]}[\{\mathbf{q}_\alpha, \dot{\nu}_\alpha, \epsilon_\alpha\}](0)\rangle = \frac{1}{N!} \prod_{n=1}^{s} N_n! n^{N_n} \prod_\alpha \delta_R^8(\mathbf{q}_\alpha + \mathbf{k}_\alpha) \delta^{\dot{\mu}_\alpha \dot{\nu}_\alpha} \delta^{\rho_\alpha \epsilon_\alpha},$$

where $\prod_{n=1}^s N_n! n^{N_n}$ is the number of elements of the centralizer subgroup C_g.

Thus, we have introduced the vertex operators that create asymptotic states corresponding to the massless particles of the type IIA string.

To describe the interaction vertex proposed by DVV [9] we need another kind of spin twist fields. Note that on the z-plane there are twist fields around which fermions obey the following boundary condition:

$$\theta(e^{2\pi i} z) = g\theta(z).$$

Consider the group element g_{IJ} transposing the fields θ_I and θ_J. Since the combination $\theta_I - \theta_J$ satisfies the Ramond boundary condition the corresponding spin field carries a representation of the Clifford algebra. The DVV interaction vertex is defined with the help of the twist field Σ_{IJ}^i transforming as a vector of $SO(8)$. In fact, Σ_{IJ}^i is a well-known spin field of the $\mathbf{R}^8/\mathbf{Z}_2$ supersymmetric orbifold sigma model.

To write down the DVV interaction vertex it is useful to come back to the Minkowskian space-time. Then the interaction is described by the translationally-invariant vertex

$$V_{int} = \frac{\lambda N}{2\pi} \sum_{I<J} \int d\tau d\sigma \left(\tau^i(\sigma_+)\Sigma^i(\sigma_+)\bar{\tau}^j(\sigma_-)\bar{\Sigma}^j(\sigma_-)\right)_{IJ},$$

where λ is a coupling constant proportional to the string coupling, and $\sigma_\pm$ are light-cone coordinates: $\sigma_\pm = \tau \pm \sigma$.

The twist field $V_{IJ}(\sigma_+, \sigma_-) = \left(\tau^i(\sigma_+)\Sigma^i(\sigma_+)\bar{\tau}^j(\sigma_-)\bar{\Sigma}^j(\sigma_-)\right)_{IJ}$ is a weight $(\frac{3}{2}, \frac{3}{2})$ conformal field and the coupling constant λ has dimension -1. As was shown in [9] the interaction vertex is space-time supersymmetric, $SO(8)$ invariant and describes an elementary string interaction. Another important property of the interaction vertex is the invariance with respect to the world-sheet parity transformation $\sigma \to -\sigma$ and an odd number of space reflections.

Performing again the Wick rotation and the conformal map onto the sphere, one gets the following expression for V_{int}

$$V_{int} = -\frac{\lambda N}{2\pi} \sum_{I<J} \int d^2z |z| V_{IJ}(z, \bar{z}),$$

where the minus sign appears because V_{IJ} has conformal dimension $(\frac{3}{2}, \frac{3}{2})$.

Thus, the action of the interacting $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model is given by the sum

$$S_{int} = S_0 + V_{int}$$

In the next section we calculate the S-matrix element corresponding to the scattering of two gravitons and show that the scattering amplitude coincides with the type IIA string scattering amplitude.

4 S-matrix element

The S-matrix element at the second order in the coupling constant λ is given by the standard formula of quantum field theory

$$\langle f|S|i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \langle f|\int d^2z_1 d^2z_2 |z_1||z_2| T\left(\mathcal{L}_{int}(z_1,\bar{z}_1)\mathcal{L}_{int}(z_2,\bar{z}_2)\right)|i\rangle, \tag{4.1}$$

where the symbol T means the time-ordering: $|z_1| > |z_2|$, and

$$\mathcal{L}_{int}(z,\bar{z}) = \sum_{I<J} V_{IJ}(z,\bar{z}).$$

The initial state $|i\rangle$ describes two gravitons with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, and polarizations ζ_1 and ζ_2, and is created by the vertex operator $V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2]$:

$$V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} V_{h^{-1}(n_0)h}[\mathbf{k}_1,\zeta_1](z,\bar{z}) V_{h^{-1}(N-n_0)h}[\mathbf{k}_2,\zeta_2](z,\bar{z}).$$

Namely,

$$|i\rangle = C_0 V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)|0\rangle.$$

Here the element g_0 is taken in the canonical block-diagonal form

$$g_0 = (n_0)(N-n_0),$$

where $n_0 < N - n_0$.

The final state $\langle f|$ describes two gravitons with momenta $\mathbf{k}_3$ and $\mathbf{k}_4$, and polarizations ζ_3 and ζ_4, and is given by the formula (see [15])

$$\langle f| = C_\infty \lim_{z_\infty\to\infty} |z_\infty|^{4\Delta_\infty}\langle 0|V_{[g_\infty]}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](z_\infty,\bar{z}_\infty).$$

The element g_∞ has the canonical decomposition

$$g_\infty = (n_\infty)(N-n_\infty), \quad n_\infty < N - n_\infty.$$

The constants C_0 and C_∞ are chosen to be equal to

$$C_0 = \sqrt{\frac{N!}{n_0(N-n_0)}}, \quad C_\infty = \sqrt{\frac{N!}{n_\infty(N-n_\infty)}}$$

that guarantees the standard normalization of the initial and final states.

After the conformal transformation $z \to \frac{z}{z_1}$ eq.(4.1) acquires the form

$$\begin{aligned}\langle f|S|i\rangle &= -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \int d^2z_1 d^2z_2 |z_1||z_2||z_1|^{2\Delta_\infty - 2\Delta_0 - 6} \\ &\times \langle f|T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(\frac{z_2}{z_1},\frac{\bar{z}_2}{\bar{z}_1})\right)|i\rangle,\end{aligned}$$

where, according to (3.24), the conformal dimensions Δ_0 and Δ_∞ of the vertex operators $V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2]$ and $V_{[g_\infty]}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4]$ are given by

$$\begin{aligned}\Delta_0 &= \frac{N}{2} + \frac{\mathbf{k}_1^2}{8n_0} + \frac{\mathbf{k}_2^2}{8(N-n_0)}, \\ \Delta_\infty &= \frac{N}{2} + \frac{\mathbf{k}_3^2}{8n_\infty} + \frac{\mathbf{k}_4^2}{8(N-n_\infty)}.\end{aligned} \tag{4.2}$$

Let us introduce the light-cone momenta of the gravitons [9] taking into account the mass-shell condition for the graviton states

$$\begin{aligned} k_1^+ &= \frac{n_0}{N}, \quad k_1^- k_1^+ - \mathbf{k}_1^2 \equiv -k_1^2 = 0, \\ k_2^+ &= \frac{N-n_0}{N}, \quad k_2^- k_2^+ - \mathbf{k}_2^2 \equiv -k_2^2 = 0, \\ k_3^+ &= -\frac{n_\infty}{N}, \quad k_3^- k_3^+ - \mathbf{k}_3^2 \equiv -k_3^2 = 0, \\ k_4^+ &= -\frac{N-n_\infty}{N}, \quad k_4^- k_4^+ - \mathbf{k}_4^2 \equiv -k_4^2 = 0. \end{aligned}$$

By using the light-cone momenta and the mass-shell condition, one can rewrite (4.2) in the form

$$\begin{aligned} \Delta_0 &= \frac{N}{2} + \frac{k_1^- + k_2^-}{8N}, \\ \Delta_\infty &= \frac{N}{2} - \frac{k_3^- + k_4^-}{8N}. \end{aligned}$$

Performing the change of variables $\frac{z_2}{z_1} = u$, one obtains

$$\begin{aligned} \langle f|S|i\rangle &= -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \int d^2 z_1 |z_1|^{2\Delta_\infty - 2\Delta_0 - 2} \\ &\times \int d^2 u |u| \langle f| T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\bar{u})\right)|i\rangle. \end{aligned}$$

The integral over z_1 is obviously divergent. To understand the meaning of this divergency one should remember that we made the Wick rotation. Coming back to the σ,τ-coordinates on the cylinder, we get for the integral over z_1

$$\int d^2 z_1 |z_1|^{2\Delta_\infty - 2\Delta_0 - 2} \to i \int d\tau d\sigma\, \mathrm{e}^{2i\tau(\Delta_\infty - \Delta_0)}.$$

Integration over σ and τ gives us the conservation law for the light-cone momenta k_i^-

$$\int d\tau d\sigma\, \mathrm{e}^{2i\tau(\Delta_\infty - \Delta_0)} = 4N(2\pi)^2 \delta(k_1^- + k_2^- + k_3^- + k_4^-).$$

Thus, the S-matrix element is equal to

$$\langle f|S|i\rangle = -i2\lambda^2 N^3 \delta(k_1^- + k_2^- + k_3^- + k_4^-) \int d^2 u |u| \langle f| T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\bar{u})\right)|i\rangle. \quad (4.3)$$

So, to find the S-matrix element one has to calculate the correlation function

$$\begin{aligned} &F(u,\bar{u}) = \langle f| T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\bar{u})\right)|i\rangle \\ &= C_0 C_\infty \sum_{I<J;K<L} \langle V_{[g_\infty]}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty) T\left(V_{IJ}(1,1)V_{KL}(u,\bar{u})\right) V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle. \end{aligned} \quad (4.4)$$

In what follows we assume for definiteness that $n_0 < n_\infty$ and $|u| < 1$.

By using the definition (3.25) of $V_{[g]}$, and taking into account that the interaction vertex is S_N-invariant, and that any correlation function of vertex operators is invariant with respect to the global action of the symmetric group

$$\langle V_{g_1} V_{g_2} \cdots V_{g_n}\rangle = \langle V_{h^{-1}g_1 h} V_{h^{-1}g_2 h} \cdots V_{h^{-1}g_n h}\rangle, \quad (4.5)$$

we rewrite the correlation function in the form

$$F(u,\bar{u}) = \frac{C_0 C_\infty}{N!} \sum_{h_\infty \in S_N} \sum_{I<J;K<L} \langle V_{h_\infty^{-1} g_\infty h_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty) V_{IJ}(1,1) V_{KL}(u,\bar{u}) V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle. \quad (4.6)$$

Let us note that the correlation function

$$\langle V_{g_1}(\infty)V_{g_2}(1,1)V_{g_3}(u,\bar{u})V_{g_4}(0,0)\rangle \tag{4.7}$$

does not vanish only if

$$g_1g_2g_3g_4 = 1 \quad \text{or} \quad g_1g_4g_3g_2 = 1. \tag{4.8}$$

It can be seen as follows. Due to the OPE (3.1) of V_g, in the limit $u \to 0$ the correlation function (4.7) reduces to the sum of three-point correlation functions $\langle V_{g_1}V_{g_2}V_{g_3g_4}\rangle$ and $\langle V_{g_1}V_{g_2}V_{g_4g_3}\rangle$. This sum does not vanish if one of the following equations is fulfilled:

$$g_1g_2g_3g_4 = 1, \quad g_1g_3g_4g_2 = 1, \quad g_1g_2g_4g_3 = 1, \quad g_1g_4g_3g_2 = 1. \tag{4.9}$$

From the other side in the limit $u \to 1$ one gets the sum of the correlation functions $\langle V_{g_1}V_{g_2g_3}V_{g_4}\rangle$ and $\langle V_{g_1}V_{g_3g_2}V_{g_4}\rangle$. This sum does not vanish if

$$g_1g_2g_3g_4 = 1, \quad g_1g_4g_2g_3 = 1, \quad g_1g_3g_2g_4 = 1, \quad g_1g_4g_3g_2 = 1. \tag{4.10}$$

Comparing eqs.(4.9) and (4.10), one obtains eq.(4.8).

Thus, every summand in (4.6) is not equal to zero in the following two cases:

$$h_\infty^{-1}g_\infty h_\infty g_0 g_{KL} g_{IJ} = 1, \qquad h_\infty^{-1}g_\infty h_\infty g_{IJ} g_{KL} g_0 = 1.$$

On the other hand, one can express the correlation functions with the second monodromy condition via the correlation functions with the first one. Indeed, since the action and the interaction vertex of the model is invariant under the world-sheet parity transformation $z \to \bar{z}$ and the space-reflection $X^3 \to -X^3$, and the graviton vertex operators $V_g[\{\mathbf{k}_\alpha, \zeta_\alpha\}]$ transform into $\tilde{V}_{g^{-1}}[\{\mathbf{k}_\alpha, \zeta_\alpha\}] \equiv V_{g^{-1}}[\{\tilde{\mathbf{k}}_\alpha, \tilde{\zeta}_\alpha\}]$ their correlation functions satisfy the following equality:

$$\langle V_{g_{IJ}g_{KL}g_0^{-1}} V_{IJ} V_{KL} V_{g_0}\rangle = \langle \tilde{V}_{g_0 g_{KL} g_{IJ}} V_{IJ} V_{KL} \tilde{V}_{g_0^{-1}}\rangle.$$

Now taking into account eqs.(4.5) and that the elements g and g^{-1} belong to the same conjugacy class, one obtains

$$\langle V_{g_{IJ}g_{KL}g_0^{-1}} V_{IJ} V_{KL} V_{g_0}\rangle = \langle \tilde{V}_{g_0^{-1} g_{K'L'} g_{I'J'}} V_{I'J'} V_{K'L'} \tilde{V}_{g_0}\rangle,$$

where $g_{I'J'} = hg_{IJ}h^{-1}$, $g_{K'L'} = hg_{KL}h^{-1}$, and the element h is such that $g_0^{-1} = h^{-1}g_0h$. It is now clear that the contribution of the terms satisfying the second monodromy condition coincides with the one of the terms satisfying the first monodromy condition after the replacement $\mathbf{k}_\alpha \to \tilde{\mathbf{k}}_\alpha$ and $\zeta_\alpha \to \tilde{\zeta}_\alpha$. It is obvious that due to the $SO(8)$ invariance the correlation function (4.4) can depend only on the scalar products of momenta and polarizations of the gravitons and, therefore, is invariant under space reflections. Thus, the contributions of the terms satisfying the first and the second monodromy conditions coincide.

Schematically, the function $F(u,\bar{u})$ given by a sum of correlation functions of twist fields can be represented as

$$\mathcal{S} = \sum_{h_\infty \in S_N} \sum_{I<J;K<L} \langle V_{h_\infty^{-1}g_\infty h_\infty} V_{IJ} V_{KL} V_{g_0}\rangle,$$

where the elements h_∞, g_{IJ}, g_{KL} solve the equation $h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0 = 1$.
We can fix the values of the indices K and L by using the action of the centralizer of g_0 and the invariance (4.5) of the correlation functions

$$\begin{aligned}\mathcal{S} = \sum_{h_\infty \in S_N} \sum_{I<J} \quad & \Big(n_0(N-n_0)\langle V_{h_\infty^{-1}g_\infty h_\infty} V_{IJ} V_{n_0 N} V_{g_0}\rangle \\ & + (N-n_0)\langle V_{h_\infty^{-1}g_\infty h_\infty} V_{IJ} V_{n_\infty N} V_{g_0}\rangle \\ & + (N-n_0)\langle V_{h_\infty^{-1}g_\infty h_\infty} V_{IJ} V_{n_0+n_\infty, N} V_{g_0}\rangle \Big) .\end{aligned} \tag{4.11}$$

The first term in (4.11) corresponds to the joining of two incoming strings and the factor $n_0(N-n_0)$ appears since in this case the index K can take n_0 values, $K=1,...,n_0$, and the index L takes $N-n_0$ values, $L=n_0+1,...,N$. To fix $K=n_0$ and $L=N$ we have to use all elements of C_{g_0}. The second and the third terms correspond to the splitting of the string of length $N-n_0$ into two strings of lengths $n_\infty - n_0$ and $N-n_\infty$, and $N-n_0-n_\infty$ and n_∞ respectively. In these cases to fix the values of K and L one should use $N-n_0$ elements of the subgroup $\mathbf{Z}_{N-n_0}$ of C_{g_0} that does not act on the cycle (n_0).

Eq.(4.11) can be further rewritten in the form

$$\begin{aligned} \mathcal{S} &= n_0(N-n_0)n_\infty(N-n_\infty)\left(\sum_{I=1}^{n_\infty}\langle V_{g_\infty(I)}V_{I,I+N-n_\infty}V_{n_0N}V_{g_0}\rangle\right. \\ &+ \sum_{I=1}^{N-n_\infty}\langle V_{g_\infty(I)}V_{I,I+n_\infty}V_{n_0N}V_{g_0}\rangle + \sum_{J=n_0+1}^{n_\infty}\langle V_{g_\infty(J)}V_{n_0J}V_{n_\infty N}V_{g_0}\rangle \\ &+ \left.\sum_{J=n_0+n_\infty+1}^{N}\langle V_{g_\infty(J)}V_{n_0J}V_{n_0+n_\infty,N}V_{g_0}\rangle\right), \end{aligned} \tag{4.12}$$

where the elements g_∞ have to be found from the equation $g_\infty g_{IJ} g_{KL} g_0 = 1$. The diagramms corresponding to these four terms are depicted in Fig.1.

So, we need to compute the correlation functions (and the same correlation functions with the interchange $u \leftrightarrow 1$)

$$G(u,\bar{u}) = \langle V_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1,1)V_{KL}(u,\bar{u})V_{g_0}[\mathbf{k}_1,\zeta_1,\mathbf{k}_2,\zeta_2](0,0)\rangle, \tag{4.13}$$

where all possible elements $g_\infty, g_{IJ}, g_{KL}, g_0$ are listed in eq.(4.12).

To calculate the correlation function (4.13) we employ the stress-energy tensor method [16]. The idea of the method is as follows. Suppose that one knows the following ratio

$$f(z,u) = \frac{\langle T(z)\phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle}{\langle \phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle},$$

where $T(z)$ is the stress-energy tensor and ϕ are primary fields. Taking into account that the OPE of $T(z)$ with any primary field has the form

$$T(z)\phi(0) = \frac{\Delta}{z^2}\phi(0) + \frac{1}{z}\partial\phi(0) + \cdots,$$

one gets a differential equation on the correlation function $G(u,\bar{u}) = \langle \phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle$

$$\partial_u \log G(u,\bar{u}) = H(u,\bar{u}),$$

where $H(u,\bar{u})$ is the second term in the decomposition of the function $f(z,u)$ in the vicinity of u

$$f(z,u) = \frac{\Delta_2}{(z-u)^2} + \frac{1}{z-u}H(u,\bar{u}) + \cdots.$$

In the same way one gets the second equation on $G(u,\bar{u})$ by using the stress-energy tensor $\bar{T}(\bar{z})$

$$\partial_{\bar{u}} \log G(u,\bar{u}) = \bar{H}(u,\bar{u}).$$

A solution of these two equations determines the correlation function $G(u,\bar{u})$ up to a constant.

It turns out that for the correlation functions we consider the functions $H(u,\bar{u})$ and $\bar{H}(u,\bar{u})$ are holomorphic and antiholomorphic functions respectively. Therefore, the correlation function $G(u,\bar{u})$ admits a factorization $G(u,\bar{u}) = G(u)\bar{G}(\bar{u})$. Moreover, since the stress-energy tensor is a sum of the bosonic and fermionic ones, the function $G(u)$ admits further factorization $G(u) = G_b(u)G_f(u)$. Thus, the correlation function $G(u,\bar{u})$ acquires the form

$$G(u,\bar{u}) = G_b(u)G_f(u)\bar{G}_b(\bar{u})\bar{G}_f(\bar{u}), \tag{4.14}$$

where, e.g., $G_b(u)$ is a contribution of the bosonic left-moving sector to the correlation function. By using the stress-energy tensor method one can compute each multiplier on the r.h.s. of (4.14) up to a constant. Since

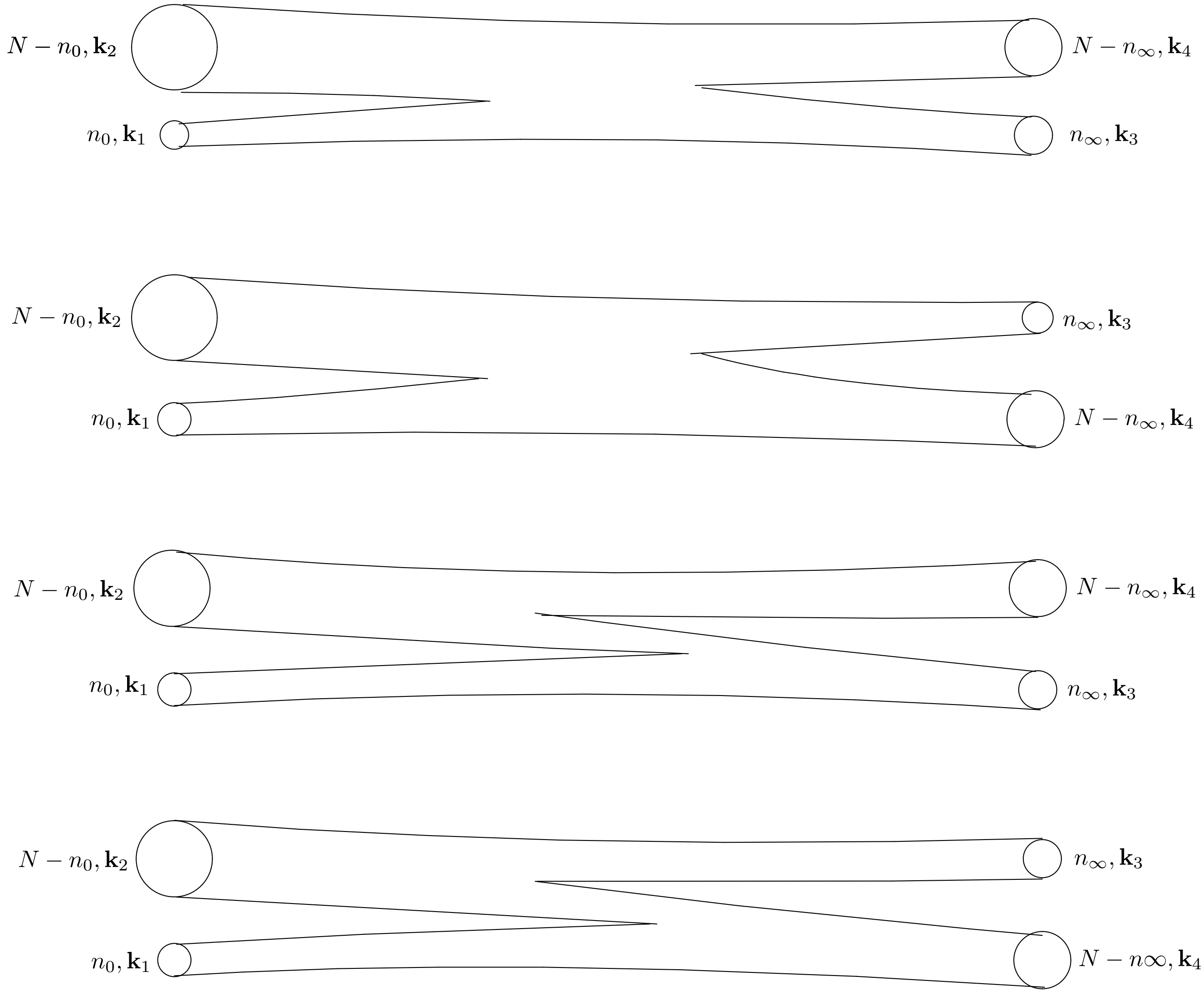

Figure 1: The diagramm representation of different correlation functions in eq.(4.32)

the vertex operators can not be represented as a tensor product of bosonic and fermionic twist fields, only the overall normalization constant for $G(u, \bar{u})$ can be found exactly. As to the individual constants, below we show that they can be determined only up to phases.

In the next two sections we calculate the contribution of the bosonic and fermionic correlation functions occuring in (4.14).

5 Bosonic correlation functions

By using the definition of the vertex operators one can see that the contribution of the bosonic left-moving sector to correlation function (4.13) is given by

$$G_b^{ij}(u) = \langle \sigma_{g_\infty}[\mathbf{k}_3/2, \mathbf{k}_4/2](\infty) \tau^i_{IJ}(1) \tau^j_{KL}(u) \sigma_{g_0}[\mathbf{k}_1/2, \mathbf{k}_2/2](0) \rangle .$$

According to the definition (3.13) of τ this correlation function can be written as the following limit

$$G_b^{ij}(u) = \lim_{z \to 1, w \to u} (z-1)^{1/2} (w-u)^{1/2} \langle \partial X^i_I(z) \partial X^j_K(w) \sigma_{g_\infty}[\mathbf{k}_3/2, \mathbf{k}_4/2](\infty) \sigma_{IJ}(1) \sigma_{KL}(u) \sigma_{g_0}[\mathbf{k}_1/2, \mathbf{k}_2/2](0) \rangle . \tag{5.1}$$

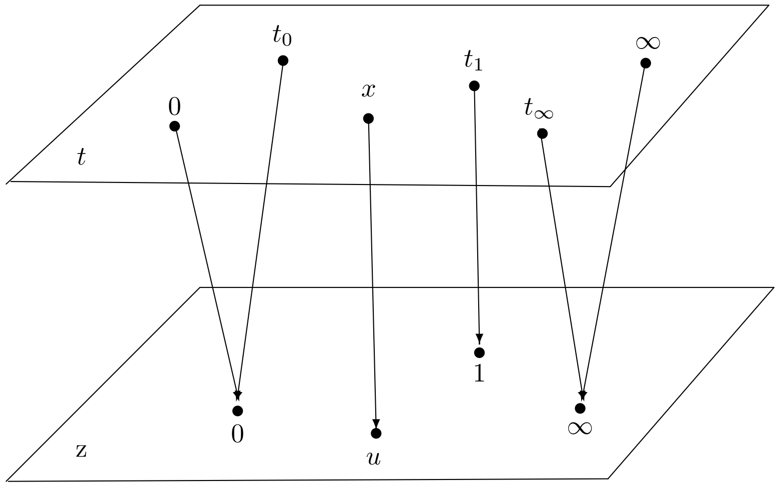

Figure 2: The N-fold covering of the z-sphere by the t-sphere.

Thus, the calculation of G_b^{ij} is reduced to calculation of the Green function

$$G_{MS}^{ij}(z,w) = \frac{\langle \partial X_M^i(z)\partial X_S^j(w)\sigma_{g_\infty}[\mathbf{k}_3/2,\mathbf{k}_4/2](\infty)\sigma_{IJ}(1)\sigma_{KL}(u)\sigma_{g_0}[\mathbf{k}_1/2,\mathbf{k}_2/2](0)\rangle}{\langle \sigma_{g_\infty}[\mathbf{k}_3/2,\mathbf{k}_4/2](\infty)\sigma_{IJ}(1)\sigma_{KL}(u)\sigma_{g_0}[\mathbf{k}_1/2,\mathbf{k}_2/2](0)\rangle}$$
$$\equiv \langle\langle \partial X_M^i(z)\partial X_S^j(w)\rangle\rangle.$$

and the correlation function

$$G_b(u) = \langle \sigma_{g_\infty}[\mathbf{k}_3/2,\mathbf{k}_4/2](\infty)\sigma_{IJ}(1)\sigma_{KL}(u)\sigma_{g_0}[\mathbf{k}_1/2,\mathbf{k}_2/2](0)\rangle. \tag{5.2}$$

We start with considering the more general correlation function

$$G(u) = \langle \sigma_{g_\infty}[\mathbf{p}_3,\mathbf{p}_4](\infty)\sigma[\mathbf{p}_5]_{IJ}(1)\sigma[\mathbf{p}_6]_{KL}(u)\sigma_{g_0}[\mathbf{p}_1,\mathbf{p}_2](0)\rangle, \tag{5.3}$$

and the corresponding Green function

$$G_{MS}^{ij}(z,w) = \frac{\langle \partial X_M^i(z)\partial X_S^j(w)\sigma_{g_\infty}[\mathbf{p}_3,\mathbf{p}_4](\infty)\sigma[\mathbf{p}_5]_{IJ}(1)\sigma[\mathbf{p}_6]_{KL}(u)\sigma_{g_0}[\mathbf{p}_1,\mathbf{p}_2](0)\rangle}{\langle \sigma_{g_\infty}[\mathbf{p}_3,\mathbf{p}_4](\infty)\sigma[\mathbf{p}_5]_{IJ}(1)\sigma[\mathbf{p}_6]_{KL}(u)\sigma_{g_0}[\mathbf{p}_1,\mathbf{p}_2](0)\rangle},$$

for D bosonic fields and arbitrary momenta $\mathbf{p}_5$ and $\mathbf{p}_6$ keeping in mind application to calculation of the fermionic correlation functions.

These Green functions have non-trivial monodromies around points $\infty, 1, u$ and 0, and, in fact, are different branches of one multi-valued function. However, this function is single-valued on the sphere that is obtained by gluing the fields X_I^i at $z=0$ and $z=\infty$. Thus to construct $G_{MS}^{ij}(z,w)$ we introduce the following map from this sphere onto the original one:

$$z = \left(\frac{t}{t_1}\right)^{n_0}\left(\frac{t-t_0}{t_1-t_0}\right)^{N-n_0}\left(\frac{t_1-t_\infty}{t-t_\infty}\right)^{N-n_\infty} \equiv u(t). \tag{5.4}$$

Here the points $t=0$ and $t=t_0$ are mapped to the point $z=0$; $t=\infty$, $t=t_\infty \to z=\infty$, $t=t_1 \to z=1$ and $t=x \to z=u$ (see Fig.2). In what follows we often use the notation Ω_A to refer to the set of the branch points: $\Omega_1=0$, $\Omega_2=t_0$, $\Omega_3=\infty$, $\Omega_4=t_\infty$, $\Omega_5=t_1$ and $\Omega_6=x$.

The map (5.4) may be viewed as the N-fold covering of the z-plain by the t-sphere on which the Green function is single-valued. The more detailed discussion of eq.(5.4) is presented in the Appendix B.

Due to the projective transformations, the positions of the points t_0, t_∞, t_1 depend on x and it is convinient to choose this dependence as follows

$$\begin{aligned} t_0 &= x-1, \\ t_\infty &= x - \frac{(N-n_\infty)x}{(N-n_0)x+n_0}, \\ t_1 &= \frac{N-n_0-n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty((N-n_0)x+n_0)}. \end{aligned}$$

This choice leads to the following expression for the rational function $u(x)$

$$\begin{aligned} u = u(x) &= (n_0-n_\infty)^{n_0-n_\infty} \frac{n_\infty^{n_\infty}}{n_0^{n_0}} \left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty} \left(\frac{x+\frac{n_0}{N-n_0}}{x-1}\right)^N \\ &\times \left(\frac{x-\frac{N-n_0-n_\infty}{N-n_0}}{x}\right)^{N-n_0-n_\infty} \left(x - \frac{n_0}{n_0-n_\infty}\right)^{n_0-n_\infty}. \end{aligned} \tag{5.5}$$

Since $n_0 < n_\infty$, the map $u(x)$ can be treated as the $2(N-n_0)$-fold covering of the u-sphere by the x-sphere, that means that an equation $u(x) = u$ has $2(N-n_0)$ different solutions. It is worthwhile to note that this number coincides with the number of nontrivial correlation functions in eq.(4.12) and, therefore different roots of eq.(5.5) correspond to different correlation functions (4.12). We see that the t-sphere can be represented as the union of $2(N-n_0)$ domains, and each domain V_{IJKL} contains the points x corresponding to the correlation function (4.13). If we take on the u-plain the appropriate system of cuts, then every root of eq.(5.5) realizes a one-to-one conformal mapping of the cut u-plain onto the corresponding domain V_{IJKL}.

Let us now choose some root of eq.(5.5). One can always cut the z-sphere and numerate the roots $t_R(z)$ of eq.(5.4) in such a way that they have the same monodromies as the fields X do. Then the Green functions are obviously not equal to zero only if the momentum conservation law $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4+\mathbf{p}_5+\mathbf{p}_6 = 0$ is fulfilled, and are given by

$$G^{ij}_{MS}(z,w) = -\delta^{ij}\frac{t'_M(z)t'_S(w)}{(t_M(z)-t_S(w))^2} - \sum_{AB} \frac{p^i_A p^j_B t'_M(z)t'_S(w)}{(t_M(z)-\Omega_A)(t_S(w)-\Omega_B)}. \tag{5.6}$$

Since $\Omega_3 = \infty$ the corresponding terms in the sum (5.6) are absent.

One can easily check that these functions have the singularity $-\frac{\delta^{ij}\delta_{MS}}{(z-w)^2}$ in the vicinity $z-w=0$ and proper monodromies around the points $z = \infty, 1, u, 0$.

Recall that the stress-energy tensor is defined as

$$T(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{N}\left(\partial X^i_I(z)\partial X^i_I(w) + \frac{1}{(z-w)^2}\right).$$

By using this definition and eq.(5.6), one gets [6]

$$\langle\langle T(z)\rangle\rangle = \sum_M \frac{D}{12}\left(\left(\frac{t''_M(z)}{t'_M(z)}\right)' - \frac{1}{2}\left(\frac{t''_M(z)}{t'_M(z)}\right)^2\right) + \sum_{AB,M}\frac{\mathbf{p}_A\mathbf{p}_B(t'_M(z))^2}{2(t_M(z)-\Omega_A)(t_M(z)-\Omega_B)}.$$

The term

$$\left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2 = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2$$

is the Schwartz derivative as one could expect from the very beginning. To get the differential equation on the correlation function (5.3) one should expand $\langle\langle T(z)\rangle\rangle$ in the vicinity of $z = u$. This expansion is given by

$$\langle\langle T(z)\rangle\rangle = \frac{\mathbf{p}_6^2 + D/4}{4}\frac{1}{(z-u)^2} + \frac{1}{u(z-u)}\left(-\frac{\mathbf{p}_6^2+D/4}{4} + \frac{9\mathbf{p}_6^2+3D/2}{16}\frac{a_1^2}{a_0^3} - \frac{\mathbf{p}_6^2+D/4}{2}\frac{a_2}{a_0^2}\right. \tag{5.7}$$

[6] If all $\mathbf{p}_i = 0$, the expectation value of $T(z)$ in the presence of twist fields can be equivalently found by mapping with $t_M(z)$ the stress-energy tensor on the t-sphere onto the z-sphere with the subsequent summation over M (see e.g.[16])

$$+\sum_{A,B=1}^{5}\frac{\mathbf{p}_A\mathbf{p}_B}{4a_0(x-\Omega_A)(x-\Omega_B)}-\sum_{A=1}^{5}\frac{\mathbf{p}_6\mathbf{p}_A}{2a_0(x-\Omega_A)^2}\left(1+\frac{3a_1(x-\Omega_A)}{a_0}\right)\Bigg)+\cdots.$$

Here the coefficients a_k are defined as follows

$$a_k=\frac{(-1)^{k-1}}{k+2}\left(\frac{n_0}{x^{k+2}}+\frac{N-n_0}{(x-t_0)^{k+2}}-\frac{N-n_\infty}{(x-t_\infty)^{k+2}}\right).$$

The first term shows that the conformal dimension of the twist field $\sigma_{KL}[\mathbf{p}_6]$ is equal to $\frac{D}{16}+\frac{\mathbf{p}_6^2}{4}$, as it should be, and the other terms lead to the following differential equation on $G(u)$

$$\begin{aligned} u\partial_u\log G(u) &= -\frac{\mathbf{p}_6^2+D/4}{4}+\frac{9\mathbf{p}_6^2+3D/2}{16}\frac{a_1^2}{a_0^3}-\frac{\mathbf{p}_6^2+D/4}{2}\frac{a_2}{a_0^2} \\ &+\sum_{A,B=1}^{5}\frac{\mathbf{p}_A\mathbf{p}_B}{4a_0(x-\Omega_A)(x-\Omega_B)}-\sum_{A=1}^{5}\frac{\mathbf{p}_6\mathbf{p}_A}{2a_0(x-\Omega_A)^2}\left(1+\frac{3a_1(x-\Omega_A)}{a_0}\right). \end{aligned} \quad (5.8)$$

It is useful to make the change of variables $u\to u(x)$. Then, performing simple but tedious calculations which are outlined in Appendix B, one obtains the following differential equation on $G(u)$

$$\begin{aligned} \partial_x\log G(u(x)) &= -\left(\frac{D}{16}+\frac{\mathbf{p}_6^2}{4}\right)\frac{d}{dx}\log u+\frac{d_0}{x}+\frac{d_1}{x-1}+\frac{d_2}{x+\frac{n_0}{N-n_0}} \\ &+\frac{d_3}{x-\frac{N-n_0-n_\infty}{N-n_0}}+\frac{d_4}{x-\frac{n_0}{n_0-n_\infty}}-d_5\left(\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}\right). \end{aligned} \quad (5.9)$$

Here

$$\alpha_i=\frac{n_0}{n_0-n_\infty}+(-1)^i\sqrt{\frac{n_0n_\infty(N-n_\infty)}{(n_0-n_\infty)^2(N-n_0)}}$$

are roots of the equation $x^2a_0=0$ and the coefficients d_i are given by the following formulas

$$\begin{aligned} d_0 &= \frac{D}{24}\left(1-\frac{N-n_\infty}{n_0}-\frac{n_0}{N-n_\infty}\right)+\frac{N-n_\infty}{2n_0}\mathbf{p}_1^2+\frac{n_0}{2(N-n_\infty)}\mathbf{p}_4^2+\mathbf{p}_1\mathbf{p}_4+\mathbf{p}_6\mathbf{p}_1+\mathbf{p}_6\mathbf{p}_4+\frac{\mathbf{p}_6^2}{2}, \\ d_1 &= \frac{D}{24}\left(1+\frac{N-n_\infty}{n_\infty}+\frac{n_\infty}{N-n_\infty}\right)-\frac{N-n_\infty}{2n_\infty}\mathbf{p}_3^2-\frac{n_\infty}{2(N-n_\infty)}\mathbf{p}_4^2+\mathbf{p}_6\mathbf{p}_3+\mathbf{p}_6\mathbf{p}_4+\mathbf{p}_3\mathbf{p}_4+\frac{\mathbf{p}_6^2}{2}, \\ d_2 &= \frac{D}{24}\left(1+\frac{N-n_0}{n_0}+\frac{n_0}{N-n_0}\right)-\frac{N-n_0}{2n_0}\mathbf{p}_1^2-\frac{n_0}{2(N-n_0)}\mathbf{p}_2^2+\mathbf{p}_1\mathbf{p}_2+\mathbf{p}_6\mathbf{p}_1+\mathbf{p}_6\mathbf{p}_2+\frac{\mathbf{p}_6^2}{2}, \\ d_3 &= \frac{D}{24}\left(1-\frac{n_\infty}{N-n_0}-\frac{N-n_0}{n_\infty}\right)+\frac{n_\infty}{2(N-n_0)}\mathbf{p}_2^2+\frac{N-n_0}{2n_\infty}\mathbf{p}_3^2+\mathbf{p}_6\mathbf{p}_2+\mathbf{p}_6\mathbf{p}_3+\mathbf{p}_2\mathbf{p}_3+\frac{\mathbf{p}_6^2}{2}, \\ d_4 &= \frac{D}{24}\left(1-\frac{n_0}{n_\infty}-\frac{n_\infty}{n_0}\right)+\frac{n_\infty}{2n_0}\mathbf{p}_1^2+\frac{n_0}{2n_\infty}\mathbf{p}_3^2+\mathbf{p}_6\mathbf{p}_1+\mathbf{p}_6\mathbf{p}_3+\mathbf{p}_1\mathbf{p}_3+\frac{\mathbf{p}_6^2}{2}, \\ d_5 &= \frac{D}{24}-\frac{4\mathbf{p}_5\mathbf{p}_6-\mathbf{p}_5^2-\mathbf{p}_6^2}{4}. \end{aligned} \quad (5.10)$$

Thus, with the account of the momentum conservation law the solution of eq.(5.9) is given by

$$G(u) = C(g_0,g_\infty)R^{D/2}\frac{x^{d_0}(x-1)^{d_1}\left(x+\frac{n_0}{N-n_0}\right)^{d_2}\left(x-\frac{N-n_0-n_\infty}{N-n_0}\right)^{d_3}\left(x-\frac{n_0}{n_0-n_\infty}\right)^{d_4}}{u^{\frac{D}{16}+\frac{\mathbf{p}_6^2}{4}}((x-\alpha_1)(x-\alpha_2))^{d_5}}. \quad (5.11)$$

Here $x=x(u)$ is the root of equation $u=u(x)$ that corresponds to given values of the indices I,J,K,L, and $C(g_0,g_\infty)$ is a normalization constant which does not depend on u.

Now we proceed with calculation of the correlation function $G_b^{ij}(u)$. Setting in (5.11) $D=8$, $\mathbf{p}_1=\mathbf{k}_1/2$, $\mathbf{p}_2=\mathbf{k}_2/2$, $\mathbf{p}_3=\mathbf{k}_3/2$, $\mathbf{p}_4=\mathbf{k}_4/2$ and $\mathbf{p}_5=\mathbf{p}_6=0$ one obtains the bosonic correlation function $G_b(u)$ (5.2) up to the normalization constant $C(g_0,g_\infty)$. Then, according to (5.1), in the limit $z\to 1$ and $w\to u$ we find

$$\lim_{z\to 1, w\to u}(z-1)^{1/2}(w-u)^{1/2}G_{IK}^{ij}(z,w)=\frac{1}{4u^{1/2}(a_0(t_1)a_0)^{1/2}}\left(-\delta^{ij}\frac{1}{(t_1-x)^2}-\sum_{AB}\frac{k_A^i k_B^j}{4(t_1-\Omega_A)(x-\Omega_B)}\right),$$

where

$$a_0(t_1)=-\frac{1}{2}\left(\frac{n_0}{t_1^2}+\frac{N-n_0}{(t_1-t_0)^2}-\frac{N-n_\infty}{(t_1-t_\infty)^2}\right).$$

Taking into account that

$$a_0(t_1)=-\frac{n_\infty^3(N-n_\infty)}{n_0(N-n_0)(n_0-n_\infty)^2}\frac{\left(x+\frac{n_0}{N-n_0}\right)^2x^2}{(x-1)^2\left(x-\frac{N-n_0-n_\infty}{N-n_0}\right)^2\left(x-\frac{n_0}{n_0-n_\infty}\right)^2}a_0,$$

and performing simple calculations one obtains

$$G_b^{ij}(u)=\frac{i}{8}\left(\frac{n_\infty n_0(N-n_\infty)}{(N-n_0)}\right)^{1/2}\frac{\langle\tau_i\tau_j\rangle}{(n_\infty-n_0)u^{1/2}(x-\alpha_1)(x-\alpha_2)}G_b(u),$$

where we introduced a concise notation

$$\begin{aligned}\langle\tau_i\tau_j\rangle=&-\delta^{ij}\frac{4x(x-1)(x+\frac{n_0}{N-n_0})(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})}{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}\\ &-\left(\frac{(x+\frac{n_0}{N-n_0})}{n_0}k_1^i+\frac{(x-\frac{N-n_0-n_\infty}{N-n_0})}{n_\infty}k_3^i+\frac{1}{N-n_0}k_4^i\right)\left((x-1)k_1^j+xk_3^j+\frac{n_0-n_\infty}{N-n_\infty}(x-\frac{n_0}{n_0-n_\infty})k_4^j\right).\end{aligned}\tag{5.12}$$

Thus, we determined the correlation function $G_b^{ij}(u)$ up to the constant occuring in $G_b(u)$. However, we are only interested in the overall normalization constant for correlation function (4.13). To determine this constant we are in need of the structure constants in the OPE (3.1). As will be shown later these structure constants can be expressed through the corresponding constants for the bosonic and fermionic twist fields. The latter can be extracted from an auxiliary correlation function

$$G_0(u)=\langle\sigma_{g_0^{-1}}[-\mathbf{p}_1,-\mathbf{p}_2](\infty)\sigma_{IJ}[-\mathbf{p}](1)\sigma_{IJ}[\mathbf{p}](u)\sigma_{g_0}[\mathbf{p}_1,\mathbf{p}_2](0)\rangle,\tag{5.13}$$

where $I=1,\dots,n_0$, $J=n_0+1,\dots,N$.
Let us note that by using the action of C_{g_0} one can fix $I=n_0$, $J=N$. This correlation function corresponds to the case $n_\infty=n_0$ and the rational function $u(x)$ is equal to

$$u(x)=\left(1+\frac{2n_0-N}{N-n_0}\frac{1}{x}\right)^{N-2n_0}\left(\frac{1+\frac{n_0}{N-n_0}\frac{1}{x}}{1-\frac{1}{x}}\right)^N.\tag{5.14}$$

The root of eq.(5.14) that corresponds to the correlation function (5.13) behaves as

$$\frac{1}{x}=\frac{1}{4n_0}(u-1)+o(u-1),\qquad \text{when}\quad u\to 1.\tag{5.15}$$

The following expression for the correlation function $G_0(u)$ can be derived from eq.(5.11) in the limit $n_\infty\to n_0$

$$G_0(u)\quad=\quad C(g_0)R^{D/2}\frac{x^{d_0}(x-1)^{d_1}\left(x+\frac{n_0}{N-n_0}\right)^{d_2}\left(x-\frac{N-2n_0}{N-n_0}\right)^{d_3}}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}}\left(x-\frac{N-2n_0}{2(N-n_0)}\right)^{d_5}},\tag{5.16}$$

where the coefficients d_i are given by eq.(5.10) with the obvious substitution $n_\infty \to n_0$, $\mathbf{p}_3 = -\mathbf{p}_1$, $\mathbf{p}_4 = -\mathbf{p}_2$, and $\mathbf{p}_6 = -\mathbf{p}_5 = \mathbf{p}$.

Taking into account the OPE

$$\sigma_{IJ}[-\mathbf{p}](1)\sigma_{IJ}[\mathbf{p}](u) = \frac{R^{-D/2}}{(1-u)^{\frac{D}{8}+\frac{\mathbf{p}^2}{2}}} + \cdots,$$

and the normalization (3.15) of two-point correlation functions, one gets

$$G_0(u) \to \frac{R^{-D/2}}{(1-u)^{\frac{D}{8}+\frac{\mathbf{p}^2}{2}}}. \tag{5.17}$$

From the other side by using eqs.(5.15) and (5.16), one derives in the limit $u \to 1$

$$G_0(u) \to C(g_0)R^{D/2}\left(\frac{1}{4n_0}(u-1)\right)^{-(d_0+d_1+d_2+d_3-d_5)} = \frac{R^{D/2}}{(u-1)^{\frac{D}{8}+\frac{\mathbf{p}^2}{2}}}C(g_0)(4n_0)^{\frac{D}{8}+\frac{\mathbf{p}^2}{2}}. \tag{5.18}$$

Comparing eqs.(5.17) and (5.18), one finds the normalization constant

$$C(g_0) = (-4n_0)^{-\frac{D}{8}-\frac{\mathbf{p}^2}{2}}.$$

Let us now consider the limit $u \to 0$. Taking into account the OPE

$$\begin{aligned}\sigma_{n_0N}[\mathbf{p}](u)\sigma_{g_0}[\mathbf{p}_1,\mathbf{p}_2](0) &= \frac{C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}+\Delta_{g_0}[\mathbf{p}_1,\mathbf{p}_2]-\Delta_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}]}}\sigma_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}](0) \\ &+ \frac{C^{g_0g_{n_0N}}_{n_0N,g_0}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}+\Delta_{g_0}[\mathbf{p}_1,\mathbf{p}_2]-\Delta_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}]}}\sigma_{g_0g_{n_0N}}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}](0) + \cdots,\end{aligned} \tag{5.19}$$

one obtains

$$\begin{aligned}&G_0(u) \to \\ &\frac{C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}+\Delta_{g_0}[\mathbf{p}_1,\mathbf{p}_2]-\Delta_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}]}}\langle\sigma_{g_0^{-1}}[-\mathbf{p}_1,-\mathbf{p}_2](\infty)\sigma_{n_0N}[-\mathbf{p}](1)\sigma_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}](0)\rangle \\ &+\frac{C^{g_0g_{n_0N}}_{n_0N,g_0}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}+\Delta_{g_0}[\mathbf{p}_1,\mathbf{p}_2]-\Delta_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}]}}\langle\sigma_{g_0^{-1}}[-\mathbf{p}_1,-\mathbf{p}_2](\infty)\sigma_{n_0N}[-\mathbf{p}](1)\sigma_{g_0g_{n_0N}}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}](0)\rangle.\end{aligned}$$

By using the global S_N invariance and the obvious symmetry property of the structure constant

$$C^{g_{n_0N}g_0}_{n_0N,g_0}(-\mathbf{p}_1,-\mathbf{p}_2,-\mathbf{p}) = C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})$$

it is not difficult to show that the correlation functions $\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle$ and $\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_0g_{n_0N}}\rangle$ are equal to $R^{D/2}C^{g_0g_{n_0N}}_{n_0N,g_0}$ and $R^{D/2}C^{g_{n_0N}g_0}_{n_0N,g_0}$ respectively:

$$\begin{aligned}\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle = \langle\sigma_{g_{n_0N}g_0}\sigma_{n_0N}\sigma_{g_0^{-1}}\rangle = \langle\sigma_{g_{n_0N}g_0^{-1}}\sigma_{n_0N}\sigma_{g_0}\rangle = R^{D/2}C^{g_0g_{n_0N}}_{n_0N,g_0} \\ \langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_0g_{n_0N}}\rangle = \langle\sigma_{g_0g_{n_0N}}\sigma_{n_0N}\sigma_{g_0^{-1}}\rangle = \langle\sigma_{g_0^{-1}g_{n_0N}}\sigma_{n_0N}\sigma_{g_0}\rangle = R^{D/2}C^{g_{n_0N}g_0}_{n_0N,g_0}.\end{aligned} \tag{5.20}$$

Moreover, the structure constants $C^{g_0g_{n_0N}}_{n_0N,g_0}$ and $C^{g_{n_0N}g_0}_{n_0N,g_0}$ are complex-conjugated to each other. Indeed, it follows from the normalization of the two-point correlation functions of the twist fields $\sigma_g[\{\mathbf{k}_\alpha\}](z)$ that the conjugate operator $(\sigma_g[\{\mathbf{k}_\alpha\}](z))^\dagger$ is given by

$$(\sigma_g[\{\mathbf{k}_\alpha\}](z))^\dagger = z^{-2\Delta_g[\{\mathbf{k}_\alpha\}]}\sigma_{g^{-1}}[\{-\mathbf{k}_\alpha\}]\left(\frac{1}{z}\right). \tag{5.21}$$

Then, from eqs.(5.20) and (5.21) one gets

$$(R^{D/2}C^{g_0 g_{n_0N}}_{n_0N,g_0})^* = \langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle^* = \langle\sigma^\dagger_{g_{n_0N}g_0}\sigma^\dagger_{n_0N}\sigma^\dagger_{g_0^{-1}}\rangle = \langle\sigma_{g_0^{-1}g_{n_0N}}\sigma_{n_0N}\sigma_{g_0}\rangle = R^{D/2}C^{g_{n_0N}g_0}_{n_0N,g_0}.$$

Thus, the correlation function $G_0(u)$ in the limit $u\to 0$ is expressed through the structure constant

$$C(n_0,\mathbf{p}_1;N-n_0,\mathbf{p}_2;\mathbf{p}) \equiv C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})$$

as follows

$$G_0(u)\to \frac{2R^{D/2}|C(n_0,\mathbf{p}_1;N-n_0,\mathbf{p}_2;\mathbf{p})|^2}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}+\Delta_{g_0}[\mathbf{p}_1,\mathbf{p}_2]-\Delta_{g_{n_0N}g_0}[\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}]}}. \tag{5.22}$$

On the other hand, taking into account that in the limit $u\to 0$ the root $x(u)$ behaves as

$$x+\frac{n_0}{N-n_0}\to -Nn_0^{\frac{N-2n_0}{N}}(N-n_0)^{\frac{2n_0-2N}{N}}u^{\frac{1}{N}},$$

one gets from eq.(5.16)

$$G_0(u,\bar u)\to 2^{\frac12\mathbf{p}^2-\frac{5}{24}D}\frac{R^{D/2}}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}-\frac{1}{N}d_2}}\frac{(N-n_0)^{-\frac{D}{24}+2\frac{n_0-N}{N}d_2+2\mathbf{p}\mathbf{p}_2+\frac12\mathbf{p}^2}n_0^{-\frac{D}{24}-2\frac{n_0}{N}d_2+2\mathbf{p}\mathbf{p}_1+\frac12\mathbf{p}^2}}{N^{\frac{D}{24}-2d_2+2\mathbf{p}(\mathbf{p}_1+\mathbf{p}_2)+\frac32\mathbf{p}^2}}. \tag{5.23}$$

Comparing eqs.(5.22) and (5.23), one obtains the following expression for the modulus of the structure constant

$$|C(n_0,\mathbf{p}_1;N-n_0,\mathbf{p}_2;\mathbf{p})| = 2^{-\frac{5D+24}{48}+\frac14\mathbf{p}^2}\frac{(N-n_0)^{-\frac{D}{48}+\frac{n_0-N}{N}d_2+\mathbf{p}\mathbf{p}_2+\frac14\mathbf{p}^2}n_0^{-\frac{D}{48}-\frac{n_0}{N}d_2+\mathbf{p}\mathbf{p}_1+\frac14\mathbf{p}^2}}{N^{\frac{D}{48}-d_2+\mathbf{p}(\mathbf{p}_1+\mathbf{p}_2)+\frac34\mathbf{p}^2}}, \tag{5.24}$$

where

$$\begin{aligned} d_2 &\equiv d_2(n_0,\mathbf{p}_1;N-n_0,\mathbf{p}_2;\mathbf{p}) \\ &= \frac{D}{24}\left(1+\frac{N-n_0}{n_0}+\frac{n_0}{N-n_0}\right)-\frac{N-n_0}{2n_0}\mathbf{p}_1^2-\frac{n_0}{2(N-n_0)}\mathbf{p}_2^2+\mathbf{p}_1\mathbf{p}_2+\mathbf{p}\mathbf{p}_1+\mathbf{p}\mathbf{p}_2+\frac{\mathbf{p}^2}{2}. \end{aligned}$$

Note that the phase of the structure constant remains to be undetermined.

It is now not difficult to express any three-point correlation function of the form $\langle\sigma_{g^{-1}g_{IJ}}\sigma_{IJ}\sigma_g\rangle$ through the structure constant $C(n,\mathbf{k};m,\mathbf{q};\mathbf{p})$. Recall that any twist field $\sigma_g[\{\mathbf{k}_\alpha\}]$ has the following decomposition into the product of the twist fields $\sigma_{(n)}[\mathbf{k}]$

$$\sigma_g[\{\mathbf{k}_\alpha\}] = \prod_{\alpha=1}^{N_{str}}\sigma_{(n_\alpha)}[\mathbf{k}_\alpha], \tag{5.25}$$

where the element g has the decomposition $(n_1)(n_2)\cdots(n_{N_{str}})$.
Then, due to eq.(5.20), the structure constant $C(n,\mathbf{k};m,\mathbf{q};\mathbf{p})$ with arbitrary n and m is equal to

$$C(n,\mathbf{k};m,\mathbf{q};\mathbf{p}) = R^{-D/2}\langle\sigma_{(-n-m)}[-\mathbf{k}-\mathbf{q}-\mathbf{p}](\infty)\sigma_{IJ}[\mathbf{p}](1)\sigma_{(n)}[\mathbf{k}]\sigma_{(m)}[\mathbf{q}](0)\rangle, \tag{5.26}$$

where $I\in(n)$ and $J\in(m)$.
By using eqs.(5.25) and (5.26), one can easily get the following expression for the three-point correlation function

$$\begin{aligned} &\langle\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](\infty)\sigma_{IJ}[\mathbf{p}](1)\sigma_g[\{\mathbf{k}_\alpha\}](0)\rangle = \langle\sigma_g[\{\mathbf{k}_\alpha\}](\infty)\sigma_{IJ}[\mathbf{p}](1)\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0)\rangle \\ &= \langle\sigma_{(-n_1-n_2)}[\mathbf{q}]\prod_{\alpha=3}^{N_{str}}\sigma_{(-n_\alpha)}[\mathbf{q}_\alpha](\infty)\sigma_{IJ}[\mathbf{p}](1)\sigma_{(n_1)}[\mathbf{k}_1]\sigma_{(n_2)}[\mathbf{k}_2]\prod_{\alpha=3}^{N_{str}}\sigma_{(n_\alpha)}[\mathbf{k}_\alpha](0)\rangle \\ &= \prod_{\alpha=3}^{N_{str}}R^{-D/2}\delta^D_R(\mathbf{q}_\alpha+\mathbf{k}_\alpha)\langle\sigma_{(-n_1-n_2)}[\mathbf{q}](\infty)\sigma_{IJ}[\mathbf{p}](1)\sigma_{(n_1)}[\mathbf{k}_1]\sigma_{(n_2)}[\mathbf{k}_2](0)\rangle \\ &= C(n_1,\mathbf{k}_1;n_2,\mathbf{k}_2;\mathbf{p})R^{-D/2}\delta^D_R(\mathbf{q}+\mathbf{k}_1+\mathbf{k}_2+\mathbf{p})\prod_{\alpha=3}^{N_{str}}R^{-D/2}\delta^D_R(\mathbf{q}_\alpha+\mathbf{k}_\alpha), \end{aligned}$$

where $I \in (n_1)$ and $J \in (n_2)$.
It is now clear that the structure constant $C^{g_{IJ}g}_{IJ,g}$ in the OPE of σ_{IJ} and σ_g is just equal to $C(n_1, \mathbf{k}_1; n_2, \mathbf{k}_2; \mathbf{p})$, and that the structure constant $C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}$ (which is complex conjugated to $C^{g_{IJ}g^{-1}g_{IJ}}_{IJ,g^{-1}g_{IJ}}$ due to eq.(5.21)) in the OPE

$$\sigma_{IJ}[\mathbf{p}](u)\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0) = \sum_{\mathbf{q}_1,\mathbf{q}_2} \frac{\delta_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}-\mathbf{p},0}}{u^{\frac{D}{16}+\frac{\mathbf{p}^2}{4}+\Delta_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}]-\Delta_g[\{\mathbf{q}_\alpha\}]}} \tag{5.27}$$
$$\times\left(C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})\sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}](0)+ \overset{*}{C}{}^{g^{-1}}_{IJ,g^{-1}g_{IJ}}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})\sigma_{g_{IJ}g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0)\right)+\cdots$$

is equal to

$$C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}) = R^{-D/2}C(n_1,\mathbf{q}_1;n_2,\mathbf{q}_2;\mathbf{p}).$$

In particular, the structure constants $C^{g_{n_\infty}Ng_0}_{n_\infty N,g_0}$ and $C^{g_{n_0+n_\infty},Ng_0}_{n_0+n_\infty,N;g_0}$, which will be used to find the overall normalization constant for correlation function (4.13) are given by

$$\begin{aligned} C^{g_{n_\infty}Ng_0}_{n_\infty N,g_0}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p}) &= R^{-D/2}C(n_\infty - n_0,\mathbf{k}_1;N-n_\infty,\mathbf{k}_2;\mathbf{p}), \\ C^{g_{n_0+n_\infty},Ng_0}_{n_0+n_\infty,N;g_0}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p}) &= R^{-D/2}C(N-n_\infty-n_0,\mathbf{k}_1;n_\infty,\mathbf{k}_2;\mathbf{p}). \end{aligned} \tag{5.28}$$

6 Fermionic correlation functions

To find the contribution of the left-moving fermions to the graviton scattering amplitude one has to compute the following correlation function of four fermion twist fields:

$$\langle \Sigma^{i_3i_4}_{g_\infty}(\infty)\Sigma^i_{IJ}(1)\Sigma^j_{KL}(u)\Sigma^{i_1i_2}_{g_0}(0)\rangle. \tag{6.1}$$

Computation of (6.1) will be again based on the stress-energy tensor method and the conformal map (5.4). Instead of N fermions $\theta_I(z)$ on the z-sphere, obeying twisted boundary conditions around the points $0, 1, \infty$ and u, on the t-sphere one has one fermion $\theta(t)$ with the Ramond boundary condition around the points Ω_A.

It is well known that the Ramond fermions are created from the NS sector by the standard spin fields (see, e.g., [17]). The simplest way to deal with correlation functions of the spin fields is to bosonize the fermions. To this end it is convinient to use the $SU(4)\times U(1)$ formalism [18]. Recall that with respect to the $SU(4)\times U(1)$ subgroup representations $\mathbf{8_V}$, $\mathbf{8_S}$ and $\mathbf{8_C}$ are decomposed as

$$\mathbf{8_S} \to \mathbf{4_{1/2}} + \mathbf{\bar{4}_{-1/2}}, \qquad \mathbf{8_C} \to \mathbf{4_{-1/2}} + \mathbf{\bar{4}_{1/2}}, \qquad \mathbf{8_V} \to \mathbf{6_0} + \mathbf{1_1} + \mathbf{1_{-1}}.$$

The corresponding basis for the fermions θ^a and their spin fields $\Sigma^{\dot a}$ and Σ^i consistent with this decomposition is given by

$$\begin{aligned} \Theta^A &= \frac{1}{\sqrt{2}}(\theta^A + i\theta^{A+4}), &\qquad \Theta^{\bar A} &= \frac{1}{\sqrt{2}}(\theta^A - i\theta^{A+4}), \\ \mathcal{S}^{\dot A} &= \frac{1}{\sqrt{2}}(\Sigma^{\dot A} + i\Sigma^{\dot A+4}), &\qquad \mathcal{S}^{\dot{\bar A}} &= \frac{1}{\sqrt{2}}(\Sigma^{\dot A} - i\Sigma^{\dot A+4}), \\ S^A &= \frac{1}{\sqrt{2}}(\Sigma^{2A-1} + i\Sigma^{2A}), &\qquad S^{\bar A} &= \frac{1}{\sqrt{2}}(\Sigma^{2A-1} - i\Sigma^{2A}), \end{aligned}$$

where $A = 1,\ldots,4$. Note that the spin fields Σ^4 and $\Sigma^{\bar 4}$ transform as $\mathbf{1_1}$ and $\mathbf{1_{-1}}$ respectively.

As usual, bosonization of the fermions and their twist fields up to cocycles is realized in terms of four bosonic fields ϕ^A as

$$\Theta^A = \mathrm{e}^{iq^A_B\phi^B}, \qquad \mathcal{S}^{\dot A} = \mathrm{e}^{iq^{\dot A}_B\phi^B}, \qquad S^A = \mathrm{e}^{i\phi^A},$$

where the weights of the spinor representations $\mathbf{8_S}$ and $\mathbf{8_C}$ are given by

$$\mathbf{q}^1 = \frac{1}{2}(-1,-1,1,1); \quad \mathbf{q}^2 = \frac{1}{2}(-1,1,-1,1); \quad \mathbf{q}^3 = \frac{1}{2}(1,-1,-1,1); \quad \mathbf{q}^4 = \frac{1}{2}(1,1,1,1);$$
$$\mathbf{q}^{\dot 1} = \frac{1}{2}(-1,1,1,1); \quad \mathbf{q}^{\dot 2} = \frac{1}{2}(-1,-1,-1,1); \quad \mathbf{q}^{\dot 3} = \frac{1}{2}(1,1,-1,1); \quad \mathbf{q}^{\dot 4} = \frac{1}{2}(1,-1,1,1).$$

The Cartan generators of $SU(4)\times U(1)$ in the bosonized form look as $H^A = i\partial\phi^A$.

Clearly, bosonization of the fermions of the orbifold model is achieved by introducing $4N$ bosonic fields and reads as

$$\Theta_I^A(z) = \mathrm{e}^{\,iq_B^A\phi_I^B(z)}.$$

Twist fields σ_g creating twisted sectors for the fields $\phi_I^A(z)$ are introduced in the same manner as in Sec.3 with the only exception that now they have the unit norm. Since, $\sigma_{(n)}$ on the z-sphere corresponds to the identity operator on the t-sphere, it is natural to assume that the spin twist fields of the orbifold model can be realized as

$$\begin{aligned} \mathcal{S}_{(n)}^{\dot A}(z) &= \mathrm{e}^{\,\frac{i}{n}\sum_{I\in(n)} q_B^{\dot A}\phi_I^B(z)}\sigma_{(n)}(z) = \sigma_{(n)}[\mathbf{q}^{\dot A}](z), \\ S_{(n)}^{A}(z) &= \mathrm{e}^{\,\frac{i}{n}\sum_{I\in(n)} \phi_I^A(z)}\sigma_{(n)}(z) = \sigma_{(n)}[\mathbf{e}^{A}](z), \end{aligned} \tag{6.2}$$

where $\mathbf{e}^A$ is a weight vector of $\mathbf{8_V}$ with components δ_B^A.

Indeed, according to (3.14), the conformal dimension of the field $\sigma_{(n)}[\mathbf{e}^A]$ is

$$\Delta_n[\mathbf{e}^A] = \frac{1}{6}\left(n - \frac{1}{n}\right) + \frac{(\mathbf{e}^A)^2}{2n}$$

and analogously for $\sigma_{(n)}[\mathbf{q}^{\dot A}]$. Since $(\mathbf{e}^A)^2 = (\mathbf{q}^{\dot A})^2 = 1$ these fields have the correct conformal dimension (3.19) of a spin twist field.

By using the bosonization rule one can now establish that the OPE of fermions with the twist fields (6.2) coincides with (3.20) up to a sign. The correct sign dependence should be restored by taking into account cocycles of fermions and twist fields. Fortunately, our method of calculation does not involve the knowledge of these cocycles.

We proceed with considering correlation function (6.1). In the $SU(4)\times U(1)$ basis correlation function (6.1) reduces to correlation functions of the form (5.3):

$$G_f(u) = \langle \sigma_{g_\infty}[\mathbf{p}_3,\mathbf{p}_4](\infty)\sigma_{IJ}[\mathbf{p}_5](1)\sigma_{KL}[\mathbf{p}_6](u)\sigma_{g_0}[\mathbf{p}_1,\mathbf{p}_2](0)\rangle, \tag{6.3}$$

where a momentum $\mathbf{p}_i$ is now some weight vector $\pm\mathbf{e}^A$. The computation of (6.3) for general values of $\mathbf{p}$ was performed in the previous Section and the answer is given by eq.(5.11). Recall that the structure constant (5.24) occuring in the OPE of twist fields was found up to a phase that may depend on $\mathbf{e}^A$.

Some comments are in order. It follows from (5.19) that the basic OPE's of the spin twist fields S_{IJ} and S_{g_0} in the bosonized form looks as

$$\begin{aligned} S_{n_0N}^A(u)S_{g_0}^{BC}(0) &= \frac{C_{n_0N,g_0}^{g_{n_0N}g_0}(\mathbf{e}^A,\mathbf{e}^B,\mathbf{e}^C)}{u^{\frac{1}{2}+\Delta_{g_0}[\mathbf{e}^B,\mathbf{e}^C]-\Delta_{g_{n_0N}g_0}[\mathbf{e}^A+\mathbf{e}^B+\mathbf{e}^C]}}\sigma_{g_{n_0N}g_0}[\mathbf{e}^A+\mathbf{e}^B+\mathbf{e}^C](0) \\ &+ \frac{C_{n_0N,g_0}^{g_0g_{n_0N}}(\mathbf{e}^A,\mathbf{e}^B,\mathbf{e}^C)}{u^{\frac{1}{2}+\Delta_{g_0}[\mathbf{e}^B,\mathbf{e}^C]-\Delta_{g_{n_0N}g_0}[\mathbf{e}^A+\mathbf{e}^B+\mathbf{e}^C]}}\sigma_{g_0g_{n_0N}}[\mathbf{e}^A+\mathbf{e}^B+\mathbf{e}^C](0)+\cdots, \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} S_{n_0N}^A(u)S_{g_0}^{\bar BC}(0) &= \frac{C_{n_0N,g_0}^{g_{n_0N}g_0}(\mathbf{e}^A,-\mathbf{e}^B,\mathbf{e}^C)}{u^{\frac{1}{2}+\Delta_{g_0}[\mathbf{e}^B,\mathbf{e}^C]-\Delta_{g_{n_0N}g_0}[\mathbf{e}^A-\mathbf{e}^B+\mathbf{e}^C]}}\sigma_{g_{n_0N}g_0}[\mathbf{e}^A-\mathbf{e}^B+\mathbf{e}^C](0) \\ &+ \frac{C_{n_0N,g_0}^{g_0g_{n_0N}}(\mathbf{e}^A,-\mathbf{e}^B,\mathbf{e}^C)}{u^{\frac{1}{2}+\Delta_{g_0}[\mathbf{e}^B,\mathbf{e}^C]-\Delta_{g_{n_0N}g_0}[\mathbf{e}^A-\mathbf{e}^B+\mathbf{e}^C]}}\sigma_{g_0g_{n_0N}}[\mathbf{e}^A-\mathbf{e}^B+\mathbf{e}^C](0)+\cdots, \end{aligned} \tag{6.5}$$

Here the conformal dimension of $\sigma_{g_0 g_{n_0 N}}[\mathbf{e}^A \pm \mathbf{e}^B + \mathbf{e}^C]$ is given by

$$\Delta_{g_{n_0 N} g_0}[\mathbf{e}^A \pm \mathbf{e}^B + \mathbf{e}^C] = \frac{N}{6} + \frac{1}{3N} + \frac{(\mathbf{e}^A \pm \mathbf{e}^B + \mathbf{e}^C)^2 - 1}{2N}.$$

Obviously, the norm of the vectors $\mathbf{e}^+ = \mathbf{e}^A + \mathbf{e}^B + \mathbf{e}^C$ and $\mathbf{e}^- = \mathbf{e}^A - \mathbf{e}^B + \mathbf{e}^C$ can be equal to $3, 5, 9$ and to $1, 3, 5$ respectively. When $(\mathbf{e}^-)^2 = 1$ the field $\sigma[\mathbf{e}^-]$ naturally corresponds to the spin twist field S^-. The appearence in (6.4), (6.5) the bosonic twist fields carrying integral momenta with non-identity norms implies the existence of new spin twist fields whose bosonic realizations (up to cocycles) are given by $\sigma_{g_{n_0 N} g_0}[\mathbf{e}^\pm]$. The origin of these fields becomes clear if one considers excited states of a given twisted sector.

To discuss the excited states it is convenient to use another basis for fermion fields ψ_I:

$$\psi_I^a(z) = \frac{1}{\sqrt{n}} \sum_{J=1}^{n} e^{-\frac{2\pi i}{n} IJ} \theta_J^a(z) = \sum_{m \in \mathbf{Z}} \theta^a_{nm-I} z^{\frac{I}{n} - m - \frac{1}{2}}$$

with the twisting property $\psi_I^a(e^{2\pi i} z) = -e^{\frac{2\pi i}{n} I} \psi_I^a(z)$.

By applying the operators ψ_I^a to the vacuum state $|\Sigma^{\dot\mu}\rangle$ of a twisted sector (n) one obtains the excited states $(\psi_I^a \Sigma^{\dot\mu})(0) = \theta^a_{-I} |\Sigma^{\dot\mu}\rangle$:

$$\psi_I^a(z) |\Sigma^{\dot\mu}\rangle = \frac{1}{z^{\frac{1}{2} - \frac{I}{n}}} \theta^a_{-I} |\Sigma^{\dot\mu}\rangle + \text{reg.}$$

The conformal dimension of the corresponding primary fields $\psi_I^a \Sigma^{\dot\mu}$ is $\Delta[\psi_I \Sigma] = \frac{n}{6} + \frac{1}{3n} + \frac{I}{n}$. In the same manner one can introduce primary fields $\psi_{I_1}^{a_1} \cdots \psi_{I_k}^{a_k} \Sigma^{\dot\mu}$ corresponding to more general excited states of a twisted sector. If $I_s + I_p \neq n$ for any s and p, then their conformal dimensions are given by

$$\Delta[\psi_{I_1} \cdots \psi_{I_k} \Sigma] = \frac{n}{6} + \frac{1}{3n} + \frac{I_1 + \cdots + I_k}{n}. \tag{6.6}$$

Let us now consider the OPE of the Z_2-twist field Σ^i with the twist field Σ^{jk} corresponding to an element g_0. The product $\Sigma^i \Sigma^{jk}$ transforms as the tensor product of three $\mathbf{8}_v$ representations of $SO(8)$ and, therefore, it can be decomposed in a set of irreducible representations, each of them realized by excited fields of a twisted sector. Thus, schematically, the first few terms of the OPE required by the $SO(8)$ symmetry read as

$$\begin{aligned} \Sigma^i(u) \cdot \Sigma^{jk}(0) \; = \; & \delta^{ij} \Sigma^k + \delta^{ij} \gamma^k_{a\dot a} \psi_I^a \Sigma^{\dot a} + \gamma^{ij}_{ab} \gamma^k_{b\dot a} \psi_I^a \Sigma^{\dot a} + \delta^{ij} \gamma^{km}_{ab} \psi_I^a \psi_J^b \Sigma^m + \gamma^{ij}_{ab} \psi_I^a \psi_J^b \Sigma^k \\ + \; & \delta^{ij} \gamma^{km}_{ab} \gamma^m_{c\dot c} \psi_I^a \psi_J^b \psi_M^c \Sigma^{\dot c} + \gamma^{ij}_{ab} \gamma^k_{c\dot c} \psi_I^a \psi_J^b \psi_M^c \Sigma^{\dot c} + \delta^{ij} \gamma^{km}_{ab} \gamma^{mp}_{cd} \psi_I^a \psi_J^b \psi_M^c \psi_N^d \Sigma^p \\ + \; & \gamma^{ij}_{ab} \gamma^{kp}_{cd} \psi_I^a \psi_J^b \psi_M^c \psi_N^d \Sigma^p + \gamma^{ip}_{ab} \gamma^{jp}_{cd} \psi_I^a \psi_J^b \psi_M^c \psi_N^d \Sigma^k + \text{cycl. perm. } (i,j,k) + \ldots . \end{aligned} \tag{6.7}$$

For the sake of simplicity we omitted here the $SO(8)$ index independent structure constants standing by each summond on the r.h.s., as well as the u-dependence. Note that in (6.7) we can take into account only nonzero indices I, J, M, N since the action of ψ_0 on a vacuum state does not produce a new excited field and, therefore, does not disturb the form of the OPE.

Comparing $\Delta_{g_{n_0 N} g_0}[\mathbf{e}^\pm]$ with (6.6) one can relate the norm of the vectors $\mathbf{e}^\pm$ with the numbers I_s of excited states in the twisted sector (N):

$$I_1 + \cdots + I_k = \frac{(\mathbf{e}^\pm)^2 - 1}{2} = \{0, 1, 2, 4\}, \tag{6.8}$$

where on the r.h.s. all possible values of the sum $I_1 + \cdots + I_k$ are indicated.

Under the $SU(4) \times U(1)$ group the γ-matrices γ^{ij} are decomposed into the matrices γ^{AB}, $\gamma^{A\bar B}$ and $\gamma^{\bar A \bar B}$. A simple analysis shows that in the $SU(4) \times U(1)$ basis the invariant OPE (6.7) acquires a form

$$\begin{aligned} S^A_{n_0 N}(u) S^{BC}_{g_0}(0) \; = \; & \frac{c_3}{u^{\Delta_2}} \gamma^{AB}_{ab} \gamma^C_{b\dot b} \psi_1^a \Sigma^{\dot b} \\ + \; & \frac{1}{u^{\Delta_3}} \left(c_5^{(1)} \gamma^{AB}_{ab} \psi_1^a \psi_1^b S^C + c_5^{(2)} \gamma^{AC}_{ab} \psi_1^a \psi_1^b S^B + c_5^{(3)} \gamma^{CB}_{ab} \psi_1^a \psi_1^b S^A \right) \\ + \; & \frac{c_9}{u^{\Delta_4}} \gamma^{As}_{[ab} \gamma^{As}_{cd]} \psi_1^a \psi_1^b \psi_1^c \psi_1^d S^A \end{aligned} \tag{6.9}$$

$$S^A_{n_0 N}(u) S^{\bar B C}_{g_0}(0) \; = \; \frac{1}{u^{\Delta_1}} \left(c_1^{(1)} \delta^{AB} S^C + c_1^{(3)} \delta^{BC} S^A \right) + \frac{c_3}{u^{\Delta_2}} \gamma^{A\bar B}_{ab} \gamma^C_{b\dot b} \psi_1^a \Sigma^{\dot b} + \frac{c_5}{u^{\Delta_3}} \gamma^{A\bar B}_{ab} \psi_1^a \psi_1^b S^A, \tag{6.10}$$

where the coefficients $\Delta_1, \ldots, \Delta_4$ are defined by the conformal symmetry and at the moment are unessential.

On the r.h.s. of eqs.(6.9) and (6.10) we have indicated only the leading singular terms of the OPE's. Now it is readily seen that they are in one to one correspondence with bosonic fields $\sigma[\mathbf{e}^+]$ arising on the r.h.s. of (6.4). Consider, for instance, eq.(6.9). Due to the facts that $\gamma^{AB} = -\gamma^{BA}$ and $\gamma^A(\gamma^A)^T = 0$ for any A, the first term in (6.9) is nonzero only if $A \neq B \neq C$. Then, according to (6.8), the operators $\gamma^{AB}_{ab}\gamma^{C}_{b\dot{b}}\psi^a_1\Sigma^{\dot{b}}$ can be identified with the fields $\sigma[\mathbf{e}^+]$ with $(\mathbf{e}^+)^2 = 3$. Next, when, e.g., $A = B \neq C$, the first nonzero term stands by the singularity $\frac{1}{u^{\Delta_3}}$ and, therefore admits the identification with $\sigma[\mathbf{e}^+]$ with $(\mathbf{e}^+)^2 = 5$. The last term in (6.9) becomes involved in the case $A = B = C$ and the corresponding operators can be regarded as $\sigma[\mathbf{e}^+]$ with $(\mathbf{e}^+)^2 = 9$. The same situation takes place for (6.10), where the three singular terms correspond to $\sigma[\mathbf{e}^-]$ with $(\mathbf{e}^-)^2 = 1, 3, 5$. Thus, the structure constants in (6.4) and (6.5) correspond to the ones in (6.9) and (6.10).

7 Normalization of four-point correlation functions

In this section we combine the results obtained in the two previous Sections and calculate the normalization constant occuring in the product of bosonic and fermionic correlation functions:

$$
\begin{aligned}
G(u) = G_b(u)G_f(u) \quad &= \quad \langle \sigma_{g_\infty}[\mathbf{k}_3/2, \mathbf{k}_4/2](\infty)\sigma_{IJ}(1)\sigma_{KL}(u)\sigma_{g_0}[\mathbf{k}_1/2, \mathbf{k}_2/2](0)\rangle \\
&\times \quad \langle \sigma_{g_\infty}[\mathbf{p}_3, \mathbf{p}_4](\infty)\sigma[\mathbf{p}_5]_{IJ}(1)\sigma[\mathbf{p}_6]_{KL}(u)\sigma_{g_0}[\mathbf{p}_1, \mathbf{p}_2](0)\rangle,
\end{aligned}
\tag{7.1}
$$

where each $\mathbf{p}_i$ coincides with some $\pm\mathbf{e}^A$. According to (5.11), one gets for $G(u)$:

$$
G(u) \quad = \quad C(g_0, g_\infty)R^4 \frac{x^{d_0}(x-1)^{d_1}\left(x + \frac{n_0}{N-n_0}\right)^{d_2}\left(x - \frac{N-n_0-n_\infty}{N-n_0}\right)^{d_3}\left(x - \frac{n_0}{n_0-n_\infty}\right)^{d_4}}{u((x-\alpha_1)(x-\alpha_2))^{d_5}}. \tag{7.2}
$$

Here the coefficients d_i are given by

$$
\begin{aligned}
d_0 &= 1 + \frac{1}{4}k_1k_4 + \mathbf{p}_1\mathbf{p}_4 + \mathbf{p}_6\mathbf{p}_1 + \mathbf{p}_6\mathbf{p}_4, \qquad d_1 = 1 + \frac{1}{4}k_3k_4 + \mathbf{p}_6\mathbf{p}_3 + \mathbf{p}_6\mathbf{p}_4 + \mathbf{p}_3\mathbf{p}_4, \\
d_2 &= 1 + \frac{1}{4}k_1k_2 + \mathbf{p}_1\mathbf{p}_2 + \mathbf{p}_6\mathbf{p}_1 + \mathbf{p}_6\mathbf{p}_2, \qquad d_3 = 1 + \frac{1}{4}k_2k_3 + \mathbf{p}_6\mathbf{p}_2 + \mathbf{p}_6\mathbf{p}_3 + \mathbf{p}_2\mathbf{p}_3, \\
d_4 &= 1 + \frac{1}{4}k_1k_3 + \mathbf{p}_6\mathbf{p}_1 + \mathbf{p}_6\mathbf{p}_3 + \mathbf{p}_1\mathbf{p}_3, \qquad d_5 = 1 - \mathbf{p}_5\mathbf{p}_6,
\end{aligned}
$$

where $k_ik_j = \mathbf{k}_i\mathbf{k}_j - \frac{1}{2}k_i^+k_j^- - \frac{1}{2}k_i^-k_j^+$.

The normalization constant $C(g_0, g_\infty)$ can be determined by factorizing $G(u)$ in the limit $u \to 0$ on three-point functions. According to eq.(5.5), $u \to 0$ in the following three cases

$$
I) \;\; x \to -\frac{n_0}{N-n_0}; \qquad II) \;\; x \to \infty; \qquad III) \;\; x \to \frac{N-n_0-n_\infty}{N-n_0}
$$

and, conversely, any root $x_M = x_M(u)$ of eq.(5.5) tends to one of these values when $u \to 0$. Evidently, these three possible asymptotics correspond to three different choices of the indices K and L in eq.(4.12).

Let us begin with the case $K = n_0$, $L = N$. By using the OPE (5.19) and the normalization (3.15) of two-point correlation functions, one gets in the limit $u \to 0$

$$
G(u) \to R^4 \frac{C(n_0, \mathbf{k}_1, \mathbf{p}_1; N-n_0, \mathbf{k}_2, \mathbf{p}_2; \mathbf{p}_6) \overset{*}{C}(n_\infty, \mathbf{k}_3, \mathbf{p}_3; N-n_\infty, \mathbf{k}_4, \mathbf{p}_4; \mathbf{p}_5)}{u^{1-\frac{d_2}{N}}}, \tag{7.3}
$$

where we have taken into account that

$$
\Delta^b_{g_0}[\mathbf{k}_1, \mathbf{k}_2] + \Delta^f_{g_0}[\mathbf{p}_1, \mathbf{p}_2] - \Delta^b_{g_{n_0 N} g_0}[\mathbf{k}_1 + \mathbf{k}_2] - \Delta^f_{g_{n_0 N} g_0}[\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_6] = -\frac{1}{N}d_2,
$$

Here $C(n_0, \mathbf{k}_1, \mathbf{p}_1; N-n_0, \mathbf{k}_2, \mathbf{p}_2; \mathbf{p})$ denotes the product of the bosonic and fermionic structure constants

$$
C(n_0, \mathbf{k}_1, \mathbf{p}_1; N-n_0, \mathbf{k}_2, \mathbf{p}_2; \mathbf{p}) = C(n_0, \mathbf{k}_1/2; N-n_0, \mathbf{k}_2/2; 0)C(n_0, \mathbf{p}_1; N-n_0, \mathbf{p}_2; \mathbf{p}).
$$

Due to (5.24) the modulus of this structure constant is equal to

$$|C(n_0,\mathbf{k}_1,\mathbf{p}_1;N-n_0,\mathbf{k}_2,\mathbf{p}_2;\mathbf{p})| = 2^{-\frac{3}{2}}\frac{(N-n_0)^{\frac{n_0-N}{N}d_2+\mathbf{p}\mathbf{p}_2}n_0^{-\frac{n_0}{N}d_2+\mathbf{p}\mathbf{p}_1}}{N^{1-d_2+\mathbf{p}(\mathbf{p}_1+\mathbf{p}_2)}},$$

where

$$\begin{aligned} d_2 &\equiv d_2(n_0,\mathbf{k}_1,\mathbf{p}_1;N-n_0,\mathbf{k}_2,\mathbf{p}_2;\mathbf{p}) = 1+\frac{1}{4}\left(\mathbf{k}_1\mathbf{k}_2-\frac{N-n_0}{2n_0}\mathbf{k}_1^2-\frac{n_0}{2(N-n_0)}\mathbf{k}_2^2\right)\\ &+ \frac{N-n_0}{2n_0}(1-\mathbf{p}_1^2)+\frac{n_0}{2(N-n_0)}(1-\mathbf{p}_2^2)+\mathbf{p}_1\mathbf{p}_2+\mathbf{p}\mathbf{p}_1+\mathbf{p}\mathbf{p}_2. \end{aligned}$$

The root $x(u)$ has the following behaviour in the vicinity of $u=0$

$$|x+\frac{n_0}{N-n_0}| \to Nn_0^{\frac{N-n_0}{N}}n_\infty^{-\frac{n_\infty}{N}}(N-n_0)^{\frac{n_0-2N}{N}}(N-n_\infty)^{\frac{n_\infty}{N}}|u|^{\frac{1}{N}}. \tag{7.4}$$

By using eqs.(7.2) and (7.4), one can easily find

$$G(u) \to \mathrm{e}^{i\varphi}\frac{C(g_0,g_\infty)}{u^{1-\frac{\mathbf{d}_2}{N}}}\frac{n_0^{d_0+d_4+\frac{N-n_0}{N}d_2-1+\mathbf{p}_5\mathbf{p}_6}N^{d_1+d_2-1+\mathbf{p}_5\mathbf{p}_6}(N-n_\infty)^{d_2+d_3+d_4-\frac{N-n_\infty}{N}d_2-1+\mathbf{p}_5\mathbf{p}_6}}{n_\infty^{\frac{n_\infty}{N}d_2}(N-n_0)^{d_0+d_1+d_3+d_4+\frac{2N-n_0}{N}d_2-2+2\mathbf{p}_5\mathbf{p}_6}(n_\infty-n_0)^{d_4-1+\mathbf{p}_5\mathbf{p}_6}}, \tag{7.5}$$

where the phase multiplier $\mathrm{e}^{i\varphi}$ depending, in particular, on a phase behaviour of the root $x(u)$ remains to be undetermined.
Comparing eqs.(7.3) and (7.5), one obtains the modulus of the normalization constant:

$$|C(g_0,g_\infty)| = 2^{-3}\frac{n_0^{\mathbf{p}_1\mathbf{p}_5}n_\infty^{\mathbf{p}_3\mathbf{p}_5}(N-n_\infty)^{\mathbf{p}_4\mathbf{p}_6}(n_\infty-n_0)^{\frac{1}{4}k_1k_3+\mathbf{p}_1\mathbf{p}_6+\mathbf{p}_3\mathbf{p}_6+\mathbf{p}_1\mathbf{p}_3+\mathbf{p}_5\mathbf{p}_6}}{(N-n_0)^{1+\frac{1}{4}k_1k_3+\mathbf{p}_1\mathbf{p}_5+\mathbf{p}_3\mathbf{p}_5+\mathbf{p}_1\mathbf{p}_3-\mathbf{p}_2\mathbf{p}_6}}. \tag{7.6}$$

Thus, we have found the normalization constant up to a phase for N correlation functions which are presented in the first and second terms of eq.(4.12).

Let us now determine the normalization constant for $n_\infty - n_0$ correlation functions of the form $\langle V_{g_\infty(J)}V_{n_0J}V_{n_\infty N}V_{g_0}\rangle$. By using the OPE (5.27) and eq.(5.28), one finds in the limit $u\to 0$

$$\begin{aligned} G(u) &\to \frac{R^4\overset{*}{C}(n_\infty-n_0,\mathbf{k}_1+\mathbf{k}_3,\mathbf{p}_1+\mathbf{p}_5+\mathbf{p}_3;N-n_\infty,\mathbf{k}_4,\mathbf{p}_4;\mathbf{p}_6)}{u^{1-\frac{1}{n_\infty-n_0}(1+\frac{1}{4}\mathbf{k}_1\mathbf{k}_3+\mathbf{p}_1\mathbf{p}_3+\mathbf{p}_3\mathbf{p}_5+\mathbf{p}_1\mathbf{p}_5)}}\times\\ &\times\ C(n_\infty-n_0,-\mathbf{k}_1-\mathbf{k}_3,-\mathbf{p}_1-\mathbf{p}_5-\mathbf{p}_3;n_0,\mathbf{k}_1,\mathbf{p}_1;\mathbf{p}_5). \end{aligned}$$

Taking into account the behaviour of the root $x(u)$ in the vicinity of $u=0$

$$|x| \to \left((n_\infty-n_0)^{n_0-n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\right)^{\frac{1}{n_\infty-n_0}}|u|^{\frac{1}{n_0-n_\infty}},$$

one obtains from eq.(7.2)

$$\begin{aligned} G(u) &\to \frac{R^4C(g_0,g_\infty)}{u^{1-\frac{1}{n_\infty-n_0}(1+\frac{1}{4}\mathbf{k}_1\mathbf{k}_3+\mathbf{p}_1\mathbf{p}_3+\mathbf{p}_3\mathbf{p}_5+\mathbf{p}_1\mathbf{p}_5)}}\\ &\times\left((n_\infty-n_0)^{n_0-n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\right)^{-\frac{1+\frac{1}{4}\mathbf{k}_1\mathbf{k}_3+\mathbf{p}_1\mathbf{p}_3+\mathbf{p}_3\mathbf{p}_5+\mathbf{p}_1\mathbf{p}_5}{n_\infty-n_0}}, \end{aligned}$$

where we have used the relation

$$d_0+d_1+d_2+d_3+d_4+2d_5 = -\left(1+\frac{1}{4}k_1k_3+\mathbf{p}_1\mathbf{p}_3+\mathbf{p}_1\mathbf{p}_5+\mathbf{p}_3\mathbf{p}_5\right).$$

Now to find the normalization constant one should take into account the following identities:

$$d_2(n_\infty - n_0, \mathbf{k}_1 + \mathbf{k}_3, \mathbf{p}_1 + \mathbf{p}_3 + \mathbf{p}_5; N - n_\infty, \mathbf{k}_4, \mathbf{p}_4; \mathbf{p}_6) = -\frac{N - n_0}{n_\infty - n_0}(1 + \frac{1}{4}k_1k_3 + \mathbf{p}_1\mathbf{p}_3 + \mathbf{p}_1\mathbf{p}_5 + \mathbf{p}_2\mathbf{p}_5)$$
$$d_2(n_\infty - n_0, -\mathbf{k}_1 - \mathbf{k}_3, -\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{p}_5; n_0, \mathbf{k}_1, \mathbf{p}_1; \mathbf{p}_5) = -\frac{n_\infty}{n_\infty - n_0}(1 + \frac{1}{4}k_1k_3 + \mathbf{p}_1\mathbf{p}_3 + \mathbf{p}_1\mathbf{p}_5 + \mathbf{p}_2\mathbf{p}_5)$$

By using these formulae one can establish that the modulus of the normalization constant is again given by (7.6).

The normalization constant for the remaining $N - n_0 - n_\infty$ correlation functions of the form

$$\langle V_{g_\infty(J)} V_{n_0 J} V_{n_0+n_\infty, N} V_{g_0} \rangle$$

can be found in the same manner and is again defined by eq.(7.6).

Now we are ready to establish relative signs of the normalization constants corresponding to different values of $\mathbf{p}$'s. Note that the $SO(8)$ invariance of the model dictates the form of the correlation function

$$G^{i_1 i_2 i_3 i_4 i_5 i_6}_{IJKL}(u) = \langle V^{i_3 i_4}_{g_\infty}[\mathbf{k}_3, \mathbf{k}_4](\infty) V^{i_5}_{IJ}(1) V^{i_6}_{KL}(u) V^{i_1 i_2}_{g_0}[\mathbf{k}_1, \mathbf{k}_2](0) \rangle,$$

where we have used the notation $V^{i_1 i_2}_{g_0}[\mathbf{k}_1, \mathbf{k}_2] = \left(\sigma[\mathbf{k}_1/2, \mathbf{k}_2/2]\Sigma^{i_1 i_2}\right)_{g_0}$.

Namely, this correlation function is decomposed into $SO(8)$ invariant rank six tensors:

$$G^{i_1 i_2 i_3 i_4 i_5 i_6}(u) = C_1(u)\delta^{i_1 i_2}\delta^{i_3 i_4}\delta^{i_5 i_6} + C_2(u)\delta^{i_1 i_2}\delta^{i_3 i_6}\delta^{i_4 i_5} + \cdots.$$

Here the total number of terms is equal to 15. In fact, each function $C_i(u)$ coincides up to a phase with the correlation function (7.1) under a particular choice of $\mathbf{p}$'s. For example, by using the $SU(4) \times U(1)$ basis, the functions $C_1(u)$ and $C_2(u)$ can be schematically expressed as

$$\begin{aligned} C_1(u) &= \langle \sigma S^{B\bar{B}} \sigma S^{C} \sigma S^{\bar{C}} \sigma S^{A\bar{A}} \rangle \sim \langle \sigma[\mathbf{e}^B, -\mathbf{e}^B]\sigma[\mathbf{e}^C]\sigma[-\mathbf{e}^C]\sigma[\mathbf{e}^A, -\mathbf{e}^A] \rangle \\ C_2(u) &= \langle \sigma S^{B\bar{C}} \sigma S^{C} \sigma S^{\bar{B}} \sigma S^{A\bar{A}} \rangle \sim \langle \sigma[\mathbf{e}^B, -\mathbf{e}^C]\sigma[\mathbf{e}^C]\sigma[-\mathbf{e}^B]\sigma[\mathbf{e}^A, -\mathbf{e}^A] \rangle, \end{aligned} \tag{7.7}$$

where one has to choose all the vectors $\mathbf{e}^A$, $\mathbf{e}^B$ and $\mathbf{e}^C$ to be different. On the other hand, if $C = B$, then one can recognize that

$$C_1(u) + C_2(u) = \langle \sigma S^{B\bar{B}} \sigma S^{B} \sigma S^{\bar{B}} \sigma S^{A\bar{A}} \rangle \sim \langle \sigma[\mathbf{e}^B, -\mathbf{e}^B]\sigma[\mathbf{e}^B]\sigma[-\mathbf{e}^B]\sigma[\mathbf{e}^A, -\mathbf{e}^A] \rangle.$$

Since we know all three correlation functions up to phases we get a nontrivial relation on $C_1(u)$ and $C_2(u)$ allowing one to determine their relative sign. Namely, from (7.2) with the account of the found normalization constants (7.6) one obtains

$$\begin{aligned} C_1(u) &\sim \frac{\mathrm{e}^{\,i\varphi_1}}{(n_\infty - n_0)(x-1)(x + \frac{n_0}{N-n_0})(x-\alpha_1)(x-\alpha_2)}, \\ C_2(u) &\sim -\frac{\mathrm{e}^{\,i\varphi_2}}{(n_\infty - n_0)(x-1)(x + \frac{n_0}{N-n_0})(x - \frac{N-n_0-n_\infty}{N-n_0})(x - \frac{n_0}{n_0-n_\infty})}, \\ C_1(u) + C_2(u) &\sim \frac{\mathrm{e}^{\,i\varphi_3}(N-n_\infty)n_\infty x}{(N-n_0)(n_\infty-n_0)^2(x-1)(x + \frac{n_0}{N-n_0})(x - \frac{N-n_0-n_\infty}{N-n_0})(x - \frac{n_0}{n_0-n_\infty})(x-\alpha_1)(x-\alpha_2)}, \end{aligned}$$

where a common multiplier coming in all three functions is omitted.

Now it can be readily seen that the last equation is satisfied only if $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$. Proceeding in the same manner we fix the relative signs of all 15 functions C_i. The final answer for the correlation function $G^{i_1 i_2 i_3 i_4 ij}_{IJKL}(u)$ reads as

$$\begin{aligned} G^{i_1 i_2 i_3 i_4 ij}_{IJKL}(u) &= -\,\mathrm{e}^{\,i\varphi}\frac{R^4}{8(N-n_0)}\left(\frac{n_\infty - n_0}{N - n_0}\right)^{\frac{1}{4}k_1k_3} \\ &\times \frac{(x - \frac{n_0}{n_0-n_\infty})^3}{u(x-\alpha_1)(x-\alpha_2)}\left(\frac{x(x - \frac{N-n_0-n_\infty}{N-n_0})}{(x - \frac{n_0}{n_0-n_\infty})}\right)^{1+\frac{1}{4}k_1k_4}\left(\frac{(x-1)(x + \frac{n_0}{N-n_0})}{(x - \frac{n_0}{n_0-n_\infty})}\right)^{1+\frac{1}{4}k_3k_4} T^{i_1 i_2 i_3 i_4 ij}_{IJKL}(u), \end{aligned} \tag{7.8}$$

where

$$
\begin{aligned}
&T^{i_1i_2i_3i_4ij}_{IJKL}(u) = \\
&= \frac{\delta^{ij}}{(n_0-n_\infty)(x-\alpha_1)(x-\alpha_2)}\left(\frac{\delta^{i_1i_2}\delta^{i_3i_4}}{(x-1)(x+\frac{n_0}{N-n_0})} - \frac{\delta^{i_1i_4}\delta^{i_2i_3}}{x(x-\frac{N-n_0-n_\infty}{N-n_0})} - \frac{N-n_0}{(n_0-n_\infty)}\frac{\delta^{i_1i_3}\delta^{i_2i_4}}{(x-\frac{n_0}{n_0-n_\infty})}\right) \\
&+\frac{\delta^{i_3i_4}}{(x-1)(x+\frac{n_0}{N-n_0})}\left(\frac{\delta^{ii_1}\delta^{ji_2}}{n_0(x-\frac{N-n_0-n_\infty}{N-n_0})} - \frac{\delta^{ii_2}\delta^{ji_1}}{(n_0-n_\infty)x(x-\frac{n_0}{n_0-n_\infty})}\right) \\
&+\frac{\delta^{i_1i_2}}{(x-1)(x+\frac{n_0}{N-n_0})}\left(\frac{N-n_0}{n_\infty(N-n_\infty)}\frac{\delta^{ii_3}\delta^{ji_4}}{x} - \frac{\delta^{ii_4}\delta^{ji_3}}{(n_0-n_\infty)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})}\right) \\
&+\frac{\delta^{i_2i_3}}{x(x-\frac{N-n_0-n_\infty}{N-n_0})}\left(\frac{\delta^{ii_4}\delta^{ji_1}}{(n_0-n_\infty)(x+\frac{n_0}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})} - \frac{N-n_0}{n_0(N-n_\infty)}\frac{\delta^{ii_1}\delta^{ji_4}}{(x-1)}\right) \\
&+\frac{\delta^{i_2i_4}}{(x-\frac{n_0}{n_0-n_\infty})}\left(\frac{N-n_0}{n_0(n_0-n_\infty)}\frac{\delta^{ii_1}\delta^{ji_3}}{(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})} - \frac{N-n_0}{n_\infty(n_0-n_\infty)}\frac{\delta^{ii_3}\delta^{ji_1}}{x(x+\frac{n_0}{N-n_0})}\right) \\
&+\frac{\delta^{i_1i_4}}{x(x-\frac{N-n_0-n_\infty}{N-n_0})}\left(\frac{\delta^{ii_2}\delta^{ji_3}}{(n_0-n_\infty)(x-1)(x-\frac{n_0}{n_0-n_\infty})} - \frac{\delta^{ii_3}\delta^{ji_2}}{n_\infty(x+\frac{n_0}{N-n_0})}\right) \\
&+\frac{\delta^{i_1i_3}}{(x-\frac{n_0}{n_0-n_\infty})}\left(\frac{N-n_0}{(N-n_\infty)(n_0-n_\infty)}\frac{\delta^{ii_2}\delta^{ji_4}}{x(x-1)} - \frac{\delta^{ii_4}\delta^{ji_2}}{(n_0-n_\infty)(x+\frac{n_0}{N-n_0})(x-\frac{N-n_0-n_\infty}{N-n_0})}\right).
\end{aligned}
$$

Certainly, the common phase φ remains to be undetermined and can depend on the indices I,J,K,L and the momenta k_i. However, we will see at a moment that this phase disappears if one takes into account the contribution of the right-moving sector. By using the world-sheet parity symmetry combined with space reflection (3.22) we get the following relation between the correlation functions of the holomorphic and antiholomorphic sectors:

$$
\begin{aligned}
&\langle V^{i_3i_4}_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)V^{i_5}_{IJ}(1)V^{i_6}_{KL}(u)V^{i_1i_2}_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0)\rangle^* \\
&= \langle \bar{V}^{\tilde{i}_3\tilde{i}_4}_{g_\infty^{-1}}[\tilde{\mathbf{k}}_3,\tilde{\mathbf{k}}_4](\infty)\bar{V}^{\tilde{i}_5}_{IJ}(1)\bar{V}^{\tilde{i}_6}_{KL}(u)\bar{V}^{\tilde{i}_1\tilde{i}_2}_{g_0^{-1}}[\tilde{\mathbf{k}}_1,\tilde{\mathbf{k}}_2](0)\rangle^* \\
&= u^{-2}\langle \bar{V}^{\tilde{i}_1\tilde{i}_2}_{g_0}[-\tilde{\mathbf{k}}_1,-\tilde{\mathbf{k}}_2](\infty)\bar{V}^{\tilde{i}_6}_{KL}(u^{-1})\bar{V}^{\tilde{i}_5}_{IJ}(1)\bar{V}^{\tilde{i}_3\tilde{i}_4}_{g_\infty}[-\tilde{\mathbf{k}}_3,-\tilde{\mathbf{k}}_4](0)\rangle \\
&= \langle \bar{V}^{\tilde{i}_3\tilde{i}_4}_{g_\infty}[-\tilde{\mathbf{k}}_3,-\tilde{\mathbf{k}}_4](\infty)\bar{V}^{\tilde{i}_5}_{IJ}(1)\bar{V}^{\tilde{i}_6}_{KL}(u)\bar{V}^{\tilde{i}_1\tilde{i}_2}_{g_0}[-\tilde{\mathbf{k}}_1,-\tilde{\mathbf{k}}_2](0)\rangle \\
&= \langle \bar{V}^{i_3i_4}_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\bar{V}^{i_5}_{IJ}(1)\bar{V}^{i_6}_{KL}(u)\bar{V}^{i_1i_2}_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0)\rangle,
\end{aligned}
$$

and we recall that $V^{i_1i_2}_{g_0}[\mathbf{k}_1,\mathbf{k}_2] = \left(\sigma[\mathbf{k}_1/2,\mathbf{k}_2/2]\Sigma^{i_1i_2}\right)_{g_0}$ and $\bar{V}^{i_1i_2}_{g_0}[\mathbf{k}_1,\mathbf{k}_2] = \left(\bar{\sigma}[\mathbf{k}_1/2,\mathbf{k}_2/2]\bar{\Sigma}^{i_1i_2}\right)_{g_0}$. Here the conjugation property (5.21) of V and the invariance of the correlation function (7.8) under the space reflection were used. Thus, correlation functions of the anti-holomorphic sector are complex-conjugated to correlation functions of the holomorphic one. Therefore, after combining these two sectors the phase ambiguity dissappears.

8 Scattering amplitude

The results obtained in the previous section allows one to determine a holomorphic contribution $G^{i_1i_2i_3i_4}_{IJKL}(u)$ to the correlation function (4.13)

$$
\begin{aligned}
G^{i_1i_2i_3i_4}_{IJKL}(u) &= G^{ij}_b(u)\langle\Sigma^{i_3i_4}_{g_\infty}(\infty)\Sigma^i_{IJ}(1)\Sigma^j_{KL}(u)\Sigma^{i_1i_2}_{g_0}(0)\rangle \\
&= \frac{i}{8}\left(\frac{n_\infty n_0(N-n_\infty)}{(N-n_0)}\right)^{1/2}\frac{x\langle\tau_i\tau_j\rangle G^{i_1i_2i_3i_4ij}_{IJKL}(u)}{(n_\infty-n_0)u^{1/2}(x-\alpha_1)(x-\alpha_2)}.
\end{aligned} \tag{8.1}
$$

Up to now we considered the correlation functions $G^{i_1i_2i_3i_4}_{IJKL}(u)$ with $|u|<1$. The correlation functions $G^{i_1i_2i_3i_4}_{IJKL}(u)$ with $|u|>1$ can be calculated in the same way. In particular, the normalization constant in this case is derived by studying the limit $u \to \infty$ and coincides with the previously found constant (7.6). We find that $G^{i_1i_2i_3i_4}_{IJKL}(u)$ with $|u| < 1$ is again given by eq.(8.1). The time-ordering, therefore, can be omitted, and to complete the computation of the S-matrix element we have to integrate the product $G^{i_1i_2i_3i_4}_{IJKL}(u)\bar{G}^{j_1j_2j_3j_4}_{IJKL}(\bar{u})$ over the complex plane. With the account of eq.(7.8) and the equality

$$\frac{1}{u}\frac{du}{dx} = \frac{(n_0 - n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}{x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})}$$

we find that the corresponding integral is given by

$$\int d^2u|u|G^{i_1i_2i_3i_4}_{IJKL}(u)\bar{G}^{j_1j_2j_3j_4}_{IJKL}(\bar{u}) = \kappa\frac{R^8}{2^{12}}\frac{n_0n_\infty(N-n_\infty)}{(N-n_0)^3}\left(\frac{n_\infty - n_0}{N-n_0}\right)^{\frac{1}{2}k_1k_3} \tag{8.2}$$
$$\times\int d^2u\left|\frac{du}{dx}\right|^{-2}\left|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\right|^{\frac{1}{2}k_1k_4}\left|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\right|^{\frac{1}{2}k_3k_4} T^{i_1i_2i_3i_4}_{IJKL}(u)T^{j_1j_2j_3j_4}_{IJKL}(\bar{u}),$$

where we have introduced a concise notation

$$T^{i_1i_2i_3i_4}_{IJKL}(u) = \langle\tau_i\tau_j\rangle T^{i_1i_2i_3i_4ij}_{IJKL}(u).$$

There is an important subtlety originating from the non-abelian nature of the orbifold model that leads to changing the overall normalization of (8.2) by some constant κ. Recall that our computation scheme relies on independent computation of boson and fermion (holomorphic and antiholomorphic) contributions to the correlation function (4.13), i.e. we regard the vertex operators as the tensor product of bosonic and fermionic, and holomorphic, and antiholomorphic twist fields. However, as was mentioned above, the absence of such a decomposition in the orbifold model has to be taken into account. A correct way of determining the normalization constant for the correlation functions should be based on the OPE (3.1) of the vertex operators and involves the knowledge of the corresponding structure constants. Namely, omitting all unessential details, the normalization constant turns out to be proportional to the product of two structure constants: $C(g_0, g_\infty) \sim \check{C}\check{C}$, while $\check{C}$ are obtained by considering an auxiliary correlation function $\langle V_{g_0^{-1}}(\infty)V_{IJ}(1)V_{IJ}(u)V_{g_0}(0)\rangle$. It's normalization is found by studying the limit $u \to 1$ and does not appeal to the tensor product structure of the vertex operators. In the limit $u \to 0$ one gets $\langle V_{g_0^{-1}}(\infty)V_{IJ}(1)V_{IJ}(u)V_{g_0}(0)\rangle \to \frac{f}{u^\Delta\bar{u}^\Delta}$, where f is some constant, which due to (3.1) is related with $\check{C}$ as $f = 2\check{C}^2$. Note that the multiplier 2 emerges namely due to the nonabelian character of the orbifold. On the other hand, the constant f is expressed through the structure constants C_{bh}, C_{fh} and C_{ba}, C_{fa} [7] found in the previous sections as $f = 2^4C^2_{bh}C^2_{fh}C^2_{ba}C^2_{fa}$, since each sector again provides the multiplier 2. Thus, the structure constant $\check{C}$ is equal to the product $\check{C} = 2^{3/2}C_{bh}C_{fh}C_{ba}C_{fa}$. Therefore, the constant κ is found to be 2^3.

Coming back to eq.(8.2), we see that under the change of variables $u \to x$ the integral acquires the form

$$\int d^2u|u|G^{i_1i_2i_3i_4}_{IJKL}(u)\bar{G}^{j_1j_2j_3j_4}_{IJKL}(\bar{u}) = \frac{R^8}{2^9}\frac{n_0n_\infty(N-n_\infty)}{(N-n_0)^3}\left(\frac{n_\infty - n_0}{N-n_0}\right)^{\frac{1}{2}k_1k_3}$$
$$\times\int_{V_{IJKL}} d^2x\left|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\right|^{\frac{1}{2}k_1k_4}\left|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\right|^{\frac{1}{2}k_3k_4} T^{i_1i_2i_3i_4}(x)T^{j_1j_2j_3j_4}(\bar{x}),$$

where we have taken into account that under this change of variables the u-sphere is mapped onto the domain V_{IJKL}.

Since the basic correlation function (4.6) is equal to the sum

$$F(u,\bar{u}) = \frac{C_0C_\infty}{N!}2n_0(N-n_0)n_\infty(N-n_\infty)\sum_{IJKL}G^{i_1i_2i_3i_4}_{IJKL}(u)\bar{G}^{j_1j_2j_3j_4}_{IJKL}(\bar{u})\zeta_1^{i_1j_1}\zeta_2^{i_2j_2}\zeta_3^{i_3j_3}\zeta_4^{i_4j_4},$$

[7] Here, e.g., C_{bh} refers to the structure constant (5.24) in the bosonic left-moving sector.

where the summation goes over the set of indices listed in eq.(4.12), the integral $\int d^2u|u|F(u,\bar{u})$ is equal to

$$\int d^2u|u|F(u,\bar{u}) = \frac{R^8}{2^8\sqrt{k_1^+k_2^+k_3^+k_4^+}}\left(\frac{n_0n_\infty(N-n_\infty)}{N(N-n_0)}\right)^2\left(\frac{n_\infty-n_0}{N-n_0}\right)^{\frac{1}{2}k_1k_3} \tag{8.3}$$
$$\times\int d^2x\left|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\right|^{\frac{1}{2}k_1k_4}\left|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\right|^{\frac{1}{2}k_3k_4} T^{i_1i_2i_3i_4}(x)T^{j_1j_2j_3j_4}(\bar{x})\zeta_1^{i_1j_1}\zeta_2^{i_2j_2}\zeta_3^{i_3j_3}\zeta_4^{i_4j_4}.$$

To discuss the Lorentz invariance of the theory, without loss of generality, we choose the polarization $\zeta^{\mu\nu}$ in the form $\zeta^\mu\zeta^\nu$. Recall that in ten dimensions a polarization of a graviton satisfies the transversality condition: $k_\mu\zeta^\mu=0$. In the light-cone gauge the polarization obeys $\zeta^+=0$ allowing to express the component ζ^- via ζ^i and k_μ as $\zeta^-=\frac{2k^i\zeta^i}{k^+}$. In our model we only deal with eight transversal polarizations ζ^i and can treat the last equation as a definition of the light-cone polarization ζ^-. An important property of the light-cone gauge is that $\zeta_1^i\zeta_2^i=\zeta_1^\mu\zeta_2^\mu\equiv(\zeta_1\zeta_2)$. Glearly, the integrand in (8.3) depends on scalar products of the transversal momenta k^i and polarizations ζ^i. It turns out that by using the light-cone momenta and polarizations k^- and ζ^- the integrand can be written via scalar products of ten-dimensional vectors. Namely, it is enough to check this assertion for the expressions $\langle\tau_i\tau_i\rangle$ and $\langle\tau_i\tau_j\rangle\zeta_a^i\zeta_b^j$. We confine ourselves with considering $\langle\tau_i\tau_j\rangle\zeta_a^i\zeta_b^j$. The result easily follows from the fact that, e.g.,

$$\begin{aligned}&\frac{(x+\frac{n_0}{N-n_0})}{n_0}k_1^i\zeta_a^i+\frac{(x-\frac{N-n_0-n_\infty}{N-n_0})}{n_\infty}k_3^i\zeta_a^i+\frac{1}{N-n_0}k_4^i\zeta_a^i\\ &=\frac{(x+\frac{n_0}{N-n_0})}{n_0}(k_1\zeta_a)+\frac{(x-\frac{N-n_0-n_\infty}{N-n_0})}{n_\infty}(k_3\zeta_a)+\frac{1}{N-n_0}(k_4\zeta_a),\end{aligned}$$

where the relation $k^i\zeta_a^i=(k\zeta_a)+\frac{1}{2}k^+\zeta_a^-$ was used.

To rewrite the integral (8.3) in the conventional form we perform the change of variables

$$\frac{n_\infty-n_0}{N-n_0}z=\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}},\qquad dz=\frac{(x-\alpha_1)(x-\alpha_2)}{\frac{n_\infty-n_0}{N-n_0}(x-\frac{n_0}{n_0-n_\infty})^2}.$$

Then, after simple but rather lengthy calculations, one arrives at the following result

$$\int d^2u|u|F(u,\bar{u})=\frac{R^8}{2^8N^2\sqrt{k_1^+k_2^+k_3^+k_4^+}}\int d^2z\,|z|^{\frac{1}{2}k_1k_4-2}\,|1-z|^{\frac{1}{2}k_3k_4-2}\,K(z,\bar{z},\zeta),$$

Here we introduced a notation

$$K(z,\bar{z},\zeta)=K^{i_1i_2i_3i_4}(z)K^{j_1j_2j_3j_4}(\bar{z})\zeta_1^{i_1j_1}\zeta_2^{i_2j_2}\zeta_3^{i_3j_3}\zeta_4^{i_4j_4},$$

where

$$\begin{aligned}&K^{i_1i_2i_3i_4}(z)\zeta_1^{i_1}\zeta_2^{i_2}\zeta_3^{i_3}\zeta_4^{i_4}=\\ &\quad z(k_3k_4)(\zeta_1\zeta_3)(\zeta_2\zeta_4)-(k_3k_4)(\zeta_2\zeta_3)(\zeta_1\zeta_4)-(k_1k_4)(\zeta_1\zeta_2)(\zeta_3\zeta_4)\\ &\quad+(1-z)\left[(\zeta_1k_4)(\zeta_3k_2)(\zeta_2\zeta_4)+(\zeta_2k_3)(\zeta_4k_1)(\zeta_1\zeta_3)+(\zeta_1k_3)(\zeta_4k_2)(\zeta_2\zeta_3)+(\zeta_2k_4)(\zeta_3k_1)(\zeta_1\zeta_4)\right]\\ &\quad+z\left[(\zeta_2k_1)(\zeta_4k_3)(\zeta_3\zeta_1)+(\zeta_3k_4)(\zeta_1k_2)(\zeta_2\zeta_4)+(\zeta_2k_4)(\zeta_1k_3)(\zeta_3\zeta_4)+(\zeta_3k_1)(\zeta_4k_2)(\zeta_1\zeta_2)\right]\\ &\quad-\left[(\zeta_1k_2)(\zeta_4k_3)(\zeta_3\zeta_2)+(\zeta_3k_4)(\zeta_2k_1)(\zeta_1\zeta_4)+(\zeta_1k_4)(\zeta_2k_3)(\zeta_3\zeta_4)+(\zeta_3k_2)(\zeta_4k_1)(\zeta_1\zeta_2)\right].\end{aligned}$$

Now one can recognize in K the standard open string kinematical factor for the four vector particle scattering.

The S-matrix element can be now found by using eq.(4.3) and by taking the limit $R\to\infty$:

$$\langle f|S|i\rangle=-i\frac{\lambda^2N\delta_{m_1+m_2+m_3+m_4,0}\delta(\sum_i k_i^-)\delta^D(\sum_i \mathbf{k}_i)}{2^7\sqrt{k_1^+k_2^+k_3^+k_4^+}}\int d^2z\,|z|^{\frac{1}{2}k_1k_4-2}\,|1-z|^{\frac{1}{2}k_3k_4-2}\,K(z,\bar{z},\zeta), \tag{8.4}$$

where we have restored δ-functions responsible for the momentum conservation law and have represented the light-cone momenta k_i^+ as $k_i^+ = \frac{m_i}{N}$.

In the limit $N \to \infty$ the combination $N\delta_{m_1+m_2+m_3+m_4,0}$ goes to $\delta(\sum_i k_i^+)$ and eq.(8.4) acquires the form

$$\langle f|S|i\rangle = -i\frac{\lambda^2 \delta^{D+2}(\sum_i k_i^\mu)}{2^8\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}}\int d^2z\, |z|^{\frac{1}{2}k_1k_4-2}\, |1-z|^{\frac{1}{2}k_3k_4-2}\, K(z,\bar{z},\zeta).$$

Taking into account that the scattering amplitude A is related to the S-matrix element as follows (see e.g. [18])

$$\langle f|S|i\rangle = -i\frac{\delta^{D+2}(\sum_i k_i^\mu)}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}}A(1,2,3,4),$$

one finally gets

$$A(1,2,3,4) = \lambda^2 2^{-8}\int d^2z\, |z|^{\frac{1}{2}k_1k_4-2}\, |1-z|^{\frac{1}{2}k_3k_4-2}\, K(z,\bar{z},\zeta)$$

that is the well-known four graviton scattering amplitude.

9 Conclusion

In this paper we developed the technique for calculating scattering amplitudes of type II string states by using the interacting $S^N\mathbf{R}^8$ orbifold sigma model. Although we considered only the four graviton scattering, our results allow one to find immediately the scattering amplitudes of any four particles. Let us stress that in our calculation we did not impose any kinematical restrictions on momenta and polarizations of gravitons and, hence, automatically obtained the Lorentz-invariant amplitude. It gives a strong evidence that the two-dimensional Yang-Mills model should possess the same invariance in the large N limit.

An interesting problem is to consider the scattering amplitudes of the heterotic string states. Recall that an important point of our construction was the cancellation of phases coming both from the left- and the right-moving sectors. The world-sheet parity transformations combined with an odd number of space reflections are the symmetry of the type IIA string theory responsible for this cancellation. A symmetry of the heterotic string theory that may be responsible for such a cancellation is unknown.

It would be of interest to trace the appearance of the loop amplitudes in the framework of the orbifold sigma model. Obviously, the one-loop amplitude requires the computation of the correlation function of four DVV interaction vertices sandwiched between the asymptotic states, which technically results in constructing the non-commutative Green functions in the presence of six twist fields. We note that cancellation of possible divergences in the amplitude may require the further perturbation of the CFT action by higher-order contact terms.

Although the string scattering amplitudes follow from the orbifold model description, the problem of a great interest is to reproduce the amplitudes directly from the SYM theory.

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Appendix A

We use the following representation of γ-matrices satisfying the relation

$$\gamma^i(\gamma^j)^T + \gamma^j(\gamma^i)^T = 2\delta^{ij}I$$

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\gamma^3 = 1 \qquad \gamma^4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^6 = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma^8 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the spinor representation $\mathbf{8_S}$ of the $SO(8)$. In this representation the algebra generators R^{ij} can be realized as

$$R^{ij} = \frac{1}{4}\theta^a \gamma^{ij}_{ab} \theta^b.$$

Here $\gamma^{ij} = \frac{1}{2}(\gamma^i(\gamma^j)^T - \gamma^j(\gamma^i)^T)$.

According to our definition of the fermions Θ:

$$\Theta^A = \frac{\theta^A + i\theta^{A+4}}{\sqrt{2}}, \quad \Theta^{\bar{A}} = \frac{\theta^A - i\theta^{A+4}}{\sqrt{2}},$$

where $A = 1, \ldots, 4$, we get for R^{ij}:

$$R^{ij} = \frac{1}{4}(\Theta^A S^{ij}_{AB} \Theta^B + \Theta^{\bar{A}} \bar{S}^{ij}_{AB} \Theta^{\bar{B}} + \Theta^{\bar{A}} \bar{T}^{ij}_{AB} \Theta^B + \Theta^A T^{ij}_{AB} \Theta^{\bar{B}},$$

where

$$T^{ij}_{AB} = \frac{1}{2}(\gamma^{ij}_{AB} + i\gamma^{ij}_{A\ B+4} - i\gamma^{ij}_{A+4\ B} + \gamma^{ij}_{A+4\ B+4})$$

and

$$S^{ij}_{AB} = \frac{1}{2}(\gamma^{ij}_{AB} - i\gamma^{ij}_{A\ B+4} - i\gamma^{ij}_{A+4\ B} - \gamma^{ij}_{A+4\ B+4}).$$

The four Cartan generators of $SO(8)$ are given by $R^{2i-1\ 2i}$. Define the $SU(4) \times U(1)$ generators acting on Θ^As':

$$J^{ij} = -\frac{i}{2}\Theta^A T^{ij}_{AB} \Theta^{\bar{B}}.$$

Then the Cartan generators H_1, H_2, H_3 of $SU(4)$ and the generator H_4 of $U(1)$ are given by

$$H^1 = J^{12}, \quad H^2 = J^{34}, \quad H^3 = J^{56}, \quad H^4 = J^{78}.$$

Since

$$T^{12} = i\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad T^{34} = i\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad T^{56} = i\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad T^{78} = iI$$

the weights of the representation $\mathbf{4_{1/2}}$ look as

$$\begin{aligned} q^1 &= \frac{1}{2}(-1,-1,1,1); \quad q^3 = \frac{1}{2}(1,-1,-1,1); \\ q^2 &= \frac{1}{2}(-1,1,-1,-1); \quad q^4 = \frac{1}{2}(1,1,1,1); \end{aligned} \tag{A.1}$$

Bosonizing the Cartan generators with the help of four bosonic fields ϕ^A as $H^A = i\partial\phi^A$ we get the following expression for fermions

$$\Theta^A = e^{iq^A_B \phi^B}.$$

Appendix B

In this Appendix we consider some properties of the map (5.4) and outline the derivation of the differential equation (5.9) for the four-point correlation functions (4.13).

Let us consider the map (5.4)

$$z = \frac{t^{n_0}(t-t_0)^{N-n_0}}{(t-t_\infty)^{N-n_\infty}} \frac{(t_1-t_\infty)^{N-n_\infty}}{t_1^{n_0}(t_1-t_0)^{N-n_0}} \equiv u(t). \tag{B.1}$$

This map is the N-fold covering of the z-sphere by the t-sphere. Obviously, it branches at the points $t = 0, t_0, t_\infty$ and ∞. To find other branch points we have to solve the following equation:

$$\begin{aligned}\frac{d\log z}{dt} &= \frac{n_0}{t} + \frac{N-n_0}{t-t_0} - \frac{N-n_\infty}{t-t_\infty} \\ &= \frac{n_\infty t^2 + ((N-n_0-n_\infty)t_0 - Nt_\infty)\,t + n_0 t_0 t_\infty}{t(t-t_0)(t-t_\infty)}.\end{aligned} \tag{B.2}$$

In general there are two different solutions t_1 and t_2 of this equation, and the map (B.1) has the following form in the vicinity of these points

$$z - z_i \sim (t-t_i)^2, \quad z_1 = 1 = u(t_1), \quad z_2 = u = u(t_2).$$

Due to the projective transformations, we can impose three relations on positions of branch points. However, we have already chosen the points 0 and ∞ as two branch points, therefore, only one relation remains to be imposed. Since the differential equation on the four-point correlation function is written with respect to the point u, it is convinient not to fix the position of the point $t_2 \equiv x$. Then, the remaining relation that leads to the rational dependence of points t_0, t_∞ and t_1 on x looks as follows

$$t_0 = x - 1. \tag{B.3}$$

The point x is supposed to be a solution of eq.(B.2). Therefore, one can immediately derive from eqs.(B.2) and (B.3) that t_∞ is expressed through the point x as

$$t_\infty = x - \frac{(N-n_\infty)x}{(N-n_0)x + n_0}. \tag{B.4}$$

The second solution of eq.(B.2) can be now easily found and is given by

$$\begin{aligned} t_1 &= \frac{N-n_0-n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty((N-n_0)x+n_0)} \\ &= \frac{n_0(x-1)\,((N-n_0)x + n_0 + n_\infty - N)}{n_\infty\,((N-n_0)x+n_0)}.\end{aligned} \tag{B.5}$$

The rational function $u(x)$ is defined by the following equation

$$u(x) = \frac{x^{n_0}(x-t_0)^{N-n_0}(t_1-t_\infty)^{N-n_\infty}}{(x-t_\infty)^{N-n_\infty} t_1^{n_0}(t_1-t_0)^{N-n_0}}. \tag{B.6}$$

By using eqs.(B.3),(B.4) and (B.5), one can derive the following relations

$$\begin{aligned} t_1 - t_0 &= \frac{(N-n_0)(x-1)\,((n_0-n_\infty)x - n_0)}{n_\infty\,((N-n_0)x+n_0)}, \\ t_1 - t_\infty &= \frac{((n_0-n_\infty)x-n_0)\,((N-n_0)x+n_0+n_\infty-N)}{n_\infty\,((N-n_0)x+n_0)}.\end{aligned}$$

Then the rational function $u(x)$ is found to be equal to

$$\begin{aligned} u = u(x) &= (n_0-n_\infty)^{n_0-n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\left(\frac{x+\frac{n_0}{N-n_0}}{x-1}\right)^{N} \\ &\times \left(\frac{x-\frac{N-n_0-n_\infty}{N-n_0}}{x}\right)^{N-n_0-n_\infty}\left(x - \frac{n_0}{n_0-n_\infty}\right)^{n_0-n_\infty}.\end{aligned} \tag{B.7}$$

To obtain the differential equation (5.9) we need to know the decomposition of the roots $t_K(z)$ and $t_L(z)$ in the vicinity of $z = u$. Let us take the logarithm of the both sides of eq.(B.1):

$$\log\frac{z}{u} = n_0\log\frac{t}{x} + (N-n_0)\log\frac{t-t_0}{x-t_0} - (N-n_\infty)\log\frac{t-t_\infty}{x-t_\infty}. \tag{B.8}$$

Decomposition of the l.h.s. of eq.(B.8) around $z = u$ and the r.h.s. of eq.(B.8) around $t = x$ gives:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{z-u}{u}\right)^k = (t-x)^2 \sum_{k=0}^{\infty} a_k (t-x)^k, \tag{B.9}$$

where the coefficients a_k are equal to

$$a_k = \frac{(-1)^{k-1}}{k+2} \left(\frac{n_0}{x^{k+2}} + \frac{N-n_0}{(x-t_0)^{k+2}} - \frac{N-n_\infty}{(x-t_\infty)^{k+2}} \right). \tag{B.10}$$

It is clear from eq.(B.9) that $t(z)$ has the following decomposition

$$t - x = \sum_{k=1}^{\infty} c_k (z-u)^{\frac{k}{2}}. \tag{B.11}$$

Substituting eq.(B.11) into eq.(B.9), one finds

$$\begin{aligned} c_1^2 &= \frac{1}{ua_0}, \quad c_2 = -\frac{a_1}{2ua_0}, \\ 2a_0c_1c_3 &= -\frac{1}{2u^2} + \frac{5a_1^2}{4u^2a_0^3} - \frac{a_2}{u^2a_0^2}. \end{aligned} \tag{B.12}$$

Next coefficients are not important for us.
Then, by using the decomposition (B.11) and eq.(B.12), one gets

$$\begin{aligned} \left(\frac{t''}{t'}\right)' &= \frac{1}{2(z-u)^2} + O(1), \\ \left(\frac{t''}{t'}\right)^2 &= \frac{1}{4(z-u)^2} + \frac{3}{z-u}\left(\frac{c_2^2}{c_1^2} - \frac{c_3}{c_1}\right) + O(1), \\ \frac{c_2^2}{c_1^2} - \frac{c_3}{c_1} &= \frac{1}{4u}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right). \end{aligned}$$

Finally, taking into account that in the set of N roots $t_M(z)$ only two roots $t_K(z)$ and $t_L(z)$ have the decomposition (B.11), we obtain eqs.(5.7) and (5.8).

The coefficients a_k can be rewritten as the following functions of x:

$$\begin{aligned} a_0 &= \frac{n_0(n_0+n_\infty-N)}{2(N-n_\infty)x^2} + \frac{n_0(N-n_0)}{(N-n_\infty)x} + \frac{(N-n_0)(n_\infty-n_0)}{2(N-n_\infty)} \\ &= \frac{(N-n_0)(n_\infty-n_0)}{2(N-n_\infty)x^2}(x-\alpha_1)(x-\alpha_2), \\ a_1 &= \frac{n_0((N-n_\infty)^2-n_0^2)}{3(N-n_\infty)^2x^3} - \frac{n_0^2(N-n_0)}{(N-n_\infty)^2x^2} \\ &- \frac{n_0(N-n_0)^2}{(N-n_\infty)^2x} + \frac{(N-n_0)((N-n_\infty)^2-(N-n_0)^2)}{3(N-n_\infty)^2}, \\ a_2 &= -\frac{n_0((N-n_\infty)^3-n_0^3)}{4(N-n_\infty)^3x^4} + \frac{n_0^3(N-n_0)}{(N-n_\infty)^3x^3} + \frac{3n_0^2(N-n_0)^2}{2(N-n_\infty)^3x^2} \\ &+ \frac{n_0(N-n_0)^3}{(N-n_\infty)^3x} - \frac{(N-n_0)((N-n_\infty)^3-(N-n_0)^3)}{4(N-n_\infty)^3}. \end{aligned} \tag{B.13}$$

To obtain the differential equation (5.9) we have to use the following important equalities on $\frac{1}{u}\frac{du}{dx}$, that can be derived by using eqs.(B.7) and (B.13)

$$\frac{1}{u}\frac{du}{dx} = \frac{n_0+n_\infty-N}{x} - \frac{N}{x-1} + \frac{N}{x+\frac{n_0}{N-n_0}}$$

$$
\begin{aligned}
& + \quad \frac{N-n_0-n_\infty}{x-\frac{N-n_0-n_\infty}{N-n_0}} + \frac{n_0-n_\infty}{x-\frac{n_0}{n_0-n_\infty}}, \\
\frac{1}{u}\frac{du}{dx} &= \frac{4(N-n_\infty)^2 x^4 a_0^2}{(N-n_0)^2(n_0-n_\infty)x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})} \\
&= \frac{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}{x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})}.
\end{aligned}
$$

Finally, to get eq.(5.9) one should use the Lagrange interpolation formula for the ratio of two polynomials

$$
\frac{P(x)}{Q(x)} = \sum_i \frac{P(x_i)}{Q'(x_i)} \frac{1}{x-x_i},
$$

where x_i are the simple roots of $Q(x)$ and $degP < degQ$.

These equalities drastically simplify the derivation of eq.(5.9).

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