Preprint typeset in JHEP style - HYPER VERSION

ITP-UU-09-09
SPIN-09-09
TCDMATH 09-09
HMI-09-05

Thermodynamic Bethe Ansatz for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Mirror Model

Gleb Arutyunov[a*†] **and Sergey Frolov**[b†]

[a] *Institute for Theoretical Physics and Spinoza Institute, Utrecht University, 3508 TD Utrecht, The Netherlands*
[b] *Hamilton Mathematics Institute and School of Mathematics, Trinity College, Dublin 2, Ireland*

Abstract: We use the string hypothesis for the mirror theory to derive the Thermodynamic Bethe Ansatz equations for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model. We further demonstrate how these equations can be used to construct the associated Y-system recently discussed in the literature, putting particular emphasis on the assumptions and the range of validity of the corresponding construction.

*Email: G.E.Arutyunov@uu.nl, frolovs@maths.tcd.ie
†Correspondent fellow at Steklov Mathematical Institute, Moscow.

Contents

1. Introduction

Recently, there has been further important progress towards understanding the finite-size spectrum of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. The conjectured quantum integrability of the string model plays a primal role in those developments.

In the uniform light-cone gauge the string sigma model is described by a two-dimensional massive integrable quantum field theory defined on a cylinder of circumference equal to the light-cone momentum P_+; the latter can be viewed as the length $L \equiv P_+$ of the string, see the review [1]. In the limit of infinite L the spectrum of the model is known to contain a set of elementary particles which transform in the bi-fundamental representation of the centrally extended superalgebra $\mathfrak{su}(2|2)$. In addition, there are Q-particles, which can be thought of as bound states of Q elementary particles [2]. The Q-particles reside into the tensor product of two $4Q$-dim atypical totally symmetric multiplets of the centrally extended $\mathfrak{su}(2|2)$ algebra. In the infinite L limit the symmetries of the model are powerful enough to determine the matrix structure of both the fundamental S-matrix [3, 4] and the S-matrices of Q-particles [5, 6]. Crossing symmetry puts further constraints [7] on the normalizations of all the S-matrices.

When L is large but finite, the multi-particle states can be approximately described by the wave function of the Bethe-type [8]. Factorizability of the multi-particle scattering together with the periodicity condition for the wave function then implies the quantization conditions for the particle momenta which are encoded into a set of the Bethe-Yang equations. These equations [9] give a proper (asymptotic) description of the string spectrum for L large, but they become less and less accurate when the value of L decreases. The effect of a finite volume manifests itself in the appearance of exponentially small corrections to the particle energies computed via the Bethe-Yang equations. Derivation of the leading corrections by means of Lüscher's approach [10] has been in the focus of recent investigations [11]-[20].

To obtain the finite L spectrum, one could try to generalize the Thermodynamic Bethe Ansatz approach (TBA) originally developed for relativistic integrable models [21]. To this end, one starts with considering the string sigma model on a torus given by the Cartesian product of two circles with circumferences L and R, respectively. In the imaginary time formalism, the circumference of any of these two circles can be regarded as the inverse temperature in a statistical field theory with the Hilbert space of states defined on the complementary circle. Interchanging the role of the two-dimensional space and time, and further taking the limit $R \to \infty$, allows one to relate the ground state energy in the original string model with the free energy (or, depending on the boundary conditions for fermions, with Witten's index) in the so-called *mirror* model; the latter is obtained from the original theory by a double Wick rotation. It has been shown that the TBA approach is also capable of accounting for the excited states [22, 23], see [24]-[32] for further results and different approaches, and also [19] for the recent important development.

If a theory is relativistic, then it coincides with its mirror. The light-cone gauge string sigma model lacks two-dimensional Lorentz invariance and, for this reason, the corresponding mirror model is a new theory that requires a separate exploration. In our previous work [34] the Bethe-Yang equations for elementary particle of the mirror theory have been derived. Also, the bound states of elementary particles have been classified and shown to comprise into the tensor product of two $4Q$-dimensional atypical totally anti-symmetric multiplets of the centrally extended $\mathfrak{su}(2|2)$ algebra. The Bethe-Yang equations for the corresponding Q-particles are easily obtained by fusing the elementary equations[1]. More recently, we noticed [36] that the mirror Bethe-Yang equations involving auxiliary roots can be interpreted as the Lieb-Wu equations [37] for an inhomogeneous Hubbard model [38, 39]. This inhomogeneous Hubbard model becomes homogeneous in the limit of infinite real momenta of the mirror Q-particles[2]. Further, we have argued that the solutions of the Bethe-Yang

[1]These equations also follow from the requirement of Yangian symmetry [40].

[2]The relationship between the equations [9] and the Lieb-Wu equations for the inhomogeneous Hubbard model was observed earlier [35]. However, for the original string model there is no value of the physical, *i.e.* real particle momenta for which the Lieb-Wu equations become homogeneous.

equations contributing in the thermodynamic limit arrange themselves into Bethe string configurations similar to the ones appearing in the Hubbard model and we have obtained the corresponding Takahashi-like equations for the real centers of the string complexes. This constitutes the string hypothesis for the $AdS_5 \times S^5$ mirror theory [36] and offers an unequivocal way to obtain the corresponding TBA equations.

The aim of the present paper is to derive the TBA equations for the $AdS_5 \times S^5$ mirror theory. This will be done by following the well-established routine, *i.e.* by passing to the thermodynamic limit description of the Takahashi equations in terms of the particle-hole densities with subsequent minimization of the free energy. By exploiting the properties of the emerging TBA kernels, we will be able to slightly simplify the initial system of the TBA equations for the particle pseudo-energies. Then we will also attempt to derive the associated Y-system [41]-[44]. Opposite to the infinite set of the coupled TBA equations, the Y-system is a local set of equations but it is obtained at the price of "wiping off" a lot of information from the original TBA system. In some cases this information can be restored by specifying the analytic properties of the Y-system, as has been nicely demonstrated in the recent work [19]. We would like to stress, however, that in our present case the success of deriving the Y-system crucially depends on the analytic properties of the so-called dressing phase [45] which constitutes a part of the TBA kernel for the Q-particles. Although in the original string theory the dressing phase is believed to be known, both in the strong coupling asymptotic expansion [46] and for a finite value of the coupling [47], its analytic properties and the actual expression in the mirror theory are currently terra incognita. To be precise, the dressing phase by [47] is represented by a double series convergent in the region $|x_1^\pm| > 1$ and $|x_2^\pm| > 1$, where $x_{1,2}^\pm$ are kinematic parameters related to the first and the second particle, respectively. This series admits an integral representation found by Dorey, Hofman and Maldacena [48] which is valid in the same region of kinematic parameters. To determine the dressing phase in the mirror region, one has to analytically continue the corresponding integral representation beyond $|x_{1,2}^\pm| > 1$.

The Y-system for the planar AdS/CFT correspondence [49] has been already conjectured [50] based on the experience with the classical discrete Hirota dynamics in the O(4) model. Very recently two independent derivations of the Y-system (and the TBA equations) have been presented [51, 52]. Although we did not attempt to make a detailed comparison of our results with those by [51, 52], a bird's eye survey reveals definite similarities but also certain differences between our findings. Relegating some of our comparative remarks to Conclusions, we would like to stress that in our opinion the question – what is the mirror dressing phase – is the most important one towards an ultimate understanding of the TBA system.

The paper is organized as follows. Section 2 deals with the thermodynamic limit of the Takahashi-like equations for the mirror model. In section 3 we derive the

TBA equations. In section 4 we partially simplify the TBA equations and discuss the construction of the associated Y-system. In the three appendices we summarize the most essential properties of the mirror kinematics, the relations and properties of the TBA kernels, and the simplified system of the TBA equations.

2. Integral equations for densities

In the thermodynamic limit we introduce densities $\rho(u)$ of particles, and densities $\bar{\rho}(u)$ of holes which depend on the real rapidity variable u. We have the following types of densities ($\alpha = 1, 2$)

1. The density $\rho_Q(u)$ of the Q-particles, $-\infty \le u \le \infty$, $Q = 1, \dots, \infty$
2. The density $\rho^{(\alpha)}_{y^-}(u)$ of the y-particles with $\text{Im}(y) < 0$, $-2 \le u \le 2$. The corresponding y-coordinate is expressed in terms of u as $y = x(u)$ where $x(u)$ is defined in (6.1)
3. The density $\rho^{(\alpha)}_{y^+}(u)$ of the y-particles with $\text{Im}(y) > 0$, $-2 \le u \le 2$. The corresponding y-coordinate is expressed in terms of u as $y = \frac{1}{x(u)}$
4. The density $\rho^{(\alpha)}_{M|vw}(u)$ of the $M|vw$-strings, $-\infty \le u \le \infty$, $M = 1, \dots, \infty$
5. The density $\rho^{(\alpha)}_{N|w}(u)$ of the $N|w$-strings, $-\infty \le u \le \infty$, $N = 1, \dots, \infty$,

and the corresponding densities of holes.

Introducing a generalized index i which runs over all the densities, one can represent the system of integral equations arising in the thermodynamic limit in the following compact form

$$\rho_i(u) + \bar{\rho}_i(u) = \frac{R}{2\pi}\frac{d\widetilde{p}_i}{du} + K_{ij} \star \rho_j(u) \,. \tag{2.1}$$

where the momentum $\widetilde{p}_i$ does not vanish only for Q-particles, and is given by (6.3). Here the summation over j is assumed, and the star product is defined by the following composition law

$$K_{ij} \star \rho_j(u) = \int \mathrm{d}u'\, K_{ij}(u, u')\rho_j(u') \,, \tag{2.2}$$

where the integration is taken over the range of u specified above. The explicit form of the kernels $K's$ is discussed below.

The star product (2.2) should be thought of as the left action of the kernels $K's$ on ρ_j. In what follows we will also need the right action which is defined as

$$\rho_j \star K_{ji}(u) = \int \mathrm{d}u'\, \rho_j(u')K_{ji}(u', u) \,. \tag{2.3}$$

Equations for Q-particle densities

To derive the integral equation for Q-particle densities, we rewrite the Bethe-Yang equation (3.1) in [36] for Q-particles in terms of the function $x(u)$ given by (6.1)

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}} S_{\mathfrak{sl}(2)}^{Q_k Q_l}(u_k,u_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{N_{y^-}^{(\alpha)}} \frac{x(u_k - i\frac{Q_k}{g}) - x(u_l^{(\alpha)})}{x(u_k + i\frac{Q_k}{g}) - x(u_l^{(\alpha)})} \sqrt{\frac{x(u_k + i\frac{Q_k}{g})}{x(u_k - i\frac{Q_k}{g})}}$$

$$\times \prod_{\alpha=1}^{2} \prod_{l=1}^{N_{y^+}^{(\alpha)}} \frac{x(u_k - i\frac{Q_k}{g}) - \frac{1}{x(u_l^{(\alpha)})}}{x(u_k + i\frac{Q_k}{g}) - \frac{1}{x(u_l^{(\alpha)})}} \sqrt{\frac{x(u_k + i\frac{Q_k}{g})}{x(u_k - i\frac{Q_k}{g})}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{xv}^{Q_k M}(u_k, v_{l,M}^{(\alpha)})\,, \quad (2.4)$$

Here the $\mathfrak{sl}(2)$ S-matrix $S_{\mathfrak{sl}(2)}^{QQ'}$ in the uniform light-cone gauge [1] with the gauge parameter $a=0$ can be written in the form [53, 54, 5, 16]

$$S_{\mathfrak{sl}(2)}^{QQ'}(u,u') = S^{QQ'}(u-u')^{-1}\,\Sigma_{QQ'}(u,u')^{-2}\,, \quad (2.5)$$

where $S^{QQ'}$ is given by

$$S^{QQ'}(u-u') = \frac{u-u'-\frac{i}{g}(Q+Q')}{u-u'+\frac{i}{g}(Q+Q')}\frac{u-u'-\frac{i}{g}(Q'-Q)}{u-u'+\frac{i}{g}(Q'-Q)} \quad (2.6)$$
$$\times \prod_{j=1}^{Q-1}\left(\frac{u-u'-\frac{i}{g}(Q'-Q+2j)}{u-u'+\frac{i}{g}(Q'-Q+2j)}\right)^2 .$$

Here $\Sigma^{QQ'}(u,u')$ is related to the dressing factor $\sigma_{QQ'}$ as follows

$$\Sigma_{QQ'}(u,u') = \sigma_{QQ'}(u,u') \prod_{j=1}^{Q}\prod_{k=1}^{Q'} \frac{1-\frac{1}{x(u+\frac{i}{g}(Q+2-2j))x(u'+\frac{i}{g}(Q'-2k))}}{1-\frac{1}{x(u+\frac{i}{g}(Q-2j))x(u'+\frac{i}{g}(Q'+2-2k))}}\,. \quad (2.7)$$

Finally, the auxiliary S-matrix is given by

$$S_{xv}^{QM}(u,u') = \frac{x(u-i\frac{Q}{g})-x(u'+i\frac{M}{g})}{x(u+i\frac{Q}{g})-x(u'+i\frac{M}{g})}\,\frac{x(u-i\frac{Q}{g})-x(u'-i\frac{M}{g})}{x(u+i\frac{Q}{g})-x(u'-i\frac{M}{g})}\,\frac{x(u+i\frac{Q}{g})}{x(u-i\frac{Q}{g})}$$
$$\times \prod_{j=1}^{M-1}\frac{u-u'-\frac{i}{g}(Q-M+2j)}{u-u'+\frac{i}{g}(Q-M+2j)}\,. \quad (2.8)$$

Taking the logarithmic derivative of (2.4) with respect to u_k, we get in the thermodynamic limit the following integral equation for the densities of Q-particles and holes

$$\rho_Q(u)+\bar\rho_Q(u) = \frac{R}{2\pi}\frac{d\widetilde{p}^Q(u)}{du} + \sum_{Q'=1}^{\infty} K_{\mathfrak{sl}(2)}^{QQ'}\star\rho_{Q'} \quad (2.9)$$
$$+\sum_{\alpha=1}^{2}\left[K_-^{Qy}\star\rho_{y^-}^{(\alpha)} + K_+^{Qy}\star\rho_{y^+}^{(\alpha)} + \sum_{M'=1}^{\infty} K_{xv}^{QM'}\star\rho_{M'|vw}^{(\alpha)}\right].$$

Here the kernels $K's$ are

$$K_{\mathfrak{sl}(2)}^{QQ'}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log S_{\mathfrak{sl}(2)}^{QQ'}(u,u')\,, \tag{2.10}$$

$$K_{-}^{Qy}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log\frac{x(u-i\frac{Q}{g})-x(u')}{x(u+i\frac{Q}{g})-x(u')}\sqrt{\frac{x(u+i\frac{Q}{g})}{x(u-i\frac{Q}{g})}}\,, \tag{2.11}$$

$$K_{+}^{Qy}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log\frac{x(u-i\frac{Q}{g})-\frac{1}{x(u')}}{x(u+i\frac{Q}{g})-\frac{1}{x(u')}}\sqrt{\frac{x(u+i\frac{Q}{g})}{x(u-i\frac{Q}{g})}}\,, \tag{2.12}$$

$$K_{xv}^{QM'}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log S_{xv}^{QM'}(u,u')\,, \tag{2.13}$$

where the operation $\star$ is defined in (2.2). Some of the kernels can be expressed in terms of the basic kernels $K_M(u)$ and $K(u,v)$, see appendix 6.2, where also many important properties of the kernels are listed.

Equations for y-particles densities with $\mathrm{Im}(y)<0$

Next, we take a $y^{(\alpha)}$-particle with the root $y_k^{(\alpha)}=x(u_k^{(\alpha)})$ and rewrite the equation (3.6) in [36] in the following form

$$-1=\prod_{l=1}^{K^{\rm I}}\frac{x(u_k^{(\alpha)})-x_l^-}{x(u_k^{(\alpha)})-x_l^+}\sqrt{\frac{x_l^+}{x_l^-}}\prod_{M=1}^{\infty}\prod_{l=1}^{N_{M|vw}^{(\alpha)}}\frac{u_k^{(\alpha)}-v_{l,M}^{(\alpha)}-i\frac{M}{g}}{u_k^{(\alpha)}-v_{l,M}^{(\alpha)}+i\frac{M}{g}}\prod_{l=1}^{N_{N|w}^{(\alpha)}}\frac{u_k^{(\alpha)}-w_{l,M}^{(\alpha)}-i\frac{M}{g}}{u_k^{(\alpha)}-w_{l,M}^{(\alpha)}+i\frac{M}{g}}\,, \tag{2.14}$$

where $x_l^{\pm}=x(u_{l,Q}\pm i\frac{Q}{g})$ for a Q-particle with the real rapidity $u_{l,Q}$.

Taking the logarithmic derivative of (2.14) with respect to $u_k^{(\alpha)}$, we get in the thermodynamic limit the following integral equation for the densities of y^--particles and holes

$$\rho_{y^-}^{(\alpha)}(u)+\bar{\rho}_{y^-}^{(\alpha)}(u)=\sum_{M'=1}^{\infty}\left[K_-^{yM'}\star\rho_{M'}-K_{vw}^{yM'}\star\left(\rho_{M'|vw}^{(\alpha)}+\rho_{M'|w}^{(\alpha)}\right)\right]\,. \tag{2.15}$$

Here the kernels $K's$ are

$$K_{-}^{yM'}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log\frac{x(u)-x(u'+i\frac{M'}{g})}{x(u)-x(u'-i\frac{M'}{g})}\sqrt{\frac{x(u'-i\frac{M'}{g})}{x(u'+i\frac{M'}{g})}}\,, \tag{2.16}$$

$$K_{vw}^{yM'}(u,u') = K_{M'}(u-u') = \frac{1}{2\pi i}\frac{d}{du}\log\frac{u-u'-i\frac{M'}{g}}{u-u'+i\frac{M'}{g}}\,. \tag{2.17}$$

Equations for y-particle densities with $\mathrm{Im}(y)>0$

In the second case $\mathrm{Im}(y)>0$ the root $y_k^{(\alpha)}=1/x(u_k^{(\alpha)})$, and we get in the thermodynamic limit the following integral equation for the densities of y^+-particles and holes

$$\rho^{(\alpha)}_{y^+}(u) + \bar{\rho}^{(\alpha)}_{y^+}(u) = \sum_{M'=1}^{\infty} \left[K_+^{yM'} \star \rho_{M'} + K_{M'} \star \left(\rho^{(\alpha)}_{M'|vw} + \rho^{(\alpha)}_{M'|w} \right) \right] . \qquad (2.18)$$

Here the kernel $K^+_{yM'}$ is

$$K_+^{yM'}(u,u') = \frac{1}{2\pi i}\frac{d}{du} \log \frac{\frac{1}{x(u)} - x(u' - i\frac{M'}{g})}{\frac{1}{x(u)} - x(u' + i\frac{M'}{g})} \sqrt{\frac{x(u' + i\frac{M'}{g})}{x(u' - i\frac{M'}{g})}} \, . \qquad (2.19)$$

Equations for vw-string densities

Then, we take a $M|vw$-string with the coordinates $u^{(\alpha)}_{k,M}$, and rewrite the equation (3.13) in [36] in the following form

$$(-1)^M = \prod_{l=1}^{K^{\rm I}} S^{Q_l M}_{xv}(u_l, u^{(\alpha)}_{k,M}) \prod_{l=1}^{N_y^{(\alpha)}} \frac{u^{(\alpha)}_{k,M} - v^{(\alpha)}_l - i\frac{M}{g}}{u^{(\alpha)}_{k,M} - v^{(\alpha)}_l + i\frac{M}{g}} \prod_{M'=1}^{\infty} \prod_{l=1}^{N^{(\alpha)}_{M'|vw}} S^{MM'}(u^{(\alpha)}_{k,K} - v^{(\alpha)}_{l,M'}) \, , \qquad (2.20)$$

where the auxiliary S-matrices are given by (2.6) and (2.8).

Taking the logarithmic derivative of (2.20) with respect to u_k, we get in the thermodynamic limit the following integral equation for the densities of vw-strings and holes

$$\rho^{(\alpha)}_{M|vw}(u) + \bar{\rho}^{(\alpha)}_{M|vw}(u) = \sum_{M'=1}^{\infty} \left[K^{MM'}_{vwx} \star \rho_{M'} - K^{MM'}_{vv} \star \rho^{(\alpha)}_{M'|vw} \right] \qquad (2.21)$$

$$- K^{My}_{wv} \star \left(\rho^{(\alpha)}_{y^-} + \rho^{(\alpha)}_{y^+} \right) .$$

Here the kernels $K's$ are

$$K^{MM'}_{vwx}(u,u') = -\frac{1}{2\pi i}\frac{d}{du} \log S^{M'M}_{xv}(u',u) \, , \qquad (2.22)$$

$$K^{MM'}_{vv}(u,u') = K_{MM'}(u-u') = \frac{1}{2\pi i}\frac{d}{du} \log S^{MM'}(u-u') \, , \qquad (2.23)$$

$$K^{My}_{wv}(u,u') = K_M(u-u') \, , \qquad (2.24)$$

and they all are positive.

Equations for w-string densities

Finally we take a $M|w$-string with the coordinates $u^{(\alpha)}_{k,M}$, and rewrite the equation (3.9) in [36] in the following form

$$(-1)^M = \prod_{l=1}^{N_y^{(\alpha)}} \frac{u^{(\alpha)}_{k,M} - v^{(\alpha)}_l + i\frac{M}{g}}{u^{(\alpha)}_{k,M} - v^{(\alpha)}_l - i\frac{M}{g}} \prod_{N=1}^{\infty} \prod_{l=1}^{N^{(\alpha)}_{N|w}} S^{KN}(u^{(\alpha)}_{k,M} - w^{(\alpha)}_{l,N}) \, . \qquad (2.25)$$

Taking the logarithmic derivative of (2.25) with respect to $u^{(\alpha)}_{k,M}$, we get in the thermodynamic limit the following integral equation for the densities of w-strings and holes

$$\rho^{(\alpha)}_{M|w}(u) + \bar{\rho}^{(\alpha)}_{M|w}(u) = K_M \star \left(\rho^{(\alpha)}_{y^-} + \rho^{(\alpha)}_{y^+}\right) - \sum_{M'=1}^{\infty} K_{MM'} \star \rho^{(\alpha)}_{M'|w} \,. \tag{2.26}$$

3. Free energy and equations for pseudo-energies

Having found the equations for the densities of particles and holes, we can proceed with deriving the integral equations delivering the minimum of the free energy per unit length for the mirror theory at temperature $T = \frac{1}{L}$. The free energy in the mirror theory determines the ground state energy of the light-cone $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory defined on the cylinder with the circumference L equal the light-cone momentum P_+. In the case we are mostly interested in $L = J$, where J is the angular momentum carried by the string rotating about the equator of S^5.

To be precise, the light-cone string theory has two different sectors. The first sector contains even winding number string states and it has fermions subject to periodic boundary conditions. This is the sector which has a BPS ground state whose energy should not receive any quantum corrections. Since the fermions are periodic, the ground state energy in fact is determined not by the free energy but by Witten's index of the mirror theory. The second sector has anti-periodic fermions and has a non-BPS ground state whose energy is determined by the mirror free energy. To describe both sectors in one go, we consider a generalized free energy [55] defined by the following equation

$$\mathcal{F}_\gamma(L) = \mathcal{E} - \frac{1}{L} S + \frac{i\gamma}{L}(N_F^{(1)} - N_F^{(2)}) \,, \tag{3.1}$$

where $\mathcal{E}$ is the energy per unit length carried by Q-particle densities

$$\mathcal{E} = \int \mathrm{d}u \sum_{Q=1}^{\infty} \widetilde{\mathcal{E}}^Q(u) \rho_Q(u) \,. \tag{3.2}$$

Here $\widetilde{\mathcal{E}}^Q(u)$ is a Q-particle energy defined in (6.4), and S is the total entropy,

$$S = \int \mathrm{d}u \left[\sum_{Q=1}^{\infty} \mathfrak{s}\left(\rho_Q\right) + \sum_{\alpha=1}^{2} \left(\mathfrak{s}\left(\rho^{(\alpha)}_{y^-}\right) + \mathfrak{s}\left(\rho^{(\alpha)}_{y^+}\right) + \sum_{M=1}^{\infty} \left(\mathfrak{s}\big(\rho^{(\alpha)}_{M|vw}\big) + \mathfrak{s}\big(\rho^{(\alpha)}_{M|w}\big) \right) \right) \right] \,,$$

where $\mathfrak{s}(\rho)$ denotes the entropy function of densities of particles and holes

$$\mathfrak{s}(\rho) = \rho \log\left(1 + \frac{\bar{\rho}}{\rho}\right) + \bar{\rho} \log\left(1 + \frac{\rho}{\bar{\rho}}\right) \,. \tag{3.3}$$

Then, $1/L$ is the temperature of the mirror theory, $i\gamma/L$ plays the role of a chemical potential, and $N_F^{(\alpha)}$ is the fermion number which counts the number of $y^{(\alpha)}$-particles which are the only fermions in our system

$$N_F^{(1)} - N_F^{(2)} = \int \mathrm{d}u\, (\rho^{(1)}_{y^-}(u) + \rho^{(1)}_{y^+}(u) - \rho^{(2)}_{y^-}(u) - \rho^{(2)}_{y^+}(u))\,. \tag{3.4}$$

The relative minus sign between $N_F^{(1)}$ and $N_F^{(2)}$ is needed for the reality of the free energy, for relativistic examples see [55]. In principle, the choice of "+" or "−" sign should not matter at $\gamma = \pi$ where (3.1) becomes Witten's index. If $\gamma = 0$ we get the usual free energy.

3.1 Derivation of the equations

Thus, we need to minimize the free energy at temperature $T = 1/L$ defined by the following equation

$$\mathcal{F}_\gamma(L) = \int \mathrm{d}u \left[\sum_{Q=1}^{\infty} \widetilde{\mathcal{E}}^Q(u)\rho_Q(u) - \frac{i\gamma}{L}\sum_{\alpha=1}^{2}(-1)^\alpha(\rho^{(\alpha)}_{y^-}(u) + \rho^{(\alpha)}_{y^+}(u)) - \frac{S}{L}\right]. \tag{3.5}$$

The consideration is standard and general. We first write the free energy in compact form as follows

$$\mathcal{F}_\gamma(L) = \int \mathrm{d}u \sum_k \left[\widetilde{\mathcal{E}}_k\, \rho_k - \frac{i\gamma_k}{L}\,\rho_k - \frac{1}{L}\mathfrak{s}(\rho_k)\right], \tag{3.6}$$

where $\widetilde{\mathcal{E}}_k$ and γ_k do not vanish only for Q- and y-particles, respectively.

Since the densities of particles and holes satisfy the Bethe equations (2.1), their variations are not independent but are subject to

$$\delta\rho_k(u) + \delta\bar{\rho}_k(u) = K_{kj} \star \delta\rho_j\,. \tag{3.7}$$

Here and in what follows the summation over the repeated indices is assumed. Expressing the variations of the densities of holes in terms of the variations of the densities of particles, one finds the variation of the entropy function

$$\delta\mathfrak{s}(\rho_k) = (\epsilon_k - i\gamma_k)\delta\rho_i + \log\left(1 + e^{i\gamma_k - \epsilon_k}\right) K_{kj} \star \delta\rho_j\,, \tag{3.8}$$

where the pseudo-energies ϵ_k are defined through

$$e^{i\gamma_k - \epsilon_k} = \frac{\rho_k}{\bar{\rho}_k}\,. \tag{3.9}$$

Then, using the extremum condition $\delta\mathcal{F}_\gamma(L) = 0$, one derives the following set of TBA equations

$$\epsilon_k = L\,\widetilde{\mathcal{E}}_k - \log\left(1 + e^{i\gamma_j - \epsilon_j}\right) \star K_{jk}\,, \tag{3.10}$$

where the right action of the kernels K_{jk} defined by (2.3) is used.

Note also that $\epsilon_{y^\pm}$ is defined only for $|u|\le 2$. In principle, one could extend it to $|u|>2$ by saying that $e^{-\epsilon_{y^\pm}}=0$ for $|u|>2$. It might, however, lead to discontinuities of the y-particles pseudo-energies.

Taking into account that the entropy function can be written in the form

$$\mathfrak{s}(\rho_k) = \frac{R}{2\pi}\frac{d\widetilde{p}_k}{du}\log\left(1+e^{i\gamma_k-\epsilon_k}\right) + (\epsilon_k - i\gamma_k)\rho_k + \log\left(1+e^{i\gamma_k-\epsilon_k}\right)K_{kj}\star\rho_j\,, \quad (3.11)$$

one finds that at the extremum (3.10) the free energy is equal to

$$\mathcal{F}_\gamma(L) = -\frac{R}{L}\int \mathrm{d}u\sum_k \frac{1}{2\pi}\frac{d\widetilde{p}_k}{du}\log\left(1+e^{i\gamma_k-\epsilon_k}\right)\,. \quad (3.12)$$

Finally, one uses that the energy of the ground state of the light-cone string theory is related to the free energy of the mirror model as

$$E_\gamma(L) = \lim_{R\to\infty}\frac{L}{R}\mathcal{F}_\gamma(L)\,, \quad (3.13)$$

and gets the following expression

$$E_\gamma(L) = -\int \mathrm{d}u\sum_{Q=1}^{\infty}\frac{1}{2\pi}\frac{d\widetilde{p}^Q}{du}\log\left(1+e^{-\epsilon_Q}\right)\,. \quad (3.14)$$

We see that for $\gamma=\pi$ the necessary condition for the ground state energy to vanish for any L is

$$e^{-\epsilon_Q}=0 \quad \text{for any} \quad Q\,. \quad (3.15)$$

It also imposes restrictions on pseudo-energies of other string configurations.

3.2 TBA equations explicitly

Here we list all TBA equations (3.10) explicitly taking into account that

$$\gamma_Q = \gamma^{(\alpha)}_{M|vw} = \gamma^{(\alpha)}_{M|w} = 0\,, \quad \gamma^{(\alpha)}_{y^\pm} = (-1)^\alpha\pi + h_\alpha\,, \quad h_\alpha = (-1)^\alpha h\,, \quad (3.16)$$

and we used this representation for $\gamma^{(\alpha)}_{y^\pm}$ to handle more efficiently the physically more interesting case with $\gamma=\pi$.

Assuming summation over repeated indices and the index α in the equation for Q-particles, the TBA equations for the pseudo-energies take the following form

- Q-particles

$$\epsilon_Q = L\,\widetilde{\mathcal{E}}_Q - \log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'Q}_{\mathfrak{sl}(2)} - \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K^{M'Q}_{vwx} \quad (3.17)$$
$$- \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^-}}\right)\star K^{yQ}_{-} - \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^+}}\right)\star K^{yQ}_{+}\,.$$

- y-particles

$$\epsilon^{(\alpha)}_{y^\pm} = -\log\left(1+e^{-\epsilon_Q}\right)\star K^{Qy}_\pm + \log\frac{1+e^{-\epsilon^{(\alpha)}_{M|vw}}}{1+e^{-\epsilon^{(\alpha)}_{M|w}}}\star K_M\,. \tag{3.18}$$

- $M|vw$-strings

$$\begin{aligned}\epsilon^{(\alpha)}_{M|vw} &= -\log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'M}_{xv} \\ &\quad + \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^-}}}\star K_M\,.\end{aligned} \tag{3.19}$$

- $M|w$-strings

$$\epsilon^{(\alpha)}_{M|w} = \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|w}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^-}}}\star K_M\,. \tag{3.20}$$

We see that (3.15) solves these equations for $h=0$ if the pseudo-energies of y-particles satisfy $e^{-\epsilon_y}=1$. A proper way to analyze these system for $h=0$ is to consider the perturbation theory in small h. A rough estimate seems to show that $\epsilon^{(\alpha)}_{y^-}\sim h$ and $e^{-\epsilon_Q}\sim h^2$. We postpone detailed analysis for future.

4. Simplifying the TBA equations

In this section we simplify the system of TBA equations (3.17-3.20) by reducing most of the equations to a local form. The local form can be also readily used to derive equations for Y-functions which coincide with (or are inverse to) exponentials of pseudo-energies. We will see that the recently conjecture Y-system [52] holds only for values of the spectral parameter u satisfying the inequality $|u|<2$. For other values of u the TBA equations for Q-particles cannot be apparently reduced to the Y-type equations.

We introduce the Y-functions as

$$Y_Q = e^{-\epsilon_Q}\,,\quad Y^{(\alpha)}_{M|vw} = e^{\epsilon^{(\alpha)}_{M|vw}}\,,\quad Y^{(\alpha)}_{M|w} = e^{\epsilon^{(\alpha)}_{M|w}}\,,\quad Y^{(\alpha)}_\pm = e^{\epsilon^{(\alpha)}_{y^\pm}}\,, \tag{4.1}$$

and use the following universal kernel

$$(K+1)^{-1}_{MN} = \delta_{MN} - s\left(\delta_{M+1,N}+\delta_{M-1,N}\right)\,,\quad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}}\,, \tag{4.2}$$

that is inverse to the kernel $K_{NQ}+\delta_{NQ}$

$$\sum_{N=1}^{\infty}(K+1)^{-1}_{MN}\left(K_{NQ}+\delta_{NQ}\right) = \delta_{MQ}\,. \tag{4.3}$$

For more properties of the inverse kernel see appendix 6.2.

TBA and Y-equations for w-strings

We begin our consideration with the simplest case of w-strings. We apply the inverse kernel (4.2) to (3.20), and get the following equation

$$\log Y_{M|w}^{(\alpha)} = I_{MN} \log(1+Y_{N|w}^{(\alpha)}) \star s + \delta_{M1} \, \log \frac{1-\frac{e^{ih\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih\alpha}}{Y_+^{(\alpha)}}} \star s \,, \tag{4.4}$$

where I_{MN} is the incidence matrix

$$I_{MN} = \delta_{M+1,N} + \delta_{M-1,N} \,, \tag{4.5}$$

and we used the following identity

$$\sum_{N=1}^{\infty} (K+1)_{MN}^{-1} \; K_N = s\, \delta_{M1} \,. \tag{4.6}$$

Since the functions $Y_\pm^\alpha$ are defined on the interval $-2 < u < 2$, the integral in the last term of eq.(4.4) is taken from -2 to 2.

To derive the Y-equations for w-strings, it is convenient to define the operator s^{-1} that acts on functions of the rapidity variable u as follows

$$(f \star s^{-1})(u) = \lim_{\epsilon\to 0^+} \left[f(u+\frac{i}{g}-i\epsilon) + f(u-\frac{i}{g}+i\epsilon) \right] . \tag{4.7}$$

It satisfies the obvious identity

$$(s\star s^{-1})(u) = \delta(u) \,.$$

The operator s^{-1} has however a large null space, and as a result in general

$$f \star s^{-1} \star s \neq f \,.$$

This also means that one may loose information by acting by the operator s^{-1} on an equation. We will see examples of such a loss in what follows.

By applying the s^{-1} operator to both sides of the equation one immediately gets the following Y-equations for w-strings

$$Y_{M|w}^{(\alpha)+}\, Y_{M|w}^{(\alpha)-} = \left(1+Y_{M-1|w}^{(\alpha)}\right) \left(1+Y_{M+1|w}^{(\alpha)}\right) \quad \text{if} \quad M\geq 2 \,, \tag{4.8}$$

$$Y_{1|w}^{(\alpha)+}\, Y_{1|w}^{(\alpha)-} = \left(1+Y_{2|w}^{(\alpha)}\right) \frac{1-\frac{e^{ih\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih\alpha}}{Y_+^{(\alpha)}}} \,, \qquad |u|\leq 2 \,, \tag{4.9}$$

$$Y_{1|w}^{(\alpha)+}\, Y_{1|w}^{(\alpha)-} = 1+Y_{2|w}^{(\alpha)} \,, \qquad |u| > 2 \,, \tag{4.10}$$

where we introduce the notation $Y^{(\alpha)\pm}_{M|w}(u) \equiv Y^{(\alpha)}_{M|w}(u \pm \frac{i}{g} \mp i0)$.

These formulae show that the form of the Y-equations for $M = 1$ is not uniform with respect to the parameter u. The reason behind is that $Y^{(\alpha)}_{\pm}$ are supported on the interval $(-2, 2)$. Only eqs.(4.8) and (4.9) have appeared in [50] and they were assumed to hold for all values of u. Obviously, the uniform expression could be achieved provided $Y^{(\alpha)}_{\pm}$ admit such an analytic continuation to the complex u-plane that $Y^{(\alpha)}_{+}(u) = Y^{(\alpha)}_{-}(u)$ for $|u| > 2$. This continuation should be, however, compatible with the whole set of TBA system. Currently it is unclear if this is indeed the case.

TBA and Y-equations for y-particles

Next we consider w-strings. Equations (3.18) for the pseudo-energies of y-particles can be written in the form

$$\log Y^{(\alpha)}_{-} = -\frac{1}{2} \log\left(1 + Y_Q\right) \star \left(K_{Qy} + K_Q\right) + \log \frac{1 + \frac{1}{Y^{(\alpha)}_{M|vw}}}{1 + \frac{1}{Y^{(\alpha)}_{M|w}}} \star K_M \,, \qquad (4.11)$$

$$\log Y^{(\alpha)}_{+} = \frac{1}{2} \log\left(1 + Y_Q\right) \star \left(K_{Qy} - K_Q\right) + \log \frac{1 + \frac{1}{Y^{(\alpha)}_{M|vw}}}{1 + \frac{1}{Y^{(\alpha)}_{M|w}}} \star K_M \,, \qquad (4.12)$$

where we used that the kernels $K^{Qy}_{\pm}$ can be expressed in terms of K_Q and the kernel K_{Qy} defined in (6.9).

By adding and subtracting, these equations can be cast in the form

$$\log \frac{Y^{(\alpha)}_{+}}{Y^{(\alpha)}_{-}} = \log(1 + Y_Q) \star K_{Qy} \,, \qquad (4.13)$$

$$\log Y^{(\alpha)}_{+} Y^{(\alpha)}_{-} = -\log\left(1 + Y_Q\right) \star K_Q + 2 \log \frac{1 + \frac{1}{Y^{(\alpha)}_{M|vw}}}{1 + \frac{1}{Y^{(\alpha)}_{M|w}}} \star K_M \,. \qquad (4.14)$$

The last term in (4.14) can be replaced by (4.26) or (4.27) reducing all non-local terms to the ones depending on Y-functions of Q-particles only.

To derive Y-equations for y-particles, we need the following identities

$$(K_{Qy} + K_Q) \star s^{-1} = 2K^{Q1}_{xv} + 2\delta_{Q1} \,, \quad K_M \star s^{-1} = K_{M1} + \delta_{M1} \,. \qquad (4.15)$$

We get with their help

$$\log Y^{(\alpha)}_{-} \star s^{-1} = -\log\left(1 + Y_1\right) - \log\left(1 + Y_Q\right) \star K^{Q1}_{xv} + \log \frac{1 + \frac{1}{Y^{(\alpha)}_{1|vw}}}{1 + \frac{1}{Y^{(\alpha)}_{1|w}}} + \log \frac{1 + \frac{1}{Y^{(\alpha)}_{M|vw}}}{1 + \frac{1}{Y^{(\alpha)}_{M|w}}} \star K_{M1} \,. \qquad (4.16)$$

Now we subtract (3.20) from (3.19) for $M=1$, and obtain

$$\log\frac{Y_{1|vw}^{(\alpha)}}{Y_{1|w}^{(\alpha)}} = -\log\left(1+Y_Q\right)\star K_{xv}^{Q1} + \log\frac{1+\frac{1}{Y_{M|vw}^{(\alpha)}}}{1+\frac{1}{Y_{M|w}^{(\alpha)}}}\star K_{M1}\,. \tag{4.17}$$

Finally, subtracting (4.17) from (4.16), we derive the following Y-equation for y_--particles

$$Y_-^{(\alpha)+}\,Y_-^{(\alpha)-} = \frac{1+Y_{1|vw}^{(\alpha)}}{1+Y_{1|w}^{(\alpha)}}\frac{1}{1+Y_1}\,. \tag{4.18}$$

Repeating the same procedure with $Y_+^{(\alpha)}$, we get

$$\begin{aligned}\log Y_+^{(\alpha)+}\,Y_+^{(\alpha)-} &= \log\left(1+Y_Q\right)\star\left(2K_{xv}^{Q1}-K_{Q1}\right) + \log\frac{1+Y_{1|vw}^{(\alpha)}}{1+Y_{1|w}^{(\alpha)}} \\ &= 2\log\left(1+Y_Q\right)\star\mathcal{K}_2^{Q1} + \log\frac{1+Y_{1|vw}^{(\alpha)}}{1+Y_{1|w}^{(\alpha)}}\,,\end{aligned} \tag{4.19}$$

where the kernel $\mathcal{K}_2^{QQ'}$ is defined in (6.22). This equation cannot be reduced to the usual local Y-system form.

We stress again that the equations for $Y_\pm^{(\alpha)}$ are valid for u being in the interval $(-2,2)$ and the analytic continuation for $|u|>2$, we have discussed after eq.(4.10), would impose a consistency condition on the functions Y_Q due to eq.(4.13).

TBA and Y-equations for vw-strings

Now we can discuss vw-strings. We apply the inverse kernel (4.2) to (3.19), use an identity

$$K_{xv}^{QQ''}\star(K+1)^{-1}_{Q''Q'} = \delta_{Q-1,Q'}s + \delta_{Q'1}K_{Qy}\star s\,, \tag{4.20}$$

and, as a result, obtain the following equation

$$\begin{aligned}\log Y_{M|vw}^{(\alpha)} = {} & I_{MN}\log(1+Y_{N|vw}^{(\alpha)})\star s + \delta_{M1}\,\log\frac{1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}}\star s \\ & -\log(1+Y_{M+1})\star s - \delta_{M1}\log(1+Y_Q)\star K_{Qy}\star s\,,\end{aligned} \tag{4.21}$$

where in $K_{Qy}\star s$ we integrate over $[-2,2]$. By using (4.13), we can rewrite (4.21) in the final local form

$$\log Y_{M|vw}^{(\alpha)} = \Big(I_{MN}\log(1+Y_{N|vw}^{(\alpha)}) + \delta_{M1}\log\frac{1-e^{-ih_\alpha}Y_-^{(\alpha)}}{1-e^{-ih_\alpha}Y_+^{(\alpha)}} - \log(1+Y_{M+1})\Big)\star s\,, \tag{4.22}$$

Now applying the s^{-1} operator to both sides of the equation, one gets the following Y-equations for vw-strings

$$Y^{(\alpha)+}_{M|vw}\,Y^{(\alpha)-}_{M|vw} = \left(1+Y^{(\alpha)}_{M-1|vw}\right)\left(1+Y^{(\alpha)}_{M+1|vw}\right)\frac{1}{1+Y_{M+1}} \quad \text{if} \quad M\geq 2\,, \tag{4.23}$$

$$Y^{(\alpha)+}_{1|vw}\,Y^{(\alpha)-}_{1|vw} = \frac{1+Y^{(\alpha)}_{2|vw}}{1+Y_2}\,\frac{1-e^{-ih_\alpha}Y^{(\alpha)}_-}{1-e^{-ih_\alpha}Y^{(\alpha)}_+}\,, \qquad |u|\leq 2\,, \tag{4.24}$$

$$Y^{(\alpha)+}_{1|vw}\,Y^{(\alpha)-}_{1|vw} = \frac{1+Y^{(\alpha)}_{2|vw}}{1+Y_2}\,, \qquad |u|> 2\,. \tag{4.25}$$

Finally, subtracting (4.4) from (4.21), we derive the following equation

$$\begin{aligned}\log\frac{1+\frac{1}{Y^{(\alpha)}_{M|vw}}}{1+\frac{1}{Y^{(\alpha)}_{M|w}}}\star K_M &= \log\frac{1+Y^{(\alpha)}_{1|vw}}{1+Y^{(\alpha)}_{1|w}}\star s \\ &+\log(1+Y_{M+1})\star s\star K_M+\log(1+Y_M)\star K_{My}\star s\star K_1\,,\end{aligned} \tag{4.26}$$

and by using (4.13), one can rewrite (4.26) in the form

$$\begin{aligned}\log\frac{1+\frac{1}{Y^{(\alpha)}_{M|vw}}}{1+\frac{1}{Y^{(\alpha)}_{M|w}}}\star K_M &= \log\frac{1+Y^{(\alpha)}_{1|vw}}{1+Y^{(\alpha)}_{1|w}}\star s \\ &+\log(1+Y_{M+1})\star s\star K_M+\log\frac{Y^{(\alpha)}_+}{Y^{(\alpha)}_-}\star s\star K_1\,.\end{aligned} \tag{4.27}$$

These equations can be used to simplify eq.(4.14) for y-particles.

TBA equations for Q-particles

At last, we discuss the most complicated case of Q-particles. Our analysis is only partially complete here because of a lack of understanding of the properties of the dressing kernel in the mirror model. Still, we will be able to demonstrate that the transition from the TBA equations to Y-equations is only possible for u taking values in the interval from -2 to 2.

Equations for the pseudo-energies of Q-particles can be written in the form

$$\begin{aligned}-\log Y_Q &= L\,\widetilde{\mathcal{E}}_Q+\log\left(1+Y_{Q'}\right)\star\left(K_{Q'Q}+2K^{\Sigma}_{Q'Q}\right)-\log\left(1+\frac{1}{Y^{(\alpha)}_{M|vw}}\right)\star K^{MQ}_{vwx} \\ &-\frac{1}{2}\log\frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}}\star K_Q-\frac{1}{2}\log\left(1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}\right)\left(1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}\right)\star K_{yQ}\,,\end{aligned} \tag{4.28}$$

where the summation over α is assumed, and we use (2.5), (2.10) and (2.23) to represent the Q-particles kernel in the following form

$$K^{QQ'}_{\mathfrak{sl}(2)}(u,u') = -K_{QQ'}(u-u')-2K^{\Sigma}_{QQ'}(u,u')\,, \tag{4.29}$$

where we introduce the dressing phase kernel

$$K^{\Sigma}_{QQ'}(u,u') = \frac{1}{2\pi i}\frac{d}{du}\log\Sigma_{QQ'}(u,u')\,. \tag{4.30}$$

We now should apply the inverse kernel (4.2) to (4.28). To this end, in addition to formula (4.6), we need the following formulae

$$\widetilde{\mathcal{E}}_{Q'}\star(K+1)^{-1}_{Q'Q} = \delta_{Q1}(\widetilde{\mathcal{E}}_1-\widetilde{\mathcal{E}}_2\star s) = \delta_{Q1}\,\check{\mathcal{E}}\star s\,. \tag{4.31}$$

Here

$$\begin{aligned}(\check{\mathcal{E}}\star s)(v) &= \int_{-\infty}^{\infty}\mathrm{d}u'\,\check{\mathcal{E}}(u')s(u'-v) \qquad (4.32)\\ &= 2\Big(\int_{-\infty}^{-2}-\int_{2}^{\infty}\Big)\mathrm{d}u'\,\log\left(\frac{1}{2}\left|u'-\sqrt{u'^2-4}\right|\right)s(u'-v)\,,\end{aligned}$$

where we introduce the function

$$\check{\mathcal{E}}(u) = 2\log\left(\frac{1}{2}\left|u-\sqrt{u^2-4}\right|\right)\left(\theta(-u-2)-\theta(u-2)\right)\,, \tag{4.33}$$

and $\theta(u)$ is the standard unit step function.

Next, we find the following action of the inverse kernel on $K_{yQ'}$

$$\begin{aligned}K_{yQ'}\star(K+1)^{-1}_{Q'Q} &= \delta_{Q1}(K_{y1}-K_{y2}\star s) = \delta_{Q1}(s+2\check{K}\star s)\,, \qquad (4.34)\\ (\check{K}\star s)(u,v) &= \int_{-\infty}^{\infty}\mathrm{d}u'\,\check{K}(u,u')s(u'-v) = \Big(\int_{-\infty}^{-2}-\int_{2}^{\infty}\Big)\mathrm{d}u'\,\bar{K}(u,u')s(u'-v)\\ &= \pm\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\,K(u,u'\pm i\epsilon)s(u'-v)\,,\end{aligned}$$

where we use the kernels

$$\check{K}(u,v) = \bar{K}(u,v)\left(\theta(-v-2)-\theta(v-2)\right)\,,\quad \bar{K}(u,v) = \frac{1}{2\pi}\frac{\sqrt{v^2-4}}{\sqrt{4-u^2}}\frac{1}{u-v}\,. \tag{4.35}$$

We also need similar formulae for $K^{MQ'}_{vwx}$

$$\begin{aligned}K^{MQ'}_{vwx}\star(K+1)^{-1}_{Q'Q} &= \delta_{M+1,Q}s+\delta_{1Q}(K^{M1}_{vwx}-K^{M2}_{vwx}\star s) = \delta_{M+1,Q}s+\delta_{1Q}\check{K}_M\star s\,,\\ (\check{K}_M\star s)(u,v) &= \int_{-\infty}^{\infty}\mathrm{d}u'\,\check{K}_M(u,u')s(u'-v) \qquad (4.36)\\ &= \Big(\int_{-\infty}^{-2}-\int_{2}^{\infty}\Big)\mathrm{d}u'\,(\bar{K}(u+\frac{i}{g}M,u')+\bar{K}(u-\frac{i}{g}M,u'))s(u'-v)\\ &= \pm\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\,(K(u+\frac{i}{g}M,u'\pm i\epsilon)+K(u-\frac{i}{g}M,u'\pm i\epsilon))s(u'-v)\,,\end{aligned}$$

where

$$\check{K}_M(u,v) = (\bar{K}(u+\frac{i}{g}M,v) + \bar{K}(u-\frac{i}{g}M,v))\,(\theta(-v-2)-\theta(v-2))\,. \quad (4.37)$$

The kernels $\check{K}(u,v)$ and $\check{K}_M(u,v)$ obviously vanish only for $|v|<2$. For other values of v they are non-trivial and because of that the Y-system would not hold for $|v|>2$.

The only formula we are missing is the one which gives the action of the inverse kernel on the dressing phase kernel: $K^{\Sigma}_{Q'Q''}\star(K+1)^{-1}_{Q''Q}=?$. Unfortunately, since the structure of the dressing phase in the mirror model is unknown, we cannot proceed. We believe that the result will be similar to what we found above for the other kernels

$$K^{\Sigma}_{Q'Q''}\star(K+1)^{-1}_{Q''Q} \stackrel{?}{=} \delta_{1Q}\check{K}^{\Sigma}_{Q'}\star s\,, \quad (4.38)$$

where the kernel $\check{K}^{\Sigma}_{Q'}(u,v)$ would vanish for $|v|<2$. Finding a formula for the kernel $\check{K}^{\Sigma}_{Q'}$ is necessary for understanding the structure of the TBA equations, and we hope to address the problem in a future publication.

Applying now the inverse kernel (4.2) to (4.28), and assuming that $Q\ge 2$, we get the following equation

$$\begin{aligned}-\log Y_Q + \log(Y_{Q-1}Y_{Q+1})\star s &= \log\frac{(1+Y_{Q-1})(1+Y_{Q+1})}{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}\star s\\ &+2\log(1+Y_{Q'})\star K^{\Sigma}_{Q'Q''}\star(K+1)^{-1}_{Q''Q}\,, \end{aligned} \quad (4.39)$$

We see that if the formula (4.38) would hold then applying the s^{-1} operator to both sides of the equation (4.39), one gets the following Y-equation for Q-particles

$$\frac{Y_Q^+\,Y_Q^-}{Y_{Q-1}Y_{Q+1}} = \frac{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{(1+Y_{Q-1})\,(1+Y_{Q+1})} \quad \text{if} \quad Q\ge 2\,. \quad (4.40)$$

This Y-equation agrees with the corresponding equation of the Y-system in [50] if one identifies $Y_Q = Y_{Q,0}\,,\ Y^{(1)}_{Q|vw}=1/Y_{Q+1,1}\,,\ Y^{(2)}_{Q|vw}=1/Y_{Q+1,-1}$.

Finally, the equation for $Q=1$ takes the form

$$\begin{aligned}-\log Y_1 &= \log(1+\frac{1}{Y_2})\star s - \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\star s \\ &+ L\,\check{\mathcal{E}}\star s - \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(1-\frac{e^{ih_1}}{Y_+^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_+^{(2)}}\right)\star\check{K}\star s\\ &- \log\left(1+\frac{1}{Y^{(1)}_{M|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{M|vw}}\right)\star\check{K}_M\star s\\ &+2\log(1+Y_Q)\star K^{\Sigma}_{QQ'}\star(K+1)^{-1}_{Q'1}\,.\end{aligned} \quad (4.41)$$

We recall that the functions $Y_\pm^\alpha$ are defined on the interval $-2 < u < 2$ and, therefore, in all the formulae where they appear the corresponding integrals are taken from -2 to 2. The equation above clearly demonstrates the absence of symmetry between y^-- and y^+-particles. It does not immediately lead to the corresponding equation in the Y-system of [50] for arbitrary values of the rapidity variable u which is the argument of the function Y_1 in (4.41).

Nevertheless, applying the s^{-1} operator to both sides of (4.41), one gets the following equation

$$\frac{Y_1^+ Y_1^-}{Y_2} = \frac{\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)}{1+Y_2}\, e^{-\Delta}\,, \tag{4.42}$$

where

$$\begin{aligned}\Delta = &\log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\big(\theta(-u-2)+\theta(u-2)\big) \\ &+ L\,\check{\mathcal{E}} - \log\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)\left(1-\frac{e^{ih_1}}{Y_+^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_+^{(2)}}\right)\star\check{K} \\ &- \log\left(1+\frac{1}{Y_{M|vw}^{(1)}}\right)\left(1+\frac{1}{Y_{M|vw}^{(2)}}\right)\star\check{K}_M + 2\log\left(1+Y_Q\right)\star\check{K}_Q^\Sigma\,.\end{aligned} \tag{4.43}$$

Here the first term on the right hand side guarantees that the second term on the right hand side of eq.(4.41) contributes only for $|u|<2$.

Since the extra contribution Δ vanishes for $|u|<2$, in this case one recovers the Y-equation for $Q=1$-particles

$$\frac{Y_1^+ Y_1^-}{Y_2} = \frac{\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)}{1+Y_2} \quad \text{if} \quad |u|<2\,. \tag{4.44}$$

This equation also agrees with the Y-system of [50] under the identification $e^{-ih_1}Y_-^{(1)} = -1/Y_{1,1}\,,\ e^{-ih_2}Y_-^{(2)} = -1/Y_{1,-1}$. Let us stress again that to derive (4.44) one uses that $\check{\mathcal{E}}(u)\,,\ \check{K}(u',u)\,,\ \check{K}_M(u',u)\,,\ \check{K}_Q^\Sigma(u',u)$ vanish for $|u|<2$, and assumes the validity of formula (4.38) for any Q'.

Taking into account the important role the Y-system played in the recent studies of the O(4) chiral model [19], it might seem reasonable to assume that only solutions of the TBA equations satisfying the extra condition

$$\Delta = 2\pi i n \qquad \text{for all} \;\; u \;\; \text{and} \;\; \text{some} \;\; n\in\mathbb{Z}, \tag{4.45}$$

encode the spectrum of strings on $\mathrm{AdS}_5\times\mathrm{S}^5$, and are relevant for the AdS/CFT correspondence. Currently, however, we suspend the claim that it is really the case.

In fact, it seems that the solution of the TBA equations leading to the vanishing ground state energy does not obey the condition, and if it is really the case then the Y-system is valid only for $|u| < 2$. We hope to return to this question in a future publication.

Let us finally mention that identifying

$$\begin{aligned}&Y_Q = Y_{Q,0}\,,\ e^{-ih_1}Y_-^{(1)} = -1/Y_{1,1}\,,\ e^{-ih_2}Y_-^{(2)} = -1/Y_{1,-1}\,,\\&e^{-ih_1}Y_+^{(1)} = -Y_{2,2}\,,\ e^{-ih_2}Y_+^{(2)} = -Y_{2,-2}\,,\\&Y_{Q|vw}^{(1)} = 1/Y_{Q+1,1}\,,\ Y_{Q|vw}^{(2)} = 1/Y_{Q+1,-1}\,,\ Y_{Q|w}^{(1)} = Y_{1,Q+1}\,,\ Y_{Q|w}^{(2)} = Y_{1,-Q-1}\,,\end{aligned}$$

one finds that all the Y-equations discussed above do match the corresponding ones in the Y-system of [50].

5. Conclusions

In this paper we have derived an infinite set of the TBA equations for the $AdS_5 \times S^5$ mirror theory. There are several obvious questions to be answered. First, one needs to rigorously establish the vanishing of Witten's index ($\gamma = \pi$) in the mirror theory, the latter equals the ground state energy in the original string model, and see if/how it implies the quantization of the light-cone momentum of string theory or, equivalently, the quantization of the temperature of the mirror model. Second, one has to elaborate on the properties of the dressing phase in the mirror theory. Third, one should find an analytic continuation of the TBA equations to account for the excited states. All these questions, of course, are not independent.

The Y-system we obtained from these equations (under certain unproved assumptions about the dressing phase!) coincides with that by [50] only for u taking values in the interval $-2 < u < 2$. For other values of u one has to assume the validity of (4.38), and impose the additional condition (4.45) on the Y-functions. The condition, however, does not seem to be compatible with the vanishing of the ground state energy, and if so the Y-system could be valid only for $|u| < 2$.

Our brief comparison to the recent results [51, 52] reveals the following. The TBA equations we derived in section 3 seem to agree with those of [51]. The detailed comparison is difficult to carry out, however, due to notation differences between the various sections in [51]. The simplified form of the TBA equations we obtained in section 4 does not appear in [51, 52]. Once again, we see that the properties of our TBA kernels are such that they admit a localized Y-system in the interval $u \in (-2, 2)$ only. We could not find any indication of this fact in [51, 52]. Such an unusual feature arises due to the TBA equation for Q-particles and it is, of course, absent in the Y-system for the homogeneous Hubbard model. Then, it is assumed in [51, 52] that the discrete Laplace operator annihilates the dressing phase kernel.

There seems to be no proof of this assumption in [52], and in the appendix 2 of [51] a proof is given based on the AFS form [45] of the dressing phase. This form, however, is not valid in the mirror region because the corresponding series is not convergent.

We also see a certain difference with the findings of [52]. First of all, none of the TBA kernels in [52] seem to contain the (square root) of the ratio $x(u + i/gQ)/x(u - i/gQ)$, while our kernels $K^{Qy}_{\pm}$ do. Further, as was argued in [34], the kinematical region of the mirror theory corresponds to $\mathrm{Im}\, x^{\pm} < 0$ (or $\mathrm{Im}\, x^{\pm} > 0$) which is apparently different from the choice $|x^+| > 1$ and $|x^-| < 1$ indicated in [50, 52]. Indeed, the region $|x^+| > 1$ and $|x^-| < 1$ is a "hour-glass" in Fig.1 of [34], while the kinematical region of the mirror theory is a "leaf", where $\mathrm{Im}\, x^{\pm} < 0$ (or $\mathrm{Im}\, x^{\pm} > 0$). Although both regions do include the physical momentum of the mirror theory, it is only the leaf which contains all the bound state solutions of the mirror theory corresponding to the first and the second BPS families [34].

In any case, in spite of all the differences and yet to be justified assumptions, the present effort of deriving the TBA equations for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror brings us closer towards the final solution of the AdS/CFT spectral problem.

Acknowledgements

The work of G. A. was supported in part by the RFBR grant 08-01-00281-a, by the grant NSh-672.2006.1, by NWO grant 047017015 and by the INTAS contract 03-51-6346. The work of S.F. was supported in part by the Science Foundation Ireland under Grant No. 07/RFP/PHYF104.

6. Appendices

6.1 Mirror dispersion and parametrizations

In this paper we express all the quantities of interest in terms of the following function of the u-plane rapidity variable

$$x(u) = \frac{1}{2}\left(u - i\sqrt{4-u^2}\right)\,, \quad \mathrm{Im}(x(u)) < 0 \;\text{ for any } u \in \mathbb{C}\,, \tag{6.1}$$

with the cuts in the u-plane running from $\pm\infty$ to ± 2 along the real lines. Our choice of the square root cut agrees with the one used in Mathematica: it goes along the negative semi-axes. One can check that with this choice of the cuts the imaginary part of $x(u)$ is negative for any $u \in \mathbb{C}$. According to [34], this function maps the u-plane with the cuts onto the physical region of the mirror theory. To describe bound states of the mirror model, one should also add either the both lower or both upper edges of the cuts to the u-plane [34]. This breaks the parity invariance of the model.

The function $x(u)$ obviously satisfies the condition

$$x(u) + \frac{1}{x(u)} = u,$$

and also the following relations

$$x(-u) = -\frac{1}{x(u)}\,, \quad (x(u))^* = \frac{1}{x(u^*)}\,.$$

The variables $x^{Q\pm}(u)$ used in [36] are expressed through $x(u)$ as follows

$$x^{Q+}(u) = x(u + \frac{iQ}{g})\,, \quad x^{Q-}(u) = x(u - \frac{iQ}{g})\,, \tag{6.2}$$

where the parameter g is the string tension, and it is related to the 't Hooft coupling λ of the dual gauge theory as $g = \frac{\sqrt{\lambda}}{2\pi}$.

The momentum $\widetilde{p}^Q$, and the energy $\widetilde{\mathcal{E}}^Q$ of a mirror Q-particle are expressed in terms of $x(u)$ as follows

$$\widetilde{p}^Q(u) = g\,x(u - \frac{iQ}{g}) - g\,x(u + \frac{iQ}{g}) + iQ\,, \tag{6.3}$$

$$\widetilde{\mathcal{E}}^Q(u) = \log\frac{x(u - \frac{iQ}{g})}{x(u + \frac{iQ}{g})} = 2\,\mathrm{arcsinh}\Big(\frac{1}{2g}\sqrt{Q^2 + \widetilde{p}^2}\Big)\,, \tag{6.4}$$

and the momentum is real, and the energy is positive for real values of u. They satisfy the relations

$$\widetilde{p}(-u) = -\widetilde{p}(u)\,, \quad (\widetilde{p}(u))^* = \widetilde{p}(u^*)\,, \qquad \widetilde{\mathcal{E}}(-u) = \widetilde{\mathcal{E}}(u)\,, \quad (\widetilde{\mathcal{E}}(u))^* = \widetilde{\mathcal{E}}(u^*)\,.$$

6.2 Kernels

Let us introduce the following kernels

$$K(u,v) = \frac{1}{2\pi i}\frac{\sqrt{4-v^2}}{\sqrt{4-u^2}}\frac{1}{u-v}\,, \tag{6.5}$$

and (c.f. [19])

$$K_M(u) = \frac{1}{2\pi i}\frac{d}{du}\log\Big(\frac{u - i\frac{M}{g}}{u + i\frac{M}{g}}\Big) = \frac{1}{\pi}\frac{gM}{M^2 + g^2u^2}\,, \quad -\infty \le M \le \infty\,. \tag{6.6}$$

The Fourier transform of the kernel is

$$\hat{K}_M(\omega) = \int_{-\infty}^{\infty} \mathrm{d}u\, e^{i\omega u} K_M(u) = \mathrm{sign}(M)\, e^{-|M\omega|/g}\,, \tag{6.7}$$

and therefore

$$\int_{-\infty}^{\infty} \mathrm{d}u\, K_M(u-u') = \mathrm{sign}(M) \quad \text{for any } u' \,. \tag{6.8}$$

Then the kernels $K_{\pm}^{Qy}$ are related to them as follows

$$\begin{aligned} &K_{-}^{Qy}(u,v) + K_{+}^{Qy}(u,v) = K_Q(u-v)\,, \\ &K_{Qy}(u,v) \equiv K_{-}^{Qy}(u,v) - K_{+}^{Qy}(u,v) = K(u-\frac{i}{g}Q,v) - K(u+\frac{i}{g}Q,v)\,, \end{aligned} \tag{6.9}$$

and the kernels $K_{\pm}^{yQ}$ as

$$K_{-}^{yQ}(u,v) - K_{+}^{yQ}(u,v) = K_Q(u-v)\,, \tag{6.10}$$

$$K_{yQ}(u,v) \equiv K_{-}^{yQ}(u,v) + K_{+}^{yQ}(u,v) = K(u,v+\frac{i}{g}Q) - K(u,v-\frac{i}{g}Q)\,. \tag{6.11}$$

The important property of the kernel $K_{yM'}^{-}$ and $K_{yM'}^{+}$ is that they are positive for $-2 \le u' \le 2$, and satisfy

$$\int_{-2}^{2} \mathrm{d}u\, K_{yQ}(u,u') = 1 \quad \text{for any } u' \,. \tag{6.12}$$

We also have

$$K_{vw}^{yM}(u,u') \equiv K_M(u-u')\,. \tag{6.13}$$

Then the kernel $K_{vv}^{MN}(u,u')$ can be expressed in terms of K_M as follows

$$\begin{aligned} &K_{vv}^{MN}(u,u') \equiv K_{MN}(u-u')\,, \\ &K_{MN}(u) = K_{M+N}(u) + K_{N-M}(u) + 2\sum_{j=1}^{M-1} K_{N-M+2j}(u)\,, \quad M,N>0\,. \end{aligned} \tag{6.14}$$

The Fourier transform of the kernel is

$$\hat{K}_{MN}(\omega) = \coth\left(\frac{|\omega|}{g}\right)\left(e^{-|M-N||\omega|/g} - e^{-(M+N)|\omega|/g}\right) - \delta_{MN}\,, \quad M,N>0\,. \tag{6.15}$$

The inverse of the kernel is

$$\begin{aligned} &\left(\hat{K}(\omega)+1\right)^{-1}_{MN} = \delta_{MN} - \hat{s}(\omega)\left(\delta_{M+1,N} + \delta_{M-1,N}\right)\,, \quad \hat{s}(\omega) = \frac{1}{2\cosh\frac{\omega}{g}}\,, \\ &\sum_{N=1}^{\infty}\left(\hat{K}(\omega)+1\right)^{-1}_{MN}\left(\hat{K}_{NQ}(\omega)+\delta_{NQ}\right) = \delta_{MQ}\,. \end{aligned} \tag{6.16}$$

Inverse Fourier of $\hat{s}(\omega)$ is

$$s(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega u}\hat{s}(\omega) = \frac{g}{4\cosh\frac{g\pi u}{2}}\,. \tag{6.17}$$

There is this interesting identity

$$\sum_{N=1}^{\infty}\Big(\hat{K}_{MN}(\omega)+\delta_{MN}\Big)^{-1}\ \hat{K}_N(\omega)=\hat{s}(\omega)\,\delta_{M1}\,. \tag{6.18}$$

One can show that

$$K_{vwx}^{NN'}(u,u')=\frac{1}{2}K_{NN'}(u-u')+\mathcal{K}_1^{NN'}(u,u')\,, \tag{6.19}$$

where

$$\begin{aligned}\mathcal{K}_1^{NN'}(u,u')=&+\frac{1}{2}\left(K(u+\frac{i}{g}N,u'+\frac{i}{g}N')-K(u-\frac{i}{g}N,u'-\frac{i}{g}N')\right)\\&+\frac{1}{2}\left(K(u-\frac{i}{g}N,u'+\frac{i}{g}N')-K(u+\frac{i}{g}N,u'-\frac{i}{g}N')\right)\,,\end{aligned} \tag{6.20}$$

and

$$K_{xv}^{NN'}(u,u')=\frac{1}{2}K_{NN'}(u-u')+\mathcal{K}_2^{NN'}(u,u')\,. \tag{6.21}$$

where

$$\begin{aligned}\mathcal{K}_2^{NN'}(u,u')=&-\frac{1}{2}\left(K(u+\frac{i}{g}N,u'+\frac{i}{g}N')-K(u-\frac{i}{g}N,u'-\frac{i}{g}N')\right)\\&+\frac{1}{2}\left(K(u-\frac{i}{g}N,u'+\frac{i}{g}N')-K(u+\frac{i}{g}N,u'-\frac{i}{g}N')\right)\,.\end{aligned} \tag{6.22}$$

Below we list various identities necessary to simplify the TBA equations. The summation over repeated indices from 1 to ∞ is assumed

$$(K+1)^{-1}_{MN}\star K_N=s\,\delta_{M1}\,. \tag{6.23}$$

$$\begin{aligned}K_{yQ'}\star(K+1)^{-1}_{Q'Q}&=\delta_{Q1}(K_{y1}-K_{y2}\star s)=\delta_{Q1}(s+2\check{K}\star s)\,,\\(\check{K}\star s)(u,v)&=\int_{-\infty}^{\infty}\mathrm{d}u'\,\check{K}(u,u')s(u'-v)=\Big(\int_{-\infty}^{-2}-\int_{2}^{\infty}\Big)\mathrm{d}u'\,\bar{K}(u,u')s(u'-v)\\&=\pm\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\,K(u,u'\pm i\epsilon)s(u'-v)\,,\end{aligned} \tag{6.24}$$

where we introduce the kernel

$$\check{K}(u,v)=\bar{K}(u,v)\,(\theta(-v-2)-\theta(v-2))\ ,\quad \bar{K}(u,v)=\frac{1}{2\pi}\frac{\sqrt{v^2-4}}{\sqrt{4-u^2}}\frac{1}{u-v}\,, \tag{6.25}$$

and $\theta(u)$ is the standard unit step function.

$$(K+1)^{-1}_{QQ'}\star K_{Q'y}=\delta_{Q1}(K_{1y}-s\star K_{2y})\,, \tag{6.26}$$

$$(K_{1y}-s\star K_{2y})(u,v)=s(u,v)\pm 2\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\,s(u-u')K(u'\mp i\epsilon,v)\,. \tag{6.27}$$

$$K_{xv}^{QQ''}\star(K+1)^{-1}_{Q''Q'}=\delta_{Q-1,Q'}s+\delta_{Q'1}K_{Qy}\star s\,,\tag{6.28}$$

$$(K+1)^{-1}_{QQ''}\star K_{xv}^{Q''Q'}=\delta_{Q-1,Q'}s+\delta_{Q1}(K_{xv}^{1Q'}-s\star K_{xv}^{2Q'})\,,\tag{6.29}$$

$$\begin{aligned}K_{xv}^{1Q'}-s\star K_{xv}^{2Q'}&=\pm\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\\&\quad\times s(u-u')(K(u'\mp i\epsilon,v+\frac{i}{g}Q')+K(u'\mp i\epsilon,v-\frac{i}{g}Q'))\,.\end{aligned}\tag{6.30}$$

$$(K+1)^{-1}_{MN'}\star K_{vwx}^{N'Q}=\delta_{M+1,Q}s+\delta_{M1}s\star K_{yQ}\,.\tag{6.31}$$

$$\begin{aligned}&K_{vwx}^{MQ'}\star(K+1)^{-1}_{Q'Q}=\delta_{M+1,Q}s+\delta_{1Q}(K_{vwx}^{M1}-K_{vwx}^{M2}\star s)=\delta_{M+1,Q}s+\delta_{1Q}\check{K}_M\star s\,,\\&(\check{K}_M\star s)(u,v)=\int_{-\infty}^{\infty}\mathrm{d}u'\,\check{K}_M(u,u')s(u'-v)\\&\qquad=\Big(\int_{-\infty}^{-2}-\int_{2}^{\infty}\Big)\mathrm{d}u'\,(\bar{K}(u+\frac{i}{g}M,u')+\bar{K}(u-\frac{i}{g}M,u'))s(u'-v)\\&\qquad=\pm\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\,(K(u+\frac{i}{g}M,u'\pm i\epsilon)+K(u-\frac{i}{g}M,u'\pm i\epsilon))s(u'-v)\,,\end{aligned}\tag{6.32}$$

where

$$\check{K}_M(u,v)=(\bar{K}(u+\frac{i}{g}M,v)+\bar{K}(u-\frac{i}{g}M,v))\,(\theta(-v-2)-\theta(v-2))\,.\tag{6.33}$$

The kernels $\check{K}(u,v)$ and $\check{K}_M(u,v)$ obviously vanish for $|v|<2$.

$$\begin{aligned}\widetilde{\mathcal{E}}_{Q'}\star(K+1)^{-1}_{Q'Q}&=\delta_{Q1}(\widetilde{\mathcal{E}}_1-\widetilde{\mathcal{E}}_2\star s)=\delta_{Q1}\,\check{\mathcal{E}}\star s\,,\\(\check{\mathcal{E}}\star s)(v)&=\int_{-\infty}^{\infty}\mathrm{d}u'\,\check{\mathcal{E}}(u')s(u'-v)\\&=2(\int_{-\infty}^{-2}-\int_{2}^{\infty})\mathrm{d}u'\,\log\left(\frac{1}{2}\left|u'-\sqrt{u'^2-4}\right|\right)s(u'-v)\,,\end{aligned}\tag{6.34}$$

where we introduce the function

$$\check{\mathcal{E}}(u)=2\log\left(\frac{1}{2}\left|u-\sqrt{u^2-4}\right|\right)(\theta(-u-2)-\theta(u-2))\,,\tag{6.35}$$

$$(K+1)^{-1}_{QQ'}\star\frac{d\widetilde{p}^{Q'}}{du}=\delta_{Q1}(\frac{d\widetilde{p}^1}{du}-s\star\frac{d\widetilde{p}^2}{du})\,,\tag{6.36}$$

$$(\frac{d\widetilde{p}^1}{du}-s\star\frac{d\widetilde{p}^2}{du})(u)=\pm\Big(\int_{-\infty}^{-2}+\int_{2}^{\infty}\Big)\mathrm{d}u'\,s(u-u')\frac{igu'}{\sqrt{4-(u'\mp i\epsilon)^2}}\tag{6.37}$$

$$=\Big(\int_{2}^{\infty}-\int_{-\infty}^{-2}\Big)\mathrm{d}u'\,s(u-u')\frac{gu'}{\sqrt{u'^2-4}}\,,\tag{6.38}$$

$$K_Q \star K_{yQ'} = \int_{-2}^{2} \mathrm{d}u'\, K_Q(u-u')K_{yQ'}(u',v) = \tag{6.39}$$
$$= \mathcal{K}_1^{QQ'}(u,v) - \frac{1}{2}K_{Q'-Q}(u-v) + \frac{1}{2}K_{Q'+Q}(u-v)\,.$$

$$K_{Qy} \star K_{Q'} = \int_{-2}^{2} \mathrm{d}u'\, K_{Qy}(u,u')K_{Q'}(u'-v) = \tag{6.40}$$
$$= \mathcal{K}_2^{QQ'}(u,v) + \frac{1}{2}K_{Q'-Q}(u-v) + \frac{1}{2}K_{Q'+Q}(u-v)\,.$$

The following formula holds for $|v|<2$

$$2\mathcal{K}_2^{Q1} \star s(u,v) = K_{Qy}(u,v) \pm 2\Big(\int_{-\infty}^{-2} + \int_{2}^{\infty}\Big)\mathrm{d}u'\, K_{Qy}(u,u'\mp i\epsilon)s(u'-v+\frac{i}{g})\,. \tag{6.41}$$

Finally

$$(K_{Qy}+K_Q)\star s^{-1} = 2K_{xv}^{Q1} + 2\delta_{Q1}\,, \quad K_M \star s^{-1} = K_{M1} + \delta_{M1}\,. \tag{6.42}$$

6.3 Simplified TBA equations and Y-system

Here for reader's convenience we list all the simplified TBA equations and also the Y-system equations. Recall that we introduce the Y-functions related to the pseudo-energies as

$$Y_Q = e^{-\epsilon_Q}\,, \quad Y_{M|vw}^{(\alpha)} = e^{\epsilon_{M|vw}^{(\alpha)}}\,, \quad Y_{M|w}^{(\alpha)} = e^{\epsilon_{M|w}^{(\alpha)}}\,, \quad Y_\pm^{(\alpha)} = e^{\epsilon_{y^\pm}^{(\alpha)}}\,, \quad \alpha = 1,2\,, \tag{6.43}$$

and assume summation over repeated indices.

The simplified TBA equations for the Y-functions take the following form

- $M|w$-strings: $M \geq 1$, $Y_{0|w}^{(\alpha)} = 0$

$$\log Y_{M|w}^{(\alpha)} = \log(1+Y_{M-1|w}^{(\alpha)})(1+Y_{M+1|w}^{(\alpha)}) \star s + \delta_{M1}\, \log \frac{1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}} \star s \tag{6.44}$$

- $M|vw$-strings: $M \geq 1$, $Y_{0|vw}^{(\alpha)} = 0$

$$\log Y_{M|vw}^{(\alpha)} = \log(1+Y_{M-1|vw}^{(\alpha)})(1+Y_{M+1|vw}^{(\alpha)}) \star s \tag{6.45}$$
$$- \log(1+Y_{M+1}) \star s + \delta_{M1} \log \frac{1-e^{-ih_\alpha}Y_-^{(\alpha)}}{1-e^{-ih_\alpha}Y_+^{(\alpha)}} \star s$$

- y-particles

$$\log \frac{Y_+^{(\alpha)}}{Y_-^{(\alpha)}} = \log(1+Y_Q) \star K_{Qy}\,, \tag{6.46}$$

$$\log Y_+^{(\alpha)} Y_-^{(\alpha)} = -\log\left(1+Y_Q\right) \star K_Q + 2\log \frac{1+\frac{1}{Y_{M|vw}^{(\alpha)}}}{1+\frac{1}{Y_{M|w}^{(\alpha)}}} \star K_M \tag{6.47}$$

- Q-particles for $Q \geq 2$

$$-\log Y_Q + \log(Y_{Q-1}Y_{Q+1}) \star s = \log \frac{(1+Y_{Q-1})(1+Y_{Q+1})}{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)} \star s$$
$$+ 2\log\left(1+Y_{Q'}\right) \star K^{\Sigma}_{Q'Q''} \star (K+1)^{-1}_{Q''Q} \quad (6.48)$$

- $Q = 1$-particle

$$-\log Y_1 = \log(1+\frac{1}{Y_2}) \star s - \log\left(1-\frac{e^{ih_1}}{Y^{(1)}_-}\right)\left(1-\frac{e^{ih_2}}{Y^{(2)}_-}\right) \star s \quad (6.49)$$
$$- \log\left(1-\frac{e^{ih_1}}{Y^{(1)}_-}\right)\left(1-\frac{e^{ih_2}}{Y^{(2)}_-}\right)\left(1-\frac{e^{ih_1}}{Y^{(1)}_+}\right)\left(1-\frac{e^{ih_2}}{Y^{(2)}_+}\right) \star \check{K} \star s$$
$$- \log\left(1+\frac{1}{Y^{(1)}_{M|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{M|vw}}\right) \star \check{K}_M \star s$$
$$+ 2\log\left(1+Y_Q\right) \star K^{\Sigma}_{QQ'} \star (K+1)^{-1}_{Q'1} + L\,\check{\mathcal{E}} \star s$$

The energy of the ground state of the light-cone gauge-fixed string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ is expressed through the Y-functions as follows

$$E_\gamma(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi}\frac{d\widetilde{p}^Q}{du} \log\left(1+Y_Q\right) . \quad (6.50)$$

With the notation $Y^{\pm}(u) \equiv Y(u \pm \frac{i}{g} \mp i0)$ for any Y-function, the Y-system equations for $u \in [-2,2]$ take the following form

- $M|w$-strings

$$Y^{(\alpha)+}_{M|w}\, Y^{(\alpha)-}_{M|w} = \left(1+Y^{(\alpha)}_{M-1|w}\right)\left(1+Y^{(\alpha)}_{M+1|w}\right) \quad \text{if} \quad M \geq 2\,, \quad (6.51)$$

$$Y^{(\alpha)+}_{1|w}\, Y^{(\alpha)-}_{1|w} = \left(1+Y^{(\alpha)}_{2|w}\right) \frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}} \quad (6.52)$$

- $M|vw$-strings

$$Y^{(\alpha)+}_{M|vw}\, Y^{(\alpha)-}_{M|vw} = \left(1+Y^{(\alpha)}_{M-1|vw}\right)\left(1+Y^{(\alpha)}_{M+1|vw}\right) \frac{1}{1+Y_{M+1}} \quad \text{if} \quad M \geq 2\,, \quad (6.53)$$

$$Y^{(\alpha)+}_{1|vw}\, Y^{(\alpha)-}_{1|vw} = \frac{1+Y^{(\alpha)}_{2|vw}}{1+Y_2}\, \frac{1-e^{-ih_\alpha}Y^{(\alpha)}_-}{1-e^{-ih_\alpha}Y^{(\alpha)}_+} \quad (6.54)$$

- y^--particles

$$Y_-^{(\alpha)+}\,Y_-^{(\alpha)-} = \frac{1+Y^{(\alpha)}_{1|vw}}{1+Y^{(\alpha)}_{1|w}}\,\frac{1}{1+Y_1} \tag{6.55}$$

- y^+-particles

$$\log Y_+^{(\alpha)+}\,Y_+^{(\alpha)-} = 2\log\left(1+Y_Q\right)\star\mathcal{K}_2^{Q1} + \log\frac{1+Y^{(\alpha)}_{1|vw}}{1+Y^{(\alpha)}_{1|w}} \tag{6.56}$$

- Q-particles for $Q\geq 2$

$$\frac{Y_Q^+\,Y_Q^-}{Y_{Q-1}Y_{Q+1}} = \frac{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{\left(1+Y_{Q-1}\right)\left(1+Y_{Q+1}\right)} \tag{6.57}$$

- $Q=1$-particle

$$\frac{Y_1^+\,Y_1^-}{Y_2} = \frac{\left(1-\frac{e^{ih_1}}{Y_-^{(1)}}\right)\left(1-\frac{e^{ih_2}}{Y_-^{(2)}}\right)}{1+Y_2}\,, \tag{6.58}$$

  where we assume the validity of the formula

$$K^\Sigma_{Q'Q''}\star(K+1)^{-1}_{Q''Q} \stackrel{?}{=} \delta_{1Q}\check{K}^\Sigma_{Q'}\star s\,. \tag{6.59}$$

References

[1] G. Arutyunov and S. Frolov, "Foundations of the $AdS_5\times S^5$ Superstring. Part I," hep-th/0901.4937.

[2] N. Dorey, "Magnon bound states and the AdS/CFT correspondence," J. Phys. A **39** (2006) 13119, hep-th/0604175.

[3] N. Beisert, "The su(2|2) dynamic S-matrix," Adv. Theor. Math. Phys. **12**, 945 (2008), hep-th/0511082.

[4] G. Arutyunov, S. Frolov and M. Zamaklar, "The Zamolodchikov-Faddeev algebra for $AdS_5\times S^5$ superstring," JHEP **0704** (2007) 002, hep-th/0612229.

[5] G. Arutyunov and S. Frolov, "The S-matrix of String Bound States," Nucl. Phys. B **804** (2008) 90 hep-th/0803.4323;

[6] G. Arutyunov, M. de Leeuw and A. Torrielli, "The Bound State S-Matrix for $AdS_5\times S^5$ Superstring," hep-th/0902.0183.

[7] R. A. Janik, "The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry," Phys. Rev. D **73** (2006) 086006, hep-th/0603038.

[8] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models," Annals Phys. **120** (1979) 253.

[9] N. Beisert and M. Staudacher, "Long-range PSU(2,2|4) Bethe ansaetze for gauge theory and strings," Nucl. Phys. B **727** (2005) 1, hep-th/0504190.

[10] M. Luscher, "Volume dependence of the tnergy spectrum in massive quantum field theories. 1. Stable particle states," Commun. Math. Phys. **104** (1986) 177. • "Volume dependence of the energy spectrum in massive quantum field theories. 2. Scattering states," Commun. Math. Phys. **105** (1986) 153.

[11] J. Ambjorn, R. A. Janik and C. Kristjansen, "Wrapping interactions and a new source of corrections to the spin-chain / string duality," Nucl. Phys. B **736** (2006) 288, hep-th/0510171.

[12] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling – the giant magnon," Phys. Rev. D **76** (2007) 126008, hep-th/0708.2208.

[13] Y. Hatsuda and R. Suzuki, "Finite-Size Effects for Dyonic Giant Magnons," Nucl. Phys. B **800** (2008) 349 hep-th/0801.0747.

[14] J. A. Minahan and O. Ohlsson Sax, "Finite size effects for giant magnons on physical strings," Nucl. Phys. B **801** (2008) 97, hep-th/0801.2064.

[15] N. Gromov, S. Schafer-Nameki and P. Vieira, "Efficient precision quantization in AdS/CFT," JHEP **0812** (2008) 013, hep-th/0807.4752.

[16] Z. Bajnok and R. A. Janik, "Four-loop perturbative Konishi from strings and finite size effects for multiparticle states," Nucl. Phys. B **807** (2009) 625, hep-th/0807.0399.

[17] Y. Hatsuda and R. Suzuki, "Finite-size effects for multi-magnon states," JHEP **0809** (2008) 025, hep-th/0807.0643.

[18] Z. Bajnok, R. A. Janik and T. Lukowski, "Four loop twist two, BFKL, wrapping and strings," hep-th/0811.4448.

[19] N. Gromov, V. Kazakov and P. Vieira, "Finite Volume Spectrum of 2D Field Theories from Hirota Dynamics," hep-th/0812.5091.

[20] M. Beccaria, V. Forini, T. Lukowski and S. Zieme, "Twist-three at five loops, Bethe Ansatz and wrapping," hep-th/0901.4864.

[21] A. B. Zamolodchikov, "Thermodynamic Bethe Ansatz in relativistic models. Scaling three state Potts and Lee-Yang models," Nucl. Phys. B **342** (1990) 695.

[22] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, "Quantum field theories in finite volume: Excited state energies," Nucl. Phys. B **489** (1997) 487, hep-th/9607099.

[23] P. Dorey and R. Tateo, "Excited states by analytic continuation of TBA equations," Nucl. Phys. B **482** (1996) 639, hep-th/9607167.

[24] P. Fendley, "Excited-state energies and supersymmetric indices," Adv. Theor. Math. Phys. **1** (1998) 210, hep-th/9706161.

[25] J. Balog and A. Hegedus, "TBA equations for excited states in the O(3) and O(4) nonlinear sigma-model," J. Phys. A **37** (2004) 1881, hep-th/0309009.

[26] A. Hegedus, "Nonlinear integral equations for finite volume excited state energies of the O(3) and O(4) nonlinear sigma-models," J. Phys. A **38** (2005) 5345, hep-th/0412125.

[27] C. Destri and H. J. de Vega, "New Thermodynamic Bethe Ansatz Equations Without Strings," Phys. Rev. Lett. **69** (1992) 2313.

[28] D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini, "Excited state Destri-De Vega equation for sine-Gordon and restricted sine-Gordon models," Phys. Lett. B **390** (1997) 243, hep-th/9608091.

[29] C. Destri and H. J. de Vega, "Non-linear integral equation and excited-states scaling functions in the sine-Gordon model," Nucl. Phys. B **504** (1997) 621, hep-th/9701107.

[30] G. Feverati, F. Ravanini and G. Takacs, "Nonlinear integral equation and finite volume spectrum of sine-Gordon theory," Nucl. Phys. B **540** (1999) 543, hep-th/9805117.

[31] G. Feverati, F. Ravanini and G. Takacs, "Scaling functions in the odd charge sector of sine-Gordon/massive Thirring theory," Phys. Lett. B **444** (1998) 442, hep-th/9807160.

[32] C. Ahn, M. Bellacosa and F. Ravanini, "Excited states NLIE for sine-Gordon model in a strip with Dirichlet boundary conditions," Phys. Lett. B **595** (2004) 537, hep-th/0312176.

[33] J. Teschner, "On the spectrum of the Sinh-Gordon model in finite volume," Nucl. Phys. B **799** (2008) 403, hep-th/0702214.

[34] G. Arutyunov and S. Frolov, "On String S-matrix, Bound States and TBA," JHEP **0712** (2007) 024, hep-th/0710.1568.

[35] N. Beisert, "The analytic Bethe Ansatz for a chain with centrally extended su(2|2) symmetry," J. Stat. Mech. **0701** (2007) P017, arXiv:nlin/0610017.

[36] G. Arutyunov and S. Frolov, "String hypothesis for the $AdS_5 \times S^5$ mirror," hep-th/0901.1417.

[37] E. H. Lieb and F. Y. Wu, “Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension,” Phys. Rev. Lett. **20** (1968) 1445.

[38] M.J. Martins, P.B. Ramos, “The quantum inverse scattering method for Hubbard-like models,” Nucl. Phys. B **522** [FS] (1998) 413.

[39] F.H.L. Essler, H. Frahm, F. Göhmann, A. Klümper, V.E. Korepin, “The one-dimensional Hubbard model,” Cambridge University Press (2005).

[40] M. de Leeuw, “The Bethe Ansatz for $AdS_5 \times S^5$ Bound States,” JHEP **0901** (2009) 005 hep-th/0809.0783.

[41] A. B. Zamolodchikov, “On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories,” Phys. Lett. B **253** (1991) 391.

[42] T. R. Klassen and E. Melzer, “Spectral flow between conformal field theories in (1+1)-dimensions,” Nucl. Phys. B **370** (1992) 511.

[43] F. Ravanini, R. Tateo and A. Valleriani, “Dynkin TBAs,” Int. J. Mod. Phys. A **8** (1993) 1707, hep-th/9207040.

[44] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, “Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz,” Commun. Math. Phys. **177** (1996) 381, hep-th/9412229.

[45] G. Arutyunov, S. Frolov and M. Staudacher, “Bethe ansatz for quantum strings,” JHEP **0410**, 016 (2004), hep-th/0406256;

[46] N. Beisert, R. Hernandez and E. Lopez, “A crossing-symmetric phase for $AdS_5 \times S^5$ strings,” JHEP **0611** (2006) 070, hep-th/0609044.

[47] N. Beisert, B. Eden and M. Staudacher, “Transcendentality and crossing,” J. Stat. Mech. **0701** (2007) P021, hep-th/0610251.

[48] N. Dorey, D. M. Hofman and J. M. Maldacena, “On the singularities of the magnon S-matrix,” Phys. Rev. D **76** (2007) 025011, hep-th/0703104.

[49] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113], hep-th/9711200.

[50] N. Gromov, V. Kazakov and P. Vieira, “Integrability for the Full Spectrum of Planar AdS/CFT,” hep-th/0901.3753.

[51] D. Bombardelli, D. Fioravanti and R. Tateo, “Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal,” hep-th/0902.3930.

[52] N. Gromov, V. Kazakov, A. Kozak and P. Vieira, “Integrability for the Full Spectrum of Planar AdS/CFT II,” hep-th/0902.4458.

[53] H. Y. Chen, N. Dorey and K. Okamura, "On the scattering of magnon boundstates," JHEP **0611** (2006) 035, hep-th/0608047.

[54] R. Roiban, "Magnon bound-state scattering in gauge and string theory," JHEP **0704** (2007) 048, hep-th/0608049.

[55] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, "A New supersymmetric index," Nucl. Phys. B **386** (1992) 405, hep-th/9204102.