Lorentz invariance and supersymmetry of four particle scattering amplitudes in the $S^N\mathbf{R}^8$ orbifold sigma model

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The $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model is expected to describe the Infrared limit of the matrix string theory. In the framework of the model the type IIA string interaction is governed by a vertex which was recently proposed by Dijkgraaf, Verlinde, and Verlinde. By using this interaction vertex, we derive all four particle scattering amplitudes directly from the orbifold model in the large N limit. [S0556-2821(99)00818-8]

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I. INTRODUCTION

To provide a heuristic basis for understanding various phenomena arising in superstrings, it was suggested that there exists a fundamental nonperturbative quantum theory in eleven dimensions, called M theory. The appropriate compactification of M theory leads to one of the five superstring theories and, in particular, the compactification on S^1 leads to the ten-dimensional type IIA superstring theory [1,2]. Although at present, we do not know how to formulate M theory as a quantum theory, it has been conjectured [3] that there is a precise equivalence between the M theory and the large N limit of the supersymmetric quantum matrix model which describes the dynamics of D particles [4].

In the original D-particle language, S^1 compactification of M theory amounts to applying a T-duality transformation along the S^1 direction, thereby turning the D particles into D strings. By adopting this approach, we can cast matrix theory into the form of the two-dimensional $\mathcal{N}=8$ maximally supersymmetric $U(N)$ Yang-Mills (YM) theory [5]. According to the matrix theory philosophy, in the limit $N\to\infty$, the YM theory should describe nonperturbative dynamics of type IIA superstrings. This is a new type of nonperturbative duality between a gauge theory and a string theory in which the string coupling constant is inversely proportional to the YM coupling constant: $g_{YM}^{-2}=\alpha' g_s^2$ [6–8]. Thus, we expect that the strong coupling expansion of the YM model describes the perturbative type IIA free string theory ($g_s=0$). Recently, it was conjectured by Dijkgraaf, Verlinde and Verlinde (DVV) [8] that in the infrared (IR) limit, the YM model reduces to the $\mathcal{N}=8$ non-Abelian $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model. The fact that the orbifold model is non-Abelian comes as no surprise, since in the IR limit, the original gauge symmetry group $U(N)$ reduces to the permutation group S_N. Furthermore, in [8] it was proposed that the string interaction in the orbifold sigma model is governed by a supersymmetric vertex of conformal dimension $(\frac{3}{2},\frac{3}{2})$. This vertex describes the elementary process of joining and splitting of strings and from the viewpoint of the gauge theory is responsible for partial restoring of the $U(N)$ gauge symmetry in some small region of space-time. With the DVV interaction vertex at hand, one is tempted to deduce string scattering amplitudes directly from the orbifold sigma model. It should be realized that this is a nontrivial problem due to the non-Abelian nature of the orbifold. Nevertheless, the necessary tools for computing tree-level diagrams were recently developed in [9,10]. In particular, the four-graviton scattering amplitudes for type IIA and IIB strings were calculated and were shown to be Lorentz invariant in the large N limit. It was also observed that the string kinematical factor exhibited manifest Lorentz invariance even at finite N.

In this paper, we complete the proof of the DVV conjecture on the level of tree diagrams by explicitly calculating all four particle scattering amplitudes for type IIA superstrings directly from the $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model and demonstrating their Lorentz and supersymmetry invariance. This provides a new consistency check on the matrix model conjecture. Furthermore, this is a new evidence of the hidden supersymmetry invariance of the matrix model and its existence is a necessary condition for the model to describe M theory.

We begin by reviewing the general formalism of the $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model developed in [10]. We define the S_N invariant vertex operators which create all massless states of type IIA string theory and which form a closed operator algebra. Following the approach of [8], we describe the DVV interaction vertex which is both

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space-time supersymmetric and $SO(8)$ invariant. Then, we construct the S-matrix to the second order in the coupling constant by sandwiching two DVV vertices in-between the asymptotic states corresponding to two incoming and two outgoing particles. As a result of this construction, we obtain the expression for the S-matrix element as the sum over specific four-point correlation functions which we explicitly list at the end of Sec. II C. The procedure for calculating these correlation functions was outlined in [10] and in Sec. II D, we summarize the main results. The appropriate scattering amplitude can then be obtained from the S-matrix element by making use of the reduction formula. Since the problem of calculating scattering amplitudes is equivalent to that of calculating all possible open string kinematical factors, it follows that to prove the DVV conjecture on the level of tree diagrams, we have to show that all kinematical factors obtained directly from the orbifold sigma model coincide with those obtained in the framework of the superstring theory. To this end, we first compute the open string kinematical factor corresponding to the scattering of two vector particles and two fermions. In the process of this calculation, we develop the necessary tools to deal with spinors and focus on the issue of the Lorentz invariance of the model. It turns out that the kinematical factor that we obtain is automatically Lorentz invariant and coincides with the well-known open string kinematical factor of the superstring theory. We then compute the remaining kinematical factors for all massless particles which make up the complete spectrum of IIA supergravity and show that they also coincide with those of the superstring theory. In conclusion, we discuss interesting problems that still remain open.

II. GENERAL FORMALISM

A. Free $S^N\mathbf{R}^8$ orbifold model

The action that defines the free $S^N\mathbf{R}^8\equiv(\mathbf{R}^8)^N/S_N$ orbifold sigma model is

$$S=\frac{1}{2\pi}\int d\tau d\sigma\sum_{I=1}^{N}\left(\partial_\tau X_I^i\partial_\tau X_I^i-\partial_\sigma X_I^i\partial_\sigma X_I^i+\frac{i}{2}\theta_I^a(\partial_\tau+\partial_\sigma)\theta_I^a+\frac{i}{2}\theta_I^{\dot a}(\partial_\tau-\partial_\sigma)\theta_I^{\dot a}\right). \tag{1}$$

Here X^i are eight real bosonic fields transforming in the $\mathbf{8}_v$ representation of the transversal group $SO(8)$ and $\theta^a,\theta^{\dot a}$ $a,\dot a=1,\dots,8$ are sixteen fermionic fields transforming in the $\mathbf{8}_s$ and $\mathbf{8}_c$ representations, respectively. As pertains to all orbifold models [11,12], the fundamental fields X^i and θ^α are allowed to have twisted boundary conditions:

$$X^i(\sigma+2\pi)=gX^i(\sigma),\quad \theta^\alpha(\sigma+2\pi)=g\theta^\alpha(\sigma), \tag{2}$$

where in the case of the $S^N\mathbf{R}^8$ orbifold model $g\in S_N$.

In the conventional QFT, the scattering amplitude to the second order in the coupling constant is extracted from the S-matrix element, schematically written as

$$\langle f|S|i\rangle\sim\langle f|\int dx_1dx_2T\{V_{int}(x_1)V_{int}(x_2)\}|i\rangle$$

by using the reduction formula. Consequently, to compute scattering amplitudes, we first need to define *in* ($|i\rangle$) and *out* ($|f\rangle$) states which are the states in the Hilbert space of the $S^N\mathbf{R}^8$ orbifold sigma model. Recall that the Hilbert space of an orbifold model decomposes into the direct sum of Hilbert spaces of twisted sectors corresponding to conjugacy classes of the discrete group defining the orbifold. The conjugacy classes of S_N are described by partitions $\{N_n\}$ of N and can be represented by

$$[g]=(1)^{N_1}(2)^{N_2}\cdots(s)^{N_s},\quad N=\sum_{n=1}^{s}nN_n, \tag{3}$$

where N_n is the multiplicity of the cyclic permutation (n) of n elements. In any conjugacy class $[g]$, there is only one element g_c that has the canonical block-diagonal form

$$g_c=\text{diag}\,(\underbrace{\omega_1,\dots,\omega_1}_{N_1\ times},\underbrace{\omega_2,\dots,\omega_2}_{N_2\ times},\dots\underbrace{\omega_s,\dots,\omega_s}_{N_s\ times}),$$

where ω_n is an $n\times n$ matrix that generates the cyclic permutation (n) of n elements. Since ω_n generates the group $\mathbf{Z}_n$, as can be easily verified, the Hilbert space $\mathcal{H}_{[g]}\equiv\mathcal{H}_{\{N_n\}}$ is decomposed into the graded N_n-fold symmetric tensor products of Hilbert spaces $\mathcal{H}_{(n)}$ which are $\mathbf{Z}_n$ invariant subspaces of the Hilbert space:

$$\mathcal{H}_{\{N_n\}}=\bigotimes_{n=1}^{s}S^{N_n}\mathcal{H}_{(n)}=\bigotimes_{n=1}^{s}\left(\underbrace{\mathcal{H}_{(n)}\otimes\cdots\otimes\mathcal{H}_{(n)}}_{N_n\ times}\right)^{S_{N_n}}.$$

The fundamental fields corresponding to the space $\mathcal{H}_{(n)}$ are $8n$ bosonic fields X_I^i and $16n$ fermionic fields θ^α with the cyclic boundary condition

$$X_I^i(\sigma+2\pi)=X_{I+1}^i(\sigma),\quad \theta_I^\alpha(\sigma+2\pi)=\theta_{I+1}^\alpha(\sigma),\qquad I=1,2,\dots,n. \tag{4}$$

As usual, states of the Hilbert space $\mathcal{H}_{(n)}$ are obtained by acting on momentum eigenstates with the string creation operators. Since the fundamental fields have twisted boundary conditions, the string creation operators have nontrivial transformation properties under the action of the group S_N. However, the space $\mathcal{H}_{(n)}$ must be $\mathbf{Z}_n$ invariant and to ensure this one has to impose the condition on the allowed states of $\mathcal{H}_{(n)}$:

$$(L_0-\overline{L_0})|\Psi\rangle=nm|\Psi\rangle,$$

where m is some integer and L_0 is the canonically normalized L_0-operator of a single long string obtained by gluing together the fields $X_I(\sigma)(\theta_I(\sigma))$ into one field $X(\sigma)(\theta(\sigma))$.

Before passing on to the construction of asymptotic states corresponding to $\mathcal{H}_{(n)}$, we note that according to [13], the Fock space of the second-quantized IIA type string is recovered in the limit $N\to\infty$, $n_i/N\to p_i^+$, where the finite ratio n_i/N is identified with the p_i^+ momentum of a long string. In this limit, the $\mathbf{Z}_n$ projection becomes the usual level-matching condition $L_0^{(i)}-\overline{L}_0^{(i)}=0$ for closed strings, while the individual p_i^- light-cone momentum is defined by means of the standard mass-shell condition $p_i^+p_i^-=L_0^{(i)}$.

B. Asymptotic states of $S^N\mathbf{R}^8$

We will consider the conformal field theory (CFT) on the sphere with coordinates $(z,\overline{z})$ obtained from the cylinder with coordinates (τ,σ) by performing the Wick rotation $\tau\to -i\tau$ followed by the map: $z=\mathrm{e}^{\tau+i\sigma}$, $\overline{z}=\mathrm{e}^{\tau-i\sigma}$.

The asymptotic states of the orbifold CFT model are obtained by acting with the S_N-invariant vertex operators on the Nevell-Schwarz (NS) vacuum $|0\rangle$ which is normalized according to

$$\langle 0|0\rangle = R^{8N}.$$

Here R is the radius of a circle onto which we compactify the string coordinates x_I^i in order to regularize the sigma model.

The most natural way to build S_N-invariant vertex operators $V_{[g]}$ is to first introduce a vertex operator V_g corresponding to a particular group element g of S_N and then sum over the conjugacy class of g. This procedure can be represented as follows:

$$V_{[g]}(z,\overline{z})=\frac{1}{N!}\sum_{h\in S_N} V_{h^{-1}gh}(z,\overline{z}). \tag{5}$$

The vertex operators $V_g(z,\overline{z})$ should be constructed from the twist fields of the orbifold model—the fields about which the fundamental fields have twisted boundary conditions. Since the monodromy conditions of the bosonic fundamental fields $X_I^i(z,\overline{z})$ are given by Eq. (4), we are led to the following definition of the bosonic twist field $\sigma_g(z,\overline{z})$:

$$X^i(ze^{2\pi i},\overline{z}e^{-2\pi i})\sigma_g(0,0)=gX^i(z,\overline{z})\sigma_g(0,0).$$

In exactly the same manner, we introduce the fermionic twist field $\Sigma_g(z,\overline{z})$.

In constructing the vertex operator $V_g(z,\overline{z})$, one is tempted to consider the tensor product of the bosonic twist field $\sigma_g(z,\overline{z})$ and the fermionic twist field $\Sigma_g(z,\overline{z})$. Although the non-Abelian nature of the orbifold sigma model does not admit the factorization into bosonic and fermionic (holomorphic and antiholomorphic) contributions, it was shown in [10] that this factorization can be assumed provided that one introduces a certain normalization constant, later denoted by κ, at the final stage of scattering amplitude calculation. Thus, we define the vertex operator $V_g(z,\overline{z})$ according to

$$V_g(z,\overline{z})=\sigma_g(z)\Sigma_g(z)\overline{\sigma_g}(\overline{z})\overline{\Sigma}_g(\overline{z}). \tag{6}$$

To clarify the meaning of the holomorphic (antiholomorphic) twist field $\sigma_g(z)(\overline{\sigma_g}(\overline{z}))$, we decompose the fundamental field $X(z,\overline{z})$ into the left- and right-moving components:

$$2X(z,\overline{z})=X(z)+X(\overline{z}),$$

so that now we can define $\sigma_g(z)$ and $\overline{\sigma}_g(\overline{z})$ according to

$$X^i(ze^{2\pi i})\sigma_g(0)=gX^i(z)\sigma_g(0)$$
$$\Leftrightarrow X^i(ze^{2\pi i})\sigma_{g^{-1}}(0)=g^{-1}X^i(z)\sigma_{g^{-1}}(0)$$

and

$$\overline{X^i}(\overline{z}e^{-2\pi i})\overline{\sigma}_g(0)=g\overline{X^i}(\overline{z})\overline{\sigma}_g(0)$$
$$\Rightarrow \overline{X^i}(\overline{z}e^{2\pi i})\overline{\sigma}_g(0)=g^{-1}\overline{X^i}(\overline{z})\overline{\sigma}_g(0).$$

Now the formal substitution $z\to\overline{z}$ leads to the conclusion that the operator σ_g is identical to the operator $\overline{\sigma}_{g^{-1}}$. For any element $g\in S_N$ with the decomposition

$$g=(n_1)(n_2)\cdots(n_{N_{str}}), \tag{7}$$

we represent $V_g(z,\overline{z})$ as the tensor product of operators each corresponding to some cycle (n_α):

$$V_g(z,\overline{z})=\otimes_{\alpha=1}^{N_{str}}V_{(n_\alpha)}(z,\overline{z}).$$

The operator, $\sigma_{(n)}(z,\overline{z})=\sigma_{(n)}(z)\overline{\sigma}_{(n)}(\overline{z})$ is a primary field [14] that creates the bosonic vacuum state of a twisted sector, labeled by (n), at the point $(z,\overline{z})$. We denote this vacuum state by $|(n)\rangle=\sigma_{(n)}(0,0)|0\rangle$. Recall that zero modes of fundamental fields θ^α form the Clifford algebra. Therefore, by triality, the vacuum state can be chosen to be the direct sum $\mathbf{8}_v\oplus\mathbf{8}_c$. Consequently, we define the primary spin fields of the holomorphic sector $\Sigma_{(n)}^i(z)$, $\Sigma_{(n)}^{\dot a}(z)$ which create the fermionic vacuum state: $|(n),\dot\mu\rangle=\Sigma_{(n)}^{\dot\mu}(0)|0\rangle$, where $\dot\mu=(i,\dot a)$. Under the world-sheet parity $z\to\overline{z}$ and the space reflection $X^3\to -X^3$, twist fields transform as follows:

$$\sigma_{(n)}(z)\leftrightarrow\overline{\sigma}_{(-n)}(\overline{z});\;\;\Sigma_{(n)}^{\dot a}(z)\leftrightarrow\overline{\Sigma}_{(-n)}^{a}(\overline{z});$$

$$\Sigma_{(n)}^{i}(z)\leftrightarrow\overline{\Sigma}_{(-n)}^{i}(\overline{z}),\;\; i\neq 3;\;\;\Sigma_{(n)}^{3}(z)\leftrightarrow -\overline{\Sigma}_{(-n)}^{3}(\overline{z}), \tag{8}$$

where $(-n)$ denotes the cycle with the reversed orientation corresponding to the element ω_n^{-1}. The third direction is singled out, since in our conventions $\gamma^3=1$ (see Appendix A).

Finally, we introduce the primary field $\sigma_g[\{\mathbf{k}_\alpha\}](z,\overline{z})$ corresponding to particles with transversal momenta $\mathbf{k}_\alpha$. Suppose that $g\in S_N$ has the decomposition (7) so that the following factorization takes place

$$\sigma_g(z,\overline{z}) = \otimes_{\alpha=1}^{N_{str}} \sigma_{(n_\alpha)}(z,\overline{z}),$$

then $\sigma_g[\{\mathbf{k}_\alpha\}](z,\overline{z})$ is defined by

$$\sigma_g[\{\mathbf{k}_\alpha\}](z,\overline{z}) = :e^{ik_\alpha^i Y_\alpha^i(z,\overline{z})/\sqrt{n_\alpha}}: \sigma_g(z,\overline{z}) \equiv \otimes_{\alpha=1}^{N_{str}} \sigma_{(n_\alpha)}[\mathbf{k}_\alpha],$$

where $n_1 = n_2 = \cdots = n_{N_1} = 1, n_{N_1+1} = n_{N_1+2} = \cdots = n_{N_1+N_2} = 2, \ldots$ and

$$Y_\alpha^i(z,\overline{z}) = \frac{1}{\sqrt{n_\alpha}} \sum_{I=1}^{n_\alpha} X_I^i(z,\overline{z}).$$

Combining the fermionic vacuum state with the vacuum state of the bosonic sector, we find 256 states that describe the complete spectrum of type IIA supergravity. In particular, the state with $k^+ = n/N$, transversal momentum $\mathbf{k}$ and polarization $\zeta^{\mu\dot{\nu}}$ is generated from the NS vacuum $|0\rangle$ by the vertex operator[1]

$$V_{(n)}[\mathbf{k},\zeta](z,\overline{z}) = \zeta^{\mu\dot{\nu}} \sigma_{(n)}[\mathbf{k}](z,\overline{z}) \Sigma_{(n)}^{\dot{\mu}}(z) \overline{\Sigma}_{(n)}^{\nu}(\overline{z}). \quad (9)$$

As was shown in [10], S_N-invariant vertex operators

$$V_{[g]}[\{\mathbf{k}_\alpha,\zeta_\alpha\}] = \frac{1}{N!} \sum_{h\in S_N} \otimes_{\alpha=1}^{N_{str}} V_{h^{-1}(n_\alpha)h}[\mathbf{k}_\alpha,\zeta_\alpha] \quad (10)$$

creating ground states, i.e., states with $\mathbf{k}_\alpha \equiv 0$, have the same conformal dimension which is a necessary condition for the orbifold sigma model to originate from the IR limit of the YM theory.

Next we turn to the description of the DVV interaction vertex. To this end, we introduce the first excited state $\tau_{(n)}(z,\overline{z})$ of the twisted sector which appears as the most singular term in the one-pion exchange (OPE)

$$\partial X_I^i(z) \sigma_{(n)}(w) = (z-w)^{-(1-\frac{1}{n})} e^{\frac{2\pi i}{n}I} \tau_{(n)}^i(w) + \cdots. \quad (11)$$

Suppose (n) is a simple transposition $(n=2)$ which exchanges X_I with X_J, then we can define the field $\tau_{IJ} \equiv \tau_{(2)}$. The DVV interaction vertex [8] is then given by

$$V_{int} = -\frac{\lambda N}{2\pi} \sum_{I<J} \int d^2z |z| (\tau^i(z)\Sigma^i(z)\overline{\tau}^j(\overline{z})\overline{\Sigma}^j(\overline{z}))_{IJ}, \quad (12)$$

where λ is a coupling constant proportional to the string coupling g_s.

The twist field $V_{IJ}(z,\overline{z}) \equiv (\tau^i(z)\Sigma^i(z)\overline{\tau}^j(\overline{z})\overline{\Sigma}^j(\overline{z}))_{IJ}$ is a weight $(\frac{3}{2},\frac{3}{2})$ conformal field and the coupling constant λ has dimension -1. As was shown in [8], this interaction vertex is space-time supersymmetric, $SO(8)$ invariant and describes the elementary string interaction. In addition, it is invariant with respect to the world-sheet parity transformation $z \to \overline{z}$ and an odd number of space reflections.

[1]In what follows, we call the wave function of a particle a polarization.

C. S-matrix element

With the account of Eq. (12), the S-matrix element to the second order in the coupling constant λ is given by the formula

$$\langle f|S|i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \langle f| \int d^2z_1 d^2z_2 |z_1||z_2| \times T(\mathcal{L}_{int}(z_1,\overline{z_1})\mathcal{L}_{int}(z_2,\overline{z_2}))|i\rangle, \quad (13)$$

where T means time-ordering: $|z_1| > |z_2|$ and

$$\mathcal{L}_{int}(z,\overline{z}) = \sum_{I<J} V_{IJ}(z,\overline{z}).$$

For the initial state $|i\rangle$, we choose the state corresponding to two incoming particles with transversal momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, polarizations ζ_1 and ζ_2, and for the final state $\langle f|$—the state corresponding to two outgoing particles with transversal momenta $\mathbf{k}_3$ and $\mathbf{k}_4$, polarizations ζ_3 and ζ_4, respectively:

$$|i\rangle = C_0 V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)|0\rangle,$$

$$\langle f| = C_\infty \lim_{z_\infty\to\infty} |z_\infty|^{4\Delta_\infty} \langle 0| V_{[g_\infty]}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4] \times (z_\infty,\overline{z_\infty}). \quad (14)$$

Recall that S_N invariant vertex operators $V_{[g]}[\{\mathbf{k}_\alpha,\zeta_\alpha\}] \times (z,\overline{z})$ were defined in Eq. (10). The elements g_0, g_∞ are chosen in the canonical block-diagonal form

$$g_0 = (n_0)(N-n_0), \quad g_\infty = (n_\infty)(N-n_\infty)$$

and to ensure proper normalization, the constants C_0 and C_∞ have to be equal to

$$C_0 = \sqrt{\frac{N!}{n_0(N-n_0)}}, \quad C_\infty = \sqrt{\frac{N!}{n_\infty(N-n_\infty)}}.$$

Following the approach of [8], we introduce the light-cone momenta of initial and final particles

$$k_1^+ = \frac{n_0}{N}, \quad k_2^+ = \frac{N-n_0}{N}, \quad k_3^+ = -\frac{n_\infty}{N}, \quad k_4^+ = -\frac{N-n_\infty}{N},$$

which satisfy the mass-shell condition: $k_a^+ k_a^- - \mathbf{k}_a \mathbf{k}_a = 0$ for each a, where $a = 1, \ldots, 4$. According to [10], the S-matrix element can be written as

$$\langle f|S|i\rangle = -i2\lambda^2 N^3 \delta(k_1^- + k_2^- + k_3^- + k_4^-)\mathcal{M}, \quad (15)$$

where the delta function results from the integral over z_1 and $\mathcal{M}$ is given by

$$\mathcal{M} = \int d^2u |u| F(u,\overline{u}). \quad (16)$$

Here we introduced a concise notation

$$F(u,\overline{u})=\langle f|T(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\overline{u}))|i\rangle$$
$$=C_0C_\infty\sum_{I<J;K<L}\langle V_{[g_\infty]}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4]$$
$$\times(\infty)T(V_{IJ}(1,1)V_{KL}(u,\overline{u}))$$
$$\times V_{[g_0]}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle. \quad (17)$$

In what follows, we assume for definiteness that $|u|<1$. From the definition (5) of $V_{[g]}$, it is clear that Eq. (17) is the sum over two conjugacy classes corresponding to group elements g_0 and g_∞. However, with the account of the invariance of the interaction vertex as well as of any correlation function constructed from vertex operators under the global action of the symmetric group, it becomes possible to reduce the sum over two conjugacy classes to the single sum:

$$F(u,\overline{u})=\frac{C_0C_\infty}{N!}\sum_{h_\infty\in S_N}\sum_{I<J;K<L}\langle V_{h_\infty^{-1}g_\infty h_\infty}(\infty)V_{IJ}(1,1)$$
$$\times V_{KL}(u,\overline{u})V_{g_0}(0,0)\rangle. \quad (18)$$

The obtained expression can be further simplified, however, to do so, we need to establish certain properties of correlation functions entering Eq. (18). To this end, we recall that the action (1) and the DVV interaction vertex are invariant under the world-sheet parity transformation $z\to\overline{z}$ combined with the space reflection $X^3 \to -X^3$, while the vertex operator $V_g[\{\mathbf{k}_\alpha,\zeta_\alpha\}](z,\overline{z})$ transforms into $\tilde{V}_{g^{-1}}[\{\mathbf{k}_\alpha,\zeta_\alpha\}]\times(z,\overline{z})\equiv V_{g^{-1}}[\{\tilde{\mathbf{k}}_\alpha,\tilde{\zeta}_\alpha\}](z,\overline{z})$, where $\tilde{k}_\alpha$, $\tilde{\zeta}_\alpha$ are the space reflected momenta and polarization, respectively, $\tilde{k}^3=-k^3$. Let us consider the correlation function $\langle V_{h_\infty^{-1}g_\infty h_\infty}V_{IJ}V_{KL}V_{g_0}\rangle$ with the monodromy condition

$$h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0=1\Rightarrow h_\infty^{-1}g_\infty h_\infty=g_0^{-1}g_{KL}g_{IJ}.$$

With the account of the world-sheet parity and the space reflection symmetries, we obtain the following equality:

$$\langle V_{g_0^{-1}g_{KL}g_{IJ}}V_{IJ}V_{KL}V_{g_0}\rangle=\langle\tilde{V}_{g_{IJ}g_{KL}g_0}V_{IJ}V_{KL}\tilde{V}_{g_0^{-1}}\rangle.$$

Due to the invariance of the correlation function under the global action of S_N and the fact that the elements g and g^{-1} belong to the same conjugacy class, we obtain

$$\langle V_{g_0^{-1}g_{KL}g_{IJ}}V_{IJ}V_{KL}V_{g_0}\rangle=\langle\tilde{V}_{g_{I'J'}g_{K'L'}g_0^{-1}}V_{I'J'}V_{K'L'}\tilde{V}_{g_0}\rangle$$

where $g_{I'J'}=hg_{IJ}h^{-1}$, $g_{K'L'}=hg_{KL}h^{-1}$, and the element h is such that $g_0^{-1}=h^{-1}g_0h$. Due to the $SO(8)$ invariance of the model, the correlation function (17) can depend only on the scalar products of momenta $\mathbf{k}_\alpha$ and polarizations ζ_α as well as on their contractions with the $SO(8)$ spin-tensor $\gamma^{ij}_{a\dot{b}}$. Obviously, all scalar products are invariant under the space reflection, while $\gamma^{ij}_{a\dot{b}}$ transforms into $\tilde{\gamma}^{ij}_{ab}$. Here $\tilde{\gamma}^i=\gamma^i$ for $i\neq3$ and $\tilde{\gamma}^3=-\gamma^3$. From the explicit form of $\gamma^{ij}_{a\dot{b}}$ given in Appendix A and with the account of $\gamma^{ij}_{ab}\equiv(\gamma^{ijT})_{\dot{a}\dot{b}}$, one can easily deduce that

$$\gamma^{ij}_{a\dot{b}}=\tilde{\gamma}^{ij}_{ab}.$$

Thus, we are justified to make the replacement $\tilde{\mathbf{k}}_\alpha\to\mathbf{k}_\alpha$ and $\tilde{\zeta}_\alpha\to\zeta_\alpha$ in the correlation function. Consequently, we arrive at the equality

$$\langle V_{g_0^{-1}g_{KL}g_{IJ}}V_{IJ}V_{KL}V_{g_0}\rangle=\langle V_{g_{I'J'}g_{K'L'}g_0^{-1}}V_{I'J'}V_{K'L'}V_{g_0}\rangle. \quad (19)$$

Now note that while the correlation function on the left-hand side of Eq. (19) corresponds to $\langle V_{h_\infty^{-1}g_\infty h_\infty}V_{IJ}V_{KL}V_{g_0}\rangle$ with the monodromy condition

$$h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0=1,$$

the correlation function on the right-hand side of Eq. (19) satisfies the monodromy condition

$$h_\infty^{-1}g_\infty h_\infty=g_{I'J'}g_{K'L'}g_0^{-1}\Rightarrow h_\infty^{-1}g_\infty h_\infty g_0g_{K'L'}g_{I'J'}=1.$$

Therefore, the contribution of terms satisfying either of the two monodromy conditions coincide. As it was shown in [10], the only nontrivial terms in Eq. (18) are those that satisfy precisely these two monodromy conditions. Consequently, we can include only terms corresponding to one of the monodromy conditions and place a factor of 2 in front of the entire expression. Using the same procedures as those in establishing Eq. (19), we now show that the correlation function $F(u,\overline{u})$ is real. To this end, we first consider the result of complex conjugating the correlation function:

$$\langle V_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1,1)V_{KL}(u,\overline{u})$$
$$\times V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle^*$$
$$=\lim_{z_\infty\to\infty}\lim_{z_0\to0}|z_\infty|^{-4\Delta_{g_\infty}[\{\mathbf{k}_3,\mathbf{k}_4\}]}|z_0|^{-4\Delta_{g_0}[\{\mathbf{k}_1,\mathbf{k}_2\}]}|u|^{-6}$$
$$\times\left\langle V_{g_0^{-1}}[-\mathbf{k}_1,\zeta_1;-\mathbf{k}_2,\zeta_2]\left(\frac{1}{z_\infty},\frac{1}{\overline{z}_\infty}\right)V_{KL}\left(\frac{1}{u},\frac{1}{\overline{u}}\right)\right.$$
$$\left.\times V_{IJ}(1,1)V_{g_\infty^{-1}}[-\mathbf{k}_3,\zeta_3;-\mathbf{k}_4,\zeta_4]\left(\frac{1}{z_0},\frac{1}{\overline{z}_0}\right)\right\rangle,$$

where we took into account the conjugating property of a vertex operator

$$(V_g[\{\mathbf{k}_\alpha\}](z))^\dagger=z^{-2\Delta_g[\{\mathbf{k}_\alpha\}]}V_{g^{-1}}[\{-\mathbf{k}_\alpha\}]\left(\frac{1}{z}\right), \quad (20)$$

and the fact that the DVV vertex is of conformal dimension $(\frac{3}{2},\frac{3}{2})$. Due to the $SO(8)$ invariance, we can make a replacement $-\mathbf{k}_\alpha \to \mathbf{k}_\alpha$ and after performing the conformal transformation $z\to 1/z$ obtain

$$\begin{aligned}&\langle V_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1,1)V_{KL}(u,\bar{u})\\&\quad\times V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle^*\\&=\langle V_{g_\infty^{-1}}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1,1)V_{KL}(u,\bar{u})\\&\quad\times V_{g_0^{-1}}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle\\&=\langle V_{g'_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{I'J'}(1,1)V_{K'L'}(u,\bar{u})\\&\quad\times V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle,\end{aligned}$$

where $h\in S_N$ is the solution of $h^{-1}g_0^{-1}h=g_0$ and

$$h^{-1}g_\infty^{-1}h=g'_\infty,\quad h^{-1}g_{IJ}h=g_{I'J'},\quad h^{-1}g_{KL}h=g_{K'L'}.$$

Now we apply this result to find the complex conjugate of $F(u,\bar{u})$:

$$\begin{aligned}F(u,\bar{u})^*&=\frac{2C_0C_\infty}{N!}\sum_{h_\infty\in S_N}{}'\sum_{I<J;K<L}\langle V_{h_\infty^{-1}g_\infty h_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4\zeta_4](\infty)V_{IJ}(1,1)V_{KL}(u,\bar{u})V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle^*\\&=\frac{2C_0C_\infty}{N!}\sum_{h_\infty\in S_N}{}'\sum_{I<J;K<L}\langle V_{h'^{-1}_\infty g_\infty h'_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4\zeta_4](\infty)V_{I'J'}(1,1)V_{K'L'}(u,\bar{u})V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle\\&=\frac{2C_0C_\infty}{N!}\sum_{h'_\infty\in S_N}{}'\sum_{I'<J';K'<L'}\langle V_{h'^{-1}_\infty g_\infty h'_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4\zeta_4](\infty)V_{I'J'}(1,1)V_{K'L'}(u,\bar{u})V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle\\&=F(u,\bar{u}),\end{aligned}$$

where

$$h^{-1}g_0^{-1}h=g_0,\quad h^{-1}h_\infty^{-1}g_\infty^{-1}h_\infty h=h'^{-1}_\infty g_\infty h'_\infty$$

and the prime in the sum over h_∞ indicates that we include only terms which satisfy the monodromy condition $h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0=1$. This completes the proof.

As was shown in [10], using the global S_N invariance of the model, one can recast $F(u,\bar{u})$ into the following form

$$\begin{aligned}F(u,\bar{u})=2N^2\sqrt{k_1^+k_2^+k_3^+k_4^+}\Bigg(&\sum_{I=1}^{n_\infty}\langle V_{g_\infty(I)}(\infty)V_{I,I+N-n_\infty}(1,1)\\&\times V_{n_0N}(u,\bar{u})V_{g_0}(0,0)\rangle+\sum_{I=1}^{N-n_\infty}\langle V_{g_\infty(I)}(\infty)\\&\times V_{I,I+n_\infty}(1,1)V_{n_0N}(u,\bar{u})V_{g_0}(0,0)\rangle\\&+\sum_{J=n_0+1}^{n_\infty}\langle V_{g_\infty(J)}(\infty)V_{n_0J}(1,1)V_{n_\infty N}(u,\bar{u})\\&\times V_{g_0}(0,0)\rangle+\sum_{J=n_0+n_\infty+1}^{N}\langle V_{g_\infty(J)}(\infty)V_{n_0J}(1,1)\\&\times V_{n_0+n_\infty,N}(u,\bar{u})V_{g_0}(0,0)\rangle\Bigg),\end{aligned}\tag{21}$$

where the elements g_∞ have to be found from the equation $g_\infty g_{IJ}g_{KL}g_0=1$.[2] To simplify the notation, we did not explicitly indicate the momenta $\mathbf{k}$ and polarizations ζ in Eq. (21).

Consequently, the S-matrix element is constructed from the correlation functions

$$\begin{aligned}G_{IJKL}(u,\bar{u})\equiv&\langle V_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1,1)\\&\times V_{KL}(u,\bar{u})V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0,0)\rangle\end{aligned}\tag{22}$$

corresponding to $|u|<1$ and the correlation functions obtained from Eq. (22) by interchanging $(u,\bar{u})\leftrightarrow(1,1)$ and therefore corresponding to $|u|>1$. Here all possible combinations of g_∞, g_{IJ} and g_{KL}, g_0 are listed in Eq. (21).

D. Correlation functions

Taking into account the definition (9) of $V_g[\mathbf{k}_\alpha,\zeta_\alpha]$ and the expression (12) for the DVV interaction vertex, we obtain the holomorphic contribution to the correlation function (22):

$$G_{IJKL}(u)=G_{IJKL}^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}\zeta_1^{\dot\mu_1}\zeta_2^{\dot\mu_2}\zeta_3^{\dot\mu_3}\zeta_4^{\dot\mu_4},$$

[2] Here we assume for definiteness that $n_0<n_\infty$.

where

$$G_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}=\langle\sigma_{g_\infty}[\mathbf{k}_3/2,\mathbf{k}_4/2](\infty)\tau_{IJ}^{i}(1)\tau_{KL}^{j}(u)\sigma_{g_0}\times[\mathbf{k}_1/2,\mathbf{k}_2/2](0)\rangle\times\langle\Sigma_{g_\infty}^{\dot{\mu}_3\dot{\mu}_4}(\infty)\Sigma_{IJ}^{i}(1)\Sigma_{KL}^{j}(u)\Sigma_{g_0}^{\dot{\mu}_1\dot{\mu}_2}(0)\rangle\equiv\langle\tau_i\tau_j\rangle(u)G_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4 ij}(u). \quad (23)$$

Without any loss of generality, we will always assume that the polarization $\zeta^{\dot{\mu}\nu}$ can be taken in the form $\zeta^{\dot{\mu}}\zeta^{\nu}$.

In the approach of [10], the calculation of the correlation function $G_{IJKL}(u)$ was based on the stress-energy tensor method [15] which requires the knowledge of the Green function for DN bosonic fields $X_I^i(z)$, $I=1,\ldots,N$, $i=1,\ldots,D$. Recall that $X_I^i(z)$ have cyclic boundary conditions (4) around the insertion points of the twist fields $\sigma_{(n)}(z)$ and therefore the corresponding Green function is N-valued. So, to find the Green function, and consequently the correlation function G_{IJKL}, one needs to construct the N-fold map from the z-plane, on which it is multi-valued, to the sphere, which we call the t-sphere, on which it is single-valued. According to [10], this map is unique, and is given by the formula

$$z=\left(\frac{t}{t_1}\right)^{n_0}\left(\frac{t-t_0}{t_1-t_0}\right)^{N-n_0}\left(\frac{t_1-t_\infty}{t-t_\infty}\right)^{N-n_\infty}\equiv u(t), \quad (24)$$

where we require the point $t=x$ to be mapped to $z=u$. Due to the projective invariance, the positions of points t_0, t_1, and t_∞ can be chosen to depend on x in a specific manner, that is $t_0=t_0(x)$, $t_1=t_1(x)$, and $t_\infty=t_\infty(x)$, and one possible choice of this dependence is described in [10]. If we make the substitution (see [10]):

$$t_0=x-1,$$

$$t_\infty=x-\frac{(N-n_\infty)x}{(N-n_0)x+n_0},$$

$$t_1=\frac{N-n_0-n_\infty}{n_\infty}+\frac{n_0x}{n_\infty}-\frac{N(N-n_\infty)x}{n_\infty[(N-n_0)x+n_0]}.$$

Eq. (24) transforms into a function of x alone which can be viewed as the $2(N-n_0)$-fold covering of the u-sphere by the x-sphere. Since the number of nontrivial correlation functions in Eq. (21) is also equal to $2(N-n_0)$, as one can easily verify, we see that the t-sphere can be represented as the union of $2(N-n_0)$ domains and each domain, denoted by V_{IJKL}, contains the point x corresponding to some correlation function from Eq. (21).

Finally note that as was shown in [10], the overall phase of $G_{IJKL}(u)$ cannot be determined and, in principle, can depend on the indices I,J,K,L. However, below we will show that the correlation function of the holomorphic sector is complex-conjugated to the correlation function of the anti-holomorphic sector. Therefore, by combining the two sectors, the phase ambiguity disappears. To prove this assertion, we have to take into account the symmetry of a correlation function under the change

$$\sigma_g[\mathbf{k}/2]\leftrightarrow\overline{\sigma}_{g^{-1}}[\tilde{\mathbf{k}}/2] \quad \text{and} \quad \Sigma_g^{\dot{\mu}}\leftrightarrow\overline{\Sigma}_{g^{-1}}^{\mu}$$

to obtain the equality

$$\langle V_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1)V_{KL}(u)\times V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0)\rangle=\langle\overline{V}_{g_\infty^{-1}}[\tilde{\mathbf{k}}_3,\tilde{\zeta}_3;\tilde{\mathbf{k}}_4,\tilde{\zeta}_4](\infty)\overline{V}_{IJ}(1)\overline{V}_{KL}(u)\times\overline{V}_{g_0^{-1}}[\tilde{\mathbf{k}}_1,\tilde{\zeta}_1;\tilde{\mathbf{k}}_2,\tilde{\zeta}_2](0)\rangle.$$

Then complex conjugating the obtained expression gives

$$\langle\overline{V}_{g_\infty^{-1}}[\tilde{\mathbf{k}}_3,\tilde{\zeta}_3;\tilde{\mathbf{k}}_4,\tilde{\zeta}_4](\infty)\overline{V}_{IJ}(1)\overline{V}_{KL}(u)\times\overline{V}_{g_0^{-1}}[\tilde{\mathbf{k}}_1,\tilde{\zeta}_1;\tilde{\mathbf{k}}_2,\tilde{\zeta}_2](0)\rangle^*=\lim_{z_\infty\to\infty}\lim_{z_0\to0}z_\infty^{-2\Delta_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4]}z_0^{-2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]}u^{-3}\times\left\langle\overline{V}_{g_0}[-\tilde{\mathbf{k}}_1,\tilde{\zeta}_1;-\tilde{\mathbf{k}}_2,\tilde{\zeta}_2]\left(\frac{1}{z_0}\right)\overline{V}_{KL}\left(\frac{1}{u}\right)\overline{V}_{IJ}(1)\times\overline{V}_{g_\infty}[-\tilde{\mathbf{k}}_3,\tilde{\zeta}_3;-\tilde{\mathbf{k}}_4,\tilde{\zeta}_4]\left(\frac{1}{z_\infty}\right)\right\rangle.$$

Because of the $SO(8)$ invariance of the correlation function, we can make the replacement $-\tilde{\mathbf{k}}\to\mathbf{k}$, $\tilde{\zeta}\to\zeta$ and after performing the conformal transformation $z\to1/z$ obtain

$$\langle V_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)V_{IJ}(1)V_{KL}(u)\times V_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0)\rangle^*=\langle\overline{V}_{g_\infty}[\mathbf{k}_3,\zeta_3;\mathbf{k}_4,\zeta_4](\infty)\overline{V}_{IJ}(1)\overline{V}_{KL}(u)\times\overline{V}_{g_0}[\mathbf{k}_1,\zeta_1;\mathbf{k}_2,\zeta_2](0)\rangle.$$

By making the formal substitution $z\to\overline{z}$, we arrive at the correlation function of the anti-holomorphic sector containing right-moving fermions instead of left-moving ones. Thus, if the anti-holomorphic sector is obtained from the holomorphic one by the substitution: $z\to\overline{z}$, left-moving fermion $\to$ right-moving fermion, then the overall phase of $G_{IJKL}(u,\overline{u})$ is irrelevant.

Now we present the solution for the correlation function $G_{IJKL}(u)$ that was found in [10]. In particular, $\langle\tau_i\tau_j\rangle(u)$ is given by

$$\langle\tau_i\tau_j\rangle(u)=-\delta^{ij}\frac{4x(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}+\langle\tau_i\tau_j\rangle_k,$$

$$\langle\tau_i\tau_j\rangle_k=-\left(\frac{[x+n_0/(N-n_0)]}{n_0}k_1^i+\frac{[x-(N-n_0-n_\infty)/(N-n_0)]}{n_\infty}k_3^i+\frac{1}{N-n_0}k_4^i\right)\times\left[(x-1)k_1^j+xk_3^j+\frac{n_0-n_\infty}{N-n_\infty}\left(x-\frac{n_0}{n_0-n_\infty}\right)k_4^j\right], \tag{25}$$

while the correlation function $G_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4 ij}(u)$ is equal to

$$G_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4 ij}(u)=\kappa^{1/2}\frac{iR^4}{2^6(n_\infty-n_0)(N-n_0)}\left(\frac{n_\infty n_0(N-n_\infty)}{(N-n_0)}\right)^{1/2}\left(\frac{n_\infty-n_0}{N-n_0}\right)^{(1/4)k_1k_3}\frac{[x-n_0/(n_0-n_\infty)]^3}{u^{3/2}(x-\alpha_1)^2(x-\alpha_2)^2}\times\left(\frac{x[x-(N-n_0-n_\infty)/(N-n_0)]}{[x-n_0/(n_0-n_\infty)]}\right)^{1+(1/4)k_1k_4}\left(\frac{(x-1)[x+n_0/(N-n_0)]}{[x-n_0/(n_0-n_\infty)]}\right)^{1+(1/4)k_3k_4}T_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4 ij}(u). \tag{26}$$

Here $T_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4 ij}(u)$ is defined in the $SU(4)\times U(1)$ basis according to

$$T_{IJKL}^{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4\mathcal{A}_5\mathcal{A}_6}(u)=C(g_0,g_\infty)\frac{x^{d_0}(x-1)^{d_1}[x+n_0/(N-n_0)]^{d_2}[x-(N-n_0-n_\infty)/(N-n_0)]^{d_3}[x-n_0/(n_0-n_\infty)]^{d_4}}{[(x-\alpha_1)(x-\alpha_2)]^{d_5}}, \tag{27}$$

the coefficients d_i are given by

$$d_0=\mathbf{p}_1\mathbf{p}_4+\mathbf{p}_6\mathbf{p}_1+\mathbf{p}_6\mathbf{p}_4,\quad d_1=\mathbf{p}_6\mathbf{p}_3+\mathbf{p}_6\mathbf{p}_4+\mathbf{p}_3\mathbf{p}_4,$$

$$d_2=\mathbf{p}_1\mathbf{p}_2+\mathbf{p}_6\mathbf{p}_1+\mathbf{p}_6\mathbf{p}_2,\quad d_3=\mathbf{p}_6\mathbf{p}_2+\mathbf{p}_6\mathbf{p}_3+\mathbf{p}_2\mathbf{p}_3,$$

$$d_4=\mathbf{p}_6\mathbf{p}_1+\mathbf{p}_6\mathbf{p}_3+\mathbf{p}_1\mathbf{p}_3,\quad d_5=-\mathbf{p}_5\mathbf{p}_6,$$

$$d_6=\mathbf{p}_1\mathbf{p}_5+\mathbf{p}_3\mathbf{p}_5+\mathbf{p}_1\mathbf{p}_3-\mathbf{p}_2\mathbf{p}_6$$

and

$$|C(g_0,g_\infty)|=\frac{n_0^{\mathbf{p}_1\mathbf{p}_5}n_\infty^{\mathbf{p}_3\mathbf{p}_5}(N-n_\infty)^{\mathbf{p}_4\mathbf{p}_6}(n_\infty-n_0)^{d_4-d_5}}{(N-n_0)^{d_6}}. \tag{28}$$

A few comments are in order. First, recall that κ was introduced in Sec. II B as a multiplicative factor which compensates for the non-Abelian nature of the orbifold. This constant is equal to 2^3 (for derivation see [10]).

Secondly, computation of the fermionic correlation function $\langle\Sigma_{g_\infty}^{\dot{\mu}_3\dot{\mu}_4}(\infty)\Sigma_{IJ}^i(1)\Sigma_{KL}^j(u)\Sigma_{g_0}^{\dot{\mu}_1\dot{\mu}_2}(0)\rangle$ was done by bosonizing the fermions [10] in the framework of the $SU(4)\times U(1)$ formalism [16] which is concisely presented in Appendix B. Here we only note that in this formalism, there is a one-to-one correspondence between the $SU(4)\times U(1)$ index $\mathcal{A}\equiv\{A,\bar{A}\}$ and the weight vector $\mathbf{p}$. Specifically, if $\mathcal{A}$ corresponds to $\mathbf{8}_v$, then $\mathbf{p}^A(\mathbf{p}^{\bar{A}})=\mathbf{e}^A(-\mathbf{e}^A)$ and if it corresponds to $\mathbf{8}_c$, then $\mathbf{p}^{\dot{A}}(\mathbf{p}^{\bar{\dot{A}}})=\mathbf{q}^{\dot{A}}(-\mathbf{q}^{\dot{A}})$, where $\pm\mathbf{e}^A$ has components δ_B^A and $\pm\mathbf{q}^{\dot{A}}$ is defined in Eq. (B1).

Before we go on to consider the scattering amplitude, let us point out the remarkable property of $\langle\tau_i\tau_j\rangle_k$, namely

$$\langle\tau_+\tau_\mu\rangle_k=0=\langle\tau_\mu\tau_+\rangle_k. \tag{29}$$

To prove this assertion, we note that the "+" light-cone component of the first factor in $\langle\tau_i\tau_j\rangle_k$ is equal to

$$\frac{[x+n_0/(N-n_0)]}{n_0}k_1^++\frac{[x-(N-n_0-n_\infty)/(N-n_0)]}{n_\infty}k_3^++\frac{1}{N-n_0}k_4^+=\frac{1}{N}\left[\left(x+\frac{n_0}{N-n_0}\right)-\left(x-\frac{N-n_0-n_\infty}{N-n_0}\right)-\frac{N-n_\infty}{N-n_0}\right]=0, \tag{30}$$

while the "+" light-cone component of the second factor is equal to

$$(x-1)k_1^++xk_3^++\frac{n_0-n_\infty}{N-n_\infty}\left(x-\frac{n_0}{n_0-n_\infty}\right)k_4^+=\frac{1}{N}\left[n_0(x-1)-n_\infty x-(n_0-n_\infty)\left(x-\frac{n_0}{n_0-n_\infty}\right)\right]=0.$$

This property will be used in establishing the Lorentz invariance of scattering amplitudes.

1. Scattering amplitude

Up to now we considered the correlation function $G^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u,\overline{u})$ correspoding to $|u|<1$. It turns out that the correlation function corresponding to $|u|>1$ is again given by Eq. (26) and so the time-ordering in Eq. (17) can be omitted. Consequently, from Eqs. (21), (22) and (23), we find that $\mathcal{M}$ is equal to

$$\mathcal{M}=2N^2\sqrt{k_1^+k_2^+k_3^+k_4^+}\sum_{IJKL}\int d^2u|u|G^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u)\overline{G}^{\nu_1\nu_2\nu_3\nu_4}_{IJKL}\times(\overline{u})\zeta_1^{\dot\mu_1\nu_1}\zeta_2^{\dot\mu_2\nu_2}\zeta_3^{\dot\mu_3\nu_3}\zeta_4^{\dot\mu_4\nu_4}.$$

Substituting Eq. (26) for the holomorphic part of the correlation function $G_{IJKL}(u,\overline{u})$ and its complex conjugate for the anti-holomorphic part so as to get rid of the phase ambiguity, we arrive at the following expression for $\mathcal{M}$:

$$\mathcal{M}=\frac{R^8}{2^8\sqrt{k_1^+k_2^+k_3^+k_4^+}}\left(\frac{n_0n_\infty(N-n_\infty)}{N(N-n_0)}\right)^2\left(\frac{n_\infty-n_0}{N-n_0}\right)^{(1/2)k_1k_3}\times\int d^2u\left|\frac{du}{dx}\right|^2\left|\frac{x[x-(N-n_0-n_\infty)/(N-n_0)]}{[x-n_0/(n_0-n_\infty)]}\right|^{(1/2)k_1k_4}\times\left|\frac{(x-1)[x+n_0/(N-n_0)]}{[x-n_0/(n_0-n_\infty)]}\right|^{(1/2)k_3k_4}\times\sum_{IJKL}T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u)T^{\nu_1\nu_2\nu_3\nu_4}_{IJKL}(\overline{u})\zeta_1^{\dot\mu_1\nu_1}\zeta_2^{\dot\mu_2\nu_2}\zeta_3^{\dot\mu_3\nu_3}\zeta_4^{\dot\mu_4\nu_4},$$

where we introduced a concise notation

$$T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u)=\langle\tau_i\tau_j\rangle T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4 ij}_{IJKL}(u). \tag{31}$$

Recall that under the transformation $u\rightarrow x$, the u-sphere is mapped onto the domain V_{IJKL}. Taking this into account and performing the change of variables [9]

$$z=\frac{x[x-(N-n_0-n_\infty)/(N-n_0)]}{[(n_\infty-n_0)/(N-n_0)][x-n_0/(n_0-n_\infty)]}\Rightarrow dz=\frac{(x-\alpha_1)(x-\alpha_2)}{[(n_\infty-n_0)/(N-n_0)][x-n_0/(n_0-n_\infty)]^2}dx, \tag{32}$$

the expression for $\mathcal{M}$ assumes the conventional form

$$\mathcal{M}=\frac{R^8}{2^8\sqrt{k_1^+k_2^+k_3^+k_4^+}}\left(\frac{n_0n_\infty(N-n_\infty)}{N(N-n_0)}\right)^2\int d^2z\left|\frac{dx}{dz}\right|^2\times|z|^{(1/2)k_1k_4}|1-z|^{(1/2)k_3k_4}T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}(z)\times T^{\nu_1\nu_2\nu_3\nu_4}(\overline{z})\zeta_1^{\dot\mu_1\nu_1}\zeta_2^{\dot\mu_2\nu_2}\zeta_3^{\dot\mu_3\nu_3}\zeta_4^{\dot\mu_4\nu_4}. \tag{33}$$

Now it follows from Eq. (15) that in the limit $R\rightarrow\infty$, the expression for the S-matrix element to the second order in the coupling constant λ is given by

$$\langle f|S|i\rangle=-i\lambda^2 2^{-8}2N\delta_{m_1+m_2+m_3+m_4,0}\delta\left(\sum_i k_i^-\right)\times\delta^D\left(\sum_i \mathbf{k}_i\right)\left(\frac{\prod_{i=1}^4 (k_i^+)^{\epsilon(\dot\mu_i)}(k_i^+)^{\epsilon(\nu_i)}}{\prod_{i=1}^4 k_i^+}\right)^{1/2}\mathcal{I}(\zeta;k) \tag{34}$$

where

$$\mathcal{I}(\zeta;k)=\left(\frac{n_0n_\infty(N-n_\infty)}{N-n_0}\right)^2\left(\prod_{i=1}^4 (k_i^+)^{-\epsilon(\dot\mu_i)}(k_i^+)^{-\epsilon(\nu_i)}\right)^{1/2}\times\int d^2z\left|\frac{dx}{dz}\right|^2|z|^{(1/2)k_1k_4}|1-z|^{(1/2)k_3k_4}T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}(z)\times T^{\nu_1\nu_2\nu_3\nu_4}(\overline{z})\zeta_1^{\dot\mu_1\nu_1}\zeta_2^{\dot\mu_2\nu_2}\zeta_3^{\dot\mu_3\nu_3}\zeta_4^{\dot\mu_4\nu_4}. \tag{35}$$

Here $\epsilon^{(\dot\mu_i)}(\epsilon^{(\nu_i)})$ is equal to 0, if $\dot\mu_i(\nu_i)$ corresponds to $\mathbf{8}_v$ and is equal to 1, if $\dot\mu_i(\nu_i)$ corresponds to $\mathbf{8}_c$ ($\mathbf{8}_s$). Also note that we have restored δ-functions responsible for the momentum conservation law and represented the light-cone momenta k_i^+ as $k_i^+=m_i/N$. In the next section, we will compute all open string kinematical factors and show that all dependence on N in $\mathcal{I}(\zeta;k)$ is absorbed into the light-cone momenta k_i^+ and, hence, we are justified to consider the limit $N\rightarrow\infty$ in Eq. (34). In this limit, the combination $N\delta_{m_1+m_2+m_3+m_4,0}$ goes to $\delta(\Sigma_i k_i^+)$ and formula (34) acquires the form

$$\langle f|S|i\rangle=-i\lambda^2\delta^{D+2}\left(\sum_i k_i^\mu\right)\left(\frac{\prod_{i=1}^4 (k_i^+)^{\epsilon(\dot\mu_i)}(k_i^+)^{\epsilon(\nu_i)}}{\prod_{i=1}^4 k_i^+}\right)^{1/2}\times 2^{-8}\mathcal{I}(\zeta;k). \tag{36}$$

In order to extract the scattering amplitude $A(1,2,3,4)$ from the S-matrix element, one needs to make use of the reduction formula, namely

$$\langle f|S|i\rangle=-i\,\delta^{D+2}\left(\sum_i \mathbf{k}_i^\mu\right)\left(\frac{\prod_{i=1}^4 (k_i^+)^{\epsilon(\dot\mu_i)}(k_i^+)^{\epsilon(\nu_i)}}{\prod_{i=1}^4 k_i^+}\right)^{1/2}\times A(1,2,3,4). \tag{37}$$

Using Eq. (36) for the S-matrix element and taking into account the reduction formula, we obtain the general expression for the four particle scattering amplitude $A(1,2,3,4)$:

$$A(1,2,3,4)=\lambda^2 2^{-8}\mathcal{I}(\zeta;k).$$

Consequently, the problem of finding $A(1,2,3,4)$ in the $S^N\mathbf{R}^8$ orbifold sigma model is reduced to the calculation of $\mathcal{I}(\zeta;k)$. In the next section, we will find that $\mathcal{I}(\zeta;k)$ can be written in the form which is standard in the superstring theory, namely

$$\mathcal{I}(\zeta;k)=K(\zeta;k)K(\zeta;k)C(s,t,u), \tag{38}$$

where

$$C(s,t,u)=-\pi\frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)}. \tag{39}$$

Here we introduced open string kinematical factors $K(\zeta;k)$ which we will show coincide with the well-known kinematical factors obtained in the framework of the superstring theory.

III. KINEMATICAL FACTORS

A. Vector particle+vector particle → fermion+fermion

To conform with the standard notation of the superstring theory, let us denote the polarization of a left-(right-) moving fermion by $u^{\dot a}(u^a)$ instead of $\zeta^{\dot a}(\zeta^a)$ preserving ζ for polarizations of massless vector particles.

As follows from Eqs. (38) and (35), in order to find the kinematical factor corresponding to two massless vector particles in the inital state (i.e., $\dot\mu_1\to i_1,\ \dot\mu_2\to i_2$) and two fermions in the final state (i.e., $\dot\mu_3\to\dot a_3,\ \dot\mu_4\to\dot a_4$), one first has to find

$$T^{i_1i_2\dot a_3\dot a_4}(z)=\langle\tau_i\tau_j\rangle(z)T^{i_1i_2\dot a_3\dot a_4ij}(z), \tag{40}$$

where the spin-tensor $T^{i_1i_2\dot a_3\dot a_4ij}(z)$ is determined up to an unknown phase by Eq. (27). The overall phase is irrelevant in our computations, since we choose the kinematical factor of the right-moving sector to coincide with that of the left-moving sector. Nevertheless, it is essential to know the relative phases, of $T^{i_1i_2\dot a_3\dot a_4ij}(z)$ for different values of $SO(8)$ indices i_m and $\dot a_n$. In order to fix these phases, we decompose the spin-tensor $T^{i_1i_2\dot a_3\dot a_4ij}(z)$ into the sum of $SO(8)$ invariant rank six spin-tensors:

$$\begin{aligned}T^{i_1i_2\dot a_3\dot a_4ij}(z)=&\tfrac14\gamma^{[i_1i_2ij]}{}_{\dot a_3\dot a_4}C_1(z)+\tfrac12\gamma^{[i_1i_2]}{}_{\dot a_3\dot a_4}\delta^{ij}C_2(z)\\&+\tfrac12\gamma^{[i_2j]}{}_{\dot a_3\dot a_4}\delta^{i_1i}C_3(z)+\tfrac12\gamma^{[i_2i]}{}_{\dot a_3\dot a_4}\delta^{i_1j}C_4(z)\\&+\tfrac12\gamma^{[i_1j]}{}_{\dot a_3\dot a_4}\delta^{i_2i}C_5(z)\\&+\tfrac12\gamma^{[i_1i]}{}_{\dot a_3\dot a_4}\delta^{i_2j}C_6(z)+\tfrac12\gamma^{[ij]}{}_{\dot a_3\dot a_4}\delta^{i_1i_2}C_7(z)\\&+\delta_{\dot a_3\dot a_4}\delta^{i_1i_2}\delta^{ij}C_8(z)+\delta_{\dot a_3\dot a_4}\delta^{i_1i}\delta^{i_2j}C_9(z)\\&+\delta_{\dot a_3\dot a_4}\delta^{i_1j}\delta^{i_2i}C_{10}(z).\end{aligned}$$

By using the $SU(4)\times U(1)$ basis, the function $C_1(z)$ and $C_2(z)$ can be determined up to a phase using the following relations:

$$T^{\bar1 2\dot4\dot4\bar3\bar4}=-C_1,$$

$$T^{\bar1\bar4\dot3\dot4\bar2 2}=-\frac12 C_1-C_2,$$

$$T^{\bar1\bar4\dot3\dot4 2\bar2}=\frac12 C_1-C_2.$$

Since we know all three functions up to a phase, we get a nontrivial equation on $C_1(z)$ and $C_2(z)$ allowing us to determine their relative sign. Namely, from Eq. (27) with the account of the normalization constant (28), one obtains

$$\begin{aligned}-C_1&\sim\frac{N-n_0}{n_\infty-n_0}\,\frac{e^{i\varphi_1}}{x[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]},\\-\frac12 C_1-C_2&\sim\frac{N-n_0}{n_\infty-n_0}\,\frac{e^{i\varphi_2}}{x(x-\alpha_1)(x-\alpha_2)},\\\frac12 C_1-C_2&\sim\frac{n_\infty(N-n_\infty)}{(n_\infty-n_0)^2}\,\frac{e^{i\varphi_3}}{[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)](x-\alpha_1)(x-\alpha_2)},\end{aligned} \tag{41}$$

where a common multiplier in all three functions was omitted. Now it can be easily verified that the last equation is satisfied only if $e^{i\varphi_1}=e^{i\varphi_2}=-e^{i\varphi_3}$. Since we proved that the overall phase is irrelevant, we can set $e^{i\varphi}=1$ and proceeding in the same manner, fix relative signs of all 10 functions $C_i(z)$. For convenience in later computations, it is useful to rewrite $T^{i_1i_2\dot a_3\dot a_4ij}(z)$ in terms of ordinary products of γ's instead of their antisymmetric combinations. This is achieved with the help of identity (A3). The final answer for $T^{i_1i_2i_3i_4ij}(z)$ is

$$
\begin{aligned}
T^{i_1 i_2 \dot{a}_3 \dot{a}_4 ij}(z) = [n_\infty(N-n_\infty)]^{-1/2} \Bigg\{ & \frac{1}{4} \frac{N-n_0}{n_\infty - n_0} \frac{(\gamma^{i_1}\gamma^{i_2}\gamma^{i}\gamma^{j})_{\dot{a}_3\dot{a}_4}}{x[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]} \\
& + \frac{1}{2} x(x-1) \frac{(\gamma^{i}\gamma^{j})_{\dot{a}_3\dot{a}_4}\delta^{i_1 i_2}}{[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]} - \frac{1}{2} \frac{N-n_0}{n_0} \frac{(\gamma^{i_2}\gamma^{j})_{\dot{a}_3\dot{a}_4}\delta^{i_1 i}}{x(x-1)[x-(N-n_0-n_\infty)/(N-n_0)]} \\
& - \frac{1}{2} \frac{N-n_\infty}{n_\infty - n_0} \frac{(\gamma^{i_2}\gamma^{i})_{\dot{a}_3\dot{a}_4}\delta^{i_1 j}}{x[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]} \\
& - \frac{1}{2} \frac{N-n_0}{n_\infty - n_0} \frac{(\gamma^{i_1}\gamma^{j})_{\dot{a}_3\dot{a}_4}\delta^{i_2 i}}{x(x-1)[x-n_0/(n_0-n_\infty)]} \\
& + \frac{1}{2} \frac{N-n_\infty}{n_\infty - n_0} \frac{(\gamma^{i_1}\gamma^{i})_{\dot{a}_3\dot{a}_4}\delta^{i_2 j}}{[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]} \\
& - \frac{1}{2} \frac{n_\infty(N-n_\infty)}{(n_\infty-n_0)^2} \frac{(\gamma^{i_1}\gamma^{i_2})_{\dot{a}_3\dot{a}_4}\delta^{ij}}{(x-\alpha_1)(x-\alpha_2)[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]} \\
& + \frac{N-n_\infty}{n_\infty-n_0} x(x-1) \frac{\delta^{i_1 j}\delta^{i_2 i}\delta_{\dot{a}_3\dot{a}_4}}{[x+n_0/(N-n_0)][x-n_0/(n_0-n_\infty)]} \\
& + \frac{N-n_\infty}{n_0} \frac{\delta^{i_1 i}\delta^{i_2 j}\delta_{\dot{a}_3\dot{a}_4}}{(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]} \\
& - \frac{n_\infty(N-n_\infty)}{(n_\infty-n_0)(N-n_0)} \frac{\delta^{i_1 i_2}\delta^{ij}\delta_{\dot{a}_3\dot{a}_4}}{(x-\alpha_1)(x-\alpha_2)(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]} \Bigg\}. \qquad (42)
\end{aligned}
$$

Next we contract $T^{i_1 i_2 \dot{a}_3 \dot{a}_4 ij}(z)$ with $\langle \tau_i \tau_j \rangle$ and substitute the result thus obtained into Eq. (40). After long and tedious calculations, we arrive at the following expression for $\mathcal{I}(\zeta;k)$:

$$
\mathcal{I}(\zeta;k) = \int d^2 z |z|^{(1/2)k_1 k_4 - 2} |1-z|^{(1/2)k_3 k_4 - 2} T[u_3,\zeta_2,\zeta_1,u_4](z) T[u_3,\zeta_2,\zeta_1,u_4](\overline{z}), \qquad (43)
$$

where

$$
\begin{aligned}
T[u_3,\zeta_2,\zeta_1,u_4](z) = \frac{N}{4n_\infty} \{ & (z-1)(\gamma^{i_1}\gamma^{i_2}\gamma^{i}\gamma^{j})_{\dot{a}_3\dot{a}_4} t^{ij} + 2(\gamma^{i}\gamma^{j})_{\dot{a}_3\dot{a}_4}\delta^{i_1 i_2} t^{ij} - 2(\gamma^{i_1}\gamma^{i})_{\dot{a}_3\dot{a}_4} p^{i i_2} - 2(\gamma^{i_2}\gamma^{i})_{\dot{a}_3\dot{a}_4} q^{i i_1} \\
& - 4\delta_{\dot{a}_3\dot{a}_4}\rho^{i_1 i_2} \} \zeta_1^{i_1}\zeta_2^{i_2} u_3^{\dot{a}_3} u_4^{\dot{a}_4}. \qquad (44)
\end{aligned}
$$

Here to simplify the notation, we introduced the following tensors:

$$
\begin{aligned}
t^{ij} &= k_3^i k_1^j + \frac{n_\infty}{N-n_\infty} k_1^i k_4^j + \frac{n_0}{N-n_\infty} k_3^i k_4^j + \frac{n_\infty(N-n_\infty)}{n_0} k_1^i k_1^j, \\
p^{i i_2} &= \frac{n_0 n_\infty}{n_\infty - n_0} \left(\frac{N-n_\infty}{n_\infty-n_0} \frac{x(x-1)}{[x-n_0/(n_0-n_\infty)]} \frac{\langle \tau_{i_2}\tau_i\rangle_k - \langle \tau_i \tau_{i_2}\rangle_k}{(x-\alpha_1)(x-\alpha_2)} + \frac{\langle \tau_{i_2}\tau_i\rangle_k}{[x-n_0/(n_0-n_\infty)]} \right), \\
q^{i i_1} &= \frac{n_\infty(N-n_0)}{n_\infty-n_0} \left(\left(x + \frac{n_0}{N-n_0} \right) \frac{\langle \tau_{i_1}\tau_i\rangle_k - \langle \tau_i \tau_{i_1}\rangle_k}{(x-\alpha_1)(x-\alpha_2)} + \frac{\langle \tau_i \tau_{i_1}\rangle_k}{[x-n_0/(n_0-n_\infty)]} \right), \\
\rho^{i_1 i_2} &= \frac{n_\infty(N-n_\infty)}{n_\infty-n_0} \left(x \frac{\langle \tau_{i_1}\tau_{i_2}\rangle_k - \langle \tau_{i_2}\tau_{i_1}\rangle_k}{(x-\alpha_1)(x-\alpha_2)} + \frac{\langle \tau_{i_2}\tau_{i_1}\rangle_k}{[x-n_0/(n_0-n_\infty)]} \right). \qquad (45)
\end{aligned}
$$

Note that we purposefully wrote these tensors in terms of the variable x, even though it presents no difficulty to express them in terms of z. The point is that by writing them in this form, we can clearly see that all ‘‘+’’ light-cone components vanish due to Eq. (29).

Next we turn to the issue of the Lorentz invariance of the theory. To this end, we introduce ten pure imaginary 32×32 Γ-matrices which satisfy the Clifford algebra $\{\Gamma^\mu,\Gamma^\nu\}=-2\eta^{\mu\nu}$. These Γ matrices are constructed as tensor products of 2×2 Pauli matrices $\sigma_i, i=1,2,3$ and 16×16 matrices γ^i $i=1,\ldots,8$:

$$\gamma^i=\begin{pmatrix} 0 & \gamma^i_{a\dot a} \\ \gamma^i_{\dot a a} & 0\end{pmatrix},$$

where $\gamma^i_{a\dot a}$ and $\gamma^i_{\dot a a}$ are defined in Eq. (A1). Light-cone components of ten-dimensional gamma matrices, i.e., $\Gamma^+=\Gamma^0+\Gamma^9$ and $\Gamma^-=\Gamma^0-\Gamma^9$, are nilpotent: $(\Gamma^+)^2=(\Gamma^-)^2=0$. Evidently, in the integrand (43), transversal components of ten-dimensional matrices will be contracted with fermion wave functions $u^{\dot a}$ (u^a). In the light-cone coordinates, the 32-component Majorana-Weyl spinor u, $\Gamma_{11}u=+u$, assumes the form $(u^{\dot a},0,0,u^a)$. This spinor satisfies the massless Dirac equation $k_\mu\Gamma^\mu u=0$, or equivalently $\overline{u}\Gamma^\mu k_\mu=0$, where $\overline{u}$ is the Dirac conjugated spinor, i.e., $\overline{u}=u^{\mathrm T}\Gamma^0$. In the chosen basis, the Dirac equation takes the form (see, e.g., [16])

$$k^+u^a+\gamma^i_{a\dot a}k^iu^{\dot a}=0, \tag{46}$$

$$k^-u^{\dot a}+\gamma^i_{\dot a a}k^iu^a=0. \tag{47}$$

The first of these equations allows one to express u^a in terms of $u^{\dot a}$:

$$u^a=-\frac{1}{k^+}\gamma^i_{a\dot a}k^iu^{\dot a}. \tag{48}$$

Therefore, eight components of $u^{\dot a}$ correspond to eight physical degrees of freedom. Upon the substitution of Eq. (48) into Eq. (47), one obtains the equation on $u^{\dot a}$ which is just the Klein-Gordon equation $k^2=0$. In order to express the integrand (44) in terms of ten-dimensional Γ-matrices and 32-component Majorana-Weyl spinors, u_i we need the following identities:

$$u_1^{\dot a}(\gamma^{i_1}\gamma^{i_2}\gamma^{i_3}\gamma^{i_4})_{\dot a\dot b}u_2^{\dot b}=\tfrac{1}{2}\overline{u_1}\Gamma^+\Gamma^{i_1}\Gamma^{i_2}\Gamma^{i_3}\Gamma^{i_4}u_2,$$

$$u_1^{\dot a}(\gamma^{i_1}\gamma^{i_2})_{\dot a\dot b}u_2^{\dot b}=-\tfrac{1}{2}\overline{u_1}\Gamma^+\Gamma^{i_1}\Gamma^{i_2}u_2,$$

$$u_1^{\dot a}\delta^{\dot a\dot b}u_2^{\dot b}=\frac{1}{2}\overline{u_1}\Gamma^+u_2,$$

which can be easily verified by using the explicit form of Γ-matrices, provided in Appendix A. Now it is straightforward to replace transversal 8×8 γ-matrices with 32×32 Γ-matrices and 8-component spinors $u^{\dot a}$, u^a with 32-component Majorana-Weyl spinors u. In addition to fermion wave functions, the integrand (43) also depends on vector polarizations. As usual, in ten dimensions, a polarization of a massless vector particle satisfies the transversality condition: $k_\mu\zeta^\mu=0$. In the light-cone gauge, the polarization obeys $\zeta^+=0$ allowing us to express the component ζ^- in terms of ζ^i and k_μ as $\zeta^-=2k^i\zeta^i/k^+$. In our model, we only deal with eight transversal polarizations ζ^i and can treat this equation as the definition of the light-cone polarization ζ^-. An important property of the light-cone gauge is that $\zeta_1^i\zeta_2^i=\zeta_1^\mu\zeta_2^\mu\equiv(\zeta_1\zeta_2)$ which is a direct consequence of $\zeta_1^+=\zeta_2^+=0$. Clearly, the integrand in Eq. (43) depends on scalar products of transversal momenta k^i with ζ^i. It turns out that by using the light-cone momenta and polarizations k^- and ζ^-, the integrand can be written via scalar products of ten-dimensional vectors. To show that this is indeed the case, we first note that $t^{i+}=t^{+i}=0$ and the same holds for all tensors in Eq. (45). This is a direct consequence of Eq. (29). Taking into account $\{\Gamma^i,\Gamma^+\}=0$ and $(\Gamma^+)^2=0$, the first term in Eq. (44) becomes

$$\begin{aligned}\overline{u_1}\Gamma^+\Gamma^{i_1}\zeta^{i_1}\Gamma^{i_2}\zeta^{i_2}\Gamma^it^{ij}\Gamma^ju_2&=\overline{u_1}\Gamma^+\Gamma^{i_1}\zeta^{i_1}\Gamma^{i_2}\zeta^{i_2}\Gamma^i(\tfrac{1}{2}\Gamma^+t^{i-}\\&\quad+t^{i\nu}\Gamma^\nu)u_2\\&=\overline{u_1}\Gamma^+\Gamma^{i_1}\zeta^{i_1}\Gamma^{i_2}\zeta^{i_2}\Gamma^it^{i\nu}\Gamma^\nu u_2\\&=\ldots\\&=\overline{u_1}\Gamma^+\Gamma\zeta_1\Gamma\zeta_2\Gamma^\mu t^{\mu\nu}\Gamma^\nu u_2.\end{aligned}$$

Proceeding in the same manner, we find

$$\overline{u_1}\Gamma^+\Gamma^i\langle\tau_i\tau_j\rangle\Gamma^ju_2=\overline{u_1}\Gamma^+\Gamma^\mu\langle\tau_\mu\tau_\nu\rangle\Gamma^\nu u_2,$$

$$\overline{u_1}\Gamma^+\Gamma^{i_1}\zeta^{i_1}\Gamma^{i_2}\zeta^{i_2}u_2=\overline{u_1}\Gamma^+\Gamma\zeta_1\Gamma\zeta_2u_2,$$

$$\overline{u_1}\Gamma^+\Gamma^{i_1}\zeta^{i_1}\Gamma^i\langle\tau_i\tau_{i_2}\rangle\zeta^{i_2}u_2=\overline{u_1}\Gamma^+\Gamma\zeta_1\Gamma^\mu\langle\tau_\mu\tau_\nu\rangle\zeta^\nu u_2.$$

Imposing the Dirac equation $k_{4\mu}\Gamma^\mu u_4=0$ and the transversality condition $k_{1\mu}\zeta_1^\mu = 0 = k_{2\mu}\zeta_2^\mu$, the expression for $T[u_3,\zeta_2,\zeta_1,u_4](z)$ acquires a particularly simple form:

$$\begin{aligned}T[u_3,\zeta_2,\zeta_1,u_4](z) &= \frac{N}{4n_\infty}\{(z-1)\tfrac{1}{2}\overline{u_3}\Gamma^+\Gamma\zeta_1\Gamma\zeta_2\Gamma k_3\Gamma k_1 u_4 \\ &\quad -\overline{u_3}\Gamma^+\Gamma k_3\Gamma k_1 u_4\zeta_1\zeta_2 + 2\overline{u_3}\Gamma^+\Gamma\zeta_1[(z-1)\Gamma k_1\zeta_2 k_3 \\ &\quad -z\Gamma k_3\zeta_2 k_1]u_4 + 2\overline{u_3}\Gamma\zeta_2[(z-1)(\Gamma k_3\zeta_1 k_4 - \Gamma k_1\zeta_1 k_3) \\ &\quad +4z\Gamma k_3\zeta_1 k_2]u_4 - 2\overline{u_3}\Gamma^+ u_4[\zeta_1 k_4\zeta_2 k_3 - z\zeta_1 k_3\zeta_2 k_4]\}.\end{aligned}$$

The last step in rendering $T[u_3,\zeta_2,\zeta_1,u_4](z)$ the Lorentz covariant form, requires us to impose the Dirac equation $\overline{u_3}\Gamma^\mu k_{3\mu} = 0$. To this end, one has to anticommute Γk_3 all the way to the left until it multiplies the spinor $\overline{u_3}$ and annihilates it. This procedure will generate additional terms due to the anticommutation relation of Γ-matrices. The appearance of these terms can be easily traced in the example below:

$$\begin{aligned}\tfrac{1}{2}\overline{u_3}\Gamma^+\Gamma\zeta_1\Gamma\zeta_2\Gamma k_3\Gamma k_1 u_4 &= -\tfrac{1}{2}\overline{u_3}\Gamma^+\Gamma\zeta_1\Gamma k_3\Gamma\zeta_2\Gamma k_1 u_4 \\ &\quad -\overline{u_3}\Gamma^+\Gamma\zeta_1 k_3\zeta_2\Gamma k_1 u_4 \\ &= \tfrac{1}{2}\overline{u_3}\Gamma^+\Gamma k_3\Gamma\zeta_1\Gamma\zeta_2\Gamma k_1 u_4 \\ &\quad +\overline{u_3}\Gamma^+\zeta_1 k_3\Gamma\zeta_2\Gamma k_1 u_4 \\ &\quad -\overline{u_3}\Gamma^+\Gamma\zeta_1 k_3\zeta_2\Gamma k_1 u_4 \\ &= -\overline{u_3}\Gamma\zeta_1\Gamma\zeta_2\Gamma k_1 u_4 k_3^+ \\ &\quad +\overline{u_3}\Gamma^+\zeta_1 k_3\Gamma\zeta_2\Gamma k_1 u_4 \\ &\quad -\overline{u_3}\Gamma^+\Gamma\zeta_1 k_3\zeta_2\Gamma k_1 u_4. \qquad (49)\end{aligned}$$

Proceeding in this fashion, it can be easily shown that all terms containing Γ^+ cancel and we arrive at the following result:

$$\begin{aligned}T[u_3,\zeta_2,\zeta_1,u_4](z) &= -\frac{1-z}{4}\overline{u_3}\Gamma\zeta_2\Gamma(k_1+k_4)\Gamma\zeta_1 u_4 \\ &\quad +\frac{z}{2}(\overline{u_3}\Gamma\zeta_1 u_4 k_1\zeta_2 - \overline{u_3}\Gamma\zeta_2 u_4 k_2\zeta_1 \\ &\quad -\overline{u_3}\Gamma k_1 u_4\zeta_1\zeta_2).\end{aligned}$$

Finally, we perform the integration over the sphere $(z,\overline{z})$ to get:

$$\mathcal{I}(\zeta;k) = K(u_3,\zeta_2,\zeta_1,u_4;k)K(u_3,\zeta_2,\zeta_1,u_4;k)C(s,t,u),$$

where

$$\begin{aligned}K(u_3,\zeta_2,\zeta_1,u_4;k) = 2^{-4}\Big\{&-\frac{s}{2}\overline{u_3}\Gamma\zeta_2\Gamma(k_1+k_4)\Gamma\zeta_1 u_4 \\ &+t(\overline{u_3}\Gamma\zeta_1 u_4 k_1\zeta_2 - \overline{u_3}\Gamma\zeta_2 u_4 k_2\zeta_1 \\ &-\overline{u_3}\Gamma k_1 u_4\zeta_1\zeta_2)\Big\}. \qquad (50)\end{aligned}$$

Now one can recognize in $K(u_3,\zeta_2,\zeta_1,u_4;k)$ the standard open string kinematical factor of the superstring theory (see [17]). Futhermore, as was mentioned earlier, all dependence on N in $K(u_1,\zeta_2,u_3,\zeta_4;k)$ was absorbed into k^+.

B. Fermion+vector particle → fermion+vector particle

The kinematical factor corresponding to a massless vector particle and a fermion in the initial state and the same type of particles in the final state is computed in complete analogy with the kinematical factor found in the previous section. In particular, here we need to determine the spin-tensor $T^{\dot{a}_1 i_2 \dot{a}_3 i_4 ij}(z)$ which we decompose into $SO(8)$ invariant rank six spin-tensors as follows

$$\begin{aligned}T^{\dot{a}_1 i_2 \dot{a}_3 i_4 ij}(z) &= \tfrac{1}{4}\gamma^{[i_2 i_4 ij]}{}_{\dot{a}_1\dot{a}_3}C_1(z) + \tfrac{1}{2}\gamma^{[ij]}{}_{\dot{a}_1\dot{a}_3}\delta^{i_2 i_4}C_2(z) \\ &\quad +\tfrac{1}{2}\gamma^{[i_2 j]}{}_{\dot{a}_1\dot{a}_3}\delta^{i_4 i}C_3(z) \\ &\quad +\tfrac{1}{2}\gamma^{[i_2 i]}{}_{\dot{a}_1\dot{a}_3}\delta^{i_4 j}C_4(z) + \tfrac{1}{2}\gamma^{[i i_4]}{}_{\dot{a}_1\dot{a}_3}\delta^{i_2 i}C_5(z) \\ &\quad +\tfrac{1}{2}\gamma^{[j i_4]}{}_{\dot{a}_1\dot{a}_3}\delta^{i_2 j}C_6(z) \\ &\quad +\tfrac{1}{2}\gamma^{[i_2 i_4]}{}_{\dot{a}_1\dot{a}_3}\delta^{ij}C_7(z) + \delta_{\dot{a}_1\dot{a}_3}\delta^{i_2 i_4}\delta^{ij}C_8(z) \\ &\quad +\delta_{\dot{a}_1\dot{a}_3}\delta^{i_2 i}\delta^{i_4 j}C_9(z) \\ &\quad +\delta_{\dot{a}_1\dot{a}_3}\delta^{i_2 j}\delta^{i_4 i}C_{10}(z).\end{aligned}$$

To fix the functions $C_i(z)$, we transform to the $SU(4)\times U(1)$ basis, as we did in the previous case. After fixing relative signs of $C_i(z)$, we arrive at the following expression for $T^{\dot{a}_1 i_2 \dot{a}_3 i_4 ij}(z)$:

$$T^{\dot a_1 i_2 \dot a_3 i_4 ij}(z)=(n_0n_\infty)^{-1/2}\Bigg\{\frac{1}{4}\frac{(\gamma^{i_1}\gamma^{i_2}\gamma^{i}\gamma^{j})_{\dot a_1\dot a_3}}{x(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]}$$
$$+\frac{1}{2}\frac{N-n_0}{n_\infty-n_0}\frac{(\gamma^{i}\gamma^{j})_{\dot a_1\dot a_3}\delta^{i_2i_4}}{x[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}$$
$$-\frac{1}{2}\frac{n_\infty}{n_\infty-n_0}\frac{(\gamma^{j}\gamma^{i_4})_{\dot a_1\dot a_3}\delta^{i_2i}}{x(x-1)[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}$$
$$-\frac{1}{2}\frac{(\gamma^{i}\gamma^{i_4})_{\dot a_1\dot a_3}\delta^{i_2j}}{x[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]}$$
$$-\frac{1}{2}\frac{n_\infty}{n_\infty-n_0}\frac{(\gamma^{i_2}\gamma^{j})_{\dot a_1\dot a_3}\delta^{i_4i}}{(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}$$
$$-\frac{1}{2}\frac{N-n_0}{N-n_\infty}\frac{(\gamma^{i_2}\gamma^{i})_{\dot a_1\dot a_3}\delta^{i_4j}}{x(x-1)[x+n_0/(N-n_0)]}+\frac{1}{2}\frac{n_0}{n_\infty-n_0}\frac{(\gamma^{i_2}\gamma^{i_4})_{\dot a_1\dot a_3}\delta^{ij}}{x[x+n_0/(N-n_0)](x-\alpha_1)(x-\alpha_2)}$$
$$-\frac{n_0}{n_\infty-n_0}\frac{\delta^{i_2j}\delta^{i_4i}\delta_{\dot a_1\dot a_3}(x-1)}{x[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}$$
$$+\frac{n_\infty(N-n_0)}{(n_\infty-n_0)(N-n_\infty)}\frac{\delta^{i_2i}\delta^{i_4j}\delta_{\dot a_1\dot a_3}[x+n_0/(N-n_0)]}{x(x-1)[x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}$$
$$+\frac{n_0(N-n_0)}{(n_\infty-n_0)^2}\frac{\delta^{i_2i_4}\delta^{ij}\delta_{\dot a_1\dot a_3}(x-1)}{x[x-n_0/(n_0-n_\infty)](x-\alpha_1)(x-\alpha_2)}\Bigg\}.$$

Next we contract $T^{\dot a_1 i_2 \dot a_3 i_4 ij}(z)$ with $\langle\tau_i\tau_j\rangle(z)$ in order to obtain $T^{\dot a_1 i_2 \dot a_3 i_4}=T^{\dot a_1 i_2 \dot a_3 i_4 ij}\langle\tau_i\tau_j\rangle$. After long calculations, we find that $\mathcal{I}(\zeta;k)$ is equal to

$$\mathcal{I}(\zeta;k)=\int d^2z|z|^{(1/2)k_1k_4-2}|1-z|^{(1/2)k_3k_4-2}T^{\dot a_1 i_2\dot a_3 i_4}(z)T^{a_1j_2a_3j_4}(\bar z)u_1^{\dot a_1a_1}\zeta_2^{i_2j_2}u_3^{\dot a_3a_3}\zeta_4^{i_4j_4},$$

where

$$T^{\dot a_1 i_2\dot a_3 i_4}(z)=\frac{1}{8}(\gamma^{i_2}\gamma^{i_4}\gamma^{i}\gamma^{j})_{\dot a_1\dot a_3}\frac{N(N-n_\infty)}{n_\infty-n_0}\frac{\langle\tau_i\tau_j\rangle_k-\langle\tau_j\tau_i\rangle_k}{(x-\alpha_1)(x-\alpha_2)}+\frac{1}{4}(\gamma^{i}\gamma^{j})_{\dot a_1\dot a_3}\delta^{i_2i_4}\frac{N(N-n_\infty)}{n_\infty-n_0}\frac{\langle\tau_i\tau_j\rangle_k-\langle\tau_j\tau_i\rangle_k}{(x-\alpha_1)(x-\alpha_2)}(z-1)$$
$$-\frac{1}{2}(\gamma^{i_2}\gamma^{i})_{\dot a_1\dot a_3}\frac{N(N-n_0)}{n_\infty-n_0}\left[\frac{\langle\tau_i\tau_{i_4}\rangle_k-\langle\tau_{i_4}\tau_i\rangle_k}{(x-\alpha_1)(x-\alpha_2)}\left(x-\frac{N-n_0-n_\infty}{N-n_0}\right)+\frac{\langle\tau_{i_4}\tau_i\rangle_k}{[x-n_0/(n_0-n_\infty)]}\right]$$
$$-\frac{1}{2}(\gamma^{i}\gamma^{i_4})_{\dot a_1\dot a_3}\frac{N(N-n_\infty)}{(n_\infty-n_0)[x-n_0/(n_0-n_\infty)]}\left[-\frac{n_\infty}{n_\infty-n_0}\frac{\langle\tau_i\tau_{i_2}\rangle_k-\langle\tau_{i_2}\tau_i\rangle_k}{(x-\alpha_1)(x-\alpha_2)}\left(x+\frac{n_0}{N-n_0}\right)+\langle\tau_i\tau_{i_2}\rangle_k\right]$$
$$+\delta_{\dot a_1\dot a_3}\frac{N(N-n_\infty)}{(n_\infty-n_0)[x-n_0/(n_0-n_\infty)]}\left(\frac{n_0}{n_\infty-n_0}\frac{\langle\tau_{i_2}\tau_{i_4}\rangle_k-\langle\tau_{i_4}\tau_{i_2}\rangle_k}{(x-\alpha_1)(x-\alpha_2)}+\frac{N}{N-n_\infty}\langle\tau_{i_2}\tau_{i_4}\rangle_k\right)$$
$$+\frac{1}{4}(\gamma^{i_2}\gamma^{i_4})_{\dot a_1\dot a_3}\frac{N}{n_0n_\infty}[(N-n_0-n_\infty)k_3k_4+Nk_1k_4]-\frac{1}{2}\delta^{i_2i_4}\delta_{\dot a_1\dot a_3}\frac{N}{n_0n_\infty}[(n_0+n_\infty)z+(N-n_0-n_\infty)]k_3k_4.$$

Note that in the last two lines, we took advantage of Eq. (29) in order to obtain Lorentz invariant scalar products. To rewrite this expression in terms of ten dimensional Γ-matrices and 32-component Majorana-Weyl spinors u_1 and u_3, we should proceed exactly as we did in the previous calculation. Namely, here we need the formulas

$$u_1^{\dot a}(\gamma^i\gamma^j\gamma^k\gamma^l)_{\dot a\dot b}u_3^{\dot b}=\tfrac{1}{2}\overline{u_1}\Gamma^+\Gamma^i\Gamma^j\Gamma^k\Gamma^l u_3,$$

$$u_1^{\dot a}(\gamma^i\gamma^j)_{\dot a\dot b}u_3^{\dot b}=-\tfrac{1}{2}\overline{u_1}\Gamma^+\Gamma^i\Gamma^j u_3,$$

$$u_1^{\dot a}\delta^{\dot a\dot b}u_3^{\dot b}=\tfrac{1}{2}\overline{u_1}\Gamma^+ u_3.$$

Taking into account these formulas as well as the property (29), the nilpotency of Γ^+ and the fact that $\{\Gamma^i,\Gamma^+\}=0$, then after some algebra, we find that $T[u_1,\zeta_2,u_3,\zeta_4] = T^{\dot a_1 i_2\dot a_3 i_4}u_1^{\dot a_1}\zeta_2^{i_2}u_3^{\dot a_3}\zeta_4^{i_4}$ is equal to

$$T[u_1,\zeta_2,u_3,\zeta_4](z)=\frac{N}{4n_0}\bigg(\frac{1}{2}\overline{u_1}\Gamma^+\Gamma\zeta_2\Gamma\zeta_4\Gamma k_1\Gamma k_4 u_3 - \overline{u_1}\Gamma^+\Gamma k_1\Gamma k_4 u_3\zeta_2\zeta_4 + \overline{u_1}\Gamma^+(\Gamma\zeta_2\Gamma^\mu\rho^{\mu\nu}\zeta_4^\nu - \Gamma\zeta_4\Gamma^\mu p_I^{\mu\nu}\zeta_2^\nu)u_3 + 2\overline{u_1}\Gamma^+u_3(\zeta_2^\mu q^{\mu\nu}\zeta_4^\nu - \zeta_4^\mu p_{II}^{\mu\nu}\zeta_2^\nu)\bigg).$$

Here for convenience, we introduced the following tensors

$$\rho^{\mu\nu}=k_4^\mu k_1^\nu - z k_1^\mu k_3^\nu-(z-1)k_1^\mu k_1^\nu,$$

$$p_I^{\mu\nu}=(z-1)k_1^\mu k_4^\nu + k_4^\mu k_1^\nu,$$

$$p_{II}^{\mu\nu}=\left(1-\frac{N}{n_\infty}\right)k_3^\mu k_1^\nu+\frac{n_0}{n_\infty}(z-1)k_3^\mu k_4^\nu+(z-1)k_1^\mu k_4^\nu,$$

$$q^{\mu\nu}=(z-1)k_1^\mu k_2^\nu+\frac{n_0}{n_\infty}(z-1)k_4^\mu k_3^\nu-\frac{N}{n_\infty}k_1^\mu k_3^\nu.$$

To cast the integrand into the Lorentz covariant form, we impose the Dirac equation $\overline{u_1}\Gamma^\mu k_{1\mu}=0$. Then all non-covariant terms, i.e., terms containing Γ^+, cancel and we obtain:

$$\mathcal{I}(\zeta;k)=\int d^2z|z|^{(1/2)k_1k_4-2}|1-z|^{(1/2)k_3k_4-2} \times T[u_1,\zeta_2,u_3\zeta_4](z)T[u_1,\zeta_2,u_3\zeta_4](\overline{z}),$$

where

$$T[u_1,\zeta_2,u_3\zeta_4](z)=\frac{z}{4}\overline{u_1}\Gamma\zeta_2\Gamma(k_3+k_4)\Gamma\zeta_4u_3 + \frac{1-z}{4}\overline{u_1}\Gamma\zeta_4\Gamma(k_2+k_3)\Gamma\zeta_2u_3.$$

Finally, we perform the integration over the sphere $(z,\overline{z})$ to get:

$$\mathcal{I}(\zeta;k)=K(u_1,\zeta_2,u_3,\zeta_4;k)K(u_1,\zeta_2,u_3,\zeta_4;k)C(s,t,u),$$

where

$$K(u_1,\zeta_2,u_3,\zeta_4;k)=2^{-4}\left(\frac{t}{2}\overline{u_1}\Gamma\zeta_2\Gamma(k_3+k_4)\Gamma\zeta_4u_3 + \frac{s}{2}\overline{u_1}\Gamma\zeta_4\Gamma(k_2+k_3)\Gamma\zeta_2u_3\right).$$

Now one can recognize in $K(u_1,\zeta_2,u_3,\zeta_4;k)$ the standard open string kinematical factor of the superstring theory (see [17]).

C. Fermion+fermion → fermion+fermion

Finally, we consider the kinematical factor corresponding to two fermions in the initial and final states. Our first task is to decompose $T^{\dot a_1\dot a_2\dot a_3\dot a_4 ij}(z)$ into $SO(8)$ invariant rank six spin-tensors. This decomposition is given by

$$\begin{aligned}T^{\dot a_1\dot a_2\dot a_3\dot a_4 ij}(z)=&\tfrac{1}{4}\gamma^{[ik]}{}_{\dot a_1\dot a_2}\gamma^{[kj]}{}_{\dot a_3\dot a_4}C_1(z)+\tfrac{1}{2}\gamma^{[ij]}{}_{\dot a_3\dot a_4}\delta_{\dot a_1\dot a_2}C_2(z)+\tfrac{1}{2}\gamma^{[ij]}{}_{\dot a_2\dot a_4}\delta_{\dot a_1\dot a_3}C_3(z)+\tfrac{1}{2}\gamma^{[ij]}{}_{\dot a_2\dot a_3}\delta_{\dot a_1\dot a_4}C_4(z)\\&+\tfrac{1}{2}\gamma^{[ij]}{}_{\dot a_1\dot a_4}\delta_{\dot a_2\dot a_3}C_5(z)+\tfrac{1}{2}\gamma^{[ij]}{}_{\dot a_1\dot a_3}\delta_{\dot a_2\dot a_4}C_6(z)+\tfrac{1}{2}\gamma^{[ij]}{}_{\dot a_1\dot a_2}\delta_{\dot a_3\dot a_4}C_7(z)+\delta_{\dot a_1\dot a_4}\delta^{\dot a_2\dot a_3}\delta^{ij}C_8(z)\\&+\delta_{\dot a_1\dot a_3}\delta^{\dot a_2\dot a_4}\delta^{ij}C_9(z)+\delta_{\dot a_1\dot a_2}\delta^{\dot a_2\dot a_4}\delta^{ij}C_{10}(z).\end{aligned}\qquad(51)$$

All other $SO(8)$ invariant spin-tensors can be expressed in terms of linear combinations of spin-tensors from Eq. (51) and therefore are not linearly independent. To see this, first note that the most general expression for such spin-tensor should be at most fourth order in γ's. Indeed, a term which is of higher than fourth order in γ's and which has only two vector indices, namely *i* and *j*, must contain contractions like $\Sigma_{k,l}\gamma^{[kl]}{}_{\dot a\dot b}\gamma^{[kl]}{}_{\dot c\dot d}$ where $\dot a,\dot b,\dot c,\dot d$ are chosen from

$\dot{a}_1,\dot{a}_2,\dot{a}_3,\dot{a}_4$. However, this contraction is just a liner combination of Kronecker deltas as follows from the identity:

$$\sum_{k,l}\gamma^{[kl]}{}_{\dot{a}\dot{b}}\gamma^{[kl]}{}_{\dot{c}\dot{d}}=8\,\delta_{\dot{a}\dot{c}}\delta_{\dot{b}\dot{d}}-8\,\delta_{\dot{a}\dot{d}}\delta_{\dot{b}\dot{c}}. \qquad (52)$$

However, in Eq. (51) we could have included spin-tensors which are fourth order in γ's and which are obtained from $\gamma^{[ik]}{}_{\dot{a}_1\dot{a}_2}\gamma^{[kj]}{}_{\dot{a}_3\dot{a}_4}$ by permuting spinor indices $\dot{a}_1,\dot{a}_2,\dot{a}_3,\dot{a}_4$. Nonetheless, with the account of the identity

$$(\gamma^i\gamma^k)_{\dot{a}\dot{b}}(\gamma^k\gamma^j)_{\dot{c}\dot{d}}=(\gamma^k\gamma^j)_{\dot{a}\dot{d}}(\gamma^i\gamma^k)_{\dot{b}\dot{c}}+2\,\delta_{\dot{a}\dot{c}}(\gamma^i\gamma^j)_{\dot{b}\dot{d}}+2\,\delta_{\dot{a}\dot{b}}(\gamma^i\gamma^j)_{\dot{c}\dot{d}} \qquad (53)$$

it becomes clear that there is only one independent spin-tensor containing all four γ's and it is represented by the first term in Eq. (51). This identity is a direct consequence of (A4). By using the $SU(4)\times U(1)$ basis, we fix all functions $C_{\dot{i}}(z)$ and their relative phases. The final answer for $T^{\dot{a}_1\dot{a}_2\dot{a}_3\dot{a}_4ij}(z)$ is given by the following expression:

$$\begin{aligned}
T^{\dot{a}_1\dot{a}_2\dot{a}_3\dot{a}_4ij}(z)=&\frac{n_0^{-1/2}(N-n_0)^{-1/2}n_\infty^{-1/2}(N-n_\infty)^{-1/2}}{n_\infty-n_0}\\
&\times\Bigg\{-\frac{(\gamma^k\gamma^i)_{\dot{a}_1\dot{a}_2}(\gamma^k\gamma^j)_{\dot{a}_3\dot{a}_4}}{4}\frac{(N-n_0)(n_\infty-n_0)(x-\alpha_1)(x-\alpha_2)}{x(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}\\
&-\frac{(\gamma^i\gamma^j)_{\dot{a}_3\dot{a}_4}\delta_{\dot{a}_1\dot{a}_2}}{2}\frac{n_0(N-n_0)}{x(x-1)[x+n_0/(N-n_0)][x-n_0/(n_0-n_\infty)]}\\
&-\frac{(\gamma^i\gamma^j)_{\dot{a}_2\dot{a}_4}\delta_{\dot{a}_1\dot{a}_3}}{2}\frac{n_0(N-n_0)}{x[x+n_0/(N-n_0)][x-n_0/(n_0-n_\infty)]}\\
&-\frac{(\gamma^i\gamma^j)_{\dot{a}_2\dot{a}_3}\delta_{\dot{a}_1\dot{a}_4}}{2}\frac{n_0(N-n_\infty)}{x[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}\\
&-\frac{(\gamma^i\gamma^j)_{\dot{a}_1\dot{a}_4}\delta_{\dot{a}_2\dot{a}_3}}{2}\frac{(N-n_0)(n_\infty)}{x[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]}\\
&-\frac{(\gamma^i\gamma^j)_{\dot{a}_1\dot{a}_3}\delta_{\dot{a}_2\dot{a}_4}}{2}\frac{(N-n_0)(N-n_\infty)}{(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)][x-n_0/(n_0-n_\infty)]}\\
&+\frac{(\gamma^i\gamma^j)_{\dot{a}_1\dot{a}_2}\delta_{\dot{a}_3\dot{a}_4}}{2}\frac{(N-n_0)(n_\infty-n_0)}{x(x-1)[x+n_0/(N-n_0)]}\\
&+\delta^{ij}\delta_{\dot{a}_1\dot{a}_4}\delta_{\dot{a}_2\dot{a}_3}\frac{n_0(N-n_\infty)(x-1)}{x[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)](x-\alpha_1)(x-\alpha_2)}\\
&-\delta^{ij}\delta_{\dot{a}_1\dot{a}_3}\delta_{\dot{a}_2\dot{a}_4}\frac{n_0(N-n_0)(N-n_\infty)(x-1)}{(n_\infty-n_0)[x+n_0/(N-n_0)][x-n_0/(n_0-n_\infty)](x-\alpha_1)(x-\alpha_2)}\\
&+\delta^{ij}\delta_{\dot{a}_1\dot{a}_4}\delta_{\dot{a}_2\dot{a}_3}\frac{n_0n_\infty(N-n_\infty)}{(n_\infty)(x-1)[x+n_0/(N-n_0)][x-n_0/(n_0-n_\infty)](x-\alpha_1)(x-\alpha_2)}\Bigg\}.
\end{aligned}$$

The contraction of $T^{\dot{a}_1\dot{a}_2\dot{a}_3\dot{a}_4ij}(z)$ with $\langle\tau_i\tau_j\rangle(z)$ is most conveniently performed, if we express $\langle\tau_i\tau_j\rangle_k$ in the form

$$\langle\tau_i\tau_j\rangle_k=\left(\frac{x}{n_0}k_1^i-\frac{1}{N-n_0}k_2^i+\frac{(x-1)}{n_\infty}k_3^i\right)\left[(x-1)k_1^i-xk_3^i+\frac{n_0-n_\infty}{N-n_\infty}\left(x-\frac{n_0}{n_0-n_\infty}\right)k_4^i\right]\equiv\tau_{ab}k_a^ik_b^j,$$

obtained from Eq. (25) by using the momentum conservation law: $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0$. Since the first term in $\langle\tau_i\tau_j\rangle$ contains δ^{ij}, its contraction with $(\gamma^k\gamma^i)_{\dot{a}_1\dot{a}_2}(\gamma^k\gamma^j)_{\dot{a}_3\dot{a}_4}$ will produce terms which are lower than fourth order in γ's and which at present do not interest us. So, consider contracting the spin-tensor $(\gamma^k\gamma^i)_{\dot{a}_1\dot{a}_2}(\gamma^k\gamma^j)_{\dot{a}_3\dot{a}_4}$, i.e., the first term in Eq. (51), with $\langle\tau_i\tau_j\rangle_k$ and fermionic polarizations $u_i^{\dot{a}_i}$:

$$\begin{aligned}
-(\gamma^k\gamma^i)_{\dot{a}_1\dot{a}_2}(\gamma^k\gamma^j)_{\dot{a}_3\dot{a}_4}u_1^{\dot{a}_1}u_2^{\dot{a}_2}u_3^{\dot{a}_3}u_4^{\dot{a}_4}\langle\tau_i\tau_j\rangle_k
&=-\tfrac{1}{4}\overline{u_1}\Gamma^k\Gamma^i\Gamma^+u_2\overline{u_3}\Gamma^k\Gamma^j\Gamma^+u_4\langle\tau_i\tau_j\rangle_k\\
&=-\tfrac{1}{4}\overline{u_1}\Gamma^\mu\Gamma^\rho\Gamma^+u_2\overline{u_3}\Gamma_\mu\Gamma^\sigma\Gamma^+u_4\tau_{24}k_2^\rho k_4^\sigma\\
&\quad-\tfrac{1}{4}\overline{u_1}\Gamma^\mu\Gamma^\rho\Gamma^+u_2\overline{u_3}\Gamma_\mu\Gamma^\sigma\Gamma^+u_4\tau_{21}k_2^\rho k_1^\sigma\\
&\quad-\tfrac{1}{4}\overline{u_1}\Gamma^\mu\Gamma^\rho\Gamma^+u_2\overline{u_3}\Gamma_\mu\Gamma^\sigma\Gamma^+u_4\tau_{31}k_3^\rho k_1^\sigma\\
&\quad-\tfrac{1}{4}\overline{u_1}\Gamma^\mu\Gamma^\rho\Gamma^+u_2\overline{u_3}\Gamma_\mu\Gamma^\sigma\Gamma^+u_4\tau_{34}k_3^\rho k_4^\sigma. \qquad (54)
\end{aligned}$$

Here again we used the property of $\langle\tau_i\tau_j\rangle_k$, namely Eq. (29), the nilpotency of Γ^+ and the fact that u satisfies the Dirac equation. After commuting Γ^ρ and Γ^σ through Γ^+ and imposing the Dirac equation, the first term in Eq. (54) becomes

$$\begin{aligned}
-\tfrac{1}{4}\overline{u_1}\Gamma^\mu\Gamma^\rho\Gamma^+u_2\overline{u_3}\Gamma_\mu\Gamma^\sigma\Gamma^+u_4\tau_{24}k_2^\rho k_4^\sigma
&=-\overline{u_1}\Gamma^\mu u_2\overline{u_3}\Gamma_\mu u_4\tau_{24}k_2^+k_4^+. \qquad (55)
\end{aligned}$$

In order to make use of the Dirac equation in the remaining three terms of Eq. (54), we are in need of the identity

$$\begin{aligned}
\overline{u_1}\Gamma^\mu\Gamma^\rho\Gamma^+u_2\overline{u_3}\Gamma_\mu\Gamma^\sigma\Gamma^+u_4=&-\overline{u_1}\Gamma^\mu\Gamma^\sigma\Gamma^+u_4\overline{u_2}\Gamma^+\Gamma^\rho\Gamma_\mu u_3\\
&-4\overline{u_1}\Gamma^\mu u_3\overline{u_4}\Gamma_\mu u_2\eta^{\rho+}\eta^{\sigma+}\\
&-2\overline{u_1}\Gamma^\mu u_3\overline{u_4}(\Gamma^\sigma\Gamma^+\Gamma_\mu\eta^{\rho+}\\
&+\Gamma_\mu\Gamma^+\Gamma^\rho\eta^{\sigma+})u_2,
\end{aligned}$$

which allows one to place Γ^ρ next to u_2 (or u_3) when it is contracted with k_2^ρ (or k_3^ρ) thereby making it possible to impose the Dirac equation. This identity just like Eq. (53) is a direct consequence of Eq. (A4). As a result of this procedure and with the account of Eqs. (54) and (55), we obtain:

$$T^{\dot{a}_1\dot{a}_2\dot{a}_3\dot{a}_4}(z)u_1^{\dot{a}_1}u_2^{\dot{a}_2}u_3^{\dot{a}_3}u_4^{\dot{a}_4}=\frac{(N-n_0)(n_\infty-n_0)}{N^2}\,\frac{(x-\alpha_1)(x-\alpha_2)}{x(x-1)[x+n_0/(N-n_0)][x-(N-n_0-n_\infty)/(N-n_0)]}$$
$$\times\left(-\overline{u_1}\Gamma^\mu u_3\overline{u_4}\Gamma_\mu u_2\frac{N-n_0}{n_\infty-n_0}\,\frac{(x-1)[x+n_0/(N-n_0)]}{[x-n_0/(n_0-n_\infty)]}+\overline{u_1}\Gamma^\mu u_2\overline{u_3}\Gamma_\mu u_4\right).$$

Substituting this result into Eq. (35), we arrive at the expression for $\mathcal{I}(\zeta;k)$

$$\mathcal{I}(\zeta;k)=\int d^2z|z|^{(1/2)k_1k_4-2}|1-z|^{(1/2)k_3k_4-2}T[u_1,u_2,u_3,u_4](z)T[u_1,u_2,u_3,u_4](\overline{z}),$$

where

$$T[u_1,u_2,u_3,u_4](z)=\frac{1-z}{4}\overline{u_1}\Gamma^\mu u_3\overline{u_4}\Gamma_\mu u_2+\frac{1}{4}\overline{u_1}\Gamma^\mu u_2\overline{u_3}\Gamma_\mu u_4=-\frac{1-z}{4}\overline{u_2}\Gamma^\mu u_3\overline{u_1}\Gamma_\mu u_4+\frac{z}{4}\overline{u_1}\Gamma^\mu u_2\overline{u_4}\Gamma_\mu u_3.$$

Finally, we perform the integration over the sphere $(z,\overline{z})$ to get:

$$\mathcal{I}(\zeta;k)=K(u_1,u_2,u_3,u_4;k)K(u_1,u_2,u_3,u_4;k)C(s,t,u),$$

where

$$K(u_1,u_2,u_3,u_4;k)=2^{-4}\left(-\frac{s}{2}\overline{u_2}\Gamma^\mu u_3\overline{u_1}\Gamma_\mu u_4+\frac{t}{2}\overline{u_1}\Gamma^\mu u_2\overline{u_4}\Gamma_\mu u_3\right).$$

We recognize in $K(u_1,u_2,u_3,u_4;k)$ the standard open string kinematic factor of the superstring theory (see [17]). For the sake of completeness below, we provide the kinematical factor corresponding to four massless vector particles which was calculated in [10].

D. Vector particle + vector particle → vector particle + vector particle

The four graviton scattering amplitude was found in [10] and is equal to

$$A(1,2,3,4)=\lambda^2 2^{-8}\mathcal{I}(\zeta;k),$$

where

$$\mathcal{I}(\zeta;k)=K(\zeta_1,\zeta_2,\zeta_3,\zeta_4;k)K(\zeta_1,\zeta_2,\zeta_3,\zeta_4;k)C(s,t,u)$$

and

$$K(\zeta_1,\zeta_2,\zeta_3,\zeta_4;k)=2^{-2}\Big(-\tfrac{1}{4}(st\zeta_1\zeta_3\zeta_2\zeta_4+su\zeta_2\zeta_3\zeta_1\zeta_4+tu\zeta_1\zeta_2\zeta_3\zeta_4)+\frac{s}{2}(\zeta_1k_4\zeta_3k_2\zeta_2\zeta_4+\zeta_2k_3\zeta_4k_1\zeta_1\zeta_3+\zeta_1k_3\zeta_4k_2\zeta_2\zeta_3+\zeta_2k_4\zeta_3k_1\zeta_1\zeta_4)+\frac{t}{2}(\zeta_2k_1\zeta_4k_3\zeta_3\zeta_1+\zeta_3k_4\zeta_1k_2\zeta_2\zeta_4+\zeta_2k_4\zeta_1k_3\zeta_3\zeta_4+\zeta_3k_1\zeta_4k_2\zeta_1\zeta_2)+\frac{u}{2}(\zeta_1k_2\zeta_4k_3\zeta_3\zeta_2+\zeta_3k_4\zeta_2k_1\zeta_1\zeta_4+\zeta_1k_4\zeta_2k_3\zeta_3\zeta_4+\zeta_3k_2\zeta_4k_1\zeta_1\zeta_2)\Big).$$

IV. CONCLUSION

In this paper, we obtained kinematical factors and therefore scattering amplitudes for all massless particles of type IIA superstrings directly from the interacting $S^N\mathbf{R}^8$ orbifold sigma model. Our kinematical factors showed to coincide with those obtained in the framework of the superstring theory. This provides further evidence of the duality between the YM theory in the IR limit and the superstring theory in the week coupling limit.

In computing the scattering amplitudes, we did not impose any kinematic restrictions on momenta and polarizations of particles. Nevertheless, the obtained kinematical factors which define scattering amplitudes exhibit manifest Lorentz invariance even at finite N. All dependence on N was absorbed into the light-cone momenta k^+.

Moreover, if one restores the dependence on the radius R_- of the compactified direction x_- (remind that N was identified with R_-), then any dependence on N disappears. Since the $S^N\mathbf{R}^8$ orbifold model can be embedded into the $S^\infty\mathbf{R}^8$ orbifold model, this suggests that the latter might have a deformed (quantum) Lorentz symmetry realized in the space of the twist fields $\Sigma^{\dot\mu}_{(n)}$. The deformation parameter seems to be identified with $exp(2\pi i/R_-)$.

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APPENDIX A

We use the following representation of γ-matrices satisfying the relation

$$\gamma^i(\gamma^j)^T+\gamma^j(\gamma^i)^T=2\delta^{ij}I$$

$$\gamma^1=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\gamma^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\gamma^3=1\quad\gamma^4=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\gamma^5=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\gamma^6=-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\gamma^7=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\gamma^8=\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

$$\Gamma^0=\sigma_2\otimes 1_{16},\tag{A1}$$

$$\Gamma^i=i\sigma_3\otimes\begin{pmatrix}0&\gamma^i\\(\gamma^i)^T&0\end{pmatrix},$$

$$i=1,\dots,8,$$

$$\Gamma^9=i\sigma_1\otimes 1_{16},$$

$$\Gamma_{11}=\Gamma^0\Gamma^1\dots\Gamma^9=\sigma_3\otimes\sigma_3\otimes 1_8.$$

By definition,

$$\Upsilon^{[\mu\nu\lambda\rho]}=\frac{1}{4!}\sum_P(-1)^{P(\mu\nu\lambda\rho)}\Upsilon^\mu\Upsilon^\nu\Upsilon^\lambda\Upsilon^\rho=\frac{1}{6}(\Upsilon^{[\mu\nu]}\Upsilon^{[\lambda\rho]}+\Upsilon^{[\lambda\rho]}\Upsilon^{[\mu\nu]}-\Upsilon^{[\mu\lambda]}\Upsilon^{[\nu\rho]}-\Upsilon^{[\nu\rho]}\Upsilon^{[\mu\lambda]}+\Upsilon^{[\mu\rho]}\Upsilon^{[\nu\lambda]}+\Upsilon^{[\nu\lambda]}\Upsilon^{[\mu\rho]}).\tag{A2}$$

Here Υ can be either γ or Γ.

In terms of ordinary products of Υ-matrices $\Upsilon^{[\mu\nu\lambda\rho]}$ is expressed as follows

$$\Upsilon^{[\mu\nu\lambda\rho]}=\Upsilon^\mu\Upsilon^\nu\Upsilon^\lambda\Upsilon^\rho-\Upsilon^\lambda\Upsilon^\rho\eta^{\mu\nu}+\Upsilon^\nu\Upsilon^\rho\eta^{\mu\lambda}-\Upsilon^\nu\Upsilon^\lambda\eta^{\mu\rho}-\Upsilon^\mu\Upsilon^\rho\eta^{\nu\lambda}+\Upsilon^\mu\Upsilon^\lambda\eta^{\nu\rho}-\Upsilon^\mu\Upsilon^\nu\eta^{\lambda\rho}+\eta^{\mu\nu}\eta^{\lambda\rho}-\eta^{\mu\lambda}\eta^{\nu\rho}+\eta^{\mu\rho}\eta^{\nu\lambda}.\tag{A3}$$

In $D=10$, Γ's satisfy the following equality (see, e.g., [16]):

$$(\Gamma^0\Gamma^\mu)_{mn}(\Gamma^0\Gamma_\mu)_{pq}+(\Gamma^0\Gamma^\mu)_{mp}(\Gamma^0\Gamma_\mu)_{qn}+(\Gamma^0\Gamma^\mu)_{mq}(\Gamma^0\Gamma_\mu)_{np}=0. \tag{A4}$$

Here it is assumed that spinor indices have definite chirality.

APPENDIX B

With respect to the $SU(4)\times U(1)$ subgroup representations, $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ are decomposed as

$$\mathbf{8}_s\rightarrow\mathbf{4}_{1/2}+\overline{\mathbf{4}}_{-1/2}, \quad \mathbf{8}_c\rightarrow\mathbf{4}_{-1/2}+\overline{\mathbf{4}}_{1/2}, \quad \mathbf{8}_v\rightarrow\mathbf{6}_0+\mathbf{1}_1+\mathbf{1}_{-1}.$$

The corresponding basis for the fermions θ^a and their spin fields[3] $\Sigma^{\dot a}$ and Σ^i consistent with this decomposition is given by

$$\Theta^A=\frac{1}{\sqrt{2}}(\theta^A+i\,\theta^{A+4}), \quad \Theta^{\bar A}=\frac{1}{\sqrt{2}}(\theta^A-i\,\theta^{A+4}),$$

$$\mathcal{S}^{\dot A}=\frac{1}{\sqrt{2}}(\Sigma^{\dot A}+i\Sigma^{\dot A+4}), \quad \mathcal{S}^{\bar{\dot A}}=\frac{1}{\sqrt{2}}(\Sigma^{\dot A}-i\Sigma^{\dot A+4}),$$

$$S^A=\frac{1}{\sqrt{2}}(\Sigma^{2A-1}+i\Sigma^{2A}), \quad S^{\bar A}=\frac{1}{\sqrt{2}}(\Sigma^{2A-1}-i\Sigma^{2A}),$$

where $A=1,\ldots,4$. Note that the spin fields Σ^4 and $\Sigma^{\bar 4}$ transform as $\mathbf{1}_1$ and $\mathbf{1}_{-1}$, respectively.

[3]See, e.g., [18] for a detailed discussion of spin fields.

Bosonization of the fermions and their twist fields up to cocycles is realized in terms of four bosonic fields ϕ^A as

$$\Theta^A=\mathrm{e}^{iq^A_B\phi^B}, \quad \mathcal{S}^{\dot A}=\mathrm{e}^{iq^{\dot A}_B\phi^B}, \quad S^A=\mathrm{e}^{i\phi^A},$$

where the weights of the spinor representations $\mathbf{8}_s$ and $\mathbf{8}_c$ are given by

$$\mathbf{q}^1=\tfrac12(-1,-1,1,1);\quad \mathbf{q}^2=\tfrac12(-1,1,-1,1);$$
$$\mathbf{q}^3=\tfrac12(1,-1,-1,1);\quad \mathbf{q}^4=\tfrac12(1,1,1,1);$$
$$\mathbf{q}^{\dot 1}=\tfrac12(-1,1,1,1);\quad \mathbf{q}^{\dot 2}=\tfrac12(-1,-1,-1,1);$$
$$\mathbf{q}^{\dot 3}=\tfrac12(1,1,-1,1);\quad \mathbf{q}^{\dot 4}=\tfrac12(1,-1,1,1). \tag{B1}$$

The Cartan generators of $SU(4)\times U(1)$ in the bosonized form look as $H^A=i\partial\phi^A$.

Bosonization of the fermions of the orbifold model is achieved by introducing $4N$ bosonic fields and reads as

$$\Theta^A_I(z)=\mathrm{e}^{iq^A_B\phi^B_I(z)}.$$

Twist fields σ_g creating twisted sectors for the fields $\phi^A_I(z)$ are introduced in the same manner as in Sec. II B. The spin twist fields of the orbifold model can be realized as

$$\mathcal{S}^{\dot A}_{(n)}(z)=\mathrm{e}^{(i/n)\sum_{I\in(n)}q^{\dot A}_B\phi^B_I(z)}\sigma_{(n)}(z)=\sigma_{(n)}[\mathbf{q}^{\dot A}](z),$$
$$S^A_{(n)}(z)=\mathrm{e}^{(i/n)\sum_{I\in(n)}\phi^A_I(z)}\sigma_{(n)}(z)=\sigma_{(n)}[\mathbf{e}^A](z), \tag{B2}$$

where $\mathbf{e}^A$ is a weight vector of $\mathbf{8}_v$ with components δ^A_B.

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