

# Exploring the mirror TBA

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**ABSTRACT:** We apply the contour deformation trick to the Thermodynamic Bethe Ansatz equations for the  $\text{AdS}_5 \times S^5$  mirror model, and obtain the integral equations determining the energy of two-particle excited states dual to  $\mathcal{N} = 4$  SYM operators from the  $\mathfrak{sl}(2)$  sector. We show that each state/operator is described by its own set of TBA equations. Moreover, we provide evidence that for each state there are infinitely-many critical values of 't Hooft coupling constant  $\lambda$ , and the excited states integral equations have to be modified each time one crosses one of those. In particular, the first critical value for the Konishi operator occurs at  $\lambda \approx 774$ . Our results disagree with the ones of Gromov, Kazakov and Vieira, and may explain the mismatch between their recent computation of the scaling dimension of the Konishi operator and the one done by Roiban and Tseytlin by using the string theory sigma model.

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## 1. Introduction

An important open problem of the AdS/CFT correspondence [1] is to understand the finite-size spectrum of the  $\text{AdS}_5 \times S^5$  superstring. Recently, there has been further significant progress in this direction. First, the four-loop anomalous dimension of

the Konishi operator was computed [2] by means of generalized Lüscher's formulae [3, 4, 2] (see also [5]-[15] for other applications of Lüscher's approach), and the result exhibits a stunning agreement with a direct field-theoretic computation [16, 17]. Second, the groundwork for constructing the Thermodynamic Bethe Ansatz (TBA) [18], which encodes the finite-size spectrum for all values of the 't Hooft coupling, has been laid down, based on the mirror theory approach<sup>1</sup> [20]. Most importantly, the string hypothesis for the mirror model was formulated [21] and used to derive TBA equations for the ground state [22]-[25]. Also, a corresponding Y-system [26] was conjectured [27], and its general solution was obtained [28]. The AdS/CFT Y-system has unusual properties, and, in particular, is defined on an infinite-genus Riemann surface [22, 29].

The derivation of the TBA equations is not yet complete, because the equations pertain only to the ground state energy (or Witten's index in the case of periodic fermions) and do not capture the energies of excited states. Therefore, one has to find a generalization of the TBA equations that can account for the complete spectrum of the string sigma model, including all excited states.

Here we continue to explore the mirror TBA approach. In particular, we will be interested in finding the TBA integral equations which describe the spectrum of string states in the  $\mathfrak{sl}(2)$  sector. An attempt in this direction has been already undertaken in [25] and the emerging integral equations have been used for numerical computation of the anomalous dimension of the Konishi operator [30]. However, the subleading term in the strong coupling expansion in this result disagrees with the result by [31] obtained by string theory means. There exists yet another prediction [32] for this subleading term, which differs from both [30] and [31]. All these results are based on certain assumptions which require further justification. This makes urgent to carefully analyze the issue of the TBA equations for excited states, and to better understand what happens on the string theory side.

In this paper we analyze two-particle states in the  $\mathfrak{sl}(2)$  sector. First, we show that each state is governed by its own set of the TBA equations. Second, we provide evidence that for each state there are infinitely-many critical values of 't Hooft coupling constant  $\lambda$ , and that the excited states integral equations have to be modified each time one of these critical values is crossed.<sup>2</sup> Performing careful analysis of two-particle states in a region between any two neighboring critical points, we propose the corresponding integral equations.

The problem of finite-size spectrum of two-dimensional integrable models has been studied in many works, see e.g. [33]-[49]. To explain our findings, we start

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<sup>1</sup>The TBA approach in the AdS/CFT spectral problem was advocated in [19] where it was used to explain wrapping effects in gauge theory.

<sup>2</sup>Existence of such critical values was observed in the excited-state TBA equations for perturbed minimal models [37]. We thank Patrick Dorey for this comment.

with recalling that for some integrable models the inclusion of excited states in the framework of the TBA approach has been achieved by applying a certain analytic continuation procedure [36, 37]. This can be understood from the fact that the convolution terms entering the integral TBA equations exhibit a singular behavior in the complex rapidity plane, the structure of these singularities does depend on the value of the coupling constant. This leads to a modification of the ground-state TBA equations, which, indeed, describe the profile and energies of excited states. Here we intend a similar strategy for the string sigma model.

To derive the TBA equations for excited states, we propose to use a contour deformation trick. In other words, we assume that the TBA equations for excited states have the same form as those for the ground state with the only exception that the integration contour in the convolution terms is different. Returning the contour back to the real rapidity line of the mirror theory, one picks up singularities of the convolution terms which leads to modification of the final equations. The original contour should be drawn in such a way, that the arising TBA equations would reproduce the large  $L$  asymptotic solution.

Recall that the TBA equations for the string mirror model [22] are written in terms of the following Y-functions:  $Y_Q$ -functions associated with  $Q$ -particle bound states, auxiliary functions  $Y_{Q|vw}$  for  $Q|vw$ -strings,  $Y_{Q|w}$  for  $Q|w$ -strings, and  $Y_{\pm}$  for  $y_{\pm}$ -particles. The Y-functions depend on the 't Hooft coupling  $\lambda$  related to the string tension  $g$  as  $\lambda = 4\pi^2 g^2$ . As we will see, the analytic structure of these Y-functions depends on  $g$  and plays a crucial role in obtaining the TBA equations for excited states.

Most conveniently, the large  $L$  asymptotic solution for the Y-functions is written in terms of certain transfer-matrices associated with an underlying symmetry group of the model [33, 34]. In the context of the string sigma model the corresponding asymptotic solution was presented in [27]. We will use this solution to check the validity of our TBA equations.

Our analysis starts from describing physical two-particle states in the  $\mathfrak{sl}(2)$  sector. It appears that for the  $\mathfrak{sl}(2)$  states the functions  $Y_{Q|vw}(u)$  play the primary role in formulating the excited state TBA equations. Analyzing the asymptotic solution, we find that each  $Y_{Q|vw}$  has four zeroes in the complex  $u$ -plane. With  $g$  changing, the zeroes change their position as well, and at certain critical values  $g = g_{cr}$  they give rise to new singularities in the TBA equations which resolution results in the appearance of new driving terms. The critical values  $g_{cr}$  are defined as values of  $g$  at which  $Y_{Q|vw}(u)$  acquires two zeros at  $u = \pm i/g_{cr}$ :  $Y_{Q|vw}(\pm i/g_{cr}) = 0$ .

We show that at weak coupling  $g \sim 0$  all two-particle states can be organized into an infinite tower of classes  $k = \text{I}, \text{II}, \dots, \infty$ , see Table 1.

Type of a state	Y-functions	Number of zeros
I	$Y_{1 vw}$	2
II	$Y_{1 vw}, Y_{2 vw}$	2+2
III	$Y_{1 vw}, Y_{2 vw}, Y_{3 vw}$	4+2+2
IV	$Y_{1 vw}, Y_{2 vw}, Y_{3 vw}, Y_{4 vw}$	4+4+2+2
$\vdots$	$\vdots$	$\vdots$
$k \rightarrow \infty$	$Y_{1 vw}, Y_{2 vw}, \dots$	4+4+ ...

Table 1. Classification of two-particle states in the  $\mathfrak{sl}(2)$ -sector at  $g \sim 0$ . The right column shows the number of zeros which the corresponding asymptotic  $Y_{M|vw}$ -functions from the middle column have in the physical strip  $|\text{Im}(u)| < 1/g$ . States of type I are called “Konishi-like”. States of type II, III, IV, ... correspond to larger value of  $\kappa$ , see section 3.

Each class is unambiguously determined by number of zeroes of  $Y_{M|vw}$ -functions in the strip  $|\text{Im}(u)| < 1/g$ . In particular, for states of type I only  $Y_{1|vw}$ -function has two zeroes in the physical strip. We call all these states “Konishi-like” because they share this property with the particular string state corresponding to the Konishi operator.

Our results disagree with that by [25, 30] in the following two aspects: First, the integral equations for excited states in the  $\mathfrak{sl}(2)$  sector do not have a universal form, even for two-particle states. Second, we face the issue of critical points. When a critical point is crossed, the compatibility of the asymptotic solution with the integral TBA equations requires modification of the latter. The equations proposed in [25] capture only type I states and only below the first critical point.

To find approximate locations of the critical values, we first solve the asymptotic Bethe-Yang equations [50] (which include the BES/BHL dressing phase [51, 52]) for some states numerically from weak to strong coupling and obtain the corresponding interpolating curve  $u \equiv u_1(g)$ , where  $u_1$  is the rapidity of an excited particle in string theory. The second particle has rapidity  $u_2 = -u_1$  due to the level matching condition. Further, we compute the Y-functions on the large  $L$  asymptotic solution corresponding to this two-particle excited state and study their analytic properties considered as functions of  $g$ , and, in particular, determine approximately the critical values.

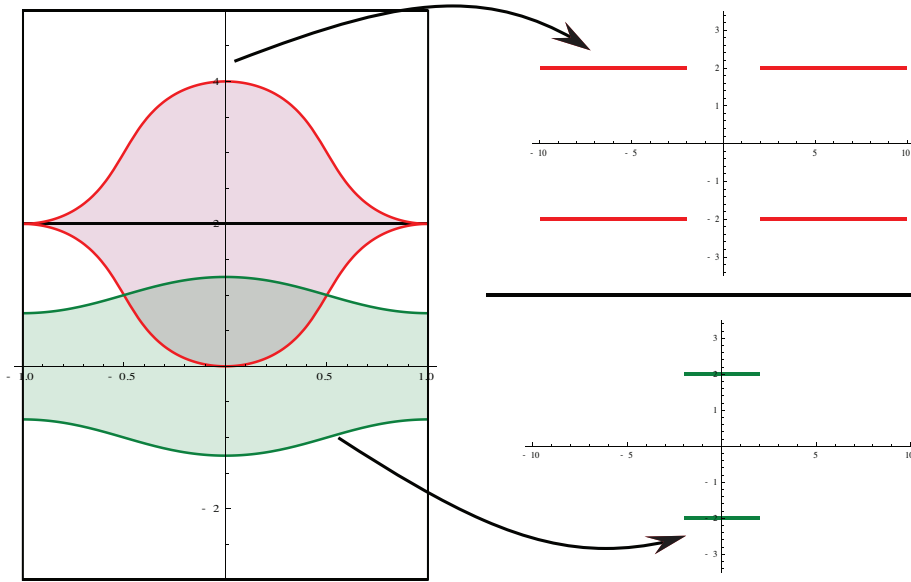
The first critical value for the Konishi operator occurs at  $\lambda \approx 774$ . In the weak-coupling region below the first critical point, the integral equations for Konishi operator we obtain seem to agree with that of [25]. However, these weak-coupling equations become inconsistent with the known large  $L$  asymptotic solution (where  $L$  is the size of the system) once the first critical point is crossed, and have to be

modified. Consequently, the derivation of the anomalous dimension for the Konishi operator at strong coupling requires re-examination. Of course, the existence of critical points is not expected to violate analyticity of the energy  $E(\lambda)$  of a string state considered as the function of  $\lambda$ , but it poses a question about the precise analytic behavior of  $E(\lambda)$  in the vicinity of critical points.

We discuss both the canonical and simplified TBA equations. The canonical equations [22, 24, 25] follow from the string hypothesis for the mirror model [21] by using the standard procedure, see e.g. [53]. The simplified equations [22, 23] obtained from the canonical ones have more close relation to the Y-system. It turns out that the simplified equations are sensitive only to the critical points defined above. In contrast, the canonical equations have to be modified when crossing not only a critical point but also what we call a subcritical point  $\bar{g}_{cr}$ . A subcritical point  $\bar{g}_{cr}$  is defined as the value of  $g$  at which the function  $Y_{Q|vw}(u)$  acquires zero at  $u = 0$ . Hence, in comparison to the canonical equations, the simplified equations exhibit a more transparent analytic structure. In addition to locality, this is yet another reason why we attribute to the simplified equations a primary importance and carry out their analysis in the main text. To study the exact Bethe equations which determine the exact, *i.e.* non-asymptotic, location of the Bethe roots, we find it advantageous to use a so-called *hybrid form* of the TBA equations for  $Y_Q$ -functions. This form is obtained by exploiting both the canonical and simplified TBA equations.

Recently, the finite-gap solutions of semi-classical string theory have been nicely derived [54] from the TBA equations [25]. This raises a question why modifications of the TBA equations we find in this paper were not relevant for this derivation. We have not studied this question thoroughly. However, one can immediately see that there is a principle difference between states with finite number of particles and semiclassical states composed of infinitely many particles. Namely, at strong coupling the rapidities of two-particle states fall inside the interval  $[-2, 2]$ , while those of semiclassical states are always outside this interval. Thus, the modification of the TBA equations discussed in this paper might not be necessary for semi-classical states. It would be important to better understand this issue.

The paper is organized as follows. In the next section we explain our criteria for a choice of the integration contour in the excited states TBA equations. In section 3 we discuss two-particle states in the  $\mathfrak{sl}(2)$  sector and the corresponding asymptotic Y-functions. By analysing analytic properties of the Y-functions, we determine the critical and subcritical values of the coupling constant both for the Konishi and for some other states. In section 4 we present the simplified TBA equations for Konishi-like states and we explain why and how their form depends on the value of the coupling constant. In section 5 we generalize this discussion to arbitrary two-particle states from the  $\mathfrak{sl}(2)$  sector. In section 6 we summarize the most essential properties of the AdS/CFT Y-system implied by the TBA equations under study.



**Figure 1:** These are the mirror and string regions on the  $z$ -torus. They are in one-to-one correspondence with the  $u$ -planes. The boundaries of the regions are mapped to the cuts.

Finally, in Conclusions we mention some interesting open problems. The definitions, treatment of canonical equations, and further technical details are relegated to eight appendices.

## 2. Contour deformation trick

The TBA equations for the  $\text{AdS}_5 \times \text{S}^5$  mirror model are written for Y-functions which depend on the real momentum of the mirror model. The energy of string excited states obviously depends on real momenta of string theory particles, and to formulate the TBA equations for excited states one also needs to continue analytically the Y-functions to the string theory kinematic region. To visualize the analytic continuation it is convenient to use the  $z_Q$ -tori because the kinematic regions of the mirror and string theory Q-particle bound states (Q-particles for short) are subregions of the  $z$ -torus, see Figure 1. In addition, the Q-particle energy, and many of the kernels appearing in the set of TBA equations are meromorphic functions on the corresponding torus. The mirror Q-particle region can be mapped onto a  $u$ -plane with the cuts running from the points  $\pm 2 \pm \frac{i}{g}Q$  to  $\pm\infty$ , and the string Q-particle region can be mapped onto a  $u_*$ -plane with the cuts connecting the points  $-2 \pm \frac{i}{g}Q$  and  $2 \pm \frac{i}{g}Q$ , see Figure 1. Since the cut structure on the planes is different for each Y-function, they cannot be considered as different sheets of a Riemann surface. The  $z_Q$ -torus can be glued either from four mirror  $u$ -planes or four string  $u_*$ -planes.

As was shown in [20],  $z$ -torus variables corresponding to real momenta of the mirror and string theory are related to each other by the shift by a quarter of the

imaginary period of the torus

$$z = z_* + \frac{\omega_2}{2}, \quad (2.1)$$

where  $z$  is the variable parametrizing the real momenta of the mirror theory, and  $z_*$  is the variable parametrizing the real momenta of the string theory. The line  $(-\infty, \infty)$  in the string  $u_*$ -plane is mapped to the interval  $\text{Re}(z_*) \in (-\frac{\omega_1}{2}, \frac{\omega_1}{2})$ ,  $\text{Im}(z_*) = \text{const}$  on the  $z$ -torus, and we choose  $z_*$  in the string region to be real. Then, the interval  $\text{Re}(z) \in (-\frac{\omega_1}{2}, \frac{\omega_1}{2})$ ,  $\text{Im}(z) = \frac{\omega_2}{2i}$  of the mirror region is mapped onto the real line of the mirror  $u$ -plane.

It is argued in [36, 37] that the TBA equations for excited states can be obtained from the ones for the ground state by analytically continuing in the coupling constants and picking up the singularity of proper convolution terms. We prefer however to employ a slightly different procedure which we refer to as the contour deformation trick. We believe it is equivalent to [36, 37]. It is based on the following assumptions

- The form of TBA equations for any excited state and the expression for the energy are universal. TBA equations for excited states differ from each other only by a choice of integration contours of convolution terms and the length parameter  $L$  which depend on a state.
- The choice of the integration contours and  $L$  is fixed by requiring that the large  $L$  solution of the excited state TBA equations be given by the generalized Lüscher formulae, that is all the  $Y$ -functions can be written in terms of the eigenvalues of the transfer matrices. The integration contour depends on the excited state under consideration, and in general on the values of 't Hooft's coupling and  $L$ .
- An excited state is completely characterized by the five charges it carries and a set of  $N$  real numbers  $p_k$  which are in one-to-one correspondence with momenta  $p_k^o$  of  $N$   $Q$ -particles in the small coupling limit  $g \rightarrow 0$ . The momenta  $p_k^o$  are found by using the one-loop Bethe equations for fundamental particles and their bound states. For finite values of  $g$  the set of  $p_k$  is determined by exact Bethe equations which state that at any  $p = p_k$  the corresponding  $Y_Q$ -functions are equal to  $-1$ .

We consider only the simplest case of two-particle excited states in the  $\mathfrak{sl}(2)$  sector of the string theory because there are no bound states in this sector and the complete two-particle spectrum can be readily classified. The physical states satisfy the level-matching condition which for two-particle states takes a very simple form:  $p_1 = -p_2$ , or  $u_1 = -u_2$ , or  $z_{*1} = -z_{*2}$  depending on the coordinates employed. The TBA equations we propose in next sections are valid only for physical states.



### 3. States and Y-functions in the $\mathfrak{sl}(2)$ sector

To fix the integration contour one should choose a state and analyze the analytic structure of the large  $L$  Y-functions which we refer to as the asymptotic Y-functions. We begin with a short discussion of two-particle states in the  $\mathfrak{sl}(2)$  sector.

#### 3.1 Bethe-Yang equations for the $\mathfrak{sl}(2)$ sector

There is only a single Bethe-Yang (BY) equation in the  $\mathfrak{sl}(2)$ -sector for two-particle physical configurations satisfying the vanishing total momentum condition  $p_1 + p_2 = 0$  that can be written in the form [55]

$$1 = e^{ipJ} S_{\mathfrak{sl}(2)}(p, -p) \implies e^{ip(J+1)} = \frac{1 + \frac{1}{x_s^{+2}}}{1 + \frac{1}{x_s^{-2}}} \sigma(p, -p)^2, \quad (3.1)$$

where  $p \equiv p_1 > 0$ ,  $J$  is the charge carried by the state,  $\sigma$  is the dressing factor, and  $x_s^\pm$  are defined in appendix 8.1. Taking the logarithm of the equation, one gets

$$ip(J+1) - \log \frac{1 + \frac{1}{x_s^{+2}}}{1 + \frac{1}{x_s^{-2}}} - 2i\theta(p, -p) = 2\pi i n, \quad (3.2)$$

where  $\theta = \frac{1}{i} \log \sigma$  is the dressing phase, and  $n$  is a positive integer because we have assumed  $p$  to be positive. As was shown in [56], at large values of  $g$  the integer  $n$  is equal to the string level of the state.

As is well known, in the small  $g$  limit the equation has the obvious solution

$$p_{J,n}^o = \frac{2\pi n}{J+1}, \quad n = 1, \dots, \left[ \frac{J+1}{2} \right], \quad (3.3)$$

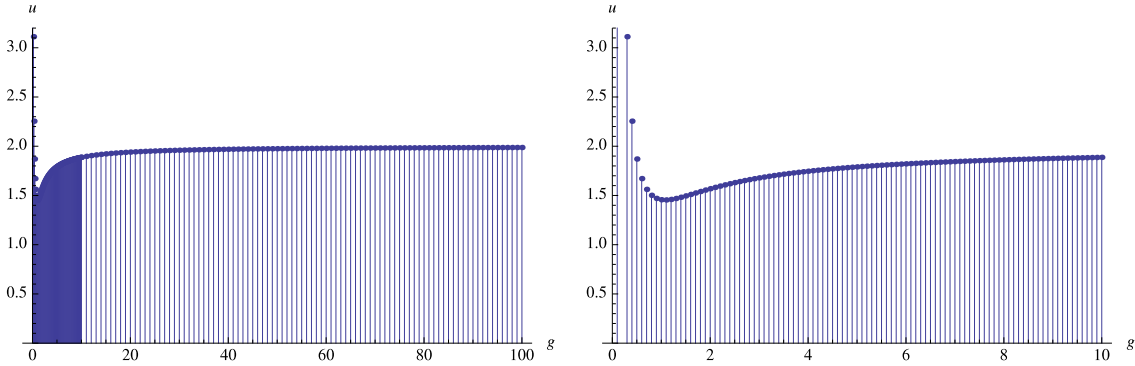
where  $[x]$  denotes the integer part of  $x$ , and the range of  $n$  is bounded because the momentum  $p$  can only take values from 0 to  $\pi$ . The corresponding rapidity variable  $u_{J,n}$  in the small  $g$  limit takes the following form

$$u_{J,n} \rightarrow \frac{1}{g} u_{J,n}^o, \quad u_{J,n}^o = \cot \frac{\pi n}{J+1}. \quad (3.4)$$

Thus, any two-particle state in the  $\mathfrak{sl}(2)$  sector is completely characterized by the two integers  $J$  and  $n$ . In particular, in the simplest  $J = 2$  case corresponding to a descendent of the Konishi state  $n$  can take only one value  $n = 1$ , and the small  $g$  solution is

$$p_{2,1}^o = \frac{2\pi}{3}, \quad u_{2,1}^o = \frac{1}{\sqrt{3}}. \quad (3.5)$$

The BY equation (3.2) can be easily solved perturbatively up to any desired order in  $g$ , and numerically up to very large values of  $g$ . We have used the BES series



**Figure 2:** These are the plots of  $u$  which solves the BY equation for the Konishi state.

representation [51] for the dressing phase for perturbative computations, and the DHM integral representation [57] for the numerical ones<sup>3</sup>. For the Konishi state, the perturbative solution up to  $g^{16}$ -th can be found in appendix 8.2.

We have solved numerically the BY equation for the Konishi state for  $\frac{1}{10} \leq g \leq 1000$  with the step  $\frac{1}{10}$  for  $\frac{1}{10} \leq g \leq 10$ , the step 1 for  $10 \leq g \leq 100$ , and the step 10 for  $100 \leq g \leq 1000$ . In Figure 2 we show the results up to  $g = 100$ . For greater values of  $g$  nothing interesting happens, and the solution can be approximated by asymptotic formulae from [56, 32, 58], see appendix 8.2 for more details. We then applied the Interpolation function in Mathematica to have  $u_{2,1}$  as a smooth function of  $g$ . Using the function, one can find that  $u_{2,1}(g)$  decreases up to  $g \sim 0.971623$ , and then begins to increase and at large  $g$  it asymptotes to  $u = 2$ .

The functions  $u_{J,n}(g)$  for other values of  $J$  and  $n$  have similar  $g$  dependence. The only exception is the case  $J = 2n - 1$  where the exact solution of the BY equation is  $p_{2n-1,n} = \pi$  and, therefore,  $u_{2n-1,n} = 0$ . The TBA equations we propose below are not in fact valid for these  $(2n - 1, n)$  states.

The perturbative and numerical solutions for  $u_{J,n}(g)$  can be easily used to analyze the behavior of Y-functions considered as functions of  $g$ . In particular it is easy to determine if some of them become negative for large enough values of  $g$ .

### 3.2 Y-functions in the $\mathfrak{sl}(2)$ sector

Let us recall that the TBA equations for the  $\text{AdS}_5 \times S^5$  mirror model involve  $Y_Q$ -functions for momentum carrying  $Q$ -particle bound states, and auxiliary functions  $Y_{Q|vw}^{(\alpha)}$  for  $Q|vw$ -strings,  $Y_{Q|w}^{(\alpha)}$  for  $Q|w$ -strings, and  $Y_{\pm}^{(\alpha)}$  for  $y_{\pm}$ -particles. The index  $\alpha = 1, 2$  reflects two  $\mathfrak{su}(2|2)$  algebras in the symmetry algebra of the light-cone string sigma model. The TBA equations [22] depend also on the parameters  $h_{\alpha}$  which take care of the periodicity condition of the fermions of the model [59]. For the  $\mathfrak{sl}(2)$  sector

<sup>3</sup>The DHM representation can be also readily used for perturbative computations.

the fermions are periodic, and from the very beginning one can set the parameters  $h_\alpha$  to 0 because there is no singularity at  $h_\alpha = 0$  in the excited states TBA equations.

For the  $\mathfrak{sl}(2)$  states there is a symmetry between the left and right  $\mathfrak{su}(2|2)$  auxiliary roots, and, therefore, all Y-functions satisfy the condition

$$Y_{\forall}^{(1)} = Y_{\forall}^{(2)} = Y_{\forall}, \quad (3.6)$$

where  $\forall$  denotes a Y-function of any kind.

The string theory spectrum in the  $\mathfrak{sl}(2)$  sector is characterized by a set of  $N$  real numbers  $z_{*k}$  or  $u_k$  corresponding to momenta of  $N$  fundamental particles in the limit  $g \rightarrow 0$ . According to the discussion above, these numbers are determined from the exact Bethe equations

$$Y_1(z_{*k}) = Y_{1*}(u_k) = -1, \quad k = 1, \dots, N, \quad (3.7)$$

where  $Y_1(z)$  is the Y-function of fundamental mirror particles considered as a function on the  $z$ -torus, and  $Y_{1*}(u)$  denotes the  $Y_1$ -function analytically continued to the string  $u$ -plane.

In the large  $J$  limit the exact Bethe equations must reduce to the BY equations, and it is indeed so because the asymptotic  $\mathfrak{sl}(2)$   $Y_Q$ -functions can be written in terms of the transfer matrices defined in appendix 8.3 as follows [2]

$$Y_Q^o(v) = e^{-J\tilde{\mathcal{E}}_Q(v)} \frac{T_{Q,1}(v|\vec{u})^2}{\prod_{i=1}^N S_{\mathfrak{sl}(2)}^{1*Q}(u_i, v)} = e^{-J\tilde{\mathcal{E}}_Q(v)} T_{Q,1}(v|\vec{u})^2 \prod_{i=1}^N S_{\mathfrak{sl}(2)}^{Q1*}(v, u_i), \quad (3.8)$$

where  $v$  is the rapidity variable of the mirror  $u$ -plane and  $\tilde{\mathcal{E}}_Q$  is the energy of a mirror  $Q$ -particle.  $S_{\mathfrak{sl}(2)}^{1*Q}$  denotes the S-matrix with the first and second arguments in the string and mirror regions, respectively.  $T_{Q,1}(v|\vec{u})$  is up to a factor the trace of the S-matrix describing the scattering on these string theory particles with a mirror  $Q$ -particle or in other words the eigenvalue of the corresponding transfer matrix. The BY equations then follow from the fact that  $\tilde{\mathcal{E}}_{1*}(u_k) = -ip_k$ , and the following normalization of  $T_{1,1}$

$$T_{1,1}(u_{*k}|\vec{u}) = 1 \implies -1 = e^{iJp_k} \prod_{i=1}^N S_{\mathfrak{sl}(2)}^{1*1*}(u_k, u_i), \quad (3.9)$$

where  $u_{*k} = u_k$ , and the star just indicates that one analytically continues  $T_{1,1}$  to the string region. Then,  $S_{\mathfrak{sl}(2)}^{1*1*}(u_k, u_i) = S_{\mathfrak{sl}(2)}(u_k, u_i)$  is the usual  $\mathfrak{sl}(2)$  sector S-matrix used in the previous subsection.

Let us also mention that  $T_{Q,1}$  has the following large  $v$  asymptotics

$$T_{Q,1}(v|\vec{u}) \rightarrow Q \left( 1 - \prod_{i=1}^N \sqrt{\frac{x_i^+}{x_i^-}} \right)^2, \quad v \rightarrow \infty, \quad (3.10)$$

and therefore it goes to 0 if the level-matching is satisfied.

Then, in the large  $J$  limit all auxiliary asymptotic  $\mathfrak{sl}(2)$  Y-functions can be written in terms of the transfer matrices as follows [27]

$$Y_-^o = -\frac{T_{2,1}}{T_{1,2}}, \quad Y_+^o = -\frac{T_{2,3}T_{2,1}}{T_{3,2}T_{1,2}}, \quad Y_{Q|vw}^o = \frac{T_{Q+2,1}T_{Q,1}}{T_{Q+1,2}}, \quad Y_{Q|w}^o = \frac{T_{1,Q+2}T_{1,Q}}{T_{2,Q+1}T_{0,Q+1}}.$$

The transfer matrices  $T_{a,s}$  can be computed in terms of  $T_{a,1}$  by using the Bazhanov-Reshetikhin formula [61], see appendix 8.3 for all the necessary explicit formulae.

An important property of the Y-functions for  $vw$ - and  $w$ -strings is that they approach their vacuum values as  $v \rightarrow \infty$

$$Y_{M|vw}(v) \rightarrow M(M+2), \quad Y_{M|w}(v) \rightarrow M(M+2), \quad v \rightarrow \infty, \quad -\frac{M}{g} < \text{Im}(v) < \frac{M}{g}.$$

Now we are ready to analyze the dependence of asymptotic Y-functions on  $g$ . Recall that they depend on the rapidities  $u_k$  which are solutions of the BY equations.

We begin with the small  $g$  limit where the effective length goes to infinity, and one can in fact trust all the asymptotic formulae. It is convenient to rescale  $v$  and  $u_k$  variables as  $v \rightarrow v/g$ ,  $u_k \rightarrow u_k/g$  because the rescaled variables are finite in this limit. Let  $\kappa \equiv u_1 = -u_2$  be the rescaled rapidity of a fundamental particle. According to the previous subsection, in the small  $g$  limit they are given by  $u_{J,n}^o$ , eq.(3.4).

The most important functions in the  $\mathfrak{sl}(2)$  case are  $Y_{M|vw}$ , and we find that for  $N = 2$  they exhibit the following small  $g$  behavior in the strip  $-M < \text{Im}(v) < M$

$$Y_{M|vw}(v) = M(M+2) \frac{[M^2 - 1 + v^2 - \kappa^2][(M+2)^2 - 1 + v^2 - \kappa^2]}{[(M+1)^2 + (v-\kappa)^2][(M+1)^2 + (v+\kappa)^2]} + \mathcal{O}(g^2). \quad (3.11)$$

The leading term has the correct large  $u$ -asymptotics and four apparent zeros at

$$v = \pm\sqrt{\kappa^2 - M^2 + 1}, \quad v = \pm\sqrt{\kappa^2 - (M+2)^2 + 1}.$$

One can see that  $Y_{1|vw}$ -function always has at least two real zeros at  $v = \pm\kappa$ . Other zeros of  $Y_{M|vw}$ -functions can be either real or purely imaginary depending on the values of  $M$  and  $\kappa$ . It appears that the form of simplified TBA equations depends on the imaginary part of these zeros, and we will see in next sections that if a pair of zeros  $v = \pm r$  fall in the strip  $|\text{Im}(r)| < 1$  then the equations should be modified.

Thus, we are lead to consider the following three possibilities

1. If  $M^2 - 2 < \kappa^2 < (M+2)^2 - 2$  then  $Y_{M|vw}$  has two zeros at  $v = \pm\sqrt{\kappa^2 - M^2 + 1}$  that are in the strip  $|\text{Im}(v)| < 1$ . In terms of the integers  $J$  and  $m$  characterizing two-particle states one gets the condition

$$\sqrt{M^2 - 2} < \cot \frac{\pi n}{J+1} < \sqrt{(M+2)^2 - 2}. \quad (3.12)$$

2. If  $\kappa^2 < M^2 - 2 \iff \cot \frac{\pi n}{J+1} < \sqrt{M^2 - 2}$  then  $Y_{M|vw}$  does not have any zeros in the strip  $|\text{Im}(v)| < 1$ .
3. If  $\kappa^2 > (M+2)^2 - 2 \iff \cot \frac{\pi n}{J+1} > \sqrt{(M+2)^2 - 2}$  then  $Y_{M|vw}$  has four zeros in the strip  $|\text{Im}(v)| < 1$ .

Some of these zeros can be real, and in fact the canonical TBA equations take different forms depending on whether the roots are real or imaginary.

Classification of two-particle states at  $g \sim 0$  is presented in Table 1. The type of a state is determined by how many zeroes of  $Y_{M|vw}$ -functions occur in the physical strip and it depends on  $J$  and  $n$ .

Consider a two-particle state with  $\kappa = u_{J,n}^o$  for some  $(J, n)$ . Table 1 shows that there exists a number  $m \geq 1$ , equal to the maximal value of  $M$  the condition (3.12) is satisfied. Then both  $Y_{m|vw}$  and  $Y_{m-1|vw}$  have two zeros, all  $Y_{k|vw}$  with  $k \leq m-2$  have four zeros, and all  $Y_{k|vw}$ -functions with  $k \geq m+1$  have no zeros in the strip  $|\text{Im}(v)| < 1$ . For example, among the states with  $(J, n = 1)$  at small coupling, the states of type I are found if and only if  $J \leq 7$ . The type II is found for  $5 \leq J \leq 7$ , type III for  $8 \leq J \leq 11$ , and type IV for  $12 \leq J \leq 14$ . In particular,  $Y_{1|vw}$  for the state  $(8, 1)$  has two real zeros and two imaginary zeros in the strip  $|\text{Im}(v)| < 1$ , and  $Y_{1|vw}$  for the state  $(J \geq 9, 1)$  has four real zeros.

As for the Konishi state with  $J = 2$  and  $n = 1$  only  $Y_{1|vw}$ -function has two zeros and all the other  $Y_{M|vw}$ -functions have no zeros at small coupling. Let us also mention that at  $g = 0$  the  $Y_{2|vw}$ -function of the state  $(5, 1)$  (and in general of any state  $(6k-1, k)$ ) has a double zero at  $v = 0$ . This double zero however is an artifact of the perturbative expansion, and in reality  $Y_{2|vw}$  has two imaginary zeros for small values of  $g$  equal to  $\approx \pm ig\sqrt{3}$ . For the state with  $J = 6$  and  $n = 1$  both  $Y_{1|vw}$  and  $Y_{2|vw}$  have two real zeros.

### 3.3 Critical values of $g$

#### Evolution of zeros

Now we would like to understand what happens with  $Y_{M|vw}$ -functions when one starts increasing  $g$ . To this end one should use numerical solutions of the BY equations discussed at the beginning of this section. We also switch back to the original  $u$  variables because they are more convenient for general values of  $g$ , and refer to the strip  $|\text{Im}(u)| < 1/g$  as the physical one.

We find that for finite  $g$  any  $Y_{k|vw}$ -function has four zeros which are either real or purely imaginary. We could not find any other complex zeros. The four zeros of  $Y_{k|vw}$  are split into two pairs, and the two zeros in a pair have opposite signs, and are either real or complex conjugate to each other. We denote the four zeroes of  $Y_{k|vw}$  by  $r_j^{(k)}$  and  $\hat{r}_j^{(k)}$ , where  $j = 1, 2$  and  $k = 1, 2, \dots$ . If the four zeros are real then  $r_j^{(k)}$

are the zeros of  $Y_{k|vw}$  which have a larger absolute value than  $\hat{r}_j^{(k)}$ . If only two zeros are real then we denote them as  $r_j^{(k)}$  and the imaginary zeros as  $\hat{r}_j^{(k)}$ . Finally, if the four zeros are imaginary then  $r_j^{(k)}$  are the ones closer to the real line than the second pair  $\hat{r}_j^{(k)}$ .

The locations of the zeros depend on  $g$ , and we should distinguish two cases. We observe first that if two zeros are real at  $g \sim 0$  then they are of order  $1/g$ , and obviously outside the interval  $[-2, 2]$ . With  $g$  increasing they start moving toward the origin, and at some value of  $g$  they reach their closest position to the origin which is inside the interval  $[-2, 2]$ . Then, for larger  $g$  they remain inside the interval but begin to move to its boundaries and reach them at  $g = \infty$ . In the second case, one considers a pair of imaginary zeros at  $g \sim 0$ . With  $g$  increasing they start moving toward the real line, and at some value of  $g$  they get to the origin and become a double zero. Then, for larger  $g$  they split and begin to move to the boundaries of the interval  $[-2, 2]$ , and reach them at  $g = \infty$ . The only exception from this behavior we find is the  $g$ -dependence of the two zeros of  $Y_{2|vw}$ -function for the states  $(6k - 1, k)$  that are equal to  $\pm i\sqrt{3}$  at small  $g$ . These zeros become real at very small value of  $g$ . Then they start moving to  $\pm 2$ , cross the boundaries of the interval  $[-2, 2]$ , and reach their maximum. After that they behave as zeros of all the other  $Y_{k|vw}$ -functions. Thus, at very large values of  $g$  all zeros of any  $Y_{k|vw}$ -function are real, inside the interval  $[-2, 2]$ , and very close to  $\pm 2$ .

The pairs of the zeros of different  $Y_{M|vw}$ -functions are not independent, and satisfy the following relations

$$\hat{r}_j^{(k-1)} = r_j^{(k+1)}, \quad k = 2, \dots, \infty. \quad (3.13)$$

Therefore, the zeros of  $Y_{M|vw}$ -functions can be written as follows

$$\{r_j^{(1)}, r_j^{(3)}\}; \{r_j^{(2)}, r_j^{(4)}\}; \dots; \{r_j^{(k-1)}, r_j^{(k+1)}\}; \{r_j^{(k)}, r_j^{(k+2)}\}; \{r_j^{(k+1)}, r_j^{(k+3)}\}; \dots \quad (3.14)$$

so that  $Y_{k|vw}$  has the zeros  $\{r_j^{(k)}, r_j^{(k+2)}\}$ .

These zeros have a natural ordering. If we assume for definiteness that the zeros with  $j = 1$  have negative real or imaginary parts, then they are ordered as

$$r_1^{(1)} \prec r_1^{(2)} \prec r_1^{(3)} \prec \dots \prec r_1^{(k)} \prec r_1^{(k+1)} \prec \dots, \quad (3.15)$$

where  $r_1^{(k)} \prec r_1^{(k+1)}$  if either  $\text{Re}(r_1^{(k)}) < \text{Re}(r_1^{(k+1)})$  or  $\text{Im}(r_1^{(k)}) > \text{Im}(r_1^{(k+1)})$ . It is important that the zeros never change the ordering they have at  $g \sim 0$ . In particular,  $Y_{1|vw}$  always has two real zeros  $r_j^{(1)} = u_j$  which are Bethe roots. They are the largest (in magnitude) zeros among all  $Y_{k|vw}$ -functions, and are the closest ones to  $\pm 2$  at large  $g$ .

Initial condition $\rightarrow$	$Y_{1 vw}, Y_{2 vw}$	2+2
	$Y_{1 vw}, Y_{2 vw}, Y_{3 vw}$	4+2+2
$g \downarrow$	$Y_{1 vw}, Y_{2 vw}, Y_{3 vw}, Y_{4 vw}$	4+4+2+2
	$Y_{1 vw}, Y_{2 vw}, Y_{3 vw}, Y_{4 vw}, Y_{5 vw}$	4+4+4+2+2
$\vdots$	$\vdots$	$\vdots$
	$Y_{1 vw}, Y_{2 vw}, \dots$	4+4+ ...

Table 2. Evolution of a two-particle states in the  $\mathfrak{sl}(2)$ -sector with respect to  $g$ . At  $g \sim 0$  a state has a certain number of  $Y_{M|vw}$ -functions with zeroes in the physical strip. Increasing the coupling, the critical points get crossed which leads to accumulation of zeroes of  $Y_{M|vw}$ 's in the physical strip. This phenomenon can be called ‘‘Y-function democracy’’.

In addition we find that the functions below have either zeros or equal to  $-1$  at locations related to  $r_j^{(k)}$

$$Y_{k|vw}(r_j^{(k+1)} \pm \frac{i}{g}) = -1, \quad Y_{k+1}(r_j^{(k+1)}) = 0, \quad k = 1, \dots, \infty, \quad Y_{\pm}(r_j^{(2)}) = 0. \quad (3.16)$$

As will be discussed in the next section the equations  $Y_{k|vw}(r_j^{(k+1)} \pm \frac{i}{g}) = -1$  lead to integral equations which play the same role as the exact Bethe equations  $Y_1(u_j) = -1$  and allow one to find the exact location of the roots  $r_j^{(k+1)}$ .

Let us finally mention that nothing special happens to  $Y_{M|w}$ -functions.

### Critical and subcritical values

Let  $m$  again be the maximum value of  $M$  the condition (3.12) is satisfied. According to the discussion above for any two-particle state there is a critical value of  $g$  such that the function  $Y_{m+1|vw}$  which had no zeros in the physical strip for small values of  $g$ , acquires two zeros at  $u = \pm i/g$ . At the same time  $Y_{m-1|vw}$  also acquires zeros at  $u = \pm i/g$ . At a slightly larger value of  $g$  the two zeros that were at  $u = \pm i/g$  collide at the origin, and  $Y_{m+1|vw}$  and  $Y_{m-1|vw}$  acquire double zeros at  $u = 0$ . Then, the double zeros split, and both  $Y_{m|vw}$  and  $Y_{m+1|vw}$  have two real zeros, and  $Y_{m-1|vw}$  has four. Increasing  $g$  more, one reaches the second critical value of  $g$  such that the functions  $Y_{m+2|vw}$  and  $Y_{m|vw}$  acquire zeros at  $u = \pm i/g$ , see Table 2.

This pattern repeats itself, and there are infinitely many critical values of  $g$  which we denote as  $g_{J,n}^{r,m}$  and define as the smallest value of  $g$  such that for a symmetric configuration of Bethe roots the function  $Y_{m+r|vw}$  acquires two zeros at  $u = \pm i/g$ . The subscript  $J, n$  denotes a state in the  $\mathfrak{sl}(2)$  sector, and they determine  $m$ .

The critical values of  $g$  can be also determined from the requirement that at  $g = g_{J,n}^{r,m}$  the function  $1 + Y_{m+r-1|vw}$  has a double zero at  $u = 0$ :  $1 + Y_{m+r-1|vw}(0, g_{J,n}^{r,m}) = 0$ .

This condition is particularly useful because the value of the Y-functions at  $u = 0$  can be found from the TBA equations, see next section for detail.

The second set of subcritical values of  $g$  can be defined as the smallest value of  $g$  such that the function  $Y_{m+r|vw}$  acquires a double zero at  $u = 0$ . They are denoted as  $\bar{g}_{J,n}^{r,m}$ , and they are always greater than the corresponding critical values:  $g_{J,n}^{r,m} < \bar{g}_{J,n}^{r,m}$ .

The locations of the critical values depend on the state under consideration, and can be determined approximately by using the asymptotic Y-functions discussed in the previous subsection. The values obtained this way are only approximate because for large enough values of  $g$  one should take into account the deviations of the Y-functions from their large  $J$  expressions. Since at very large values of  $g$  the contribution of  $Y_Q$ -functions to the exact energy is expected to be negligible,<sup>4</sup> these values should also be accurate enough for any  $J$  at strong coupling.

We will see in next sections that the critical values  $g_{J,n}^{r,m}$  play a crucial role in formulating excited states simplified TBA equations which take different form in each of the intervals  $g_{J,n}^{r,m} < g < g_{J,n}^{r+1,m}$ ,  $r = 0, 1, \dots$  where  $g_{J,n}^{0,m} = 0$ . The second set of  $\bar{g}_{J,n}^{r,m}$  is not important for the simplified equations. The canonical TBA equations however require both sets because they take different form in each of the intervals  $g_{J,n}^{r,m} < g < \bar{g}_{J,n}^{r,m}$ ;  $\bar{g}_{J,n}^{r,m} < g < g_{J,n}^{r+1,m}$ ,  $r = 0, 1, \dots$

Strictly speaking the integration contour in TBA equations also depends on  $g$  and the state under consideration. Nevertheless it appears that in simplified TBA equations the contour can be chosen to be the same for all values of  $g$ , and even for all two-particle states from the  $\mathfrak{sl}(2)$  sector if one allows its dynamical deformation. That means that with increasing  $g$  the contour should be deformed in such a way that it would not hit any singularity. This also shows that one should not expect any kind of non-analyticity in the energy of a state at a critical value of  $g$ . What may happen is that the critical values are the inflection points of the energy.

### Critical values of $g$ for the Konishi state

In this subsection we discuss in detail the critical values for Konishi state. To analyze the dependence of Y-functions on  $g$  one should first solve the BY equations with  $J = 2$ ,  $n = 1$ , and then plug the  $u_j$ 's obtained into the expressions for Y-functions from appendix 8.3.

Solving the equations

$$Y_{r+1|vw}\left(\pm \frac{i}{g}, g\right) = 0, \quad r = 1, 2, \dots, \quad (3.17)$$

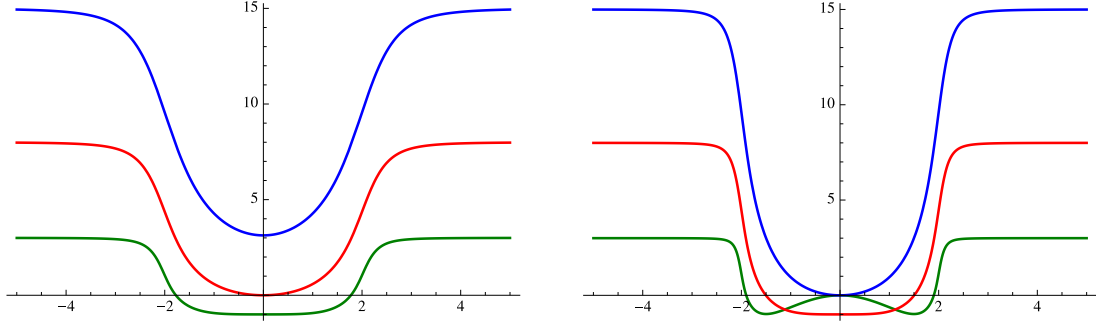
we find that there are 7 critical values of  $g$  for  $g < 100$

$$g_{2,1}^{(r,1)} = \{4.429, 11.512, 21.632, 34.857, 51.204, 70.680, 93.290\}. \quad (3.18)$$

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<sup>4</sup>It follows from the results of [56] where it was shown that the leading large  $g$  asymptotics of the energy of multi-particle states with a finite number of particles follows from the BY equations.





**Figure 3:** On the left and right pictures  $Y_{1|vw}$ ,  $Y_{2|vw}$  and  $Y_{3|vw}$  are plotted for the Konishi state at  $\bar{g}_{cr}^{(1)} \approx 4.5$  and  $\bar{g}_{cr}^{(2)} \approx 11.5$ , respectively.  $Y_{2|vw}$  touches the  $u$ -axis at  $g = \bar{g}_{cr}^{(1)}$ , and has two real zeros for  $\bar{g}_{cr}^{(1)} < g < \bar{g}_{cr}^{(2)}$ .

Note that the distance between the critical values increases with  $g$ . The first critical value is distinguished because only  $Y_{2|vw}(\pm i/g, g)$  vanishes there. For all the other critical values the function  $Y_{r-1|vw}(\pm i/g, g)$  also is equal to zero

$$Y_{r+1|vw}(\pm \frac{i}{g}, g_{2,1}^{(r,1)}) = 0 \implies Y_{r-1|vw}(\pm \frac{i}{g}, g_{2,1}^{(r,1)}) = 0, \quad \text{for } r = 2, 3, \dots \quad (3.19)$$

Then, solving the equations

$$Y_{r+1|vw}(0, g) = 0, \quad r = 1, 2, \dots, \quad (3.20)$$

one finds the following 7 subcritical values of  $g$  for  $g < 100$

$$\bar{g}_{2,1}^{(r,1)} = \{4.495, 11.536, 21.644, 34.864, 51.209, 70.684, 93.292\}. \quad (3.21)$$

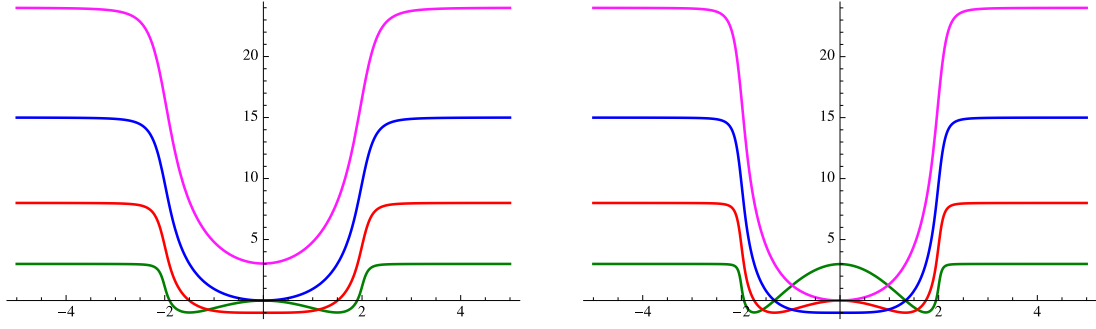
Note that the distance between a critical value and a corresponding subcritical one decreases with  $g$ . Again, at the first subcritical value only  $Y_{2|vw}(0, g)$  vanishes. For all the other subcritical values the function  $Y_{r-1|vw}(0, g)$  also acquires an extra double zero

$$Y_{r+1|vw}(0, \bar{g}_{2,1}^{(r,1)}) = 0 \implies Y_{r-1|vw}(0, \bar{g}_{2,1}^{(r,1)}) = 0, \quad \text{for } r = 2, 3, \dots \quad (3.22)$$

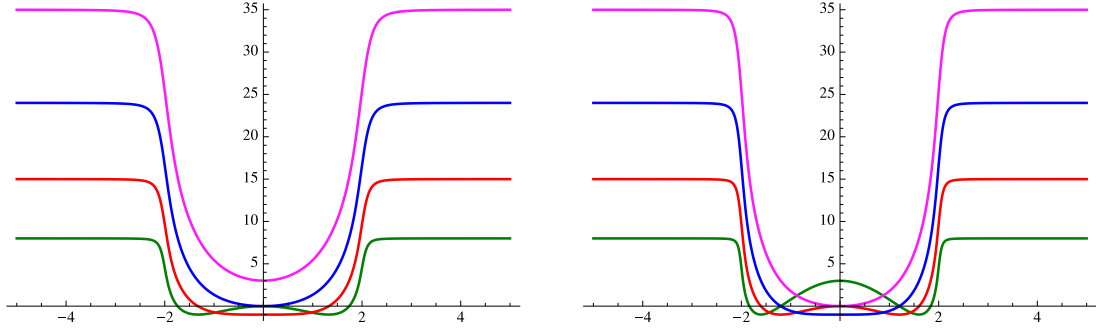
Once  $g$  crosses a subcritical value  $\bar{g}_{2,1}^{(r,1)}$  the corresponding double zeros at  $u = 0$  split, and each of the functions  $Y_{r-1|vw}(0, g)$  and  $Y_{r+1|vw}(0, g)$  acquires two symmetrically located zeros. As a result at infinite  $g$  all the  $Y_{M|vw}$ -functions have four real zeros. Moreover, one can also see that if  $g$  is between two subcritical values then all these zeros for all the functions are inside the interval  $[-2, 2]$  and approach  $\pm 2$  as  $g \rightarrow \infty$ .

In Figures 3-5, we show several plots of  $Y_{M|vw}$ -functions for the Konishi state.

In what follows a Konishi-like state refers to any two-particle state for which only  $Y_{1|vw}$ -function has two real zeros and all the other  $Y_{M|vw}$ -functions have no zeros in the physical strip at small coupling.



**Figure 4:** On the left and right pictures  $Y_{1|vw}$ ,  $Y_{2|vw}$ ,  $Y_{3|vw}$  and  $Y_{4|vw}$  are plotted for the Konishi state at  $\bar{g}_{cr}^{(2)} \approx 11.5$  and  $\bar{g}_{cr}^{(3)} \approx 21.6$ , respectively.  $Y_{1|vw}$  and  $Y_{3|vw}$  touch the  $u$ -axis at  $g = \bar{g}_{cr}^{(2)}$ , and  $Y_{2|vw}$  and  $Y_{4|vw}$  touch it at  $g = \bar{g}_{cr}^{(3)}$ .  $Y_{1|vw}$  has four real zeros for  $g > \bar{g}_{cr}^{(2)}$ .



**Figure 5:** On the left picture and right pictures  $Y_{2|vw}$ ,  $Y_{3|vw}$ ,  $Y_{4|vw}$  and  $Y_{5|vw}$  are plotted for the Konishi state at  $\bar{g}_{cr}^{(3)} \approx 21.6$  and  $\bar{g}_{cr}^{(4)} \approx 34.9$ , respectively.  $Y_{2|vw}$  and  $Y_{4|vw}$  touch the  $u$ -axis at  $g = \bar{g}_{cr}^{(3)}$ , and  $Y_{3|vw}$  and  $Y_{5|vw}$  touch it at  $g = \bar{g}_{cr}^{(4)}$ .  $Y_{2|vw}$  has four real zeros for  $g > \bar{g}_{cr}^{(3)}$ .

### Critical values of $g$ for some states

Here we analyze the  $g$ -dependence of  $Y$ -functions for several other states.

We begin with the state with  $J = 5$  and  $n = 1$ . This is the state with the lowest value of  $J$  such that both  $Y_{1|vw}$  and  $Y_{2|vw}$  have two zeros in the physical strip at small  $g$ . The critical and subcritical values are determined by the equations

$$Y_{r+2|vw}(\pm \frac{i}{g}, g) = 0, \quad Y_{r+2|vw}(0, \bar{g}) = 0, \quad r = 1, 2, \dots, \quad (3.23)$$

and we find the following values for  $g < 100$

$$\begin{aligned} g_{5,1}^{r,2} &= \{6.707, 15.458, 27.233, 42.107, 60.101, 81.222\} \\ \bar{g}_{5,1}^{r,2} &= \{6.764, 15.479, 27.244, 42.114, 60.105, 81.225\}. \end{aligned} \quad (3.24)$$

Next, we consider the state with  $J = 8$  and  $n = 1$ . This is the state with the lowest value of  $J$  such that  $Y_{1|vw}$  has four zeros,  $Y_{2|vw}$  has two real zeros and  $Y_{3|vw}$

has two imaginary zeros in the physical strip at small  $g$ . Therefore, the critical and subcritical values are determined by the equations

$$Y_{r+3|vw}(\pm \frac{i}{g}, g) = 0, \quad Y_{r+2|vw}(0, \bar{g}) = 0, \quad r = 1, 2, \dots \quad (3.25)$$

We find the following 6 critical and 7 subcritical values of  $g$  for  $g < 100$

$$\begin{aligned} g_{8,1}^{r,3} &= \{ \quad, 9.157, 19.561, 32.985, 49.505, 69.143, 91.909 \} \\ \bar{g}_{8,1}^{r,3} &= \{ 0.116, 9.207, 19.580, 32.995, 49.511, 69.148, 91.912 \}. \end{aligned} \quad (3.26)$$

The reason why  $g_{8,1}^{1,3}$  is so small is that the two imaginary roots of  $Y_{3|vw}$  that are in the physical strip at  $g = 0$  reach the real line very quickly.

One might think that the first critical value increases with  $J$ . It is not so as one can see on the example of the state with  $J = 9$  and  $n = 1$ . This is again the state such that  $Y_{1|vw}$  has four zeros, and  $Y_{2|vw}$  and  $Y_{3|vw}$  have two zeros at small  $g$ , and it has the following 6 critical values of  $g$  for  $g < 100$

$$\begin{aligned} g_{9,1}^{r,3} &= \{ 6.970, 16.982, 29.935, 45.968, 65.114, 87.384 \} \\ \bar{g}_{9,1}^{r,3} &= \{ 7.052, 17.006, 29.947, 45.976, 65.119, 87.388 \}. \end{aligned} \quad (3.27)$$

## 4. TBA equations for Konishi-like states

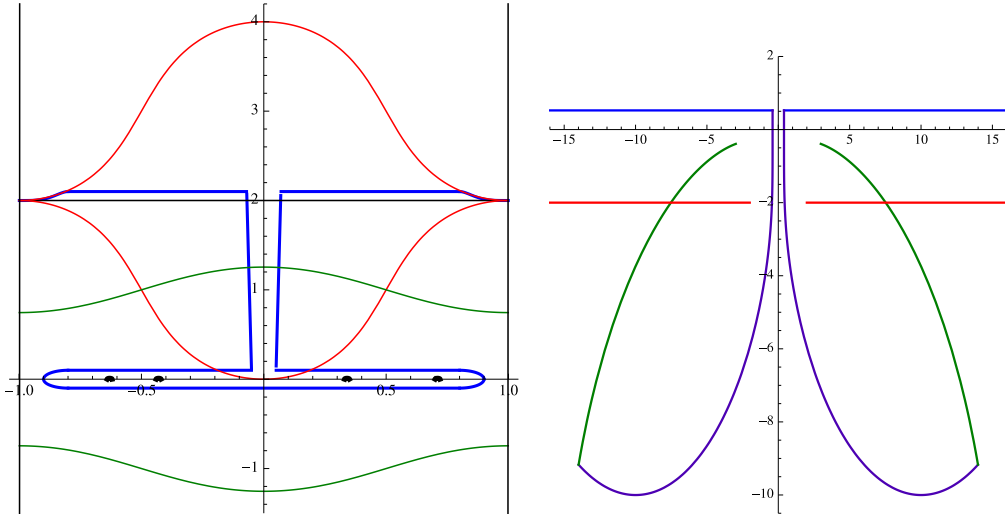
As was discussed above to formulate excited state TBA equations one should choose an integration contour, take it back to the real line of the mirror plane, and then check that the resulting TBA equations are solved by the large  $L$  expressions for Y-functions. We begin our analysis with the simplest case of a Konishi-like state which appears however to be quite general and allows one to understand the structure of the TBA equations for any two-particle  $\mathfrak{sl}(2)$  sector state. To simplify the notations we denote the critical values of the state under consideration as  $g_{cr}^{(r)}$ .

### 4.1 Excited states TBA equations: $g < g_{cr}^{(1)}$

Let us stress that the equations below are valid only for physical states satisfying the level-matching condition. Since some terms in the equations below have the same form for any  $N$  we keep an explicit dependence on  $N$  in some of the formulae.

#### Integration contour

The integration contour for all Y-functions but  $Y_{\pm}$  is chosen in such a way that it lies a little bit above the interval  $\text{Re}(z) \in (-\frac{\omega_1}{2}, \frac{\omega_1}{2})$ ,  $\text{Im}(z) = \frac{\omega_2}{2i}$  in the middle of the mirror theory region, and penetrates to the string theory region in the small vicinity of  $z = \omega_2/2$  (the centre point of the mirror region). Then, it goes along the both sides of the vertical line to the string region real line, and encloses all the points  $z_{*k}$  so



**Figure 6:** On the left picture the integration contour on the  $z$ -torus is shown. On the right picture a part of the integration contour on the mirror  $u$ -plane going from  $\pm\infty$  a little bit above the real line to the origin, and down to the string region is plotted. The green curves correspond to the horizontal lines on the  $z$ -torus just above the real line of the string region. The red semi-lines are the cuts of the mirror  $u$ -plane, and on the  $z$ -torus they are mapped to the part of the boundary of the mirror region that lies in the string region.

that they lie between the mirror theory line and the integration contour, see Figure 6. In the mirror  $u$ -plane the contour lies above the real line, then it goes down at  $u = \pm 0$ , reaches a minimum value and turns back to the real line. In the equations involving the functions  $Y_Q$ ,  $Y_{Q|vw}$  and  $Y_{Q|w}$  it crosses the cuts of the mirror  $u$ -plane with  $\text{Im}(u) = -\frac{Q}{g}$ , and enters another sheet, see Figure 2. It is worth mentioning that the contour does not cross any additional cuts  $Y$ -functions have on the  $z$ -torus.

Then, one uses the TBA equations for the ground state energy, and taking the integration contour back to the mirror region interval  $\text{Im}(z) = \frac{\omega_2}{2i}$ , picks up  $N$  extra contributions of the form  $-\log S(z_*, z)$  from any term  $\log(1 + Y_1) \star K$ , where  $S(w, z)$  is the S-matrix corresponding to the kernel  $K$ :  $K(w, z) = \frac{1}{2\pi i} \frac{d}{dw} \log S(w, z)$ . In addition, one also gets contributions of the form  $-\log S(w, z)$  from the imaginary zeros of  $1 + Y_{M|vw}$  located below the real line of the mirror  $u$ -plane, see (3.16).

Finally, the integration contour for  $Y_{\pm}$ -functions should be deformed so that the points  $u_k^- = u_k - \frac{i}{g}$  of the mirror  $u$ -plane lie between the interval  $[-2, 2]$  of the mirror theory line and the contour. Then, the terms of the form  $\log(1 - Y_+) \star K$  would produce extra contributions of the form  $+\log S(u_k^-, z)$  because  $Y_+(u_k^-) = \infty$ . In fact, this is important only if one uses the simplified TBA equations because in the canonical TBA equations  $Y_{\pm}$ -functions appear only in the combination  $1 - \frac{1}{Y_{\pm}}$ . Note also that  $Y_{\pm}$ -functions analytically continued to the whole mirror  $u$ -plane have a cut  $(-\infty, -2] \cup [2, \infty)$ , and they should satisfy the following important equality which,

as was shown in [22], is necessary for the fulfillment of the Y-system

$$Y_+(u \pm i0) = Y_-(u \mp i0) \quad \text{for } u \in (-\infty, -2] \cup [2, \infty). \quad (4.1)$$

This equality shows that one can glue the two  $u$ -planes along the cuts, and then  $Y_{\pm}$ -functions can be thought of as two branches of one analytic function defined on the resulting surface (with extra cuts in fact). We will see that the equality (4.1) indeed follows from the TBA equations.

### Simplified TBA equations

Using this procedure and the simplified TBA equations for the ground state derived in [22, 23], one gets the following set of integral equations for Konishi-like states and  $g < g_{cr}^{(1)}$

- $M|w$ -strings:  $M \geq 1$ ,  $Y_{0|w} = 0$

$$\log Y_{M|w} = \log(1 + Y_{M-1|w})(1 + Y_{M+1|w}) \star s + \delta_{M1} \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} s. \quad (4.2)$$

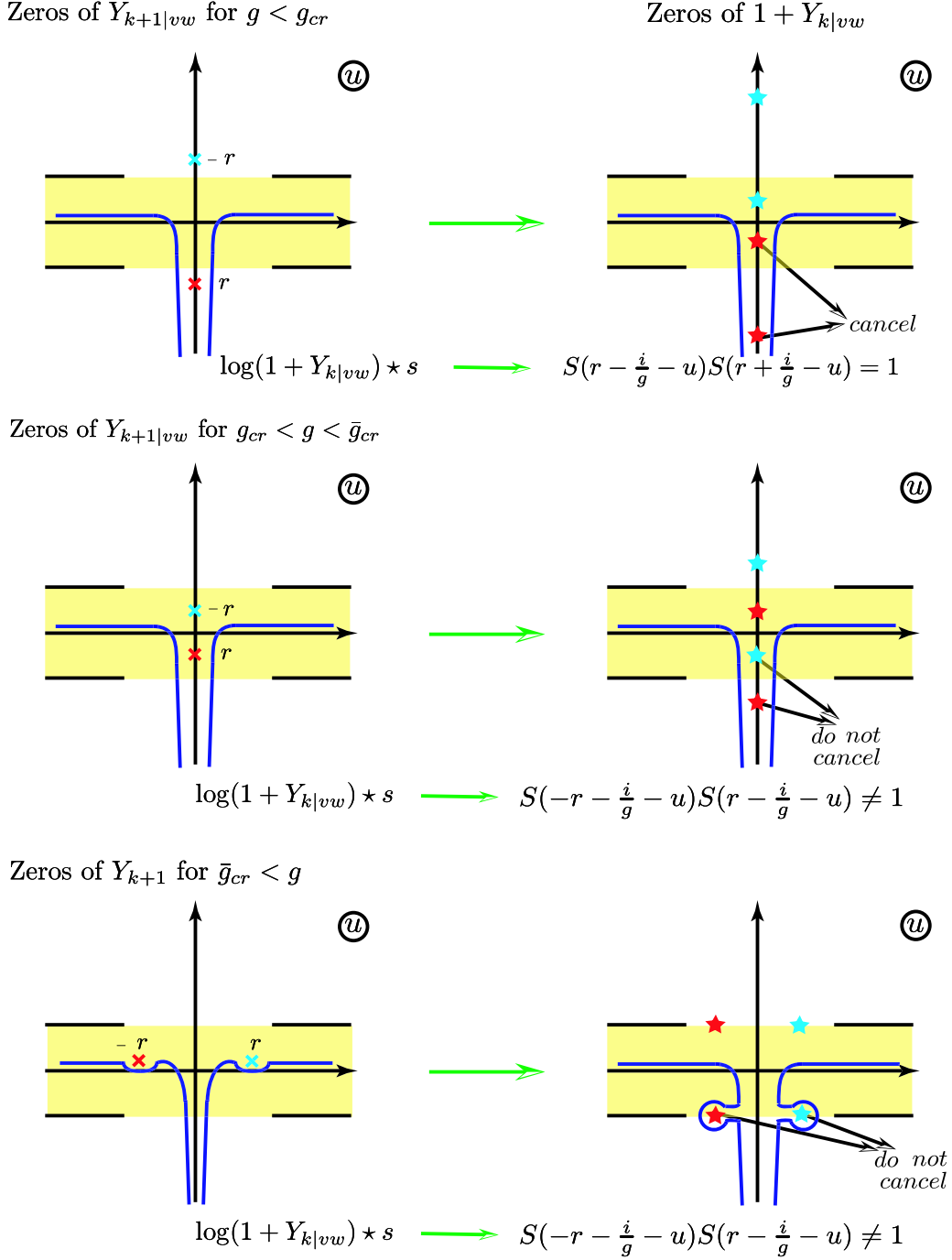
These equations coincide with the ground state ones.

- $M|vw$ -strings:  $M \geq 1$ ,  $Y_{0|vw} = 0$

$$\begin{aligned} \log Y_{M|vw}(v) = & -\delta_{M1} \sum_{j=1}^N \log S(u_j^- - v) - \log(1 + Y_{M+1}) \star s \\ & + \log(1 + Y_{M-1|vw})(1 + Y_{M+1|vw}) \star s + \delta_{M1} \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s, \end{aligned} \quad (4.3)$$

where  $u_j^- \equiv u_j - \frac{i}{g}$ , and the kernel  $s$  and the corresponding S-matrix  $S$  are defined in appendix 8.1. For  $M = 1$  the first term is due to our choice of the integration contour for  $Y_{\pm}$ -functions, and the pole of  $Y_+$  at  $u = u_j^-$ .

It is worth pointing out that there is no extra contribution in (4.3) from any term of the form  $\log(1 + Y_{k|vw}) \star s$ . In general such a term leads to a contribution equal to  $-\log S(r_1 - v)S(r_2 - v)$  where  $r_1$  and  $r_2$  are the two zeros of  $1 + Y_{k|vw}$  with negative imaginary parts. To explain this, we notice that if the zeros  $r_j^{(k+1)}$  of  $Y_{k+1|vw}$  lie outside the physical strip, then  $r_j$  are  $r_1 = r_1^{(k+1)} - \frac{i}{g}$  and  $r_2 = r_1^{(k+1)} + \frac{i}{g}$ , where  $\text{Im}(r_1^{(k+1)}) < -\frac{1}{g}$ . Since  $S(r - \frac{i}{g})S(r + \frac{i}{g}) = 1$  the term  $\log(1 + Y_{k|vw}) \star s$  does not contribute, see Figure 7.



**Figure 7:** In the upper pictures the positions of zeros of  $Y_{k+1|vw}$  and of  $1 + Y_{k|vw}$  are shown for  $g < g_{cr}$ . The contributions of  $\log(1 + Y_{k|vw}) \star s$  to the TBA equations cancel out for this case. The two pictures in the middle correspond to the situation when two zeros of  $Y_{k+1|vw}$  enter the physical region  $|\text{Im}u| < \frac{1}{g}$  (depicted in yellow). Finally, the two pictures at the bottom are drawn for the case when two zeros of  $Y_{k+1|vw}$  are on the real line which corresponds to  $g_{cr} < \bar{g}_{cr} < g$ . When  $g > g_{cr}$  the contributions of  $\log(1 + Y_{k|vw}) \star s$  do not cancel anymore and lead to modification of the corresponding TBA equations.

On the other hand, if the zeros  $r_j^{(k+1)}$  of  $Y_{k+1|vw}$  lie inside the physical strip then  $r_j$  are related to  $r_j^{(k+1)}$  as  $r_j = r_j^{(k+1)} - \frac{i}{g}$ , and the term  $\log(1 + Y_{k|vw}) \star s$  leads to the extra contribution equal to  $-\sum_j \log S(r_j^{(k+1)-} - v)$  where  $r_j^{(k+1)-} \equiv r_j^{(k+1)} - \frac{i}{g}$ , see Figure 7.

We conclude therefore that at weak coupling and only for Konishi-like states no term  $\log(1 + Y_{k|vw}) \star s$  gives an extra contribution to the TBA equations for  $vw$ -strings.

Let us also mention that the poles of  $S(u_j^- - v)$  cancel the zeros of  $Y_{1|vw}(v)$  at  $v = u_j$ , and (4.3) is compatible with the reality condition for Y-functions.

•  $y$ -particles <sup>5</sup>

$$\log \frac{Y_+}{Y_-}(v) = - \sum_{j=1}^N \log S_{1*y}(u_j, v) + \log(1 + Y_Q) \star K_{Qy}, \quad (4.4)$$

$$\begin{aligned} \log Y_+ Y_-(v) &= - \sum_{j=1}^N \log \frac{(S_{xv}^{1*1})^2}{S_2} \star s(u_j, v) \\ &+ 2 \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s - \log(1 + Y_Q) \star K_Q + 2 \log(1 + Y_Q) \star K_{xv}^{Q1} \star s, \end{aligned} \quad (4.5)$$

where we use the following notation

$$\log \frac{(S_{xv}^{1*1})^2}{S_2} \star s(u_j, v) \equiv \int_{-\infty}^{\infty} dt \log \frac{S_{xv}^{1*1}(u_j, t)^2}{S_2(u_j - t)} s(t - v).$$

Then,  $S_{1*y}(u_j, v) \equiv S_{1y}(z_{*j}, v)$  is a shorthand notation for the S-matrix with the first and second arguments in the string and mirror regions, respectively. The same convention is used for other S-matrices. Both arguments of the kernels in these formulae are in the mirror region.

Taking into account that under the analytic continuation through the cut  $|v| > 2$  the S-matrix  $S_{1*y}$  and the kernel  $K_{Qy}$  transforms as  $S_{1*y} \rightarrow 1/S_{1*y}$  and  $K_{Qy} \rightarrow -K_{Qy}$ , one gets that the functions  $Y_{\pm}$  are indeed analytic continuations of each other and, therefore, the equality (4.1) does hold.

It can be easily checked that the term on the first line in (4.5) is real, and this makes obvious that the equations for  $Y_{\pm}$ -functions are also compatible with the reality of Y-functions. The origin of this term can be readily understood if one uses the following identity

$$- \log \frac{(S_{xv}^{1*1})^2}{S_2} \star s(u_j, v) = \log S_1(u_j - v) - 2 \log S_{xv}^{1*1} \star s(u_j, v) \quad (4.6)$$

<sup>5</sup>The equation for  $Y_+ Y_-$  follows from eq.(4.14) and (4.26) of [22], and the identity

$K_{Qy} \hat{\star} K_1 = K_{xv}^{Q1} - K_{Q-1}$ . Another identity  $K_{xv}^{Q1} \star s = \frac{1}{2} K_{Qy} + \frac{1}{2} K_Q - \delta_{Q1} s - K_{Qy}^{ms} \check{\star} \tilde{s}$ , where  $\tilde{s}(u) \equiv s(u - \frac{i}{g})$  is useful in deriving Y-system equations for  $Y_{\pm}$ .

which holds up to a multiple of  $2\pi i$ , and since  $S_{xv}^{1*1}(u, v)$  has a zero at  $u = v$  the integration contour in the second term on the r.h.s. of (4.6) runs above the real line. Then, the term  $\log S_1(u_j - v)$  comes from the term  $-\log(1 + Y_1) \star K_1$ , and the second term come from  $2 \log(1 + Y_1) \star K_{xv}^{Q1} \star s$ .

Eq.(4.5) is very useful for checking the TBA equations in the large  $J$  limit where one gets

$$\log Y_+ Y_- = - \sum_{j=1}^N \log \frac{(S_{xv}^{1*1})^2}{S_2} \star s(u_j, v) + 2 \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s.$$

- $Q$ -particles for  $Q \geq 3$

$$\log Y_Q = \log \frac{\left(1 + \frac{1}{Y_{Q-1|vw}}\right)^2}{\left(1 + \frac{1}{Y_{Q-1}}\right)\left(1 + \frac{1}{Y_{Q+1}}\right)} \star s \quad (4.7)$$

- $Q = 2$ -particle

$$\log Y_2 = \sum_{j=1}^N \log S(u_j - v) - \log\left(1 + \frac{1}{Y_1}\right)\left(1 + \frac{1}{Y_3}\right) \star s + 2 \log \left(1 + \frac{1}{Y_{1|vw}}\right) \star s \quad (4.8)$$

In fact by using the p.v. prescription, one gets

$$\log Y_2 = \log \frac{\left(1 + \frac{1}{Y_{1|vw}}\right)^2}{\left(1 + \frac{1}{Y_1}\right)\left(1 + \frac{1}{Y_3}\right)} \star_{p.v.} s \quad (4.9)$$

which makes obvious the reality of Y-functions.

- $Q = 1$ -particle

$$\begin{aligned} \log Y_1 &= \sum_{j=1}^N \log \check{\Sigma}_{1*}^2 \check{S}_1 \check{\star} s(u_j, v) - L \check{\mathcal{E}} \check{\star} s + \log \left(1 - \frac{1}{Y_-}\right)^2 Y_2 \hat{\star} s \\ &\quad - 2 \log \left(1 - \frac{1}{Y_-}\right) \left(1 - \frac{1}{Y_+}\right) Y_2 \hat{\star} \check{K} \check{\star} s + \log Y_1 \star \check{K}_1 \check{\star} s \\ &\quad - \log(1 + Y_Q) \star (2\check{K}_Q^\Sigma + \check{K}_Q + \check{K}_{Q-2}) \check{\star} s - \log(1 + Y_2) \star s. \end{aligned} \quad (4.10)$$

All the kernels appearing here are defined in appendix 8.1, and we also assume that  $\check{K}_0 = 0$  and  $\check{K}_{-1} = 0$ . The reality of this equation follows from the reality of

$$\frac{S_{ss}(u - \frac{i}{g}, v)^2}{\check{S}_1(u, v)} = \frac{x_s(u - \frac{i}{g}) - x_s(v)}{x_s(u - \frac{i}{g}) - \frac{1}{x_s(v)}} \frac{x_s(u + \frac{i}{g}) - x_s(v)}{x_s(u + \frac{i}{g}) - \frac{1}{x_s(v)}} = S_{ss}(u - \frac{i}{g}, v) S_{ss}(u + \frac{i}{g}, v),$$



which appears if one uses the representation (8.24) for  $\check{\Sigma}_{1*}$ . Note that  $\check{S}_1(u, v)$  is defined through the kernel  $\check{K}_1(u, v)$  by the integral

$$\check{S}_1(u, v) = \exp \left( 2\pi i \int_{-\infty}^u du' \check{K}_1(u', v) \right) = \frac{S_{ss}(u - \frac{i}{g}, v)}{S_{ss}(u + \frac{i}{g}, v)}, \quad (4.11)$$

and it differs from the naive formula

$$S_{ms}(u - \frac{i}{g}, v) S_{ms}(u + \frac{i}{g}, v) = \check{S}_1(u, v) x_s(v)^2, \quad (4.12)$$

which one could write by using the expression (8.16) for the kernel  $\check{K}_1$ .

We see that the reality of Y-functions is a trivial consequence of these equations. Moreover, in the large  $L$  limit the simplified TBA equations do not involve infinite sums at all. As a result, they can be easily checked numerically with an arbitrary precision. We have found that for Konishi-like states the integral equations are solved at the large  $L$  limit by the asymptotic Y-functions given in terms of transfer matrices if the length parameter  $L$  is related to the charge  $J$  carried by a string state as

$$L = J + 2.$$

We expect that for  $N$ -particle state the relation would be  $L = J + N$ .

There is another form of the TBA equations for  $Q$ -particles which is obtained by combining the simplified and canonical TBA equations. We refer to this form as the hybrid one, and the equations can be written as follows

- Hybrid equations for  $Q$ -particles

$$\begin{aligned} \log Y_Q(v) = & - \sum_{j=1}^N \left( \log S_{\text{sl}(2)}^{1*Q}(u_j, v) - 2 \log S \star K_{vwx}^{1Q}(u_j^-, v) \right) \\ & - L \tilde{\mathcal{E}}_Q + \log(1 + Y_{Q'}) \star \left( K_{\text{sl}(2)}^{Q'Q} + 2s \star K_{vwx}^{Q'-1, Q} \right) \\ & + 2 \log(1 + Y_{1|vw}) \star s \hat{\star} K_{yQ} + 2 \log(1 + Y_{Q-1|vw}) \star s \\ & - 2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{1Q} + \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_Q + \log \left( 1 - \frac{1}{Y_-} \right) \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_{yQ}, \end{aligned} \quad (4.13)$$

where  $K_{vwx}^{0, Q} = 0$ , and  $Y_{0|vw} = 0$ , and we use the notation

$$\log S \star K_{vwx}^{1Q}(u_j^-, v) = \int_{-\infty}^{\infty} dt \log S(u_j^- - t - i0) \star K_{vwx}^{1Q}(t + i0, v). \quad (4.14)$$

The first term on the first line of (4.13) comes from  $\log(1 + Y_{Q'}) \star K_{\text{sl}(2)}^{Q'Q}$ , and the second one from  $-2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{1Q}$ . Eq.(4.13) is derived in appendix 8.4.

The energy of the multiparticle state is obtained in the same way by taking the integration contour back to the real mirror momentum line, and is given by

$$\begin{aligned}
E_{\{n_k\}}(L) &= \sum_{k=1}^N i\tilde{p}^1(z_{*k}) - \int dz \sum_{Q=1}^{\infty} \frac{1}{2\pi} \frac{d\tilde{p}^Q}{dz} \log(1 + Y_Q) \\
&= \sum_{k=1}^N \mathcal{E}_k - \int du \sum_{Q=1}^{\infty} \frac{1}{2\pi} \frac{d\tilde{p}^Q}{du} \log(1 + Y_Q), \tag{4.15}
\end{aligned}$$

where

$$\mathcal{E}_k = igx^-(z_{*k}) - igx^+(z_{*k}) - 1 = igx_s^-(u_k) - igx_s^+(u_k) - 1, \tag{4.16}$$

is the energy of a fundamental particle in the string theory, see appendix 8.1 for definitions and conventions.

For practical computations the analytic continuation from the mirror region to the string one reduces to the substitution  $x^{Q\pm}(u) \rightarrow x_s^{Q\pm}(u) \equiv x_s(u \pm \frac{i}{g}Q)$  in all the kernels and S-matrices. Then, as was discussed above the string theory spectrum is characterized by a set of  $N$  real numbers  $u_k$  (or  $z_{*k}$ ) satisfying the exact Bethe equations (3.7). We assume for definiteness that  $u_k$  are ordered as  $u_1 < \dots < u_N$ .

Finally, to derive exact Bethe equations one should analytically continue  $Y_1$  given by either eq.(4.10) or (4.13). We find that it is simpler and easier to handle the exact Bethe equations derived from the hybrid equation (4.13) for  $Y_1$ . In the appendix 8.6 we also derive exact Bethe equations from the canonical equation for  $Y_1$ .

## Exact Bethe equations

Now we need to derive the integral form of the exact Bethe equations (3.7). Let us note first of all that at large  $L$  eq.(3.7) reduces to the BY equations for the  $\mathfrak{sl}(2)$ -sector by construction, and the integral form of (3.7) should be compatible with this requirement.

To derive the exact Bethe equations we take the logarithm of eq.(3.7), and analytically continue the variable  $z$  of  $Y_1(z)$  in eq.(4.13) to the point  $z_{*k}$ . On the mirror  $u$ -plane it means that we go from the real  $u$ -line down below the line with  $\text{Im}(u) = -\frac{1}{g}$  without crossing any cut, then turn back, cross the cut with  $\text{Im}(u) = -\frac{1}{g}$  and  $|\text{Re}(u)| > 2$ , and go back to the real  $u$ -line, see Figure 2. As a result we should make the following replacements  $x(u - \frac{i}{g}) \rightarrow x_s(u - \frac{i}{g}) = x(u - \frac{i}{g})$ ,  $x(u + \frac{i}{g}) \rightarrow x_s(u + \frac{i}{g}) = 1/x(u + \frac{i}{g})$  in the kernels appearing in (4.13).

The analytic continuation depends on the analytic properties of the kernels and Y-functions, and its detailed consideration can be found in appendix 8.5. As shown there, the resulting exact Bethe equations for a string theory state from the  $\mathfrak{sl}(2)$

sector can be cast into the following integral form

$$\begin{aligned}
\pi i(2n_k + 1) &= \log Y_{1*}(u_k) = iL p_k - \sum_{j=1}^N \log S_{\mathfrak{sl}(2)}^{1*1*}(u_j, u_k) \\
&+ 2 \sum_{j=1}^N \log \text{Res}(S) \star K_{vwx}^{11*}(u_j^-, u_k) - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} \\
&+ \log(1 + Y_Q) \star \left( K_{\mathfrak{sl}(2)}^{Q1*} + 2s \star K_{vwx}^{Q-1,1*} \right) + 2 \log(1 + Y_{1|vw}) \star (s \hat{\star} K_{y1*} + \tilde{s}) \\
&- 2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{11*} + \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_1 + \log \left( 1 - \frac{1}{Y_-} \right) \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_{y1*},
\end{aligned} \tag{4.17}$$

where we use the notations

$$\log \text{Res}(S) \star K_{vwx}^{11*}(u^-, v) = \int_{-\infty}^{+\infty} dt \log \left[ S(u^- - t)(t - u) \right] K_{vwx}^{11*}(t, v), \tag{4.18}$$

$$\tilde{s}(u) = s(u^-). \tag{4.19}$$

The integration contours in the formulae above run a little bit above the Bethe roots  $u_j$ ,  $p_k = i\tilde{\mathcal{E}}_Q(z_{*k}) = -i \log \frac{x_s(u_k + \frac{i}{g})}{x_s(u_k - \frac{i}{g})}$  is the momentum of the  $k$ -th particle, and the second argument in all the kernels in (4.17) is equal to  $u_k$ . The first argument we integrate with respect to is the original one in the mirror region.

Taking into account that the BY equations for the  $\mathfrak{sl}(2)$  sector have the form

$$\pi i(2n_k + 1) = iJ p_k - \sum_{j=1}^N \log S_{\mathfrak{sl}(2)}^{1*1*}(u_j, u_k),$$

and that  $Y_Q$  is exponentially small at large  $J$ , we conclude that if the analytic continuation has been done correctly then up to an integer multiple of  $2\pi i$  the following identities between the asymptotic Y-functions should hold

$$\begin{aligned}
\mathcal{R}_k &\equiv iN p_k + 2 \sum_{j=1}^N \log \text{Res}(S) \star K_{vwx}^{11*}(u_j^-, u_k) - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} \\
&+ 2 \log(1 + Y_{1|vw}) \star (s \hat{\star} K_{y1*} + \tilde{s}) - 2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{11*} \\
&+ \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_1 + \log \left( 1 - \frac{1}{Y_-} \right) \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_{y1*} = 0.
\end{aligned} \tag{4.20}$$

For  $N = 2$  and  $u_1 = -u_2$  one gets one equation, and by using the expressions for the Y-functions from appendix 8.3 one can check numerically<sup>6</sup> that it does hold for any real value of  $u_1$  such that only  $Y_{1|vw}$  has two zeros for real  $u$ .

<sup>6</sup>Since  $K_{vwx}^{11*}(u, v)$  has a pole at  $u = v$  with the residue equal to  $-\frac{1}{2\pi i}$  the terms of the form  $2f \star K_{vwx}^{11*}$  can be represented as  $2f \star K_{vwx}^{11*} = 2f \star_{p.v.} K_{vwx}^{11*} + f(u_k)$  which is useful for numerics.

## 4.2 Excited states TBA equations: $g_{cr}^{(1)} < g < g_{cr}^{(2)}$

In this subsection we consider the TBA equations for values of  $g$  in the first critical region  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$ . In this region in addition to the two real zeros of  $Y_{1|vw}$  at  $u_j$ , two zeros of  $Y_{2|vw}$  enter the physical strip. We denote these zeros  $r_j$ .<sup>7</sup>

### Simplified TBA equations

We first notice that for  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$  the function  $Y_{2|vw}$  has two zeros in the physical strip. Therefore, as was discussed above, the contribution to the simplified TBA equations coming from the zeros of  $Y_{2|vw}$  and  $1 + Y_{1|vw}$  does not vanish. If  $g < \bar{g}_{cr}^{(1)}$  no deformation of the integration contour is needed because all these zeros are on the imaginary line of the mirror region. As  $g$  approaches  $\bar{g}_{cr}^{(1)}$  the two zeros of  $Y_{2|vw}$  approach  $u = 0$ , and at  $g = \bar{g}_{cr}^{(1)}$  they both are at the origin. As  $g > \bar{g}_{cr}^{(1)}$  the zeros become real, located symmetrically, and they push the integration contour to be a little bit below them. In addition to this at  $g = \bar{g}_{cr}^{(1)}$  the two zeros of  $1 + Y_{1|vw}$  with the negative imaginary part reach the point  $u = -i/g$ , and as  $g > \bar{g}_{cr}^{(1)}$  they begin to move along the line  $\text{Im}(u) = -1/g$  in opposite directions. As a result, the integration contour should be deformed in such a way that the two zeros of  $1 + Y_{1|vw}$  would not cross it. Thus, the zeros of  $1 + Y_{1|vw}$  always lie between the real line and the integration contour, and the terms of the form  $\log(1 + Y_{1|vw}) \star K$  produce the usual contribution once one takes the contour back to the real line. Let us also mention that the points  $r_j^-$  of the mirror  $u$ -plane are mapped to the upper boundary of the string region on the  $z$ -torus.

Using this integration contour, one gets the following set of simplified TBA equations for Konishi-like states and  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$

- $M|w$ -strings: their equations coincide with the ground state ones (4.2).

- $M|vw$ -strings:  $M \geq 1$ ,  $Y_{0|vw} = 0$

$$\begin{aligned} \log Y_{M|vw}(v) = & -\delta_{M1} \sum_{j=1}^N \log S(u_j^- - v) - \delta_{M2} \sum_{j=1}^2 \log S(r_j^- - v) \\ & + \log(1 + Y_{M-1|vw})(1 + Y_{M+1|vw}) \star s + \delta_{M1} \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s - \log(1 + Y_{M+1}) \star s. \end{aligned} \quad (4.21)$$

The first term is due to the pole of  $Y_+$  at  $u = u_j^-$ , and the second term is due to the zeros of  $1 + Y_{1|vw}$  at  $u = r_j^-$ .

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<sup>7</sup>Canonical TBA equations take different forms depending on the reality of the zeros, and, therefore, one has to divide the region into two subregions:  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$  and  $\bar{g}_{cr}^{(1)} < g < g_{cr}^{(2)}$ , see the next subsection and appendix 8.6 for detail.

- $y$ -particles

$$\log \frac{Y_+}{Y_-}(v) = - \sum_{j=1}^N \log S_{1*y}(u_j, v) + \log(1 + Y_Q) \star K_{Qy}, \quad (4.22)$$

$$\begin{aligned} \log Y_+ Y_-(v) = & - \sum_{j=1}^N \log \frac{(S_{1*x}^{1*1})^2}{S_2} \star s(u_j, v) - 2 \sum_{j=1}^2 \log S(r_j^- - v) \\ & + 2 \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s - \log(1 + Y_Q) \star K_Q + 2 \log(1 + Y_Q) \star K_{xv}^{Q1} \star s, \end{aligned} \quad (4.23)$$

where the second term on the second line is due to the zeros of  $1 + Y_{1|vw}$  at  $u = r_j^-$ .

- $Q$ -particles for  $Q \geq 3$

$$\log Y_Q = \log \frac{\left(1 + \frac{1}{Y_{Q-1|vw}}\right)^2}{\left(1 + \frac{1}{Y_{Q-1}}\right)\left(1 + \frac{1}{Y_{Q+1}}\right)} \star_{p.v.} s. \quad (4.24)$$

In fact the p.v. prescription, see appendix 8.7, is not really needed here because for  $Q = 3$  the double zero of  $Y_2$  cancels the zeros of  $Y_{2|vw}$ , and for  $Q \geq 4$  everything is regular. Thus, the formula works no matter if the roots  $r_j$  are real or imaginary.

- $Q = 2$ -particle

$$\log Y_2 = -2 \sum_{j=1}^2 \log S(r_j^- - v) + \log \frac{\left(1 + \frac{1}{Y_{1|vw}}\right)^2}{\left(1 + \frac{1}{Y_1}\right)\left(1 + \frac{1}{Y_3}\right)} \star_{p.v.} s \quad (4.25)$$

This makes obvious the reality of  $Y$ -functions because the double zero of  $Y_2$  at  $v = r_j$  is cancelled by the pole of  $S(r_j^- - v)$ .

### Hybrid equations

One can easily see that the simplified equation for  $Q = 1$ -particles is the same as eq.(4.10) in the weak coupling region  $g < g_{cr}^{(1)}$ . Thus, we will only discuss the hybrid equations for  $Q$ -particles. Strictly speaking, their form is sensitive to whether the zeros of  $Y_{2|vw}$  are complex or real, and the first critical region is divided into two subregions:  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$  and  $\bar{g}_{cr}^{(1)} < g < g_{cr}^{(2)}$ . On the other hand, to derive the exact Bethe equations we only need the hybrid  $Q = 1$  equation which, as we will see, takes the same form in both subregions, and, moreover, for any  $g > g_{cr}^{(1)}$ .

For  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$  the function  $Y_{2|vw}$  has two complex conjugate zeros in the physical strip which, therefore, lie on the opposite sides of the real axis of the mirror  $u$ -plane. Thus, the zero  $r_1$  with the negative imaginary part lies between the integration contour and the real line of the mirror region. Taking the integration contour back to the real line produces an extra contribution from this zero. Since  $1 + Y_{1|vw}(r_j^-) = 0$ ,

and  $Y_{\pm} = 0$ , the terms  $2 \log(1 + Y_{1|vw}) \star s \hat{\star} K_{y1}$  and  $\log(1 - \frac{1}{Y_-})(1 - \frac{1}{Y_+}) \hat{\star} K_{y1}$  in the hybrid equation (4.13) lead to the appearance of  $-2 \log S \hat{\star} K_{y1}(r_j^-, v)$  and  $2 \log S_{yQ}(r_1, v)$ ,<sup>8</sup> respectively. The integration contour for the  $\hat{\star}$ -convolution in the term  $-2 \log S \hat{\star} K_{y1}(r_j^-, v)$  is the one for  $Y_{\pm}$ -functions and it should be taken to the mirror region too. Since the S-matrix  $S(r_1^- - t)$  has a pole at  $t = r_1$ , this produces the extra term  $-2 \log S_{yQ}(r_1, v)$  that exactly cancels the previous contribution from  $\log(1 - \frac{1}{Y_-})(1 - \frac{1}{Y_+}) \hat{\star} K_{y1}$ . Thus, the only additional term that appears in the hybrid TBA equation (4.13) for  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$  is  $-2 \sum_j \log S \hat{\star} K_{y1}(r_j^-, v)$ .

As  $g > \bar{g}_{cr}^{(1)}$  the two zeros of  $Y_{\pm}$  (and  $Y_{2|vw}$ ) become real and located a little bit above the integration contour. Therefore, the only extra contribution to the TBA equation comes from the term  $2 \log(1 + Y_{1|vw}) \star s \hat{\star} K_{y1}$  and it is again is  $-2 \sum_j \log S \hat{\star} K_{y1}(r_j^-, v)$ .

Thus, we see that in the first critical region  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$  the hybrid  $Q = 1$  equation takes the following form independent of whether  $g < \bar{g}_{cr}^{(1)}$  or  $g > \bar{g}_{cr}^{(1)}$

- Hybrid  $Q = 1$  equation for  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$

$$\begin{aligned}
\log Y_1(v) = & - \sum_{j=1}^N \left( \log S_{s(2)}^{1 \star 1}(u_j, v) - 2 \log S \star K_{vwx}^{11}(u_j^-, v) \right) - 2 \sum_{j=1}^2 \log S \hat{\star} K_{y1}(r_j^-, v) \\
& - L \tilde{\mathcal{E}}_1 + \log(1 + Y_Q) \star \left( K_{s(2)}^{Q1} + 2 s \star K_{vwx}^{Q-1,1} \right) + 2 \log(1 + Y_{1|vw}) \star s \hat{\star} K_{y1} \\
& - 2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{11} + \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_1 + \log \left( 1 - \frac{1}{Y_-} \right) \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_{y1}.
\end{aligned} \tag{4.26}$$

Note also that for  $g > \bar{g}_{cr}^{(1)}$  one can also use the p.v. prescription in the terms  $\log S \hat{\star} K_{y1}(r_j^-, v)$  and  $\log(1 - \frac{1}{Y_-})(1 - \frac{1}{Y_+}) \hat{\star} K_{y1}$  because the extra terms  $\pm \log S_{y1}$  cancel each other.

**Exact Bethe equations:**  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$

The exact Bethe equations are obtained by analytically continuing  $\log Y_1$  in (4.26)

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<sup>8</sup>Note that the S-matrix  $S_{yQ}$  is normalized as  $S_{yQ}(\pm 2, v) = 1$ .

following the same route as for the small  $g$  case, and they take the following form

$$\begin{aligned}
\pi i(2n_k + 1) &= \log Y_{1*}(u_k) = iL p_k - \sum_{j=1}^N \log S_{\text{sl}(2)}^{1*1*}(u_j, u_k) \\
&+ 2 \sum_{j=1}^N \log \text{Res}(S) \star K_{vwx}^{11*}(u_j^-, u_k) - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} \\
&\quad - 2 \sum_{j=1}^2 \left( \log S \hat{\star} K_{y1*}(r_j^-, u_k) - \log S(r_j - u_k) \right) \\
&+ \log(1 + Y_Q) \star \left( K_{\text{sl}(2)}^{Q1*} + 2s \star K_{vwx}^{Q-1,1*} \right) + 2 \log(1 + Y_{1|vw}) \star (s \hat{\star} K_{y1*} + \tilde{s}) \\
&- 2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{11*} + \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_1 + \log \left( 1 - \frac{1}{Y_-} \right) \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_{y1*},
\end{aligned} \tag{4.27}$$

We recall that in the formulae above the integration contours run a little bit above the Bethe roots  $u_j$ , and below the dynamical roots  $r_j$ . The exact Bethe equation also have the same form no matter whether  $g < \bar{g}_{cr}^{(1)}$  or  $g > \bar{g}_{cr}^{(1)}$ .

We conclude again that the consistency with the BY equations requires fulfillment of identities similar to (4.20) that we have checked numerically.

### Integral equations for the roots $r_j$

In the first critical region the exact Bethe equations should be supplemented by integral equations which determine the exact location of the roots  $r_j$ . Let us recall that at  $r_j^\pm = r_j \pm \frac{i}{g}$  the function  $Y_{1|vw}$  satisfies the relations

$$Y_{1|vw}(r_j^\pm) = -1. \tag{4.28}$$

These equations can be used to find the roots  $r_j$ . They can be brought to an integral form by analytically continuing the simplified TBA equation (4.21) for  $Y_{1|vw}$  to the points  $r_j^\pm$ . The analytic continuation is straightforward, and one gets for roots with nonpositive imaginary parts

$$\prod_{k=1}^2 S(u_k - r_j) \exp \left( \log(1 + Y_{2|vw}) \star \tilde{s} + \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} \tilde{s} - \log(1 + Y_2) \star \tilde{s} \right) = -1, \tag{4.29}$$

where we used the exponential form of the TBA equation, and took into account that  $Y_{2|vw}(r_j) = Y_2(r_j) = Y_\pm(r_j) = 0$ . One can easily check that if  $r_j$  is imaginary, that is  $g < \bar{g}_{cr}^{(1)}$ , then the l.h.s. of (4.29) is positive, and the exponent of the r.h.s. is real. As a result, taking the logarithm of (4.29) does not produce any new mode number. If  $r_j$  is real, that is  $g > \bar{g}_{cr}^{(1)}$ , then both sides of (4.29) are on the unit circle. Their phases however are small (as they should be due to the continuity of the roots as functions of  $g$ ), and one does not get any mode number either.

Let us also mention that eq.(4.29) can be also used to find the critical values of  $g$ . If one sets  $r_j = -i/g$ , one gets an integral equation for  $g_{cr}^{(1)}$ , and the case of  $r_j = 0$  gives the one for  $\bar{g}_{cr}^{(1)}$ .

### 4.3 Excited states TBA equations: $g_{cr}^{(m)} < g < g_{cr}^{(m+1)}$

In this subsection we propose the TBA equations for Konishi-like states for the general case where  $g_{cr}^{(m)} < g < g_{cr}^{(m+1)}$  and  $m \geq 2$ . Then, the functions  $Y_{1|vw}, \dots, Y_{m-1|vw}$  have four zeros, and  $Y_{m|vw}$  and  $Y_{m+1|vw}$  have two zeros in the physical strip. The only two roots that can be imaginary if  $g < \bar{g}_{cr}^{(m)}$  are  $r_j^{(m+1)}$ . As was discussed in section 3, these zeros can be written in the form (3.14)

$$\{u_j, r_j^{(3)}\}, \{r_j^{(2)}, r_j^{(4)}\}, \dots, \{r_j^{(m-1)}, r_j^{(m+1)}\}, \{r_j^{(m)}\}, \{r_j^{(m+1)}\}, \quad (4.30)$$

where we indicate only those zeros which are in the physical strip and important for formulating the TBA equations. We also recall that the functions below have zeros at locations related to  $r_j^{(k)}$

$$Y_{\pm}(r_j^{(2)}) = 0, \quad 1 + Y_{k|vw}(r_j^{(k+1)} \pm \frac{i}{g}) = 0, \quad Y_{k+1}(r_j^{(k+1)}) = 0, \quad k = 1, \dots, m. \quad (4.31)$$

#### Simplified TBA equations: $g_{cr}^{(m)} < g < g_{cr}^{(m+1)}$

The necessary modification of the integration contour is obvious, and simplified equations take the following form

- $M|w$ -strings: their equations coincide with the ground state ones (4.2).
- $M|vw$ -strings:  $M \geq 1$ ,  $Y_{0|vw} = 0$

$$\begin{aligned} \log Y_{M|vw}(v) = & - \sum_{j=1}^2 \left[ \log S(r_j^{(M),-} - v) + \log S(r_j^{(M+2),-} - v) \right] \\ & + \log(1 + Y_{M-1|vw})(1 + Y_{M+1|vw}) \star s + \delta_{M1} \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s - \log(1 + Y_{M+1}) \star s, \end{aligned} \quad (4.32)$$

where we identify  $r_j^{(1)} \equiv u_j$ , and assume that  $\log S(r_j^{(k),-} - v)$  is absent in the sum on the first line if  $k \geq m + 2$ . For  $M = 1$  the term  $\log S(u_j - v)$  is due to the pole of  $Y_+$  at  $u = u_j^-$ , and the second term is due to the zero of  $1 + Y_{2|vw}$  at  $u = r_j^{(3),-}$ . For  $M \geq 2$  both terms on the first line are due to the zeros of  $1 + Y_{k|vw}$  at  $u = r_j^{(k+1),-}$  for  $k = 1, \dots, m$ .

- $y$ -particles: their equations coincide with eqs.(4.22)and (4.23) for the first critical region  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$ .



- $Q$ -particles for  $Q \geq 3$

$$\log Y_Q = -2 \sum_{j=1}^2 \log S(r_j^{(Q),-} - v) + \log \frac{\left(1 + \frac{1}{Y_{Q-1|vw}}\right)^2}{\left(1 + \frac{1}{Y_{Q-1}}\right)\left(1 + \frac{1}{Y_{Q+1}}\right)} \star_{p.v.} s. \quad (4.33)$$

The p.v. prescription is again not really needed because the four zeros  $\{r_j^{(Q-1)}, r_j^{(Q+1)}\}$  of  $Y_{Q-1|vw}$  are cancelled by the double zeros of  $Y_{Q-1}$  and  $Y_{Q+1}$ , and the formula works no matter if the roots  $r_j^{(m+1)}$  are real or imaginary.

The equations below coincide with the corresponding eqs.(4.25),(4.10),(4.26) and (4.27) for the first critical region  $g_{cr}^{(1)} < g < g_{cr}^{(2)}$  with the replacement  $r_j \mapsto r_j^{(2)}$ .

- Equations for  $Q = 2$  and  $Q = 1$ -particles.
- Hybrid  $Q = 1$  equation.
- Exact Bethe equations.

According to these results, only some of the simplified equations for  $Q$ -particles and  $vw$ -strings change their form when crossing the  $m$ -th critical point ( $m > 1$ ). However, since all  $Y$ -functions are coupled with each other, the existence of higher critical points still affects the equations above but in a less direct way.

**Integral equations for the roots  $r_j^{(k)}$ :  $g_{cr}^{(m)} < g < g_{cr}^{(m+1)}$**

In this region the exact Bethe equations should be supplemented by integral equations which determine the exact location of the roots  $r_j^{(k)}$ ,  $k = 2, \dots, m+1$ . The integral equations are obtained by analytically continuing the simplified TBA equations (4.32) for  $Y_{k|vw}$  to the points  $r_j^{(k+1)\pm}$ , and using the conditions

$$Y_{k|vw}(r_j^{(k+1)\pm}) = -1. \quad (4.34)$$

The analytic continuation is again straightforward, and one gets for roots with non-positive imaginary parts the following equations

$$r_j^{(2)} : \prod_{n=1}^2 S(u_n - r_j^{(2)}) S(r_n^{(3)} - r_j^{(2)}) \times \quad (4.35)$$

$$\times \exp \left( \log(1 + Y_{2|vw}) \star \tilde{s} + \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} \tilde{s} - \log(1 + Y_2) \star \tilde{s} \right) = -1,$$

$$r_j^{(k+1)} : \prod_{n=1}^2 S(r_n^{(k)} - r_j^{(k+1)}) S(r_n^{(k+2)} - r_j^{(k+1)}) \times \quad (4.36)$$

$$\times \exp \left( \log(1 + Y_{k-1|vw})(1 + Y_{k+1|vw}) \star \tilde{s} - \log(1 + Y_{k+1}) \star \tilde{s} \right) = -1,$$

The continuity condition of the roots as functions of  $g$  again guarantees that one does not get any new mode numbers.

## 5. TBA equations for arbitrary two-particle $\mathfrak{sl}(2)$ states

Our consideration of the TBA equations for Konishi-like states can be easily used to formulate TBA equations for an arbitrary two-particle state from the  $\mathfrak{sl}(2)$  sector. One starts again with analyzing the structure of zeros of  $Y_{M|vw}$ -functions at weak coupling. Then, as was discussed in section 3, for every state  $(J, n)$  there is a number  $m$  such that the first  $m - 1$   $Y_{k|vw}$ -functions have four zeros, both  $Y_{m|vw}$  and  $Y_{m+1|vw}$  have two zeros, and all  $Y_{k|vw}$ -functions with  $k \geq m + 2$  have no zeros in the physical strip. This is exactly the structure of zeros of  $Y_{M|vw}$ -functions for a Konishi-like state with the coupling constant being in the  $m$ -th critical region:  $g_{cr}^{(m)} < g < g_{cr}^{(m+1)}$ .

Thus, at weak coupling the simplified TBA equations for this  $(J, n)$  state have exactly the same form as the TBA equations for a Konishi-like state in the  $m$ -th critical region. As to the canonical TBA and exact Bethe equations, one should take into account that real zeros  $r_j^{(k)}$  of  $Y_{k|vw}$ -functions are located outside the interval  $[-2, 2]$  for small  $g$  for most of the  $\mathfrak{sl}(2)$  states. In particular, if the zeros  $r_j^{(2)}$  of  $Y_{2|vw}$  are outside the interval then the corresponding zeros  $r_j^{(2)-}$  of  $1 + Y_{1|vw}$  are located on the cuts of  $S_{vw}^{1Q}(r_j^{(2)-}, v)$ , and since the integration contour runs below these zeros, one should add  $+i0$  to them in all the expressions. In addition, the integration contour for  $Y_{\pm}$ -functions runs over the interval  $[-2, 2]$  (and a little bit above it), and if  $|r_j^{(2)}| > 2$  then there is no singularity of the integrand and the p.v. prescription is not necessary. Strictly speaking, this means that in addition to the critical and subcritical values of  $g$  we discussed above, one should also consider such values of  $g$  that the zeros  $r_j^{(2)}$  of  $Y_{2|vw}$  are equal to  $\pm 2$ , and distinguish the cases with the zeros being inside and outside the interval  $[-2, 2]$ . It is straightforward to do it, and we refrain from presenting explicit formulae for these cases here.

Increasing  $g$ , one reaches the first critical value  $g_{J,n}^{1,m+1}$  of the  $(J, n)$  state, and the TBA equations would have to be modified. It is in fact clear that in the region  $g_{J,n}^{r,m+1} < g < g_{J,n}^{r+1,m+1}$  the TBA equations for the  $(J, n)$  state coincide with the ones for a Konishi-like state in the  $m + r$ -th critical region.

## 6. Remarks on the Y-system

For reader's convenience, in this section we summarize the most essential properties of the Y-system implied by the TBA equations under study.

The kernel  $s$  has the uniquely defined inverse  $s^{-1}$  which acts as the following operator

$$(f \star s^{-1})(u) = \lim_{\epsilon \rightarrow 0^+} \left[ f\left(u + \frac{i}{g} - i\epsilon\right) + f\left(u - \frac{i}{g} + i\epsilon\right) \right].$$

Applying  $s^{-1}$  to the set of the simplified TBA equations, one can easily derive the Y-system equations [27] for  $-2 < u < 2$ . Since the Y-functions appear to be non-analytic, for  $|u| > 2$  one gets instead new equations which, as we will explain below,

encode the jump discontinuities of the Y-functions across the cuts [22]. To exemplify this statement, it is enough to consider  $Y_{1|w}$  and  $Y_1$ -functions.

We start with eq.(4.2) for  $Y_{1|w}$ . Applying  $s^{-1}$  to this equation, one finds

$$Y_{1|w}^{(\alpha)}\left(u + \frac{i}{g} - i0\right)Y_{1|w}^{(\alpha)}\left(u - \frac{i}{g} + i0\right) = \left(1 + Y_{2|w}^{(\alpha)}\right)\frac{1 - \frac{1}{Y_-^{(\alpha)}}}{1 - \frac{1}{Y_+^{(\alpha)}}}(u), \quad |u| < 2 \quad (6.1)$$

$$Y_{1|w}^{(\alpha)}\left(u + \frac{i}{g} - i0\right)Y_{1|w}^{(\alpha)}\left(u - \frac{i}{g} + i0\right) = 1 + Y_{2|w}^{(\alpha)}(u), \quad |u| > 2. \quad (6.2)$$

We stress that these equations are unambiguously *derived* from the TBA equation and the  $\epsilon$ -prescription on the left hand side is fixed by that of  $s^{-1}$ . We recall that in the TBA equations the functions  $Y_{\pm}$  have their support on  $[-2, 2]$ .

The TBA equation (4.2) for  $Y_{1|w}$  shows that this function has branch points located at  $u = \pm 2 \pm \frac{i}{g}$ . Since we are dealing with the mirror  $u$ -plane, it is natural to choose the cuts to run from  $\pm\infty$  to  $\pm 2 \pm \frac{i}{g}$  parallel to the real axis. In the same way the TBA equations show that various Y-functions have branch points located at  $\pm 2 \pm \frac{i}{g}Q$ ,  $Q = 0, 1, 2, \dots, \infty$ , and, therefore, all the cuts of all the Y-functions can be chosen to be outside the strip  $|\operatorname{Re} u| < 2$ , and running parallel to the real axis. Then, Y-functions are analytic in the strip  $|\operatorname{Re} u| < 2$ , in eq.(6.1) the  $\epsilon$ -prescription can be dropped, and  $u$  can be considered as a complex variable taking values in the strip

$$Y_{1|w}^{(\alpha)}\left(u + \frac{i}{g}\right)Y_{1|w}^{(\alpha)}\left(u - \frac{i}{g}\right) = \left(1 + Y_{2|w}^{(\alpha)}\right)\frac{1 - \frac{1}{Y_-^{(\alpha)}}}{1 - \frac{1}{Y_+^{(\alpha)}}}(u), \quad |\operatorname{Re} u| < 2. \quad (6.3)$$

Thus, we see that with this (and only this) choice of the cuts the Y-system takes its standard form.

In fact, analytically continuing Y-functions outside the strip  $|\operatorname{Re} u| < 2$ , one concludes that eq.(6.3) is valid for all complex values of  $u$  but those which belong to the cuts. Approaching then the real axis for  $|u| > 2$ , say, from above, one arrives at the following prescription [29]<sup>9</sup>

$$Y_{1|w}^{(\alpha)}\left(u + \frac{i}{g} + i0\right)Y_{1|w}^{(\alpha)}\left(u - \frac{i}{g} + i0\right) = \left(1 + Y_{2|w}^{(\alpha)}\right)\frac{1 - \frac{1}{Y_-^{(\alpha)}}}{1 - \frac{1}{Y_+^{(\alpha)}}}(u + i0), \quad |u| > 2. \quad (6.4)$$

Such an *analytically continued* equation can be used to determine the jump discontinuities of the Y-functions across the cut. From the compatibility of the equation

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<sup>9</sup>Naively one could try to define a new “inverse” operator  $s_G^{-1}$ ,  $(f \star s_G^{-1})(u) = \lim_{\epsilon \rightarrow 0^+} [f(u + \frac{i}{g} + i\epsilon) + f(u - \frac{i}{g} + i\epsilon)]$ , and claim that applying it to the simplified TBA equation, one gets eq.(6.4) for all values of  $u$ . The problem with this is that the operator  $s_G^{-1}$  is not inverse to  $s$ . In fact it annihilates  $s$ :  $s \star s_G^{-1} = 0$ .

(6.4) with eq.(6.2), we find that

$$\frac{Y_{1|w}^{(\alpha)}(u + \frac{i}{g} + i0)}{Y_{1|w}^{(\alpha)}(u + \frac{i}{g} - i0)} = \frac{1 - \frac{1}{Y_-^{(\alpha)}}}{1 - \frac{1}{Y_+^{(\alpha)}}}(u + i0), \quad |u| > 2, \quad (6.5)$$

and analogously

$$\frac{Y_{1|w}^{(\alpha)}(u + \frac{i}{g} - i0)}{Y_{1|w}^{(\alpha)}(u + \frac{i}{g} + i0)} = \frac{1 - \frac{1}{Y_-^{(\alpha)}}}{1 - \frac{1}{Y_+^{(\alpha)}}}(u - i0), \quad |u| > 2. \quad (6.6)$$

We also find that these discontinuity equations are consistent with the relation (4.1) which follows from the TBA equations for  $Y_{\pm}$ .

Thus, the TBA equations tell us that, because of the absence of analyticity, the Y-system equations must be supplemented by proper jump discontinuity conditions. This was one of the important observations made in [22].

Finally, we recall that for  $Y_1$  one finds

$$Y_1(u + \frac{i}{g} + i0)Y_1(u + \frac{i}{g} - i0) = \frac{\left(1 - \frac{1}{Y_-^{(1)}}\right)\left(1 - \frac{1}{Y_-^{(2)}}\right)}{1 + \frac{1}{Y_2}}, \quad |u| < 2 \quad (6.7)$$

$$Y_1(u + \frac{i}{g} + i0)Y_1(u + \frac{i}{g} - i0) = \frac{e^{-\Delta}}{1 + \frac{1}{Y_2}}, \quad |u| > 2,$$

where the explicit form of the quantity  $\Delta$  is given in [22, 23] for the ground state case, and in the excited state case it can be extracted from eq.(4.10) or obtained from the ground state one by using the contour deformation trick. Once again, the second equation here determines the jump discontinuity of  $Y_1$  across the cut. However, a new feature here is that the jump discontinuity, that is  $\Delta$ , does depend on a state under consideration!

To summarize, the Y-system exhibits the following properties dictated by the underlying TBA equations

- The Y-system is not analytic on the  $u$ -plane and for this reason it must be supplemented by jump discontinuity conditions;
- In general, jump discontinuities depend on a state of interest and they can be only fixed by the TBA equations;
- Different Y-functions have different cut structure. In total there are infinitely many cuts on the mirror theory  $u$ -plane with the branch points located at  $\pm 2 \pm \frac{i}{g}Q$ ,  $Q = 0, 1, 2, \dots, \infty$ . As a result, the Y-system lives on a Riemann surface of infinite genus.

Needless to say, these intricate analyticity properties discovered in [22, 29] render the AdS/CFT Y-system rather different from its known relativistic cousins and, for this reason, make it much harder to solve.

## 7. Conclusions

In this paper we have analyzed the TBA equations for the  $\text{AdS}_5 \times \text{S}^5$  mirror model. We provided an evidence that for any excited string state and, therefore, for any  $\mathcal{N} = 4$  SYM operator there could be infinitely many critical values of 't Hooft's coupling constant at which the TBA equations have to be modified. It is a demanding but also a rather challenging problem to locate the exact position of the critical points and it is similar in spirit to determination of the exact positions of the Bethe roots.

One could also speculate about the physical origin of critical points. It is known that the  $\mathfrak{sl}(2)$  sector is closed to any order of perturbation theory. One possibility would be that the first critical value of  $g$  of an  $\mathfrak{sl}(2)$  state is the one where this state begins to mix with states from other sectors of the theory. It would be interesting to understand if this is indeed the case.

The TBA equations and the contour deformation trick we have formulated allow one to discuss many interesting problems, and we list some of them below.

Concerning the issue of critical points, it would be very interesting to analyze the TBA equations, and compute numerically the scaling dimension of the Konishi operator in a vicinity of and beyond a critical point. The simplified TBA equations seem to be much better suited for such an analysis than the canonical ones.

Then, one should solve analytically the TBA equations for two-particle states at large  $g$ . The large  $g$  expansion should contain no  $\log g$  which follow from the BY equations [62, 58]. It should also fix the coefficient of the subleading  $1/g$  term in the energy expansion. There are currently two different predictions for the coefficient [32, 31] obtained by using some string theory methods and relying on certain assumptions. Thus, it is important to perform a rigorous string theory computation of this coefficient.

Recently, the TBA approach has been applied to obtain the 5-loop anomalous dimension of the Konishi operator by a combination of analytical and numerical means [70] and the corresponding result was found to be in a perfect agreement with the one based on the generalization of Lüscher's formulae [15]. This constitutes an important test of the TBA equations we propose in this paper (the hybrid equations). It would be nice to support this numerical agreement by an analytic proof.

The TBA equations we have proposed are not valid for the two-particle  $(J, \frac{J+1}{2})$  state. In the semi-classical string limit  $g \rightarrow \infty$  and  $J/g$  fixed it should correspond to the folded string rotating in  $\text{S}^2$  [63]. It would be interesting to analyze these states along the lines of our paper, write TBA equations, and solve them.

We have discussed only two-particle states. It is certainly important to generalize our analysis to arbitrary  $N$ -particle  $\mathfrak{sl}(2)$  states. For  $J = 2$  the lowest energy  $N$ -particle state is dual to the twist-two operator that plays an important role in field theory, see e.g. [64].

It would be also of interest to consider the one-particle case at large  $g$  and finite  $J/g$ . Its energy should match the string theory result [65].

Let us finally mention that one should also consider other sectors and exhibit the  $\mathfrak{psu}(2, 2|4)$  invariance of the spectrum.

## Acknowledgements

The work of G. A. was supported in part by the RFBR grant 08-01-00281-a, by the grant NSh-672.2006.1, by NWO grant 047017015 and by the INTAS contract 03-51-6346. The work of S.F. was supported in part by the Science Foundation Ireland under Grants No. 07/RFP/PHYF104 and 09/RFP/PHY2142. The work of R.S. was supported by the Science Foundation Ireland under Grants No. 07/RFP/PHYF104.

## 8. Appendices

### 8.1 Kinematical variables, kernels and S-matrices

All kernels and S-matrices we are using are expressed in terms of the function  $x(u)$

$$x(u) = \frac{1}{2}(u - i\sqrt{4 - u^2}), \quad \text{Im } x(u) < 0, \quad (8.1)$$

which maps the  $u$ -plane with the cuts  $[-\infty, -2] \cup [2, \infty]$  onto the physical region of the mirror theory, and the function  $x_s(u)$

$$x_s(u) = \frac{u}{2} \left( 1 + \sqrt{1 - \frac{4}{u^2}} \right), \quad |x_s(u)| \geq 1, \quad (8.2)$$

which maps the  $u$ -plane with the cut  $[-2, 2]$  onto the physical region of the string theory.

The momentum  $\tilde{p}^Q$  and the energy  $\tilde{\mathcal{E}}_Q$  of a mirror  $Q$ -particle are expressed in terms of  $x(u)$  as follows

$$\tilde{p}_Q = gx\left(u - \frac{i}{g}Q\right) - gx\left(u + \frac{i}{g}Q\right) + iQ, \quad \tilde{\mathcal{E}}_Q = \log \frac{x\left(u - \frac{i}{g}Q\right)}{x\left(u + \frac{i}{g}Q\right)}. \quad (8.3)$$

The kernels act from the right, and the three types of star operations used in this paper are defined as follows

$$\begin{aligned} f \star K(v) &\equiv \int_{-\infty}^{\infty} du f(u) K(u, v), & f \hat{\star} K(v) &\equiv \int_{-2}^2 du f(u) K(u, v), \\ f \check{\star} K(v) &\equiv \left( \int_{-\infty}^{-2} + \int_2^{\infty} \right) du f(u) K(u, v). \end{aligned} \quad (8.4)$$

The TBA equations discussed in this paper involve convolutions with a number of kernels which we specify below, see also [22] for more details. First, the following universal kernels appear in the TBA equations

$$\begin{aligned}
s(u) &= \frac{1}{2\pi i} \frac{d}{du} \log S(u) = \frac{g}{4 \cosh \frac{\pi g u}{2}}, \quad S(u) = -\tanh\left[\frac{\pi}{4}(ug - i)\right], \\
K_Q(u) &= \frac{1}{2\pi i} \frac{d}{du} \log S_Q(u) = \frac{1}{\pi} \frac{gQ}{Q^2 + g^2 u^2}, \quad S_Q(u) = \frac{u - \frac{iQ}{g}}{u + \frac{iQ}{g}}, \\
K_{MN}(u) &= \frac{1}{2\pi i} \frac{d}{du} \log S_{MN}(u) = K_{M+N}(u) + K_{N-M}(u) + 2 \sum_{j=1}^{M-1} K_{N-M+2j}(u), \\
S_{MN}(u) &= S_{M+N}(u) S_{N-M}(u) \prod_{j=1}^{M-1} S_{N-M+2j}(u)^2 = S_{NM}(u). \tag{8.5}
\end{aligned}$$

Then, the kernels  $K_{\pm}^{Qy}$  are related to the scattering matrices  $S_{\pm}^{Qy}$  of  $Q$ - and  $y_{\pm}$ -particles in the usual way

$$\begin{aligned}
K_{-}^{Qy}(u, v) &= \frac{1}{2\pi i} \frac{d}{du} \log S_{-}^{Qy}(u, v), \quad S_{-}^{Qy}(u, v) = \frac{x(u - \frac{iQ}{g}) - x(v)}{x(u + \frac{iQ}{g}) - x(v)} \sqrt{\frac{x(u + \frac{iQ}{g})}{x(u - \frac{iQ}{g})}}, \\
K_{+}^{Qy}(u, v) &= \frac{1}{2\pi i} \frac{d}{du} \log S_{+}^{Qy}(u, v), \quad S_{+}^{Qy}(u, v) = \frac{x(u - \frac{iQ}{g}) - \frac{1}{x(v)}}{x(u + \frac{iQ}{g}) - \frac{1}{x(v)}} \sqrt{\frac{x(u + \frac{iQ}{g})}{x(u - \frac{iQ}{g})}}.
\end{aligned}$$

These kernels can be expressed in terms of the kernel  $K_Q$ , and the kernel

$$K(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S(u, v) = \frac{1}{2\pi i} \frac{\sqrt{4 - v^2}}{\sqrt{4 - u^2}} \frac{1}{u - v}, \quad S(u, v) = \frac{x(u) - x(v)}{x(u) - 1/x(v)}, \tag{8.6}$$

as follows

$$K_{\mp}^{Qy}(u, v) = \frac{1}{2} \left( K_Q(u - v) \pm K_{Qy}(u, v) \right), \tag{8.7}$$

where  $K_{Qy}$  is given by

$$\begin{aligned}
K_{Qy}(u, v) &= \frac{1}{2\pi i} \frac{d}{du} \log S_{Qy}(u, v) = K\left(u - \frac{i}{g}Q, v\right) - K\left(u + \frac{i}{g}Q, v\right), \tag{8.8} \\
S_{Qy}(u, v) &= \frac{S_{-}^{Qy}(u, v)}{S_{+}^{Qy}(u, v)} = \frac{x(u - \frac{i}{g}Q) - x(v)}{x(u - \frac{i}{g}Q) - \frac{1}{x(v)}} \frac{x(u + \frac{i}{g}Q) - \frac{1}{x(v)}}{x(u + \frac{i}{g}Q) - x(v)} = \frac{S(u - \frac{i}{g}Q, v)}{S(u + \frac{i}{g}Q, v)}.
\end{aligned}$$

The S-matrices  $S_{\pm}^{Qy}$  and  $S_{Qy}$  (and kernels  $K_{\pm}^{Qy}$  and  $K_{Qy}$ ) can be easily continued to the string region by using the substitution  $x(u \pm \frac{i}{g}Q) \rightarrow x_s(u \pm \frac{i}{g}Q)$ , and the resulting S-matrices are denoted as  $S_{\pm}^{Q*y}$  and  $S_{Q*y}$ . Notice that  $S_{Q*y}(-\infty, v) = 1$ . One can

also replace  $x(v)$  by  $x_s(v)$  in the formulae above. Then, one gets the S-matrices and kernels which are denoted as  $K_{Qy}^{ms}$  and so on.

The following kernels and S-matrices are similar to  $K_{\pm}^{Qy}$

$$K_{-}^{yQ}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{-}^{yQ}(u, v), \quad S_{-}^{yQ}(u, v) = \frac{x(u) - x(v + i\frac{Q}{g})}{x(u) - x(v - i\frac{Q}{g})} \sqrt{\frac{x(v - i\frac{Q}{g})}{x(v + i\frac{Q}{g})}},$$

$$K_{+}^{yQ}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{+}^{yQ}(u, v), \quad S_{+}^{yQ}(u, v) = \frac{\frac{1}{x(u)} - x(v - i\frac{Q}{g})}{\frac{1}{x(u)} - x(v + i\frac{Q}{g})} \sqrt{\frac{x(v + i\frac{Q}{g})}{x(v - i\frac{Q}{g})}}.$$

They satisfy the following relations

$$K_{\pm}^{yQ}(u, v) = \frac{1}{2} \left( K_{yQ}(u, v) \mp K_Q(u - v) \right),$$

$$K_{yQ}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{yQ}(u, v) = K(u, v + \frac{i}{g}Q) - K(u, v - \frac{i}{g}Q)$$

$$S_{yQ}(u, v) = S_{-}^{yQ}(u, v) S_{+}^{yQ}(u, v) = \frac{x(u) - x(v + i\frac{Q}{g})}{x(u) - x(v - i\frac{Q}{g})} \frac{\frac{1}{x(u)} - x(v - i\frac{Q}{g})}{\frac{1}{x(u)} - x(v + i\frac{Q}{g})}$$

$$= \frac{S(u, v + \frac{i}{g}Q) x(v - i\frac{Q}{g})}{S(u, v - \frac{i}{g}Q) x(v + i\frac{Q}{g})}. \quad (8.9)$$

It is worth mentioning that  $S_{yQ}(\pm 2, v) = 1$ .

Next, we define the following S-matrices and kernels

$$S_{xv}^{QM}(u, v) = \frac{x(u - i\frac{Q}{g}) - x(v + i\frac{M}{g})}{x(u + i\frac{Q}{g}) - x(v + i\frac{M}{g})} \frac{x(u - i\frac{Q}{g}) - x(v - i\frac{M}{g})}{x(u + i\frac{Q}{g}) - x(v - i\frac{M}{g})} \frac{x(u + i\frac{Q}{g})}{x(u - i\frac{Q}{g})}$$

$$\times \prod_{j=1}^{M-1} \frac{u - v - \frac{i}{g}(Q - M + 2j)}{u - v + \frac{i}{g}(Q - M + 2j)}, \quad (8.10)$$

$$K_{xv}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{xv}^{QM}(u, v),$$

and

$$S_{vwx}^{QM}(u, v) = \frac{x(u - i\frac{Q}{g}) - x(v + i\frac{M}{g})}{x(u - i\frac{Q}{g}) - x(v - i\frac{M}{g})} \frac{x(u + i\frac{Q}{g}) - x(v + i\frac{M}{g})}{x(u + i\frac{Q}{g}) - x(v - i\frac{M}{g})} \frac{x(v - i\frac{M}{g})}{x(v + i\frac{M}{g})}$$

$$\times \prod_{j=1}^{M-1} \frac{u - v - \frac{i}{g}(M - Q + 2j)}{u - v + \frac{i}{g}(M - Q + 2j)}, \quad (8.11)$$

$$K_{vwx}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{vwx}^{QM}(u, v).$$

Then, the  $\mathfrak{sl}(2)$  S-matrix  $S_{\mathfrak{sl}(2)}^{QM}$  in the uniform light-cone gauge [59] with the gauge parameter  $a = 0$  can be written in the form

$$S_{\mathfrak{sl}(2)}^{QM}(u, v) = S^{QM}(u - v)^{-1} \Sigma_{QM}(u, v)^{-2}, \quad (8.12)$$



where  $\Sigma^{QM}$  is the improved dressing factor [66]. The corresponding  $\mathfrak{sl}(2)$  and dressing kernels are defined as usual

$$K_{\mathfrak{sl}(2)}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{\mathfrak{sl}(2)}^{QM}(u, v), \quad K_{QM}^{\Sigma}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log \Sigma_{QM}(u, v). \quad (8.13)$$

The analytically continued  $\mathfrak{sl}(2)$  S-matrix is given by

$$S_{\mathfrak{sl}(2)}^{1*M}(u, v) = \frac{1}{S_{1M}(u-v)\Sigma_{1*M}(u, v)^2},$$

where the improved dressing factor is given by [66]

$$\begin{aligned} \frac{1}{i} \log \Sigma_{1*M}(u, v) &= \Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-) - \Phi(y_1^-, y_2^+) + \Phi(y_1^-, y_2^-) \\ &+ \frac{1}{2} (\Psi(y_2^+, y_1^+) + \Psi(y_2^-, y_1^+) - \Psi(y_2^+, y_1^-) - \Psi(y_2^-, y_1^-)) \\ &+ \frac{1}{2i} \log \frac{(y_1^- - y_2^+)(y_1^- - \frac{1}{y_2^-})(y_1^+ - \frac{1}{y_2^-})}{(y_1^+ - y_2^+)(y_1^- - \frac{1}{y_2^+})^2}. \end{aligned} \quad (8.14)$$

Here  $y_1^{\pm} = x_s(u \pm \frac{i}{g})$  are parameters of a fundamental particle in string theory, and  $y_2^{\pm} = x(v \pm \frac{i}{g}M)$  are parameters of an  $M$ -particle bound state in the mirror theory, see [66] for details.

Next, we introduce the following kernel and S-matrix

$$\bar{K}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{ms}(u, v) = \frac{1}{2\pi} \frac{\sqrt{1 - \frac{4}{v^2}}}{\sqrt{4 - u^2}} \frac{v}{u - v}, \quad S_{ms}(u, v) = \frac{x(u) - x_s(v)}{x(u) - \frac{1}{x_s(v)}}.$$

With the help of this kernel we define<sup>10</sup>

$$\check{K}(u, v) = \bar{K}(u, v) [\theta(-v - 2) + \theta(v - 2)], \quad (8.15)$$

$$\check{K}_Q(u, v) = [\bar{K}(u + \frac{i}{g}Q, v) + \bar{K}(u - \frac{i}{g}Q, v)] [\theta(-v - 2) + \theta(v - 2)], \quad (8.16)$$

where  $\theta(u)$  is the standard unit step function. Obviously, both  $\check{K}$  and  $\check{K}_Q$  vanish for  $v$  being in the interval  $(-2, 2)$  and are equal to (twice) the jump discontinuity of the kernels  $K$  and  $K_{Qy}$  across the real semi-lines  $|v| > 2$ .

We also use

$$K_{ss}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{ss}(u, v) = \frac{1}{2\pi i} \frac{\sqrt{1 - \frac{4}{v^2}}}{\sqrt{1 - \frac{4}{u^2}}} \frac{v}{(u - v)}, \quad S_{ss}(u, v) = \frac{x_s(u) - x_s(v)}{x_s(u) - \frac{1}{x_s(v)}},$$

---

<sup>10</sup>The definitions of the kernels  $\check{K}$  and  $\check{K}_Q$  differ by the sign from the ones used in [22].

and define  $\check{S}_Q(u, v)$  as

$$\check{S}_Q(u, v) = \frac{S_{ss}(u - \frac{i}{g}Q, v)}{S_{ss}(u + \frac{i}{g}Q, v)}, \quad \check{K}_Q(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log \check{S}_Q(u, v), \quad (8.17)$$

to ensure that  $\check{S}_Q(-\infty, v) = 1$ .

The quantity  $\check{\mathcal{E}}$  is defined as

$$\check{\mathcal{E}}(u) = \log \frac{x(u - i0)}{x(u + i0)} = 2 \log |x_s(u)| \neq 0 \quad \text{for } |u| > 2. \quad (8.18)$$

The TBA equations for  $Y_Q$ -particles involve the kernel

$$\check{K}_Q^\Sigma = \frac{1}{2\pi i} \frac{\partial}{\partial u} \log \check{\Sigma}_Q = -K_{Qy} \hat{\star} \check{I}_0 + \check{I}_Q \quad (8.19)$$

where

$$\check{I}_Q = \sum_{n=1}^{\infty} \check{K}_{2n+Q}(u, v) = K_\Gamma^{[Q+2]}(u - v) + 2 \int_{-2}^2 dt K_\Gamma^{[Q+2]}(u - t) \check{K}(t, v), \quad (8.20)$$

$$K_\Gamma^{[Q]}(u) = \frac{1}{2\pi i} \frac{d}{du} \log \frac{\Gamma[\frac{Q}{2} - \frac{i}{2}gu]}{\Gamma[\frac{Q}{2} + \frac{i}{2}gu]} = \frac{g\gamma}{2\pi} + \sum_{n=1}^{\infty} \left( K_{2n+Q-2}(u) - \frac{g}{2\pi n} \right). \quad (8.21)$$

The kernel (8.19) is related to the dressing kernel  $K_{QM}^\Sigma$  as follows

$$\check{K}_Q^\Sigma(u, v) = K_{Q1}^\Sigma(u, v + \frac{i}{g} - i0) + K_{Q1}^\Sigma(u, v - \frac{i}{g} + i0) - K_{Q2}^\Sigma(u, v). \quad (8.22)$$

The analytically continued kernel  $\check{K}_{1*}^\Sigma$  is given by

$$\check{K}_{1*}^\Sigma(u, v) = -K_{1*y} \hat{\star} \check{I}_0 - K_{ss}(u - \frac{i}{g}, v). \quad (8.23)$$

Finally, integrating (8.23) over the first argument, one gets

$$\log \check{\Sigma}_{1*}(u, v) = -\log S_{1*y} \hat{\star} \check{I}_0 - \log S_{ss}(u - \frac{i}{g}, v). \quad (8.24)$$

Note that the first term here is real, but the last one is not.

## 8.2 Solution of the Bethe-Yang equation for the Konishi state

### Perturbative solution

The equation (3.2) can be solved in perturbative theory, and one gets up to  $g^{16}$

$$\begin{aligned}
p = & \frac{2\pi}{3} - \frac{\sqrt{3}g^2}{4} + \frac{9\sqrt{3}g^4}{32} - \frac{3\sqrt{3}}{8}g^6 (\zeta(3) + 1) + \frac{g^8\sqrt{3}(960\zeta(3) + 960\zeta(5) + 671)}{1024} \\
& + g^{10}\sqrt{3} \left( -\frac{141\zeta(3)}{64} - \frac{309\zeta(5)}{128} - \frac{315\zeta(7)}{128} - \frac{3807}{2560} \right) \\
& + g^{12}\sqrt{3} \left( \frac{2799\zeta(3)}{512} + \frac{9\zeta(3)^2}{64} + \frac{1527\zeta(5)}{256} + \frac{3339\zeta(7)}{512} + \frac{441\zeta(9)}{64} + \frac{7929}{2048} \right) \\
& + g^{14}\sqrt{3} \left( -\frac{30015\zeta(3)}{2048} - \frac{81\zeta(3)^2}{128} - \frac{7929\zeta(5)}{512} - \frac{45\zeta(3)\zeta(5)}{64} - \frac{17127\zeta(7)}{1024} \right. \\
& \qquad \qquad \qquad \left. - \frac{1197\zeta(9)}{64} - \frac{10395\zeta(11)}{512} - \frac{303837}{28672} \right) \\
& + g^{16}\sqrt{3} \left( \frac{340785\zeta(3)}{8192} + \frac{2349\zeta(3)^2}{1024} + \frac{350505\zeta(5)}{8192} + \frac{891\zeta(3)\zeta(5)}{256} + \frac{225\zeta(5)^2}{256} \right. \\
& \left. + \frac{183519\zeta(7)}{4096} + \frac{945\zeta(3)\zeta(7)}{512} + \frac{100863\zeta(9)}{2048} + \frac{230175\zeta(11)}{4096} + \frac{127413\zeta(13)}{2048} + \frac{15543873}{524288} \right).
\end{aligned} \tag{8.25}$$

Approximately one gets

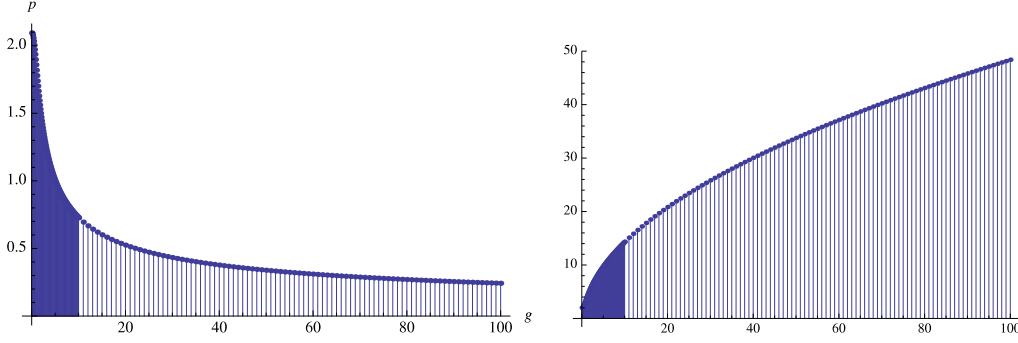
$$\begin{aligned}
p = & 2.0944 - 0.433013g^2 + 0.487139g^4 - 1.43028g^6 + 4.77062g^8 \\
& - 15.7964g^{10} + 52.5014g^{12} - 176.638g^{14} + 602.45g^{16}.
\end{aligned} \tag{8.26}$$

The corresponding expansion of the  $u$ -variable is given by

$$\begin{aligned}
u = & \frac{1}{\sqrt{3}g} \left[ 1 + 2g^2 - \frac{5g^4}{4} + g^6 \left( \frac{7}{4} + \frac{3\zeta(3)}{4} \right) - \frac{1}{128}g^8(461 + 144\zeta(3) + 240\zeta(5)) \right. \\
& + g^{10} \left( \frac{1133}{128} + \frac{63\zeta(3)}{32} + \frac{189\zeta(5)}{64} + \frac{315\zeta(7)}{64} \right) \\
& - g^{12} \left( \frac{23835}{1024} + \frac{1167\zeta(3)}{256} + \frac{9\zeta(3)^2}{64} + \frac{729\zeta(5)}{128} + \frac{2079\zeta(7)}{256} + \frac{441\zeta(9)}{32} \right) \\
& + g^{14} \left( \frac{64731}{1024} + \frac{3429\zeta(3)}{256} + \frac{9\zeta(3)^2}{128} + \frac{897\zeta(5)}{64} + \frac{45}{64}\zeta(3)\zeta(5) + \frac{8559\zeta(7)}{512} \right. \\
& \qquad \qquad \qquad \left. + \frac{189\zeta(9)}{8} + \frac{10395\zeta(11)}{256} \right) \\
& + g^{16} \left( -\frac{1441077}{8192} - \frac{176445\zeta(3)}{4096} + \frac{405\zeta(3)^2}{512} - \frac{169371\zeta(5)}{4096} - \frac{477}{512}\zeta(3)\zeta(5) \right. \\
& \left. - \frac{225\zeta(5)^2}{256} - \frac{87417\zeta(7)}{2048} - \frac{945}{512}\zeta(3)\zeta(7) - \frac{51975\zeta(9)}{1024} - \frac{147015\zeta(11)}{2048} - \frac{127413\zeta(13)}{1024} \right) \left. \right].
\end{aligned} \tag{8.27}$$

Approximately one gets

$$\begin{aligned}
u = & \frac{0.57735}{g} + 1.1547g - 0.721688g^3 + 1.53087g^5 - 3.98263g^7 \\
& + 11.1101g^9 - 32.8297g^{11} + 101.602g^{13} - 325.587g^{15}.
\end{aligned} \tag{8.28}$$



**Figure 8:** Numerical solution of the BYE for the Konishi operator. The momentum was computed for  $\frac{1}{10} \leq g \leq 10$  with the step  $\Delta g = \frac{1}{10}$ , and for  $10 \leq g \leq 100$  with  $\Delta g = 1$ . On the right picture the asymptotic dimension of the Konishi operator is plotted. It approaches  $2\sqrt{2\pi g}$  as expected from [56].

The dimension of the Konishi operator is

$$\begin{aligned}
\Delta = & 2 + 3g^2 - 3g^4 + \frac{21g^6}{4} + g^8 \left( -\frac{705}{64} - \frac{9\zeta(3)}{8} \right) + g^{10} \left( \frac{6627}{256} + \frac{135\zeta(3)}{32} + \frac{45\zeta(5)}{16} \right) \quad (8.29) \\
& + g^{12} \left( -\frac{67287}{1024} - \frac{27\zeta(3)}{2} - \frac{1377\zeta(5)}{128} - \frac{945\zeta(7)}{128} \right) \\
& + g^{14} \left( \frac{359655}{2048} + \frac{10899\zeta(3)}{256} + \frac{27\zeta(3)^2}{128} + \frac{18117\zeta(5)}{512} + \frac{7371\zeta(7)}{256} + \frac{1323\zeta(9)}{64} \right) \\
& + g^{16} \left( -\frac{7964283}{16384} - \frac{278505\zeta(3)}{2048} - \frac{621\zeta(3)^2}{512} - \frac{58491\zeta(5)}{512} - \frac{135}{128}\zeta(3)\zeta(5) \right. \\
& \quad \left. - \frac{198207\zeta(7)}{2048} - \frac{20979\zeta(9)}{256} - \frac{31185\zeta(11)}{512} \right) \\
& + g^{18} \left( \frac{22613385}{16384} + \frac{3600585\zeta(3)}{8192} + \frac{1539\zeta(3)^2}{256} + \frac{1520127\zeta(5)}{4096} + \frac{7101\zeta(3)\zeta(5)}{1024} \right. \\
& \left. + \frac{675\zeta(5)^2}{512} + \frac{2605095\zeta(7)}{8192} + \frac{2835\zeta(3)\zeta(7)}{1024} + \frac{573237\zeta(9)}{2048} + \frac{1002375\zeta(11)}{4096} + \frac{382239\zeta(13)}{2048} \right).
\end{aligned}$$

Approximately one gets

$$\begin{aligned}
\Delta_{\text{Konishi}} = & 2 + 3g^2 - 3g^4 + 5.25g^6 - 12.3679g^8 + 33.8743g^{10} - 100.537g^{12} \quad (8.30) \\
& + 313.532g^{14} - 1011.73g^{16} + 3348.11g^{18}.
\end{aligned}$$

We have also solved the BY equation numerically for small values of  $g$ , and its numerical solution perfectly agrees with the analytic one. The perturbative solution works very well at least up to  $g = \frac{1}{5}$ . For  $g = \frac{1}{5}$  the difference between the analytic solution and numerical one is  $\approx 5 \times 10^{-10}$ .

### Numerical solution for $\frac{1}{10} \leq g \leq 100$

The BY equation can be solved numerically up to very large values of  $g$ , and one gets the plot on Figure 7. for the momentum  $p$  as a function of  $g$ . For large values

of  $g \sim 100$  the momentum is approximated by

$$p_{\text{AFS}} = \sqrt{\frac{2\pi}{g} - \frac{1}{g}}, \quad (8.31)$$

with a good precision as expected from [56, 32]. For  $g = 100$  the difference between the numerical solution and the AFS formula is equal to  $-0.0016902$ . If one uses the following asymptotic expression from [58]

$$p_{\text{RS}} = \sqrt{\frac{2\pi}{g} - \frac{1}{g}} + \frac{0.931115 + 0.199472 \log(g)}{g^{3/2}}, \quad (8.32)$$

that is one includes the next subleading terms then the agreement is much better and for  $g = 100$  the difference between the numerical solution and the Rej-Spill formula is equal to  $0.0001587$ . To match the coefficients in  $p_{\text{RS}}$  one has to solve the BY equation for larger values of  $g$ .

The corresponding plots of the  $u$ -variable are shown on Figure 3, and were discussed in section 3.

### 8.3 Transfer matrices and asymptotic Y-functions

Here we remind the construction of the asymptotic Y-functions in terms of transfer-matrices corresponding to various representations of the centrally extended  $\mathfrak{su}(2|2)$  superalgebra. Consider  $K^I$  physical particles (excited states) of string theory characterized by the  $u_*$ -plane rapidities  $u_1, \dots, u_N$ ,  $N \equiv K^I$  or, equivalently, by the  $p_1, \dots, p_N$  physical momenta. Each of these particles transforms in the fundamental representation of  $\mathfrak{su}(2|2)$ . Consider also a single auxiliary particle with rapidity  $v$  corresponding to a bound state (atypical) representation of  $\mathfrak{su}(2|2)$  with the bound state number  $a$ . Scattering the bound state representation through the chain of  $N$  physical particles gives rise to the following monodromy matrix

$$\mathbb{T}(v|\vec{u}) = \prod_{i=1}^N \mathbb{S}_{0i}(u, u_i).$$

Here  $\mathbb{S}_{0i}(v, u_i)$  is the S-matrix describing the scattering of the auxiliary particle in the bound state representation with a physical particle with rapidity  $u_i$ . The transfer-matrix  $T_a(v|\vec{u})$  corresponding to this scattering process is defined as the trace of  $\mathbb{T}(u|\vec{u})$  over the auxiliary space  $T_a(v|\vec{u}) = \text{tr}_0 \mathbb{T}(v|\vec{u})$ . We are mostly interested in the situation where the auxiliary particle is in the mirror theory. An eigenvalue of this transfer matrix for an anti-symmetric bound state representation of the mirror

particle is given by the formula found in [67]

$$\begin{aligned}
T_a(v | \vec{u}) &= \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x^-}{y_i - x^+} \sqrt{\frac{x^+}{x^-}} + \tag{8.33} \\
&+ \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x^-}{y_i - x^+} \sqrt{\frac{x^+}{x^-}} \left[ \frac{x^+ + \frac{1}{x^+} - y_i - \frac{1}{y_i}}{x^+ + \frac{1}{x^+} - y_i - \frac{1}{y_i} - \frac{2ia}{g}} \right] \prod_{i=1}^{K^{\text{I}}} \left[ \frac{(x^- - x_i^-)(1 - x^- x_i^+)}{(x^+ - x_i^-)(1 - x^+ x_i^+)} \frac{x^+}{x^-} \right] \\
&+ \sum_{k=1}^{a-1} \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x^-}{y_i - x^+} \sqrt{\frac{x^+}{x^-}} \left[ \frac{x^+ + \frac{1}{x^+} - y_i - \frac{1}{y_i}}{x^+ + \frac{1}{x^+} - y_i - \frac{1}{y_i} - \frac{2ik}{g}} \right] \left\{ \prod_{i=1}^{K^{\text{I}}} \lambda_+(v, u_i, k) + \prod_{i=1}^{K^{\text{I}}} \lambda_-(v, u_i, k) \right\} \\
&- \sum_{k=0}^{a-1} \prod_{i=1}^{K^{\text{II}}} \frac{y_i - x^-}{y_i - x^+} \sqrt{\frac{x^+}{x^-}} \left[ \frac{x^+ - \frac{1}{x^+} - y_i - \frac{1}{y_i}}{x^+ - \frac{1}{x^+} - y_i - \frac{1}{y_i} - \frac{2ik}{g}} \right] \prod_{i=1}^{K^{\text{I}}} \frac{x^+ - x_i^+}{x^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \left[ 1 - \frac{2ik}{v - u_i + \frac{i}{g}(a-1)} \right] \times \\
&\times \left\{ \prod_{i=1}^{K^{\text{III}}} \frac{w_i - x^+ - \frac{1}{x^+} + \frac{i(2k-1)}{g}}{w_i - x^+ - \frac{1}{x^+} + \frac{i(2k+1)}{g}} + \prod_{i=1}^{K^{\text{II}}} \frac{y_i + \frac{1}{y_i} - x^+ - \frac{1}{x^+} + \frac{2ik}{g}}{y_i + \frac{1}{y_i} - x^+ - \frac{1}{x^+} + \frac{2i(k+1)}{g}} \prod_{i=1}^{K^{\text{III}}} \frac{w_i - x^+ - \frac{1}{x^+} + \frac{i(2k+3)}{g}}{w_i - x^+ - \frac{1}{x^+} + \frac{i(2k+1)}{g}} \right\}.
\end{aligned}$$

Eigenvalues are parametrized by solutions of the auxiliary Bethe equations:

$$\begin{aligned}
\prod_{i=1}^{K^{\text{I}}} \frac{y_k - x_i^-}{y_k - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} &= \prod_{i=1}^{K^{\text{III}}} \frac{w_i - y_k - \frac{1}{y_k} + \frac{i}{g}}{w_i - y_k - \frac{1}{y_k} - \frac{i}{g}}, \tag{8.34} \\
\prod_{i=1}^{K^{\text{II}}} \frac{w_k - y_i - \frac{1}{y_i} + \frac{i}{g}}{w_k - y_i - \frac{1}{y_i} - \frac{i}{g}} &= \prod_{i=1, i \neq k}^{K^{\text{III}}} \frac{w_k - w_i + \frac{2i}{g}}{w_k - w_i - \frac{2i}{g}}.
\end{aligned}$$

In the formulae above

$$v = x^+ + \frac{1}{x^+} - \frac{i}{g}a = x^- + \frac{1}{x^-} + \frac{i}{g}a,$$

and  $v$  takes values in the mirror theory  $v$ -plane, so  $x^\pm = x(v \pm \frac{i}{g}a)$  where  $x(v)$  is the mirror theory  $x$ -function. As was mentioned above,  $u_j$  take values in string theory  $u$ -plane, and therefore  $x_j^\pm = x_s(u_j \pm \frac{i}{g})$  where  $x_s(u)$  is the string theory  $x$ -function. Finally, the quantities  $\lambda_\pm$  are

$$\begin{aligned}
\lambda_\pm(v, u_i, k) &= \frac{1}{2} \left[ 1 - \frac{(x_i^- x^+ - 1)(x^+ - x_i^+)}{(x_i^- - x^+)(x^+ x_i^+ - 1)} + \frac{2ik}{g} \frac{x^+(x_i^- + x_i^+)}{(x_i^- - x^+)(x^+ x_i^+ - 1)} \right. \\
&\quad \left. \pm \frac{ix^+(x_i^- - x_i^+)}{(x_i^- - x^+)(x^+ x_i^+ - 1)} \sqrt{4 - \left( u - \frac{i(2k-a)}{g} \right)^2} \right]. \tag{8.35}
\end{aligned}$$

For the  $\mathfrak{sl}(2)$ -sector  $K^{\text{II}} = 0 = K^{\text{III}}$  and the expression above simplifies to

$$\begin{aligned}
T_a(v | \vec{u}) &= 1 + \prod_{i=1}^{K^{\text{I}}} \frac{(x^- - x_i^-)(1 - x^- x_i^+)}{(x^+ - x_i^-)(1 - x^+ x_i^+)} \frac{x^+}{x^-} \tag{8.36} \\
&- 2 \sum_{k=0}^{a-1} \prod_{i=1}^{K^{\text{I}}} \frac{x^+ - x_i^+}{x^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \left[ 1 - \frac{2ik}{v - u_i + \frac{i}{g}(a-1)} \right] + \sum_{m=\pm} \sum_{k=1}^{a-1} \prod_{i=1}^{K^{\text{I}}} \lambda_m(v, u_i, k).
\end{aligned}$$

For a two-particle physical state, the last formula appears to coincide up to a gauge transformation with [68]

$$T_a(v | \vec{u}) = \frac{x^+}{x^-} \left[ (1+a) \prod_{i=1}^2 \frac{x^- - x_i^-}{x^+ - x_i^-} + (a-1) \prod_{i=1}^2 \frac{x^- - x_i^+}{x^+ - x_i^-} \frac{x_i^- - \frac{1}{x^+}}{x_i^+ - \frac{1}{x^+}} \right. \\ \left. - a \prod_{i=1}^2 \frac{x^- - x_i^+}{x^+ - x_i^-} - a \prod_{i=1}^2 \frac{x^- - x_i^-}{x^+ - x_i^-} \frac{x_i^- - \frac{1}{x^+}}{x_i^+ - \frac{1}{x^+}} \right], \quad (8.37)$$

which is nothing else but an eigenvalue of the transfer matrix evaluated on the fermionic vacuum and continued to the mirror region for the auxiliary variable. In numerical computations formula (8.37) works much faster than (8.36) and, therefore, we use it for constructing the asymptotic Y-functions.

The transfer matrices  $T_{a,s}$  are used to introduce Y-functions  $Y_{a,s}$  which solve Y-system equations in the following standard way [33, 34, 27]

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}} \quad (8.38)$$

The transfer matrices  $T_{a,s}$  solve the so-called Hirota equations [60] and they can be computed via  $T_{a,1} \equiv T_a(v | \vec{u})$  through the Bazhanov-Reshetikhin formula [61] which we present here for the asymptotic solution

$$T_{a,s} = \det_{1 \leq i, j \leq s} T_{a+i-j} \left( v + \frac{i}{g} (s+1-i-j) | \vec{u} \right). \quad (8.39)$$

If here  $a+i-j < 0$  then the corresponding element  $T_{a+i-j} \left( v + \frac{i}{g} (s+1-i-j) | \vec{u} \right)$  is regarded as zero. Also,  $T_0 \left( v + \frac{i}{g} (s+1-i-j) | \vec{u} \right) = 1$ . In other words, asymptotic  $T_{a,1}$  and  $T_{1,s}$  satisfy the boundary conditions:  $T_{0,s} = 1 = T_{a,0}$ ,  $T_{a<0,s} = 0$  and  $T_{a,s<0} = 0$ .

Here for reader's convenience we recall the relationship of the Y-functions introduced in [22] and those of [27]. For the auxiliary Y-functions we have

$$Y_-^{(1)} = -\frac{1}{Y_{1,1}} = -\frac{T_{2,1} T_{0,1}}{T_{1,2} T_{1,0}}, \quad Y_-^{(2)} = -\frac{1}{Y_{1,-1}} = -\frac{T_{2,-1} T_{0,-1}}{T_{1,0} T_{1,-2}}, \\ Y_+^{(1)} = -Y_{2,2} = -\frac{T_{2,3} T_{2,1}}{T_{3,2} T_{1,2}}, \quad Y_+^{(2)} = -Y_{2,-2} = -\frac{T_{2,-1} T_{2,-3}}{T_{3,-2} T_{1,-2}}, \\ Y_{Q|vw}^{(1)} = \frac{1}{Y_{Q+1,1}} = \frac{T_{Q+2,1} T_{Q,1}}{T_{Q+1,2} T_{Q+1,0}}, \quad Y_{Q|vw}^{(2)} = \frac{1}{Y_{Q+1,-1}} = \frac{T_{Q+2,-1} T_{Q,-1}}{T_{Q+1,0} T_{Q+1,-2}}, \\ Y_{Q|w}^{(1)} = Y_{1,Q+1} = \frac{T_{1,Q+2} T_{1,Q}}{T_{2,Q+1} T_{0,Q+1}}, \quad Y_{Q|w}^{(2)} = Y_{1,-Q-1} = \frac{T_{1,-Q} T_{1,-Q-2}}{T_{2,-Q-1} T_{0,-Q-1}}. \quad (8.40)$$

The asymptotic functions  $Y_Q(v)$  in the  $\mathfrak{sl}(2)$ -sector are given by (3.8). Note that in this expression the transfer matrix naturally comes with a prefactor

$$S_Q(v | \vec{u}) = \prod_{i=1}^{K^1} \sqrt{S_{\mathfrak{sl}(2)}^{Q1^*}(v, u_i)} = \prod_{i=1}^{K^1} \frac{1}{\sqrt{S_{\mathfrak{sl}(2)}^{1^*Q}(u_i, v)}}. \quad (8.41)$$

This prefactor can be split as follows

$$S_Q(v | \vec{u}) = \prod_{i=1}^{K^1} \frac{1}{\sqrt{S_{\text{sl}(2)}^{1*Q}(u_i, v) h_Q(u_i, v)}} \prod_{k=1}^{K^1} \sqrt{h_Q(u_k, v)}, \quad (8.42)$$

where  $h_Q(u, v)$  is introduced in (8.97). This quantity satisfies the equation

$$h_Q\left(u, v + \frac{i}{g}\right) h_Q\left(u, v - \frac{i}{g}\right) = h_{Q+1}(u, v) h_{Q-1}(u, v)$$

and is a pure phase when  $u$  and  $v$  are in the string and mirror regions, respectively. The splitting (8.42) can be used to introduce the normalized transfer-matrix  $\tilde{T}_{Q,1}(v)$

$$\tilde{T}_{Q,1}(v | \vec{u}) \equiv \prod_{k=1}^{K^1} \sqrt{h_Q(u_k, v)} T_Q(v | \vec{u})$$

which renders the corresponding  $\tilde{Y}_Q = e^{-J\tilde{\mathcal{E}}_Q} \tilde{T}_{Q,1}^2(v | \vec{u})$  real. The functions  $\tilde{Y}_Q$  represent a simple and useful tool for checking the TBA equations.

#### 8.4 Hybrid equations for $Y_Q$ -functions

In this appendix we derive the hybrid TBA equations for  $Y_Q$ -functions. We discuss only the ground state TBA equations because the excited states equations can be obtained by using the contour deformation trick.

The canonical TBA equation for  $Q$ -particles reads

$$\begin{aligned} \log Y_Q &= -L \tilde{\mathcal{E}}_Q + \log(1 + Y_{Q'}) \star K_{\text{sl}(2)}^{Q'Q} \\ &+ 2 \log\left(1 + \frac{1}{Y_{M'|vw}}\right) \star K_{vw}^{M'Q} + 2 \log\left(1 - \frac{1}{Y_-}\right) \hat{\star} K_-^{yQ} + 2 \log\left(1 - \frac{1}{Y_+}\right) \hat{\star} K_+^{yQ}. \end{aligned} \quad (8.43)$$

To derive the hybrid equation we need to compute the first sum on the second line of this equation. It is done by using the simplified equations for  $Y_{M|vw}$

$$\begin{aligned} \log Y_{M|vw} &= \log(1 + Y_{M-1|vw})(1 + Y_{M+1|vw}) \star s \\ &- \log(1 + Y_{M+1}) \star s + \delta_{M1} \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s. \end{aligned} \quad (8.44)$$

Let us assume that we have some kernels  $\mathbf{K}_M$  which satisfy the following identities

$$\mathbf{K}_M - s \star (\mathbf{K}_{M+1} + \mathbf{K}_{M-1}) = \delta \mathbf{K}_M \quad (M \geq 2), \quad \mathbf{K}_1 - s \star \mathbf{K}_2 = \delta \mathbf{K}_1, \quad (8.45)$$

where  $\delta \mathbf{K}_M$  are known kernels. Then, applying the kernel  $\mathbf{K}_M$  to both sides of (8.44), and taking the sum over  $M$  from 1 to  $\infty$ , we obtain the formula

$$\begin{aligned} \sum_{M=1}^{\infty} \log\left(1 + \frac{1}{Y_{M|vw}}\right) \star \mathbf{K}_M &= \sum_{M=1}^{\infty} \log(1 + Y_{M|vw}) \star \delta \mathbf{K}_M \\ &+ \sum_{M=1}^{\infty} \log(1 + Y_{M+1}) \star s \star \mathbf{K}_M - \log \frac{1 - Y_-}{1 - Y_+} \star \mathbf{K}_1. \end{aligned} \quad (8.46)$$



Now choosing the kernel  $K_M$  to be  $K_{vwx}^{MQ}$  and using the formula (6.31) from [22]

$$\begin{aligned} K_{vwx}^{MQ} - s \star (K_{vwx}^{M+1,Q} + K_{vwx}^{M-1,Q}) &= \delta_{M+1,Q} s \quad (M \geq 2), \\ K_{vwx}^{1Q} - s \star K_{vwx}^{2Q} &= s \hat{\star} K_{yQ} + \delta_{2,Q} s, \end{aligned} \quad (8.47)$$

and the identity (8.46), we get

$$\begin{aligned} \sum_{M=1}^{\infty} \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star K_{vwx}^{MQ} &= \log(1 + Y_{1|vw}) \star s \hat{\star} K_{yQ} + \delta_{M+1,Q} \log(1 + Y_{M|vw}) \star s \\ &+ \sum_{M=1}^{\infty} \log(1 + Y_{M+1}) \star s \star K_{vwx}^{MQ} - \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star K_{vwx}^{1Q}. \end{aligned} \quad (8.48)$$

Finally, substituting this formula into (8.43), we get the hybrid ground state TBA equation.

### 8.5 Analytic continuation of $Y_1(v)$

To derive the exact Bethe equations (4.17) one has to analytically continue  $Y_1(z)$  in eq.(4.13) to the point  $z_{*k}$ . On the  $v$ -plane it means that we go from the real  $v$ -line down below the line with  $\text{Im}(v) = -\frac{1}{g}$  without crossing any cut, then turn back, cross the cut with  $\text{Im}(v) = -\frac{1}{g}$  and  $|\text{Re}(v)| > 2$ , and go back to the real  $v$ -line. As a result we should make the following replacements  $x(v - \frac{i}{g}) \rightarrow x_s(v - \frac{i}{g}) = x(v - \frac{i}{g})$ ,  $x(v + \frac{i}{g}) \rightarrow x_s(v + \frac{i}{g}) = 1/x(v - \frac{i}{g})$  in the kernels appearing in (3.7).

The analytic continuation depends on the analytic properties of the kernels and Y-functions. Let us consider the terms in eq.(4.13) order by order

#### Terms with $Y_{\pm}(v)$ -functions

For the analytic continuation of the last two terms in (4.13) it is convenient to use the kernels  $K_{\pm}^{y1}$  given by

$$K_-^{y1}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_-^{y1}(u, v), \quad S_-^{y1}(u, v) = \frac{x(u) - x(v + \frac{i}{g})}{x(u) - x(v - \frac{i}{g})} \sqrt{\frac{x(v - \frac{i}{g})}{x(v + \frac{i}{g})}}, \quad (8.49)$$

$$K_+^{y1}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_+^{y1}(u, v), \quad S_+^{y1}(u, v) = \frac{\frac{1}{x(u)} - x(v - \frac{i}{g})}{\frac{1}{x(u)} - x(v + \frac{i}{g})} \sqrt{\frac{x(v + \frac{i}{g})}{x(v - \frac{i}{g})}}. \quad (8.50)$$

It is clear from these equations that  $K_+^{y1}(u, v)$  remains regular in the analytic continuation.  $K_-^{y1}(u, v)$  on the other hand has a pole at  $v = u - \frac{i}{g}$  and behaves as

$$K_-^{y1}(u, v) = \frac{1}{2\pi i} \frac{1}{u - v - \frac{i}{g}} + \text{regular at } v \sim u - \frac{i}{g}. \quad (8.51)$$

Thus, one needs to analyze the analytic continuation of a function defined by the following integral for real  $v$

$$F(v) = \frac{1}{2\pi i} \int_{-2}^2 du f(u) \frac{1}{u - v - \frac{i}{g}}. \quad (8.52)$$

The consideration is the same as the one for the dressing phase in [66], and after  $v$  crosses the interval  $[-2 - \frac{i}{g}, 2 - \frac{i}{g}]$  one gets the following expression for  $F(v)$

$$F(v) = f(v + \frac{i}{g}) + \frac{1}{2\pi i} \int_{-2}^2 du f(u) \frac{1}{u - v - \frac{i}{g}}, \quad \text{Im}(v) < -\frac{1}{g}. \quad (8.53)$$

Then we go back to the real  $v$ -line but we do not cross the interval  $[-2 - \frac{i}{g}, 2 - \frac{i}{g}]$ , and therefore (8.53) remains the same. However, we should also analytically continue  $f(v + \frac{i}{g})$  back to real values of  $v$ . Thus we conclude that the analytic continuation of

$$\log\left(1 - \frac{1}{Y_-}\right) \hat{\star} K_-^{y1} + \log\left(1 - \frac{1}{Y_+}\right) \hat{\star} K_+^{y1} \quad (8.54)$$

is given by

$$\log\left(1 - \frac{1}{Y_{*-}(v + \frac{i}{g})}\right) + \log\left(1 - \frac{1}{Y_-}\right) \hat{\star} K_-^{y1*} + \log\left(1 - \frac{1}{Y_+}\right) \hat{\star} K_+^{y1*}, \quad (8.55)$$

where  $Y_{*-}(v)$  is the  $Y_-(v)$ -function analytically continued to the upper half-plane through the cut  $|v| > 2$ .

As was discussed in section 4,  $Y_{*-}$  coincides with  $Y_+$ , and, therefore, to find  $Y_{*-}(v + \frac{i}{g})$  one can just use the analytic continuation of the TBA equation (8.69) for  $Y_+(v)$  to  $Y_+(v + \frac{i}{g})$ . Since the S-matrix  $S_-^{1y}(u_k, v)$  has a pole at  $v = u_k + \frac{i}{g}$ , see appendix 8.1, one concludes that  $Y_{*-}(u_k + \frac{i}{g}) = \infty$ , and, therefore, the contribution of the first term in (8.55) vanishes in the exact Bethe equation  $Y_{1*}(u_k) = -1$ .

Nothing dangerous happens with the term  $\log \frac{1-Y_-}{1-Y_+} \hat{\star} s \star K_{vwx}^{11}$  because there is no singularity in the analytic continuation process of  $K_{vwx}^{11}$  until we get back to the real  $v$ -line. Then, we get for the analytically continued  $K_{vwx}^{11*}$

$$K_{vwx}^{11*}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{vwx}^{11*}(u, v), \quad (8.56)$$

$$S_{vwx}^{11*}(u, v) = \frac{x(u - \frac{i}{g}) - x_s(v + \frac{i}{g})}{x(u + \frac{i}{g}) - x_s(v - \frac{i}{g})} \frac{\frac{1}{x(u - \frac{i}{g})} - x_s(v - \frac{i}{g})}{\frac{1}{x(u + \frac{i}{g})} - x_s(v + \frac{i}{g})},$$

and it shows a pole at  $v = u$

$$K_{vwx}^{11*}(u, v) = -\frac{1}{2\pi i} \frac{1}{u - v} + \text{regular at } v \sim u. \quad (8.57)$$

Since we integrate over the line which is above the real line, the pole is not dangerous if the function we integrate with the kernel is regular as it is the case for the case under consideration.

### Terms with $Y_{M|vw}$ -functions

The analytic continuation of the term  $\log(1 + Y_{1|vw}) \star s \hat{\star} K_{y1}$  is given by

$$\log(1 + Y_{1|vw}) \star (s \hat{\star} K_{y1} + \tilde{s}) \quad (8.58)$$

because  $K_{y1}(u, v)$  has a pole at  $v = u - \frac{i}{g}$ .

### Terms with $Y_Q$ -functions

The kernel  $K_{\text{sl}(2)}^{Q1}$  is given by

$$\begin{aligned} K_{\text{sl}(2)}^{Q1}(u, v) &= -K_{Q1}(u - v) - 2K_{Q1}^\Sigma(u, v) \\ &= -K_{Q-1}(u - v) - K_{Q+1}(u - v) - 2K_{Q1}^\Sigma(u, v), \end{aligned} \quad (8.59)$$

Since  $K_{Q1}^\Sigma(u, v)$  is a holomorphic function if  $u$  is in the mirror region and  $v$  is in the string one, only  $K_{Q1}$  can cause any problem with the analytic continuation. Moreover, we see immediately that only the  $Q = 2$  case should be treated with care. It is easy to show however that since the integral over  $u$  is taken from  $-\infty$  to  $\infty$  the analytic continuation does not give any extra term because  $v$  crosses the line  $\text{Im}(u) = -\frac{1}{g}$  twice. This is the difference of this case from the  $Y_\pm$  one. The term with  $K_{vwx}^{Q'-1,1}(v)$  is harmless too because  $s$  is regular.

### The term $\log S \star K_{vwx}^{11}(u_j^-, v)$

This term is similar to  $\log\left(1 + \frac{1}{Y_{1|vw}}\right) \star K_{vwx}^{11}$  considered above, and its analytic continuation is given by

$$\log \text{Res}(S) \star K_{vwx}^{11*}(u_j^-, u_k) - \sum_{j=1}^N \log\left(u_j - u_k - \frac{2i}{g}\right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+}. \quad (8.60)$$

Summing up all the contributions, one gets the exact Bethe equations (4.17) and (8.72). We have shown that the r.h.s. of (4.17) is purely imaginary, and, therefore, the exact Bethe equations can be also written in the form

$$\begin{aligned} \pi(2n_k + 1) &= L p_k + i \sum_{j=1}^N \log S_{\text{sl}(2)}^{1*1*}(u_j, u_k) \\ &+ 2 \sum_{j=1}^N \log \text{Res}(S) \star \text{Im} K_{vwx}^{11*}(u_j^-, u_k) - 2 \sum_{j=1}^N \text{Im} \log\left(u_j - u_k - \frac{2i}{g}\right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} \\ &- 2 \log(1 + Y_Q) \star (\text{Im} K_{Q1}^\Sigma - s \star \text{Im} K_{vwx}^{Q-1,1*}) - 2i \log(1 + Y_{1|vw}) \star (\tilde{s} + s \hat{\star} K_{y1*}) \\ &- 2 \log \frac{1 - Y_-}{1 - Y_+} \hat{\star} s \star \text{Im} K_{vwx}^{11*} - i \log\left(1 - \frac{1}{Y_-}\right) \left(1 - \frac{1}{Y_+}\right) \hat{\star} K_{y1*}. \end{aligned} \quad (8.61)$$

## Analytic continuation of the canonical TBA equation for $Y_1$

Let us also consider the analytic continuation of the canonical TBA equation for  $Y_1$ . Then, the kernel  $K_{vwx}^{M1}$  is given by

$$K_{vwx}^{M1}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{vwx}^{M1}(u, v), \quad (8.62)$$

$$S_{vwx}^{M1}(u, v) = \frac{x(u - \frac{i}{g}M) - x(v + \frac{i}{g}) \frac{1}{x(u - \frac{i}{g}M)} - x(v - \frac{i}{g})}{x(u + \frac{i}{g}M) - x(v - \frac{i}{g}) \frac{1}{x(u + \frac{i}{g}M)} - x(v + \frac{i}{g})},$$

and nothing dangerous happens for  $M > 1$ . So, we just have to consider the  $M = 1$  case. Since  $Y_{1|vw}(u_j) = 0$ , and we should take care of the resulting log-singularity before the analytic continuation.

Introducing  $Z$ -functions as in (8.73), one gets for the term in the canonical TBA equation (8.68)

$$2 \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star K_{vwx}^{M1} = 2 \log \frac{M^2}{M^2 - 1} \star K_{vwx}^{M-1,1} - 2 \log \prod_{j=1}^N (u - u_j) \star K_{vwx}^{11}$$

$$+ 2 \log Z_{M|vw} \star K_{vwx}^{M1} \quad (8.63)$$

$$= 2 \log 2 - 2 \sum_{j=1}^N \log \left( u_j - v - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x^-}}{x_j^- - \frac{1}{x^+}} + 2 \log Z_{M|vw} \star K_{vwx}^{M1},$$

where we used  $\log \frac{M^2}{M^2 - 1} \star K_{vwx}^{M-1,1} = \log 2$ , and the following formula

$$\int_{-\infty}^{\infty} dt \log(t + i0 - u_j) K_{vwx}^{11}(t, v) = \log \left( u_j - v - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x^-}}{x_j^- - \frac{1}{x^+}}.$$

The analytic continuation in  $v$  then gives

$$2 \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star K_{vwx}^{M1*}(v) = 2 \log 2 - 2 \sum_{j=1}^N \log \left( u_j - v - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x^-}}{x_j^- - \frac{1}{x^+}}$$

$$+ 2 \log Z_{M|vw} \star_{p.v.} K_{vwx}^{M1*} + \log Z_{1|vw}(v).$$

The equation seems to coincide with the one derived in [30] after one performs the parity transformation  $u_j \rightarrow -u_j$ , and sets  $g = 2$ .

## 8.6 Canonical TBA equations

**Canonical TBA equations:**  $g < g_{cr}^{(1)}$

The canonical ground state TBA equations [22] are derived from the string hypothesis for the mirror model [21] by following a textbook route, see e.g. [53], and can be written in the form

- $Q$ -particles

$$V_Q \equiv \log Y_Q + L \tilde{\mathcal{E}}_Q - \log(1 + Y_{Q'}) \star K_{\text{sl}(2)}^{Q'Q} - 2 \log \left( 1 + \frac{1}{Y_{M'|vw}} \right) \star K_{vw}^{M'Q} \\ - 2 \log \left( 1 - \frac{1}{Y_-} \right) \hat{\star} K_-^{yQ} - 2 \log \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_+^{yQ} = 0, \quad (8.64)$$

- $y$ -particles

$$V_{\pm} \equiv \log Y_{\pm} + \log(1 + Y_Q) \star K_{\pm}^{Qy} - \log \frac{1 + \frac{1}{Y_{M'|vw}}}{1 + \frac{1}{Y_{M|w}}} \star K_M = 0. \quad (8.65)$$

- $M|vw$ -strings

$$V_{M|vw} \equiv \log Y_{M|vw} + \log(1 + Y_{Q'}) \star K_{xv}^{Q'M} \\ - \log \left( 1 + \frac{1}{Y_{M'|vw}} \right) \star K_{M'M} - \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_M = 0. \quad (8.66)$$

- $M|w$ -strings

$$V_{M|w} \equiv \log Y_{M|w} - \log \left( 1 + \frac{1}{Y_{M'|w}} \right) \star K_{M'M} - \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \hat{\star} K_M = 0. \quad (8.67)$$

Applying the contour deformation trick to the canonical TBA equations, one gets the following set of integral equations for Konishi-like states and  $g < g_{cr}^{(1)}$

- $Q$ -particles :  $V_Q = - \sum_{*} \log S_{\text{sl}(2)}^{1*Q}(u_j, v), \quad (8.68)$

- $y$ -particles :  $V_{\pm} = \sum_{*} \log S_{\pm}^{1*y}(u_j, v), \quad (8.69)$

- $M|vw$ -strings :  $V_{M|vw} = \sum_{*} \log S_{xv}^{1*M}(u_j, v), \quad (8.70)$

- $M|w$ -strings :  $V_{M|w} = 0. \quad (8.71)$

Here summation over repeated indices is assumed. The sums in the formulae run over the set of  $N$  particles, all  $Y$ -functions depend on the real  $u$  variable (or  $z$ ) of the mirror region. We recall that  $S_{\text{sl}(2)}^{1*Q}$  and  $S_{xv}^{1*M}$  denote the  $S$ -matrices with the first and second arguments in the string and mirror regions, respectively, and both arguments of the kernels in these formulae are in the mirror region.

Then, the integrals are taken over the interval  $[-2, 2]$  for convolutions involving  $Y_{\pm}$ , and over the horizontal line that is a little bit above the real  $u$  line (or the interval  $\text{Re}(z) \in (-\frac{\omega_1}{2}, \frac{\omega_1}{2}), \text{Im}(z) = \frac{\omega_2}{2i}$  on the  $z$ -torus) for all other convolutions.

The reason why one should choose the integration contour to run a little bit above the real line of the mirror  $u$ -plane is that  $Y_{1|vw}$  has zeros at  $u = u_k$ , and, therefore, the terms  $\log\left(1 + \frac{1}{Y_{1|vw}}\right) \star K$  with any kernel  $K$  should be treated carefully. The prescription above for the integration contour guarantees the reality of all Y-functions as we show in appendix 8.7. The log  $S$ -term in the equation for  $vw$ -strings is in fact necessary to cancel the corresponding singularity on the l.h.s. of this equation.

One can show that the imaginary zeros of  $Y_{k|vw}$  and  $1 + Y_{k|vw}$  do not contribute to the canonical equations for Konishi-like states at weak coupling. In appendix 8.8 we also show that the simplified equations can be derived from the canonical ones following the same route as in [22, 23].

The canonical TBA equations are rather complicated and involve infinite sums that makes the high-precision numerical tests very time-consuming. We have checked numerically that for Konishi-like states they are solved at the large  $L$  limit by the asymptotic Y-functions if  $L = J + 2$ .

Let us stress again that the TBA equations above are valid only up to the first critical value of  $g$  where the function  $Y_{1|vw}(u)$  has real zeros only for  $u = u_k$ , and  $Y_{M|vw}$ -functions with  $M \geq 1$  do not have other zeros in the physical strip.

**Exact Bethe equations:**  $g < g_{cr}^{(1)}$

To derive the exact Bethe equations we take the logarithm of eq.(3.7), and analytically continue the variable  $z$  of  $Y_1(z)$  in eq.(8.68) to the point  $z_{*k}$ . The analytic continuation is similar to the one in section 4, and its detailed consideration can be found in appendix 8.5. As shown there, the resulting exact Bethe equations for a string theory state from the  $\mathfrak{sl}(2)$  sector can be cast into the following integral form

$$\begin{aligned} \pi i(2n_k + 1) &= \log Y_{1*}(u_k) = iL p_k - \sum_{j=1}^N \log S_{\mathfrak{sl}(2)}^{1*1*}(u_j, u_k) + \log(1 + Y_Q) \star K_{\mathfrak{sl}(2)}^{Q1*} \\ &+ 2 \log 2 - 2 \sum_{j=1}^N \log\left(u_j - u_k - \frac{2i}{g}\right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} + 2 \log Z_{M|vw} \star K_{vw}^{M1*} \\ &+ 2 \log\left(1 - \frac{1}{Y_-}\right) \hat{\star} K_-^{y1*} + 2 \log\left(1 - \frac{1}{Y_+}\right) \hat{\star} K_+^{y1*}. \end{aligned} \quad (8.72)$$

Here the integration contours run a little bit above the Bethe roots  $u_j$ ,  $p_k = i\tilde{\mathcal{E}}_Q(z_{*k}) = -i \log \frac{x_s(u_k + \frac{i}{g})}{x_s(u_k - \frac{i}{g})}$  is the momentum of the  $k$ -th particle, and the second argument in all the kernels in this equation is equal to  $u_k$  but the first argument we integrate with respect to is the original one in the mirror region. The Z-functions are defined in the same way as in [30]

$$1 + \frac{1}{Y_{1|vw}} \equiv Z_{1|vw} \frac{4}{3} \frac{1}{\prod_{j=1}^N (u - u_j)}, \quad 1 + \frac{1}{Y_{M|vw}} \equiv Z_{M|vw} \frac{(M+1)^2}{M(M+2)}, \quad (8.73)$$

they are positive for real  $u$ , and  $Z_{M|vw}$ ,  $M = 2, 3, \dots$  asymptote to 1 at  $u \rightarrow \infty$ .

Taking into account that the BY equations for the  $\mathfrak{sl}(2)$  sector have the form

$$\pi i(2n_k + 1) = iJ p_k - \sum_{j=1}^N \log S_{\mathfrak{sl}(2)}^{1^*1^*}(u_j, u_k),$$

and that  $Y_Q$  is exponentially small at large  $J$ , we conclude that if the analytic continuation has been done correctly then up to an integer multiple of  $2\pi i$  the following identities between the asymptotic Y-functions should hold

$$\begin{aligned} \mathcal{R}_k \equiv & iN p_k + 2 \log 2 - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} \right) + 2 \log Z_{M|vw} \star K_{vw}^{M1^*} \\ & + 2 \log \left( 1 - \frac{1}{Y_-} \right) \hat{\star} K_-^{y1^*} + 2 \log \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_+^{y1^*} = 0. \end{aligned} \quad (8.74)$$

For  $N = 2$  and  $u_1 = -u_2$  one gets one equation, and by using the expressions for the Y-functions from appendix 8.3 one can check numerically that it does hold for any real value of  $u_1$  such that only  $Y_{1|vw}$  has two zeros for real  $u$ .

We believe that the canonical TBA equations (8.68-8.71) and eqs.(8.72) are equivalent to the ones proposed in [30] for states having the vanishing total momentum but a detailed comparison is hard to perform due to different notations and conventions. We see however that they are definitely different for physical states which satisfy the level-matching condition but do not have the vanishing total momentum.

**Canonical TBA equations:**  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$

Here is the set of canonical TBA equations for Konishi-like states and  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$

- $Q$ -particles : 
$$V_Q = - \sum_* \log S_{\mathfrak{sl}(2)}^{1^*Q}(u_j, v) - 2 \sum_{j=1}^2 \log S_{vw}^{1Q}(r_j^-, v) + 2 \log S_{vw}^{2Q}(r_1, v) + 2 \log S_{yQ}(r_1, v), \quad (8.75)$$
- $y$ -particles : 
$$V_{\pm} = \sum_{j=1}^N \log S_{\pm}^{1^*y}(u_j, v) - \sum_{j=1}^2 \log S_1(r_j^- - v) + \log S_2(r_1 - v),$$
- $M|vw$ -strings : 
$$V_{M|vw} = \sum_{j=1}^N \log S_{xv}^{1^*M}(u_j, v) - \sum_{j=1}^2 \log S_{1M}(r_j^-, v) + \log S_{2M}(r_1, v),$$
- $M|w$ -strings :  $V_{M|w} = 0,$

where  $2 \log S_{yQ}(r_1, v)$  appears due to the imaginary zero  $r_1$  of  $Y_{\pm}$ , and we take into account that the S-matrix  $S_{yQ}$  is normalized as  $S_{yQ}(\pm 2, v) = 1$ .

**Exact Bethe equations:**  $g_{cr}^{(1)} < g < \bar{g}_{cr}^{(1)}$

The exact Bethe equations are obtained by analytically continuing  $\log Y_1$  in (8.76) following the same route as for the small  $g$  case, and they take the following form

$$\begin{aligned}
\pi i(2n_k + 1) = \log Y_{1^*}(u_k) &= iL p_k - \sum_{j=1}^N \log S_{st(2)}^{1^*1^*}(u_j, u_k) + \log(1 + Y_Q) \star K_{st(2)}^{Q1^*} \\
&- 2 \sum_{j=1}^2 \log S_{vwx}^{11^*}(r_j^-, u_k) + 2 \log S_{vwx}^{21^*}(r_1, u_k) + 2 \log S_{y1^*}(r_1, u_k) \\
&- 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} + 2 \log 2 + 2 \log Z_{M|vw} \star K_{vwx}^{M1^*} \\
&+ 2 \log \left( 1 - \frac{1}{Y_-} \right) \hat{\star} K_-^{y1^*} + 2 \log \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_+^{y1^*}. \tag{8.76}
\end{aligned}$$

We conclude again that the consistency with the BY equations requires the fulfillment of the identities between the asymptotic Y-functions similar to (8.74) that we have checked numerically for the Konishi-like states.

**Canonical TBA equations:**  $\bar{g}_{cr}^{(1)} < g < g_{cr}^{(2)}$

The canonical TBA equations for Konishi-like states and  $\bar{g}_{cr}^{(1)} < g < g_{cr}^{(2)}$  take the form

$$\begin{aligned}
\bullet \text{ } Q\text{-particles : } V_Q &= - \sum_{j=1}^N \log S_{st(2)}^{1^*Q}(u_j, v) - 2 \sum_{j=1}^2 \log S_{vwx}^{1Q}(r_j^-, v), \tag{8.77} \\
\bullet \text{ } y\text{-particles : } V_{\pm} &= \sum_{j=1}^N \log S_{\pm}^{1^*y}(u_j, v) - \sum_{j=1}^2 \log S_1(r_j^- - v), \\
\bullet \text{ } M|vw\text{-strings : } V_{M|vw} &= \sum_{j=1}^N \log S_{xv}^{1^*M}(u_j, v) - \sum_{j=1}^2 \log S_{1M}(r_j^-, v), \\
\bullet \text{ } M|w\text{-strings : } V_{M|w} &= 0,
\end{aligned}$$

The equations for  $Y_Q$ -particles differ from (8.76) by the absence of the term  $2 \log S_{vwx}^{2Q}(r_1, v)$ . The p.v. prescription however would give instead additional terms due to the real zeros  $r_j$  of  $Y_{2|vw}$  and  $Y_{\pm}$ .



**Exact Bethe equations:**  $\bar{g}_{cr}^{(1)} < g < g_{cr}^{(2)}$

Analytically continuing  $\log Y_1$  in (8.77), one gets the exact Bethe equations

$$\begin{aligned}
\pi i(2n_k + 1) &= \log Y_{1^*}(u_k) = iL p_k - \sum_{j=1}^N \log S_{\text{sl}(2)}^{1^*1^*}(u_j, u_k) + \log(1 + Y_Q) \star K_{\text{sl}(2)}^{Q1^*} \\
&- 2 \sum_{j=1}^2 \log S_{vw}^{11^*}(r_j^-, u_k) + 2 \log 2 - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k}}{x_j^- - x_k^+} \\
&+ 2 \log Z_{M|vw} \star K_{vw}^{M1^*} + 2 \log \left( 1 - \frac{1}{Y_-} \right) \hat{\star} K_-^{y1^*} + 2 \log \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_+^{y1^*}. \quad (8.78)
\end{aligned}$$

We recall that the integration contours should run a little bit above the Bethe roots  $u_j$ , and below the dynamical roots  $r_j$ , and the consistency with the BY equations leads to identities of the form (8.74) that we have checked numerically.

**Canonical TBA equations:**  $g_{cr}^{(m)} < g < \bar{g}_{cr}^{(m)}$

The necessary modification of the integration contour follows the one for the first critical region, and the integration contour runs a little bit above the Bethe roots  $u_j$ , below the zeros  $r_j^{(k)}$ ,  $k = 1, \dots, m$ , and between the zeros  $r_j^{(m+1)}$ ,  $\text{Im}(r_1^{(m+1)}) < 0$ . All the roots  $r_j^{(k)} - \frac{i}{g}$  are between the integration contour and the real line of the mirror region. The contour for  $Y_{\pm}$  functions lies above the zeros  $r_j^{(2)}$ .

Here are the canonical TBA equations for Konishi-like states and  $g_{cr}^{(m)} < g < \bar{g}_{cr}^{(m)}$

- $Q$ -particles : 
$$\begin{aligned}
V_Q &= - \sum_{j=1}^N \log S_{\text{sl}(2)}^{1^*Q}(u_j, v) - 2 \sum_{k=1}^{m-1} \sum_{j=1}^2 \log S_{vw}^{kQ}(r_j^{(k+1)-}, v) \\
&+ 2 \log \frac{S_{vw}^{m-1,Q}(r_1^{(m+1)}, v) S_{vw}^{m+1,Q}(r_1^{(m+1)}, v)}{S_{vw}^{m,Q}(r_1^{(m+1)-}, v)}, \quad (8.79)
\end{aligned}$$
- $y$ -particles : 
$$\begin{aligned}
V_{\pm} &= \sum_{j=1}^N \log S_{\pm}^{1^*y}(u_j, v) - \sum_{k=1}^{m-1} \sum_{j=1}^2 \log S_k(r_j^{(k+1)-} - v) \\
&+ \log \frac{S_{m-1}(r_1^{(m+1)} - v) S_{m+1}(r_1^{(m+1)} - v)}{S_m(r_1^{(m+1)-} - v)},
\end{aligned}$$
- $M|vw$ -strings : 
$$\begin{aligned}
V_{M|vw} &= \sum_{j=1}^N \log S_{xv}^{1^*M}(u_j, v) - \sum_{k=1}^{m-1} \sum_{j=1}^2 \log S_{kM}(r_j^{(k+1)-}, v) \\
&+ \log \frac{S_{m-1,M}(r_1^{(m+1)}, v) S_{m+1,M}(r_1^{(m+1)}, v)}{S_{m,M}(r_1^{(m+1)-}, v)},
\end{aligned}$$
- $M|w$ -strings :  $V_{M|w} = 0$ ,

**Exact Bethe equations:**  $g_{cr}^{(m)} < g < \bar{g}_{cr}^{(m)}$

The exact Bethe equations take the following form

$$\begin{aligned}
\pi i(2n_k + 1) = \log Y_{1*}(u_k) &= iL p_k - \sum_{j=1}^N \log S_{\text{st}(2)}^{1*1*}(u_j, u_k) + \log(1 + Y_Q) \star K_{\text{st}(2)}^{Q1*} \\
&- 2 \sum_{k=1}^m \sum_{j=1}^2 \log S_{vw}^{k1*}(r_j^{(k+1)-}, u_k) + 2 \log \frac{S_{vw}^{m-1,1*}(r_1^{(m+1)}, u_k) S_{vw}^{m+1,1*}(r_1^{(m+1)}, u_k)}{S_{vw}^{m1*}(r_1^{(m+1)-}, u_k)} \\
&+ 2 \log 2 - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} + 2 \log Z_{M|vw} \star K_{vw}^{M1*} \\
&+ 2 \log \left( 1 - \frac{1}{Y_-} \right) \hat{\star} K_-^{y1*} + 2 \log \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_+^{y1*}, \quad (8.80)
\end{aligned}$$

where  $\tilde{x}_j^- \equiv x(r_j^{(3)} - \frac{i}{g})$ , and the integration contours run just above the Bethe roots, and below the zeros  $r_j^{(k)}$ ,  $k = 1, \dots, m$ . The functions  $Z_{M|vw}$  are defined as in (8.73).

The consistency with the BY equations again implies that the sum of the terms on the last three lines of (8.80) is equal to  $-2ip_k$  that we have checked numerically for the Konishi-like states.

**Canonical TBA equations:**  $\bar{g}_{cr}^{(m)} < g < g_{cr}^{(m+1)}$

In this case all the zeros  $r_j^{(k)}$ ,  $k = 1, \dots, m+1$  are real, the integration contour runs below all of them but the Bethe roots  $u_j$ , and the canonical TBA equations for Konishi-like states and  $g_{cr}^{(m)} < g < \bar{g}_{cr}^{(m)}$  take the following form

- $Q$ -particles :  $V_Q = - \sum_{j=1}^N \log S_{\text{st}(2)}^{1*Q}(u_j, v) - 2 \sum_{k=1}^m \sum_{j=1}^2 \log S_{vw}^{kQ}(r_j^{(k+1)-}, v), \quad (8.81)$
- $y$ -particles :  $V_{\pm} = \sum_{j=1}^N \log S_{\pm}^{1*y}(u_j, v) - \sum_{k=1}^m \sum_{j=1}^2 \log S_k(r_j^{(k+1)-} - v),$
- $M|vw$ -strings :  $V_{M|vw} = \sum_{j=1}^N \log S_{xv}^{1*M}(u_j, v) - \sum_{k=1}^m \sum_{j=1}^2 \log S_{kM}(r_j^{(k+1)-}, v),$
- $M|w$ -strings :  $V_{M|w} = 0.$

**Exact Bethe equations:**  $\bar{g}_{cr}^{(m)} < g < g_{cr}^{(m+1)}$

The exact Bethe equations take the following form

$$\begin{aligned}
\pi i(2n_k + 1) &= \log Y_{1^*}(u_k) = iL p_k - \sum_{j=1}^N \log S_{\mathfrak{sl}(2)}^{1^*1^*}(u_j, u_k) + \log(1 + Y_Q) \star K_{\mathfrak{sl}(2)}^{Q1^*} \\
&- 2 \sum_{k=1}^m \sum_{j=1}^2 \log S_{vwx}^{k1^*}(r_j^{(k+1),-}, u_k) + 2 \log 2 - 2 \sum_{j=1}^N \log \left( u_j - u_k - \frac{2i}{g} \right) \frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+} \\
&+ 2 \log Z_{M|vw} \star K_{vwx}^{M1^*} + 2 \log \left( 1 - \frac{1}{Y_-} \right) \hat{\star} K_-^{y1^*} + 2 \log \left( 1 - \frac{1}{Y_+} \right) \hat{\star} K_+^{y1^*}. \quad (8.82)
\end{aligned}$$

The consistency with the BY equations again implies the fulfillment of identities of the form (8.74) that we have checked numerically for the Konishi-like states.

## 8.7 Reality of Y-functions

In this appendix we show that the reality condition for Y-functions is compatible with the canonical TBA equations for Konishi-like states. We consider only the weak coupling region. The generalization to other regions and general  $\mathfrak{sl}(2)$  states is straightforward.

To start with we introduce the principle value prescription for the integrals involving  $\log f(u)$  where  $f(u)$  is real for real  $u$ , and has first-order zeros (or poles) in the interval  $[a, b]$  at  $u_k$ ,  $k = 1, \dots, N$

$$\log f \star_{p.v} K \equiv \lim_{\epsilon \rightarrow 0} \sum_{k=0}^N \int_{u_k + \epsilon}^{u_{k+1} - \epsilon} du \log |f(u)| K(u, v), \quad (8.83)$$

where  $u_0 = a$ ,  $u_{N+1} = b$ . In the cases of interest  $a = -\infty$ ,  $b = \infty$  or  $a = -2$ ,  $b = 2$ .

Assuming for definiteness that  $f(u)$  has  $N$  zeros, and  $f(\infty) > 0$ , one can write

$$f(u) = \tilde{f}(u) \prod_{k=1}^N (u - u_k), \quad (8.84)$$

where  $\tilde{f}(u) > 0$  for any  $u \in \mathbf{R}$ .

Then the convolution terms of the form  $\log f \star K$  with the integration contour running a little bit above the real line can be written as follows

$$\begin{aligned}
\log f \star K &= \int_a^b du \log f(u + i0) K(u + i0, v) \quad (8.85) \\
&= \int_a^b du \log \tilde{f}(u) K(u, v) + \sum_{k=1}^N \int_a^b du \log(u - u_k + i0) K(u + i0, v) \\
&= \log f \star_{p.v} K + \pi i \sum_{k=1}^N \int_a^{u_k} du K(u, v) = \log f \star_{p.v} K + \frac{1}{2} \sum_{k=1}^N \log \frac{S(u_k, v)}{S(a, v)},
\end{aligned}$$

where  $\log S(u, v) \equiv 2\pi i \int^u du' K(u', v)$ , and it can differ from the S-matrix defining the kernel  $K$  by a function of  $v$ . It is convenient to choose the normalization  $S(a, v) = 1$ , and most of our S-matrices satisfy this condition.

Let us now use (8.85) to show the reality of Y-functions. We start with (8.69) that we write as follows

$$\begin{aligned} \log \frac{Y_+}{Y_-}(v) &= - \sum_{j=1}^N \log S_{1*y}(u_j, v) + \log(1 + Y_Q) \star K_{Qy}, \\ \log Y_+ Y_-(v) &= \sum_{j=1}^N \log S_1(u_j - v) - \log(1 + Y_Q) \star K_Q + 2 \log \frac{1 + \frac{1}{Y_{M|vw}}}{1 + \frac{1}{Y_{M|w}}} \star K_M, \end{aligned} \quad (8.86)$$

where

$$S_{Q*y}(u, v) = \frac{x_s(u - \frac{i}{g}Q) - x(v)}{x_s(u - \frac{i}{g}Q) - \frac{1}{x(v)}} \frac{x_s(u + \frac{i}{g}Q) - \frac{1}{x(v)}}{x_s(u + \frac{i}{g}Q) - x(v)}, \quad (8.87)$$

and we used that

$$S_{Qy}(u, v) = \frac{S_-^{Qy}(u, v)}{S_+^{Qy}(u, v)}, \quad S_Q(u - v) = S_-^{Qy}(u, v) S_+^{Qy}(u, v) = \frac{u - v - \frac{i}{g}Q}{u - v + \frac{i}{g}Q}, \quad (8.88)$$

and that  $S_Q(u - v)$  analytically continued to the string-mirror region is equal to itself. Taking into account that

$$2 \log \frac{1 + \frac{1}{Y_{M|vw}}}{1 + \frac{1}{Y_{M|w}}} \star K_M = 2 \log \frac{1 + \frac{1}{Y_{M|vw}}}{1 + \frac{1}{Y_{M|w}}} \star_{p.v.} K_M - \sum_{k=1}^N \log S_1(u_k - v), \quad (8.89)$$

we see that the equations for  $Y_{\pm}$  can be written as

$$\log \frac{Y_+}{Y_-}(v) = - \sum_{j=1}^N \log S_{1*y}(u_j, v) + \log(1 + Y_Q) \star K_{Qy}, \quad (8.90)$$

$$\log Y_+ Y_-(v) = - \log(1 + Y_Q) \star K_Q + 2 \log \frac{1 + \frac{1}{Y_{M|vw}}}{1 + \frac{1}{Y_{M|w}}} \star_{p.v.} K_M. \quad (8.91)$$

Assuming now that  $Y_Q$ ,  $Y_{M|vw}$  and  $Y_{M|w}$  are real, the reality of  $Y_{\pm}$  just follows from the reality and positivity of  $S_{Q*y}(u, v)$  that can be easily checked by using (8.87).

Next we consider  $Y_{M|vw}$ . By using the p.v. prescription we write (8.70) as follows

$$\begin{aligned} \log Y_{M|vw}(v) &= \frac{1}{2} \sum_{j=1}^N \log \frac{S_{xv}^{1*M}(u_j, v)^2}{S^{1M}(u_j - v)} - \log(1 + Y_{Q'}) \star K_{xv}^{Q'M} \\ &\quad + \log \left( 1 + \frac{1}{Y_{M'|vw}} \right) \star_{p.v.} K_{M'M} + \log \frac{1 - \frac{e^{ih\alpha}}{Y_-}}{1 - \frac{e^{ih\alpha}}{Y_+}} \star K_M, \end{aligned} \quad (8.92)$$

where

$$S_{xv}^{1*M}(u, v) = \frac{x_s(u - \frac{i}{g}) - x(v + \frac{i}{g}M)}{x_s(u + \frac{i}{g}) - x(v - \frac{i}{g}M)} \frac{x_s(u + \frac{i}{g}) - \frac{1}{x(v + \frac{i}{g}M)}}{x_s(u - \frac{i}{g}) - \frac{1}{x(v - \frac{i}{g}M)}}, \quad (8.93)$$

$$S^{1M}(u) = \frac{u - \frac{i}{g}(1 + M)}{u + \frac{i}{g}(1 + M)} \frac{u - \frac{i}{g}(M - 1)}{u + \frac{i}{g}(M - 1)}. \quad (8.94)$$

Thus, we get

$$\begin{aligned} \frac{S_{xv}^{1*M}(u, v)^2}{S^{1M}(u - v)} &= \frac{x_s(u - \frac{i}{g}) - x(v - \frac{i}{g}M)}{x_s(u + \frac{i}{g}) - x(v + \frac{i}{g}M)} \frac{x_s(u + \frac{i}{g}) - \frac{1}{x(v + \frac{i}{g}M)}}{x_s(u - \frac{i}{g}) - \frac{1}{x(v - \frac{i}{g}M)}} \\ &\times \frac{x_s(u - \frac{i}{g}) - x(v + \frac{i}{g}M)}{x_s(u + \frac{i}{g}) - x(v - \frac{i}{g}M)} \frac{x_s(u + \frac{i}{g}) - \frac{1}{x(v - \frac{i}{g}M)}}{x_s(u - \frac{i}{g}) - \frac{1}{x(v + \frac{i}{g}M)}}, \end{aligned} \quad (8.95)$$

which is obviously real and positive, and at  $M = 1$  it has a double zero at  $v = u$  as it should be.

Finally we consider  $Y_Q$ -functions. We write (8.68) as follows

$$\begin{aligned} \log Y_Q(v) &= -L \tilde{\mathcal{E}}_Q - \sum_{j=1}^N \log S_{\mathfrak{sl}(2)}^{1*Q}(u_j, v) S_{vwx}^{1Q}(u_j, v) + \log(1 + Y_{Q'}) \star K_{\mathfrak{sl}(2)}^{Q'Q} \\ &+ \log\left(1 + \frac{1}{Y_{M'vw}}\right) \star_{p.v.} K_{vwx}^{M'Q} + \log\left(1 - \frac{e^{ih_\alpha}}{Y_-}\right) \star K_-^{yQ} + \log\left(1 - \frac{e^{ih_\alpha}}{Y_+}\right) \star K_+^{yQ}. \end{aligned} \quad (8.96)$$

Here

$$\begin{aligned} S_{vwx}^{1Q}(u, v) &= \frac{x(u - \frac{i}{g}) - x(v + \frac{i}{g}Q)}{x(u - \frac{i}{g}) - x(v - \frac{i}{g}Q)} \frac{x(u + \frac{i}{g}) - x(v + \frac{i}{g}Q)}{x(u + \frac{i}{g}) - x(v - \frac{i}{g}Q)} \frac{x(v - \frac{i}{g}Q)}{x(v + \frac{i}{g}Q)} \\ &= \frac{x_s(u - \frac{i}{g}) - x(v + \frac{i}{g}Q)}{x_s(u - \frac{i}{g}) - x(v - \frac{i}{g}Q)} \frac{1 - \frac{1}{x_s(u + \frac{i}{g})x(v + \frac{i}{g}Q)}}{1 - \frac{1}{x_s(u + \frac{i}{g})x(v - \frac{i}{g}Q)}} = h_Q(u, v), \end{aligned} \quad (8.97)$$

where  $h_Q(u, v)$  is the function that appears in the crossing relations, see [66]. Note that  $S_{vwx}^{1Q}$  is unitary:  $(S_{vwx}^{1Q})^* = 1/S_{vwx}^{1Q} = S_{vwx}^{1Q}/h_Q^2$ .

To prove the reality of  $S_{\mathfrak{sl}(2)}^{1*Q}(u, v)S_{vwx}^{1Q}(u, v)$  we use the crossing relation.<sup>11</sup> To this end it is convenient to go to the  $z$ -torus. Then, the crossing relation for  $S_{\mathfrak{sl}(2)}^{1Q}$  can be written in the form [69]

$$S_{\mathfrak{sl}(2)}^{1Q}(z_1 - \frac{\omega_2}{2}, z_2) S_{\mathfrak{sl}(2)}^{1Q}(z_1 + \frac{\omega_2}{2}, z_2) = \frac{1}{h_Q(u, v)^2}, \quad (8.98)$$

<sup>11</sup>It follows closely the proof of the unitarity of the mirror S-matrix [20].

and we have

$$\begin{aligned} \left( S_{\mathfrak{sl}(2)}^{1*Q}(u, v) \right)^* &\equiv \left( S_{\mathfrak{sl}(2)}^{1Q}(z_{*1}, z_2) \right)^* = \left( S_{\mathfrak{sl}(2)}^{1Q}\left(z_1 - \frac{\omega_2}{2}, z_2\right) \right)^* = \frac{1}{S_{\mathfrak{sl}(2)}^{1Q}\left(z_1 + \frac{\omega_2}{2}, z_2\right)} \\ &= S_{\mathfrak{sl}(2)}^{1*Q}(u, v) h_Q(u, v)^2, \end{aligned} \quad (8.99)$$

where we have chosen  $z_k$  to be real, and used the generalized unitarity condition for the mirror model. Taking into account (8.97), one concludes that  $S_{\mathfrak{sl}(2)}^{1*Q} S_{vw}^{1Q}$  is real. It is also possible to show that the product is positive by representing  $S_{\mathfrak{sl}(2)}^{1*Q} S_{vw}^{1Q}$  as a product  $ss^*$ , where  $s \sim \sigma$ .

## 8.8 From canonical to simplified TBA equations

In this appendix we discuss the derivation of the simplified TBA equations from the canonical ones for Konishi-like states in the weak coupling region. Since it basically repeats the one done in [22, 23] we will be sketchy.

### Simplifying the TBA equations for $vw$ -strings

To derive the simplified equation (4.3) for  $vw$ -strings from the canonical one, one applies the inverse kernel  $(K+1)_{MN}^{-1}$  to (8.70) and uses the following identity

$$\log S_{xv}^{1*Q} \star (K+1)_{QM}^{-1} = \delta_{M1} \left( \log S(u-v + \frac{i}{g} - i0) + \log S_{1*y} \star s \right). \quad (8.100)$$

Note also that one should understand  $\log S_{xv}^{1*Q}$  as

$$\log S_{xv}^{1*Q}(u, v) = \log(v-u) + \log \frac{S_{xv}^{1*Q}(u, v)}{v-u}. \quad (8.101)$$

Then, even though the last formula is valid up to a multiple of  $i\pi$ , it agrees exactly with Mathematica choice of branches.

### Simplifying the TBA equations for $Q$ -particles

We want to apply  $(K+1)_{MN}^{-1}$  to the equation (8.68). This requires computing

$$\begin{aligned} \log S_{\mathfrak{sl}(2)}^{1*M} \star (K+1)_{MQ}^{-1} &= -\log S_{1M} \star (K+1)_{MQ}^{-1} - 2 \log \Sigma_{1*M} \star (K+1)_{MQ}^{-1} \\ &= -\delta_{Q2} \log S(u-v) - 2\delta_{Q1} \log \check{\Sigma}_{1*} \check{\star} s. \end{aligned} \quad (8.102)$$

As was shown in [66], the improved dressing factor is a holomorphic function of the first argument in the string region, and the second one in the intersection of the region  $\{|y_2^+| < 1, |y_2^-| > 1\}$  with the mirror region  $\text{Im } y_2^\pm < 0$ , which includes the real momentum line of the mirror theory. This immediately implies that

$$\log \Sigma_{1*M} \star (K+1)_{MQ}^{-1} = 0 \quad \text{for } Q \neq 1. \quad (8.103)$$

To find  $\log \Sigma_{1_*M} \star (K+1)_{M1}^{-1}$ , and, therefore,  $\check{\Sigma}_{1_*}$  we start with the kernel  $K_{1_*M}^\Sigma(u, v)$ , and use the following identity derived in [23]

$$K_{Q'M}^\Sigma \star (K+1)_{MQ}^{-1} = \delta_{1Q} \check{K}_{Q'}^\Sigma \star s, \quad (8.104)$$

where the kernel  $\check{K}_Q^\Sigma(u, v)$  does not vanish for  $|v| < 2$ , and is given by (8.19). Setting  $Q' = 1 = Q$  in (8.104) and analytically continuing the first argument to the string region, one gets

$$K_{1_*M}^\Sigma \star (K+1)_{MQ}^{-1} = \delta_{1Q} \check{K}_{1_*}^\Sigma \star s, \quad (8.105)$$

where the analytically continued kernel  $\check{K}_{1_*}^\Sigma$  is given by (8.23). To obtain eq.(8.23) from (8.19) one uses the following formula

$$\begin{aligned} \check{I}_0\left(u + \frac{i}{g}, v\right) &= \check{I}_1(u, v) - \frac{1}{2\pi i} \frac{1}{u - v - \frac{i}{g}} - \frac{1}{\pi i} \int_{-2}^2 dt \frac{1}{u - t - \frac{i}{g}} \check{K}(t, v) \\ &= \check{I}_1(u, v) - K_{ss}\left(u - \frac{i}{g}, v\right). \end{aligned} \quad (8.106)$$

Finally, integrating (8.23) over the first argument, one gets (8.24).

The remaining part of the derivation of the simplified equations for Q-particles repeats [23].

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