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Antisymmetric tensor field on AdS_5.

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Abstract

By using the Hamiltonian version of the AdS/CFT correspondence, we compute the two-point Green function of a local operator in $D = 4$ $\mathcal{N} = 4$ super Yang-Mills theory, which corresponds to a massive antisymmetric tensor field of the second rank on the AdS_5 background. We discuss the conformal transformations induced on the boundary by isometries of AdS_5.

The recent Maldacena's conjecture [1] relates the large N limit of certain conformal theories in d-dimensions with classical supergravity on the product of anti de Sitter space AdS_{d+1} with a compact manifold. According to [2, 3] the precise relation consists in existing the correspondence between supergravity fields and the set of local CFT operators. Then the generating functional of the connected Green functions of the CFT operators is identified with the on-shell value of the supergravity action. With this identification at hand, the AdS/CFT correspondence was recently tested by explicit computation of some two- and three-point correlation functions of local operators in $D = 4$ $\mathcal{N} = 4$ super Yang-Mills theory, which correspond to scalar, vector, symmetric tensor and spinor fields on the AdS_5 background [4]-[12].

$D = 4$ $\mathcal{N} = 4$ super Yang-Mills is related to the S^5 compactification of $D = 10$ IIB supergravity. Except the fields mentioned above, the spectrum of the compactified theory also contains the massive antisymmetric tensor fields of the second rank [13, 14]. These fields obey first-order differential equations and their bulk action vanishes on shell. Thus, the bulk action is not enough to compute the CFT Green functions and one has to add some boundary terms. This is quite similar to the case of fermions on the AdS background [5]. The origin of boundary terms in the AdS/CFT correspondence was recently clarified in [15], where it was shown that they appear in passing from the Hamiltonian description of the bulk action to the Lagrangian one. The idea was to treat the coordinate in the bulk direction as the time and to present the bulk action in the form $\int(p\dot{q} - H(p,q) + \textit{total derivative})$. Here a choice of coordinates and momenta is dictated by the transformation properties of gravity fields under isometries of AdS. In the Hamiltonian formulation the total derivative term should be omitted while from the Lagrangian point of view it can be compensated by adding to the bulk action a proper boundary term.

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In this note we demonstrate how this general approach works in the case of antisymmetric tensor fields of the second rank and compute the two-point function of the corresponding local CFT operators.

We start with the following action for a massive complex antisymmetric tensor field of the second rank[1]

$$S = -\int d^5x \left(\frac{i}{2} \varepsilon^{\mu\nu\rho\lambda\sigma} a^*_{\mu\nu} \partial_\rho a_{\lambda\sigma} + \sqrt{-g} m a^*_{\mu\nu} a^{\mu\nu} \right). \tag{1}$$

Here $g = -x_0^{-10}$ is the determinant of the AdS metric: $ds^2 = \frac{1}{x_0^2}(dx_0^2 + \eta_{ij} dx^i dx^j)$. Because of infrared divergencies one should regularize the action by cutting AdS_5 space off at $x_0 = \varepsilon$ and leaving the part $x_0 \geq \varepsilon$. We use the convention $\varepsilon_{01234} = -\varepsilon^{01234} = 1$.

Action (1) vanishes on shell and, therefore, can not produce the two-point functions in the boundary CFT. According to the general scheme discussed above, we need to rewrite action (1) in the form suitable for passing to the Hamiltonian formulation. To this end one has to establish a proper set of variables that can be treated as coordinates and their conjugate momenta. It can be done by studying solutions of equations of motion coming from (1)

$$\frac{i}{2} \varepsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho a_{\lambda\sigma} + \sqrt{-g} m a^{\mu\nu} = 0. \tag{2}$$

Acting on (2) with an operator $-\frac{i}{2}\varepsilon_{\mu\nu\rho\lambda\sigma}\nabla^\rho + \sqrt{-g} m g_{\mu\lambda} g_{\nu\sigma}$ we arrive at the second-order equation

$$\nabla^\rho \left(\nabla_\rho a_{\mu\nu} - \nabla_\mu a_{\rho\nu} + \nabla_\nu a_{\rho\mu} \right) - m^2 a_{\mu\nu} = 0. \tag{3}$$

The last equation implies the constraint $\nabla^\mu a_{\mu\nu} = 0$ and, therefore, can be written in the form

$$\nabla^\rho \nabla_\rho a_{\mu\nu} + (6 - m^2) a_{\mu\nu} = 0. \tag{4}$$

Specifying (4) for a_{ij}, we obtain

$$x_0^2 \partial_0^2 a_{ij} + x_0 \partial_0 a_{ij} + x_0^2 \Box a_{ij} - m^2 a_{ij} - 2x_0 (\partial_i a_{0j} - \partial_j a_{0i}) = 0, \tag{5}$$

where $\Box = \eta^{ij}\partial_i\partial_j$. The derivatives $\partial_i a_{0j}$ can be expressed from (2):

$$\partial_i a_{0j} - \partial_j a_{0i} = \partial_0 a_{ij} + \frac{i}{2} x_0^{-1} m \varepsilon_{ijkl} a^{kl}, \tag{6}$$

where in the last formula and below the indices are raised with respect to the Minkowski metric. Therefore, eq.(5) reduces to

$$x_0^2 \partial_0^2 a_{ij} - x_0 \partial_0 a_{ij} + x_0^2 \Box a_{ij} - m^2 a_{ij} - i m \varepsilon_{ijkl} a^{kl} = 0, \tag{7}$$

To solve (7) we introduce the projections $a^\pm_{ij}$ on the (anti)self-dual parts of a_{ij}:

$$a^\pm_{ij} = \frac{1}{2} \left(a_{ij} \pm \frac{i}{2} \varepsilon_{ijkl} a^{kl} \right), \qquad a^\pm_{ij} = \pm \frac{i}{2} \varepsilon_{ijkl} a^{\pm kl}. \tag{8}$$

[1] It follows from [13, 14] that antisymmetric tensor fields arising in S^5 compactification of IIB supergravity are classified by complex representations of $SO(6)$.

Then eq.(7) splits into equations for a^+_{ij} and a^-_{ij}:

$$x_0^2\partial_0^2 a^\pm_{ij} - x_0\partial_0 a^\pm_{ij} - m(m\pm 2)a^\pm_{ij} + x_0^2\Box a^\pm_{ij} = 0. \tag{9}$$

Momentum space solutions of (9) for $\vec{k}^2>0$ obeying the boundary conditions $a^\pm_{ij}(\varepsilon,\vec{k}) = a^\pm_{ij}(\vec{k})$ and vanishing at $x_0=\infty$ read as[2]

$$a^\pm_{ij}(x_0,\vec{k}) = \frac{x_0}{\varepsilon}\frac{K_{m\pm 1}(x_0 k)}{K_{m\pm 1}(\varepsilon k)}a^\pm_{ij}(\vec{k}), \tag{10}$$

where $K_{m\pm 1}$ is the Mackdonald function and $k=|\vec{k}|$.

However, we can not assign arbitrary boundary values for both $a^+_{ij}(\vec{k})$ and $a^-_{ij}(\vec{k})$ since these components are related to each other. To find this relation, note that the components a_{0i} can be directly found from (2):

$$a_{0i} = -\frac{i}{2m}x_0\varepsilon_{ijkl}\partial^j a^{kl}. \tag{11}$$

Then substituting into (6) one obtains the constraint

$$\frac{x_0}{m}\left(\partial_j\partial^k(a^+_{ik}-a^-_{ik}) - \partial_i\partial^k(a^+_{jk}-a^-_{jk})\right) = \partial_0 a_{ij} + \frac{m}{x_0}(a^+_{ij}-a^-_{ij}), \tag{12}$$

which after projecting on its (anti)self-dual part results into the following equations

$$\pm\frac{x_0}{2m}\left(\Box a^\pm_{ij} + \Box a^\mp_{ij} + 2(\partial_i\partial^k a^\mp_{jk} - \partial_j\partial^k a^\mp_{ik})\right) = \partial_0 a^\pm_{ij} \pm \frac{m}{x_0}a^\pm_{ij}. \tag{13}$$

With the solution for a_{ij} at hand one can compute the derivative $\partial_0 a_{ij}$. By using the following properties of the Mackdonald function

$$K_{\nu+1}(z) - K_{\nu-1}(z) = \frac{2\nu}{z}K_\nu, \quad K_{\nu+1}(z) + K_{\nu-1}(z) = -2K'_\nu(z)$$

one finds

$$\begin{aligned}
\partial_0 a^+_{ij}(x_0,\vec{k}) &= -\left(\frac{m}{x_0} + \frac{k^2 x_0}{2m}\right)a^+_{ij}(x_0,\vec{k}) + \frac{k^2 x_0^2}{2m\varepsilon}\frac{K_{m-1}(x_0 k)}{K_{m+1}(\varepsilon k)}a^+_{ij}(\vec{k}), \\
\partial_0 a^-_{ij}(x_0,\vec{k}) &= \left(\frac{m}{x_0} + \frac{k^2 x_0}{2m}\right)a^-_{ij}(x_0,\vec{k}) - \frac{k^2 x_0^2}{2m\varepsilon}\frac{K_{m+1}(x_0 k)}{K_{m-1}(\varepsilon k)}a^-_{ij}(\vec{k}).
\end{aligned} \tag{14}$$

Finally, substituting eqs.(10) and (14) into (13) we find the relation between $a^-_{ij}(\vec{k})$ and $a^+_{ij}(\vec{k})$:

$$a^-_{ij}(\vec{k}) = -\frac{K_{m-1}(\varepsilon k)}{K_{m+1}(\varepsilon k)}\left(a^+_{ij}(\vec{k}) + 2\frac{(k_i a^+_{jl}(\vec{k}) - k_j a^+_{il}(\vec{k}))k^l}{k^2}\right). \tag{15}$$

[2] As was noted in [16] for $\vec{k}^2<0$ there are two independent solutions regular in the interior. However, both of them are nonvanishing at infinity. A proper account of these solutions may be achieved by introducing an additional boundary at $x_0 = 1/\varepsilon$ and requiring the vanishing of the solution on this boundary. Then the solution is unique and in the limit $\varepsilon\to 0$ delivers the same contribution to the two-point function as the solution for $\vec{k}^2>0$ does. In the sequal, we restrict ourselves to the case $\vec{k}^2>0$.

In the sequal, we restrict ourselves to the case $m > 0$. When $\varepsilon \to 0$ the ratio $\frac{K_{m-1}(\varepsilon k)}{K_{m+1}(\varepsilon k)}$ behaves as $(\varepsilon k)^2$ for $m \geq 1$ and as $(\varepsilon k)^{2m}$ for $0 < m < 1$. Thus, if we keep a^+_{ij} finite in the limit $\varepsilon \to 0$, then a^-_{ij} tends to zero. Otherwise, keeping of a^-_{ij} finite leads to divergency of a^+_{ij}. Therefore, only the a^+_{ij} component can couple on the boundary with the CFT operator $\mathcal{O}_{ij}$. This conclusion can be also verified by considering the conformal transformations of a^+_{ij} on the boundary induced by isometries of AdS.

Denote by ξ^a a Killing vector of the background metric. Under diffeomorphisms generated by ξ the antisymmetric tensor a_{ij} transforms as follows

$$\delta a_{ij} = \xi^\rho \partial_\rho a_{ij} + a_{i\rho}\partial_j \xi^\rho - a_{j\rho}\partial_i \xi^\rho \tag{16}$$

Note that the Killing vectors of the AdS background can be written as

$$\begin{aligned} \xi^0 &= x_0(A_k x^k + D), \\ \xi^i &= -\frac{x_0^2 - \varepsilon^2}{2} A^i + \left(-\frac{1}{2}(A^i x^2 - 2x^i A_k x^k) + D x^i + \Lambda^i_j x^j + P^i \right), \end{aligned} \tag{17}$$

where A^i, D, Λ^i_j, P^i generate on the boundary special conformal transformations, dilatations, Lorentz transformations and shifts respectively. Since $\partial_i \xi^0 \sim x_0$ and a_{0i}, a^-_{ij} tend to zero when $\varepsilon \to 0$, in this limit one finds the following transformation law for the boundary value of a^+_{ij}:

$$\delta a^+_{ij} = \xi^k \partial_k a^+_{ij} + \xi^0 \partial_0 a^+_{ij} + \frac{1}{2}\left(a^+_{ik}(\partial_j \xi^k - \partial^k \xi_j) - a^+_{jk}(\partial_i \xi^k - \partial^k \xi_i) + \partial_k \xi^k a^+_{ij} \right).$$

Recalling that $\partial_0 a^+_{ij} = -\frac{m}{x_0} a^+_{ij} + O(1)$ and taking into account the explicit form of the Killing vectors we finally arrive at

$$\delta a^+_{ij} = \xi^k \partial_k a^+_{ij} + (2 - m)(A_k x^k + D) a^+_{ij} + a^+_{ik} \mathbf{\Lambda}^k{}_j - a^+_{jk} \mathbf{\Lambda}^k{}_i, \tag{18}$$

where $\mathbf{\Lambda}^{ij} = \Lambda^{ij} + x^i A^j - x^j A^i$. Eq.(18) is nothing but the standard transformation law for an antisymmetric tensor with the conformal weight $2 - m$ under the conformal mappings. Thus, on the boundary a^+_{ij} couples to the operator of conformal dimension $\Delta = 2 + m$. In particular, for $m = 1$ the antisymmetric tensor field a^+_{ij} transforms in $\mathbf{6}_c$ irrep of $SU(4)$ and couples on the boundary to the following YM operator [17]:

$$\mathcal{O}^{AB}_{ij} = \bar{\psi}^A \sigma_{ij} \bar{\psi}^B + 2i\phi^{AB} F^+_{ij}$$

that obviously has the conformal weight 3.

It is clear from the discussion above that a^+_{ij} plays the role of the coordinate. Now rewriting action (1) in the form $\int (p\dot{q} - H(p,q))$ we get

$$\begin{aligned} S &= -\int d^5x \left((a^-_{ij})^* \partial_0 a^+_{ij} + \partial_0 (a^+_{ij})^* a^-_{ij} + i\varepsilon^{ijkl}(a^*_{0i}\partial_j a_{kl} - a^*_{ij}\partial_k a_{0l}) + \frac{m}{x_0}(a^*_{ij}a^{ij} + 2a^*_{0i}a^{0i}) \right) \\ &+ \int d^5x \partial_0 \left((a^+_{ij})^* a^{-ij} \right). \end{aligned} \tag{19}$$

The last term in (19) is a total derivative, which is omitted in passing to the Hamiltonian formulation. Thus, the action one should use in computing the Green functions is given by

$$\mathbf{S} = -\int d^5x \left((a^-_{ij})^* \partial_0 a^+_{ij} + \partial_0 (a^+_{ij})^* a^-_{ij} + i\varepsilon^{ijkl}(a^*_{0i}\partial_j a_{kl} - a^*_{ij}\partial_k a_{0l}) + \frac{m}{x_0}(a^*_{ij}a^{ij} + 2a^*_{0i}a^{0i}) \right).$$

In the Lagrangian picture the total derivative term can be compensated by adding to action (19) the following boundary term

$$I = \int d^4x (a^+_{ij})^* a^{-ij}. \tag{20}$$

Thus, the on-shell value of $\mathbf{S}$ is given by

$$\mathbf{S} = -\int d^4k \frac{K_{m-1}(\varepsilon k)}{K_{m+1}(\varepsilon k)} (a^{+ij})^* \left(a^+_{ij} + 2\frac{(k_i a^+_{jl} - k_j a^+_{il})k^l}{k^2} \right). \tag{21}$$

When $\varepsilon \to 0$ and for m integer one finds

$$\frac{K_{m-1}(\varepsilon k)}{K_{m+1}(\varepsilon k)} = \frac{(-1)^m}{2^{2m-1}(m-1)!m!} (\varepsilon k)^{2m} \log \varepsilon k + ...,$$

while for non-integer m:

$$\frac{K_{m-1}(\varepsilon k)}{K_{m+1}(\varepsilon k)} = -\frac{\Gamma(2-m)}{2^{2m}(m-1)\Gamma(m+1)} (\varepsilon k)^{2m} + ...,$$

where in both cases we indicated only the first non-analytical term. Hence, from (21) we deduce the two-point function of $\mathcal{O}$ in the boundary CFT:

$$\langle \bar{\mathcal{O}}^{ij}(\vec{k}) \mathcal{O}^{kl}(\vec{q}) \rangle = -\delta(\vec{k}+\vec{q}) \frac{(-1)^m}{2^{2m-2}(m-1)!m!} (\varepsilon k)^{2m} \log \varepsilon k \left(\eta^{i[k}\eta^{l]j} + 2\frac{(k^i \eta^{j[k} - k^j \eta^{i[k})k^{l]}}{k^2} \right)$$

and a similar result for m non-integer. The last expression exhibits the structure of the correlation function for an antisymmetric tensor field of the conformal weight $2+m$ in the $D=4$, $\mathcal{N}=4$ SYM theory.

Note that on shell instead of (20) one can use the following boundary term

$$I = \frac{1}{2} \int d^4x a^*_{ij} a^{ij}.$$

Finally, we remark that in the case $m < 0$ the component a^-_{ij} should be regarded as the coordinate that leads to the change of the sign in the last formula.

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