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The S-matrix of String Bound States

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Abstract: We find the S-matrix which describes the scattering of two-particle bound states of the light-cone string sigma model on $AdS_5 \times S^5$. We realize the M-particle bound state representation of the centrally extended $\mathfrak{su}(2|2)$ algebra on the space of homogeneous (super)symmetric polynomials of degree M depending on two bosonic and two fermionic variables. The scattering matrix $\mathbb{S}^{MN}$ of M- and N-particle bound states is a differential operator of degree $M+N$ acting on the product of the corresponding polynomials. We require this operator to obey the invariance condition and the Yang-Baxter equation, and we determine it for the two cases $M=1, N=2$ and $M=N=2$. We show that the S-matrices found satisfy *generalized physical unitarity, CPT invariance, parity transformation rule and crossing symmetry.* Although the dressing factor as a function of *four* parameters $x_1^+, x_1^-, x_2^+, x_2^-$ is universal for scattering of any bound states, it obeys a crossing symmetry equation which depends on M and N.

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1. Introduction and summary

It has been recognized in recent years that the S-matrix approach provides a powerful tool to study the spectra of both the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring and the dual gauge theory [1]-[8]. In the physical (light-cone) gauge the Green-Schwarz string sigma-model [9] is equivalent to a non-trivial massive integrable model of eight bosons and eight fermions [10, 11]. The corresponding scattering matrix, which arises in the infinite-volume limit, is determined by global symmetries almost uniquely [8, 12], up to an overall dressing phase [4] and the choice of a representation basis. A functional form of the dressing phase in terms of local conserved charges has been bootstrapped [4] from the knowledge of the classical finite-gap solutions [13]. Combining this form with the crossing symmetry requirement [14], one is able to find an exact, i.e. non-perturbative in the coupling constant, solution for the dressing phase [15] which exhibits remarkable interpolation properties from strong to weak coupling [16]. The leading finite-size corrections to the infinite-volume spectrum are encoded into a set of asymptotic Bethe Ansatz equations [6] based on this S-matrix.

In more detail, a residual symmetry algebra of the light-cone Hamiltonian $\mathbb{H}$ factorizes into two copies of the superalgebra $\mathfrak{su}(2|2)$ centrally extended by two central charges, the latter depend on the operator $\mathbb{P}$ of the world-sheet momentum [17]. Correspondingly, the S-matrix factorizes into a product of two S-matrices $\mathbb{S}^{AA}$, each of them scatters two fundamental supermultiplets A. Up to a phase, the S-matrix $\mathbb{S}^{AA}$ is determined from the invariance condition which schematically reads as

$$\mathbb{S}^{AA} \cdot \mathbb{J}_{12} = \mathbb{J}_{21} \cdot \mathbb{S}^{AA} \,,$$

where $\mathbb{J}_{12}$ is a symmetry generator in the two-particle representation.

As in some quantum integrable models, it turns out that, in addition to the supermultiplet of fundamental particles, the string sigma-model contains an infinite number of bound states [18]. They manifest themselves as poles of the multi-particle S-matrix built over $\mathbb{S}^{AA}$. The representation corresponding to a bound state of M fundamental particles constitutes a short $4M$- dimensional multiplet of the centrally extended $\mathfrak{su}(2|2)$ algebra [19, 20]. It can be obtained from the M-fold tensor product of the fundamental multiplets by projecting it on the totally symmetric component. A complete handle on the string asymptotic spectrum and the associated Bethe equations requires, therefore, the knowledge of the S-matrices which describe the scattering of bound states. This is also a demanding problem for understanding the finite-size string spectrum within the TBA approach [21].

A well-known way to obtain the S-matrices in higher representations from a fundamental S-matrix is to use the fusion procedure [22]. A starting point is a multi-particle S-matrix

$$\mathbb{S}_{a1}(p,p_1)\mathbb{S}_{a2}(p,p_2)\ldots\mathbb{S}_{aM}(p,p_M)\,,$$

where the (complex) momenta $p_1, \ldots, p_M$ provide a solution of the M-particle bound state equation and the index a denotes an auxiliary space being another copy of the fundamental irrep. Next, one symmetrises the indices associated to the matrix spaces $1, \ldots, M$ according to the Young pattern of a desired representation. The resulting object must give (up to normalization) an S-matrix for scattering of fundamental particles with M-particle bound states. A proof of the last statement uses the fact that at a pole the residue of the S-matrix is degenerate and it coincides with a projector on a (anti-)symmetric irrep corresponding to the two-particle bound state. This is, e.g., what happens for the rational S/R-matrices based on the $\mathfrak{gl}(n|m)$ superalgebras. The fusion procedure for the corresponding transfer-matrices and the associated Baxter equations have been recently worked out in [23, 24].

The fundamental S-matrix $\mathbb{S}^{AA}$ behaves, however, differently and this difference can be traced back to the representation theory of the centrally extended $\mathfrak{su}(2|2)$. Let $\mathscr{V}^A$ be a four-dimensional fundamental multiplet. It characterizes by the values of the central charges which are all functions of the particle momentum p. For the generic values of p_1 and p_2 the tensor product $\mathscr{V}^A(p_1) \otimes \mathscr{V}^A(p_2)$ is an irreducible long multiplet [8, 19]. In the special case when momenta p_1 and p_2 satisfy the two-particle bound state equation the long multiplet becomes reducible. With the proper normalization of the S-matrix the invariant subspace coincides with the null space

$$\mathbb{S}^{AA}(p_1, p_2)\, \mathscr{V}^{\text{ker}} = 0\,, \qquad \mathscr{V}^{\text{ker}} \subset \mathscr{V}^A \otimes \mathscr{V}^A\,.$$

Indeed, if $v \in \mathscr{V}^{\text{ker}}$, then $\mathbb{J}_{12} v \in \mathscr{V}^{\text{ker}}$ due to the intertwining property of the S-matrix: $\mathbb{S}^{AA} \cdot \mathbb{J}_{12} v = \mathbb{J}_{21} \cdot \mathbb{S}^{AA} v = 0$. Being reducible at the bound state point, the multiplet $\mathscr{V}^A \otimes \mathscr{V}^A$ is, however, *indecomposable*. In particular, there is no invariant projector on either $\mathscr{V}^{\text{ker}}$ or on its orthogonal completion. The bound state representation we are interested in corresponds to a factor representation on the quotient space $\mathscr{V}^A / \mathscr{V}^{\text{ker}}$. In this respect, it is not clear how to generalize the known fusion procedure to the present case.[1]

The absence of apparent fusion rules motivates us to search for other ways to determine the scattering matrices of bound states. An obvious suggestion would be to follow the same invariance argument for the bound state S-matrices as for the fundamental S-matrix together with the requirement of factorized scattering, the latter being equivalent to the Yang-Baxter (YB) equations. A serious technical problem of this approach is, however, that the dimension of the M-particle bound state representation grows as $4M$, so that the S-matrix $\mathbb{S}^{MN}$ for scattering of the M- and N-particle bound states will have rank $16MN$. Even for small values of M and N working with such big matrices becomes prohibitory complicated. Although, we are ultimately interested not in the S-matrices themselves but rather in eigenvalues

[1] The usual fusion procedure works, however, for the rank one sectors of $\mathbb{S}^{AA}$, in which case it becomes trivial. This will be discussed later on.

of the associated transfer-matrices, we fist have to make sure that the corresponding scattering matrices do exist and they satisfy the expected properties.

The aim of this paper is to develop a new operator approach to deal with the bound state representations. This approach provides an efficient tool to solve both the invariance conditions and the YB equations and, therefore, to determine the S-matrices $\mathbb{S}^{MN}$, at least for sufficiently low values of M and N.

Our construction relies on the observation that the M-particle bound state representation $\mathscr{V}^M$ of the centrally extended $\mathfrak{su}(2|2)$ algebra can be realized on the space of homogeneous (super)symmetric polynomials of degree M depending on two bosonic and two fermionic variables, w_a and θ_α, respectively. Thus, the representation space is identical to an irreducible short superfield $\Phi_M(w,\theta)$. In this realization the algebra generators are represented by linear differential operators $\mathbb{J}$ in w_a and θ_α with coefficients depending on the representation parameters (the particle momenta). More generally, we will introduce a space $\mathscr{D}_M$ dual to $\mathscr{V}^M$, which can be realized as the space of differential operators preserving the homogeneous gradation of $\Phi_M(w,\theta)$. The S-matrix $\mathbb{S}^{MN}$ is then defined as an element of

$$\mathrm{End}(\mathscr{V}^M\otimes\mathscr{V}^N)\approx\mathscr{V}^M\otimes\mathscr{V}^N\otimes\mathscr{D}_M\otimes\mathscr{D}_N .$$

On the product of two superfields $\Phi_M(w^1,\theta^1)\Phi_N(w^2,\theta^2)$ it acts as a differential operator of degree $M+N$. We require this operator to obey the following intertwining property

$$\mathbb{S}^{MN}\cdot\mathbb{J}_{12}=\mathbb{J}_{21}\cdot\mathbb{S}^{MN}\,,$$

which is the same invariance condition as before but now implemented for the bound state representations. For two $\mathfrak{su}(2)$ subgroups of $\mathfrak{su}(2|2)$ this condition literally means the invariance of the S-matrix, while for the supersymmetry generators it involves the brading (non-local) factors to be discussed in the main body of the paper. Thus, the S-matrix is an $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$-invariant element of $\mathrm{End}(\mathscr{V}^M\otimes\mathscr{V}^N)$ and it can be expanded over a basis of invariant differential operators Λ_k:

$$\mathbb{S}^{MN}=\sum_k a_k\Lambda_k\,.$$

As we will show, it is fairly easy to classify the differential $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariants Λ_k. The coefficients a_k are then partially determined from the remaining invariance conditions with the supersymmetry generators. It turns out, however, that if the tensor product $\mathscr{V}^M\otimes\mathscr{V}^N$ has m irreducible components then $m-1$ coefficients a_k together with an overall scale are left undetermined from the invariance conditions and to find them one has to make use of the YB equations. This completes the discussion of our general strategy for a search of $\mathbb{S}^{MN}$. The operator formalism can be easily implemented in the *Mathematica* program and it reduces enormously a computational time, therefore, we give it a preference in comparison to the matrix approach.

Having established a general framework, we will apply it to construct explicitly the operators $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$, which are the S-matrices for scattering processes involving the fundamental multiplet A and a multiplet B corresponding to the two-particle bound state. We will show that $\mathbb{S}^{AB}$ is expanded over a basis of 19 invariant differential operators Λ_k and all the corresponding coefficients a_k up to an overall normalization are determined from the invariance condition. Construction of $\mathbb{S}^{BB}$ involves 48 operators Λ_k. This time, two of a_k's remain undetermined by the invariance condition, one of them corresponding to an overall scale. As to the second coefficient, we find it by solving the YB equations. It is remarkable that with one coefficient we managed to satisfy two YB equations: One involving $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$, and the second involving $\mathbb{S}^{BB}$ only. This gives an affirmative answer to the question about the existence of the scattering matrices for bound states.

Recall that the universal cover of the parameter space describing the fundamental representation of the centrally extended $\mathfrak{su}(2|2)$ is an elliptic curve (a generalized rapidity torus with real and imaginary periods $2\omega_1$ and $2\omega_2$, respectively) [14]. Since the particle energy and momentum are elliptic functions with the modular parameter $-4g^2$, where g is the coupling constant, the fundamental S-matrix $\mathbb{S}^{AA}$ can be viewed as a function of two variables z_1 and z_2 with values in the elliptic curve. Analogously, the M-particle bound state representation can be uniformized by an elliptic curve but with another modular parameter $-4g^2/M^2$. Correspondingly, in general $\mathbb{S}^{MN}(z_1, z_2)$ is a function on a product of two tori with *different* modular parameters.

In a physical theory, in addition to the YB equation, the operators $\mathbb{S}$ must satisfy a number of important analytic properties. Regarding $\mathbb{S}$ as a function of generalized rapidity variables, we list them below.

- *Generalized Physical Unitarity*

$\mathbb{S}(z_1^*, z_2^*)^\dagger \cdot$ [redacted] $2)$

- *CPT Invariance*

$\mathbb{S}(z_1, z_2)^T = \mathbb{S}(z_1, z$ [redacted]

- *Parity Transformation Rule*

$\mathbb{S}^{-1}(z_1, z_2) = \mathbb{S}(-z_1,$ [redacted]

- *Crossing Symmetry*

$\mathbb{S}^{c_1}(z_1, z_2)\, \mathbb{S}(z_1 + dTT(d\ T\ n$ [redacted]

the S-matrix $\mathbb{S}^{MN}$ without appealing to the YB equations. Concerning the universal R-matrix, would it exist, one could use it to deduce all the bound states S-matrices $\mathbb{S}^{MN}$ and the fusion procedure would not be required. Due to non-invertibility of the Cartan matrix for $\mathfrak{su}(2|2)$, existence of the universal R-matrix remains an open problem. On the other hand, at the classical level [32] one is able to identify a universal analogue of the classical r-matrix [33, 34]. It is of interest to verify if the semi-classical limit of our S-matrices agrees with this universal classical r-matrix [35].

As was explained in [21], to develop the TBA approach for $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring one has to find the scattering matrices for bound states of the accompanying mirror theory. The bound states of this theory are "mirrors" of those for the original string model. Moreover, the scattering matrix of the fundamental mirror particles is obtained from $\mathbb{S}^{AA}$ by the double Wick rotation. The same rotation must also relate the S-matrices of the string and mirror bound states, which on the rapidity tori should correspond to shifts by the imaginary quarter-periods.

The leading finite-size corrections [36]-[38] to the dispersion relation for fundamental particles (giant magnons [39]) and bound states can be also derived by applying the perturbative Lüscher approach [40], which requires the knowledge of the fundamental S-matrix [41]-[45]. It is quite interesting to understand the meaning of the bound state S-matrices in the Lüscher approach and use them, e.g., to compute the corrections to the dispersion relations corresponding to string bound states.

Let us mention that our new S-matrices might also have an interesting physical interpretation outside the framework of string theory and the AdS/CFT correspondence [46]. Up to normalization, $\mathbb{S}^{AA}$ coincides [19, 25] with the Shastry R-matrix [47] for the one-dimensional Hubbard model. The operators $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ might have a similar meaning for higher Hubbard-like models describing the coupling of the Hubbard electrons to matter fields. Also, the representation of the S-matrices in the space of symmetric polynomials we found provides a convenient starting point for a search of possible q-deformations [48]. The space of symmetric polynomials admits a natural q-deformation with the corresponding symmetry algebra realized by difference operators. It would be interesting to find the q-deformed versions of $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ along these lines.

2. Bound state representations

In this section we discuss the atypical totally symmetric representations of the centrally extended $\mathfrak{su}(2|2)$ algebra which are necessary to describe bound states of the light-cone string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. These representations are $2M|2M$-dimensional, and they are parameterized by four parameters a, b, c, d which are meromorphic functions on a rapidity torus.

2.1 Centrally extended $\mathfrak{su}(2|2)$ algebra

The centrally extended $\mathfrak{su}(2|2)$ algebra which we will denote $\mathfrak{su}(2|2)_{\mathcal{C}}$ was introduced in [8]. It is generated by the bosonic rotation generators $\mathbb{L}_a{}^b$, $\mathbb{R}_\alpha{}^\beta$, the supersymmetry generators $\mathbb{Q}_\alpha{}^a$, $\mathbb{Q}^{\dagger}{}_a^{\alpha}$, and three central elements $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$ subject to the following relations

$$
\begin{aligned}
&\left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] = \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c\,, && \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma\,,\\
&\left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] = -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c\,, && \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma\,,\\
&\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}^{\dagger}{}_b^{\beta}\} = \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H}\,,\\
&\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\, \mathbb{C}\,, && \{\mathbb{Q}^{\dagger}{}_a^{\alpha}, \mathbb{Q}^{\dagger}{}_b^{\beta}\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, \mathbb{C}^\dagger\,.
\end{aligned}
\tag{2.1}
$$

Here the first two lines show how the indices c and γ of any Lie algebra generator transform under the action of $\mathbb{L}_a{}^b$ and $\mathbb{R}_\alpha{}^\beta$. For the $\mathrm{AdS}_5\times\mathrm{S}^5$ string model the central element $\mathbb{H}$ is hermitian and is identified with the world-sheet light-cone Hamiltonian, and the supersymmetry generators $\mathbb{Q}_\alpha{}^a$ and $\mathbb{Q}^{\dagger}{}_a^{\alpha}$. The central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ are hermitian conjugate to each other: $(\mathbb{Q}_\alpha{}^a)^\dagger = \mathbb{Q}^{\dagger}{}_a^{\alpha}$. It was shown in [17] that the central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ are expressed through the world-sheet momentum $\mathbb{P}$ as follows

$$
\mathbb{C} = \frac{i}{2} g\, (e^{i\mathbb{P}} - 1) e^{2i\xi}\,, \quad \mathbb{C}^\dagger = -\frac{i}{2} g\, (e^{-i\mathbb{P}} - 1) e^{-2i\xi}\,, \quad g = \frac{\sqrt{\lambda}}{2\pi}\,. \tag{2.2}
$$

The phase ξ is an arbitrary function of the central elements. It reflects an external U(1) automorphism of the algebra (2.1): $\mathbb{Q} \to e^{i\xi}\mathbb{Q}$, $\mathbb{C} \to e^{2i\xi}\mathbb{C}$. In our paper [12] the choice $\xi = 0$ has been made in order to match with the gauge theory spin chain convention by [8] and to facilitate a comparison with the perturbative string theory computation of the S-matrix performed in [49]. We use the same choice of ξ in the present paper too.

Without imposing the hermiticity conditions on the generators of the algebra the U(1) automorphism of the algebra can be extended to the external $\mathfrak{sl}(2)$ automorphism [19], which acts on the supersymmetry generators as follows

$$
\begin{aligned}
&\widetilde{\mathbb{Q}}_\alpha{}^a = u_1\, \mathbb{Q}_\alpha{}^a + u_2\, \epsilon^{ac}\, \mathbb{Q}^{\dagger}{}_c^{\gamma}\, \epsilon_{\gamma\alpha}\,, && \widetilde{\mathbb{Q}}^{\dagger}{}_a^{\alpha} = v_1\, \mathbb{Q}^{\dagger}{}_a^{\alpha} + v_2\, \epsilon^{\alpha\beta}\, \mathbb{Q}_\beta{}^b\, \epsilon_{ba}\,,\\
&\mathbb{Q}_\alpha{}^a = v_1\, \widetilde{\mathbb{Q}}_\alpha{}^a - u_2\, \epsilon^{ac}\, \widetilde{\mathbb{Q}}^{\dagger}{}_c^{\gamma}\, \epsilon_{\gamma\alpha}\,, && \mathbb{Q}^{\dagger}{}_a^{\alpha} = u_1\, \widetilde{\mathbb{Q}}^{\dagger}{}_a^{\alpha} - v_2\, \epsilon^{\alpha\beta}\, \widetilde{\mathbb{Q}}_\beta{}^b\, \epsilon_{ba}\,,
\end{aligned}
\tag{2.3}
$$

where the coefficients may depend on the central charges and must satisfy the $\mathfrak{sl}(2)$ condition

$$
u_1 v_1 - u_2 v_2 = 1\,.
$$

Then, by using the commutation relations (2.1), we find that the transformed generators satisfy the same relations (2.1) with the following new central charges

$$
\begin{aligned}
&\widetilde{\mathbb{H}} = (1 + 2u_2 v_2)\mathbb{H} - 2u_1 v_2 \mathbb{C} - 2u_2 v_1 \mathbb{C}^\dagger\,,\\
&\widetilde{\mathbb{C}} = u_1^2 \mathbb{C} + u_2^2 \mathbb{C}^\dagger - u_1 u_2 \mathbb{H}\,, \quad \widetilde{\mathbb{C}}^\dagger = v_1^2 \mathbb{C}^\dagger + v_2^2 \mathbb{C} - v_1 v_2 \mathbb{H}\,.
\end{aligned}
\tag{2.4}
$$

If we now require that the new central charges $\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{C}}^\dagger$ vanish, while the transformed supercharges $\widetilde{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}^\dagger$ are hermitian conjugate to each other, we find

$$\begin{aligned} u_1 = v_1 &= \frac{1}{\sqrt{2}}\sqrt{1+\frac{\mathbb{H}}{\sqrt{\mathbb{H}^2-4\mathbb{C}\mathbb{C}^\dagger}}}\,, \\ u_2 &= \frac{\mathbb{C}}{\sqrt{\mathbb{H}^2-4\mathbb{C}\mathbb{C}^\dagger}}\frac{1}{u_1}\,, \qquad v_2 = \frac{\mathbb{C}^\dagger}{\sqrt{\mathbb{H}^2-4\mathbb{C}\mathbb{C}^\dagger}}\frac{1}{v_1}\,. \end{aligned} \tag{2.5}$$

With this choice of the parameters u_i, v_i, the new Hamiltonian takes the following simple form

$$\widetilde{\mathbb{H}} = \sqrt{\mathbb{H}^2-4\mathbb{C}\mathbb{C}^\dagger}\,.$$

We see that any irreducible representation of the centrally-extended algebra with $\widetilde{\mathbb{H}} \neq 0$ can be obtained from a representation of the usual $\mathfrak{su}(2|2)$ algebra with zero central charges $\mathbb{C} = \mathbb{C}^\dagger = 0$. This will play an important role in our derivation of the bound state scattering matrices.

2.2 The atypical totally symmetric representation

The atypical totally symmetric representation[4] of $\mathfrak{su}(2|2)_{\mathcal{C}}$ which describes M-particle bound states of the light-cone string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ has dimension $2M|2M$ and it can be realized on the graded vector space with the following basis

- a tensor symmetric in a_i: $|e_{a_1\ldots a_M}\rangle$, where $a_i = 1,2$ are bosonic indices which gives $M+1$ bosonic states
- a symmetric in a_i and skew-symmetric in α_i: $|e_{a_1\ldots a_{M-2}\alpha_1\alpha_2}\rangle$, where $\alpha_i = 3,4$ are fermionic indices which gives $M-1$ bosonic states. The total number of bosonic states is $2M$.
- a tensor symmetric in a_i : $|e_{a_1\ldots a_{M-1}\alpha}\rangle$ which gives $2M$ fermionic states.

We denote the corresponding vector space as $\mathscr{V}^M(p,\zeta)$ (or just $\mathscr{V}^M$ if the values of p and ζ are not important), where $\zeta = e^{2i\xi}$. For non-unitary representations the parameters p and ζ are arbitrary complex numbers which parameterize the values of the central elements (charges) on this representation: $\mathbb{H}|e_i\rangle = H|e_i\rangle$, $\mathbb{C}|e_i\rangle = C|e_i\rangle$, $\mathbb{C}^\dagger|e_i\rangle = \overline{C}|e_i\rangle$, where $|e_i\rangle$ stands for any of the basis vectors. The bosonic generators act in the space in the canonical way

$$\begin{aligned} \mathbb{L}_a{}^b|e_{c_1c_2\cdots c_M}\rangle &= \delta^b_{c_1}|e_{ac_2\cdots c_M}\rangle + \delta^b_{c_2}|e_{c_1ac_3\cdots c_M}\rangle + \cdots - \frac{M}{2}\delta^b_a|e_{c_1c_2\cdots c_M}\rangle \\ \mathbb{L}_a{}^b|e_{c_1\cdots c_{M-2}\gamma_1\gamma_2}\rangle &= \delta^b_{c_1}|e_{ac_2\cdots c_{M-2}\gamma_1\gamma_2}\rangle + \delta^b_{c_2}|e_{c_1ac_3\cdots c_{M-2}\gamma_1\gamma_2}\rangle + \cdots - \frac{M-2}{2}\delta^b_a|e_{c_1\cdots c_{M-2}\gamma_1\gamma_2}\rangle \\ \mathbb{L}_a{}^b|e_{c_1c_2\cdots c_{M-1}\gamma}\rangle &= \delta^b_{c_1}|e_{ac_2\cdots c_{M-1}\gamma}\rangle + \delta^b_{c_2}|e_{c_1ac_3\cdots c_{M-1}\gamma}\rangle + \cdots - \frac{M-1}{2}\delta^b_a|e_{c_1\cdots c_{M-1}\gamma}\rangle\,, \end{aligned}$$

[4] Denoted by $\langle M-1,0;\vec{C}\rangle$ in [19].

and similar formulas for $\mathbb{R}_\alpha{}^\beta$

$$\begin{aligned}
&\mathbb{R}_\alpha{}^\beta|e_{c_1c_2\cdots c_M}\rangle = 0\\
&\mathbb{R}_\alpha{}^\beta|e_{c_1c_2\cdots c_{M-2}\gamma_1\gamma_2}\rangle = \delta^\beta_{\gamma_1}|e_{c_1\cdots c_{M-2}\alpha\gamma_2}\rangle + \delta^\beta_{\gamma_2}|e_{c_1\cdots c_{M-2}\gamma_1\alpha}\rangle - \delta^\beta_\alpha|e_{c_1\cdots c_{M-2}\gamma_1\gamma_2}\rangle\\
&\mathbb{R}_\alpha{}^\beta|e_{c_1c_2\cdots c_{M-1}\gamma}\rangle = \delta^\beta_\gamma|e_{c_1\cdots c_{M-1}\alpha}\rangle - \frac{1}{2}\delta^\beta_\alpha|e_{c_1\cdots c_{M-1}\gamma}\rangle\,.
\end{aligned}$$

Then the most general action of supersymmetry generators compatible with the $\mathfrak{su}(2)$ symmetry is of the form

$$\begin{aligned}
&\mathbb{Q}_\alpha{}^a|e_{c_1c_2\cdots c_M}\rangle = a_1\left(\delta^a_{c_1}|e_{c_2\cdots c_M\alpha}\rangle + \delta^a_{c_2}|e_{c_1\cdots c_M\alpha}\rangle + \cdots\right)\,,\\
&\mathbb{Q}_\alpha{}^a|e_{c_1c_2\cdots c_{M-2}\gamma_1\gamma_2}\rangle = b_2\epsilon^{ac_{M-1}}\left(\epsilon_{\alpha\gamma_1}|e_{c_1\cdots c_{M-1}\gamma_2}\rangle - \epsilon_{\alpha\gamma_2}|e_{c_1\cdots c_{M-1}\gamma_1}\rangle\right)\,,\\
&\mathbb{Q}_\alpha{}^a|e_{c_1c_2\cdots c_{M-1}\gamma}\rangle = b_1\epsilon^{ac_M}\epsilon_{\alpha\gamma}|e_{c_1\cdots c_M}\rangle + a_2\left(\delta^a_{c_1}|e_{c_2\cdots c_{M-1}\alpha\gamma}\rangle + \delta^a_{c_2}|e_{c_1\cdots c_{M-1}\alpha\gamma}\rangle + \cdots\right)\,,
\end{aligned}$$

where the constants a_1, a_2, b_1, b_2 are functions of g, p and ζ. The action of $\mathbb{Q}^{\dagger\alpha}_a$ is given by similar formulas

$$\begin{aligned}
&\mathbb{Q}^{\dagger\alpha}_a|e_{c_1c_2\cdots c_M}\rangle = c_1\epsilon^{\alpha\gamma}\left(\epsilon_{ac_1}|e_{c_2\cdots c_M\gamma}\rangle + \epsilon_{ac_2}|e_{c_1\cdots c_M\gamma}\rangle + \cdots\right)\,,\\
&\mathbb{Q}^{\dagger\alpha}_a|e_{c_1c_2\cdots c_{M-2}\gamma_1\gamma_2}\rangle = d_2\left(\delta^\alpha_{\gamma_1}|e_{c_1\cdots c_{M-2}a\gamma_2}\rangle - \delta^\alpha_{\gamma_2}|e_{c_1\cdots c_{M-2}a\gamma_1}\rangle\right)\,,\\
&\mathbb{Q}^{\dagger\alpha}_a|e_{c_1c_2\cdots c_{M-1}\gamma}\rangle = d_1\delta^\alpha_\gamma|e_{c_1\cdots c_{M-1}a}\rangle + c_2\epsilon^{\alpha\rho}\left(\epsilon_{ac_1}|e_{c_2\cdots c_{M-1}\rho\gamma}\rangle + \epsilon_{ac_2}|e_{c_1\cdots c_{M-1}\rho\gamma}\rangle + \cdots\right)\,.
\end{aligned}$$

The familiar fundamental representation [8, 19] corresponds to $M = 1$.

The constants a_i, b_i, c_i, d_i are not arbitrary. They obey the constraints which follow from the requirement that the formulae above give a representation of $\mathfrak{su}(2|2)_c$

$$\begin{aligned}
&a_1d_1 - b_1c_1 = 1\,, \quad a_2d_2 - b_2c_2 = 1\\
&b_1d_2 = b_2d_1\,, \quad c_1a_2 = a_1c_2\,.
\end{aligned}$$

These relations show that one can always rescale the basis vectors in such a way that the parameters with the subscript 2 would be equal to those with the subscript 1

$$a_2 = a_1 \equiv \mathsf{a}\,, \quad b_2 = b_1 \equiv \mathsf{b}\,, \quad c_2 = c_1 \equiv \mathsf{c}\,, \quad d_2 = d_1 \equiv \mathsf{d}\,.$$

It is this choice we will make till the end of the paper. Thus, we have four parameters subject to the following universal M-independent constraint

$$\mathsf{ad} - \mathsf{bc} = 1\,. \tag{2.6}$$

The values of central charges, however, depend on M, and they are given by

$$\frac{H}{M} = \mathsf{ad} + \mathsf{bc}\,, \quad \frac{C}{M} = \mathsf{ab}\,, \quad \frac{\overline{C}}{M} = \mathsf{cd}\,. \tag{2.7}$$

We see that if we replace $H/M \to H$ and $C/M \to C$ we just obtain the relations of the fundamental representation. In terms of the parameters $g\,, x^\pm$ [19] this replacement is equivalent to the substitution $g \to g/M$ in the defining relation for $x^\pm$. As a result, we obtain the following convenient parametrization of $\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}$ in terms of[5] g, x^+, x^-, ζ, η

$$\mathsf{a}=\sqrt{\frac{g}{2M}}\eta\,,\ \mathsf{b}=\sqrt{\frac{g}{2M}}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-}-1\right)\,,\ \mathsf{c}=-\sqrt{\frac{g}{2M}}\frac{\eta}{\zeta x^+}\,,\ \mathsf{d}=\sqrt{\frac{g}{2M}}\frac{x^+}{i\eta}\left(1-\frac{x^-}{x^+}\right)\,. \tag{2.8}$$

Here the parameters $x^\pm$ satisfy the M-dependent constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2M}{g} i\,, \tag{2.9}$$

which follows from $\mathsf{ad}-\mathsf{bc}=1$, and they are related to the momentum p as

$$\frac{x^+}{x^-} = e^{ip}\,.$$

The values of the central charges can be found by using eq.(2.7)

$$\begin{aligned}
&H = M + \frac{ig}{x^+} - \frac{ig}{x^-} = igx^- - igx^+ - M\,, \quad H^2 = M^2 + 4g^2\sin^2\frac{p}{2}\,,\\
&C = \frac{i}{2}g\zeta\left(\frac{x^+}{x^-}-1\right) = \frac{i}{2}g\zeta\left(e^{ip}-1\right)\,, \quad \overline{C} = \frac{g}{2i\zeta}\left(\frac{x^-}{x^+}-1\right) = \frac{g}{2i\zeta}\left(e^{-ip}-1\right)\,.
\end{aligned} \tag{2.10}$$

Let us stress that according to eq.(2.2) the central charges C and $\overline{C}$ are functions of the string tension g, and the world-sheet momenta p is independent of M. This explains the rescaling $g \to g/M$ in eqs.(2.8).

The totally symmetric representation is completely determined by the parameters g, x^+, x^-, ζ, and M. The parameter η simply reflects a freedom in the choice of the basis vectors $|e_i\rangle$. For a non-unitary representation it can be set to unity by a proper rescaling of $|e_i\rangle$. As was shown in [12, 21], the string theory singles out the following choice of η and ζ

$$\zeta = e^{2i\xi}\,, \quad \eta = e^{i\xi}\, e^{\frac{i}{4}p}\,\sqrt{ix^- - ix^+}\,, \tag{2.11}$$

where the parameter ξ should be real for unitary representations. For a single symmetric representation the parameter ζ is equal to 1. The S-matrix, however, acts in the tensor product $\mathscr{V}^M(p_1, e^{ip_2}) \otimes \mathscr{V}^N(p_2, 1) \sim \mathscr{V}^M(p_1, 1) \otimes \mathscr{V}^N(p_2, e^{ip_1})$ of the representations, see [12] for a detailed discussion.

The string choice guarantees that the S-matrix satisfies the usual (non-twisted) YB equation and the generalized unitarity condition [21].

[5]The parameter ζ in [19] should be rescaled as $\zeta \to -i\zeta$ to match our definition.

2.3 The rapidity torus

The other important property of the string choice for the phase η is that it is only with this choice the parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ are meromorphic functions of the torus rapidity variable z [14]. Let us recall that the dispersion formula

$$\left(\frac{H}{M}\right)^2 - 4\left(\frac{g}{M}\right)^2 \sin^2\frac{p}{2} = 1 \tag{2.12}$$

suggests the following natural parametrization of the energy and momentum in terms of Jacobi elliptic functions

$$p = 2\,\mathrm{am}\, z\,, \qquad \sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \qquad H = M\,\mathrm{dn}(z,k)\,, \tag{2.13}$$

where we introduced the elliptic modulus[6] $k = -4g^2/M^2 < 0$. The corresponding torus has two periods $2\omega_1$ and $2\omega_2$, the first one is real and the second one is imaginary

$$2\omega_1 = 4\mathrm{K}(k)\,, \qquad 2\omega_2 = 4i\mathrm{K}(1-k) - 4\mathrm{K}(k)\,,$$

where $\mathrm{K}(k)$ stands for the complete elliptic integral of the first kind. The rapidity torus is an analog of the rapidity plane in two-dimensional relativistic models. Note, however, that the elliptic modulus and the periods of the torus depend on the dimension of the symmetric representation.

In this parametrization the real z-axis is the physical one because for real values of z the energy is positive and the momentum is real due to

$$1 \le \mathrm{dn}(z,k) \le \sqrt{k'}\,, \qquad z \in \mathbb{R}\,,$$

where $k' \equiv 1-k$ is the complementary modulus.

The representation parameters $x^\pm$, which are subject to the constraint (2.9) can be also expressed in terms of Jacobi elliptic functions as

$$x^\pm = \frac{M}{2g}\Big(\frac{\mathrm{cn}\, z}{\mathrm{sn}\, z} \pm i\Big)(1 + \mathrm{dn}\, z)\,. \tag{2.14}$$

Then, as was mentioned above, the S-matrix acts in the tensor product of two symmetric representations with parameters $\zeta = e^{2i\xi}$ equal to either e^{ip_2} or e^{ip_1}, and, therefore, the factor $e^{i\xi}$ which appears in the expression (2.11) for η is a meromorphic function of the torus rapidity variable z. Thus, if the parameter η is a meromorphic function of z then the parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ of the symmetric representation also are meromorphic functions of z. Indeed, as was shown in [21], one can resolve the branch cut ambiguities of η by means of the following relation

$$e^{\frac{i}{4}p}\sqrt{ix^-(p) - ix^+(p)} = \frac{\sqrt{2M}}{\sqrt{g}}\frac{\mathrm{dn}\,\frac{z}{2}\big(\mathrm{cn}\,\frac{z}{2} + i\,\mathrm{sn}\,\frac{z}{2}\mathrm{dn}\,\frac{z}{2}\big)}{1 + \frac{4g^2}{M^2}\mathrm{sn}^4\frac{z}{2}} \equiv \eta(z,M) \tag{2.15}$$

[6] Our convention for the elliptic modulus is the same as accepted in the *Mathematica* program, e.g., $\mathrm{sn}(z,k) = \mathrm{JacobiSN}[z,k]$.

valid in the region $-\frac{\omega_1}{2} < \operatorname{Re} z < \frac{\omega_1}{2}$ and $i\,\omega_2 < \operatorname{Im} z < -i\,\omega_2$. We conclude, therefore, that with the choice of η made in [21] the parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ of the symmetric representation are meromorphic functions of the torus rapidity variable z. As a consequence, the S-matrix is also a meromorphic function of z_1, z_2 (up to a non-meromorphic scalar factor).

3. S-matrix in the superspace

In this section we identify the totally symmetric representations with the $2M|2M$-dimensional graded vector space of monomials of degree M of two bosonic and two fermionic variables. The generators of the centrally extended $\mathfrak{su}(2|2)$ algebra are realized as differential operators acting in this space. The S-matrix is naturally realized in this framework as a differential $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant operator in the tensor product of two representations.

3.1 Operator realization of $\mathfrak{su}(2|2)_c$

It is well-known that the $\mathfrak{su}(2)$ algebra can be realized by differential operators acting in the vector space of analytic functions of two bosonic variables w_1, w_2. An irreducible representation of spin j is then identified with the vector subspace of monomials of degree $2j$.

A similar realization also exists for the centrally extended $\mathfrak{su}(2|2)$ algebra. To this end we introduce a vector space of analytic functions of two bosonic variables w_a, and two fermionic variables θ_α. Since any such a function can be expanded into a sum

$$\Phi(w,\theta) = \sum_{M=0}^{\infty} \Phi_M(w,\theta)$$

of homogeneous symmetric polynomials of degree M

$$\Phi_M(w,\theta) = \phi^{c_1\dots c_M} w_{c_1}\cdots w_{c_M} + \phi^{c_1\dots c_{M-1}\gamma} w_{c_1}\cdots w_{c_{M-1}}\theta_\gamma + \\ + \phi^{c_1\dots c_{M-2}\gamma_1\gamma_2} w_{c_1}\cdots w_{c_{M-2}}\theta_{\gamma_1}\theta_{\gamma_2} \qquad (3.1)$$

the vector space is in fact isomorphic to the direct sum of all totally symmetric representations of $\mathfrak{su}(2|2)_c$.

Then, the bosonic and fermionic generators of $\mathfrak{su}(2|2)_c$ can be identified with the following differential operators acting in the space

$$
\begin{aligned}
\mathbb{L}_a{}^b &= w_a \frac{\partial}{\partial w_b} - \frac{1}{2}\delta_a^b w_c \frac{\partial}{\partial w_c}\,, & \mathbb{R}_\alpha{}^\beta &= \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2}\delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma} \\
\mathbb{Q}_\alpha{}^a &= \mathsf{a}\,\theta_\alpha \frac{\partial}{\partial w_a} + \mathsf{b}\,\epsilon^{ab}\epsilon_{\alpha\beta} w_b \frac{\partial}{\partial \theta_\beta}\,, & \mathbb{Q}_a^{\dagger\,\alpha} &= \mathsf{d}\,w_a \frac{\partial}{\partial \theta_\alpha} + \mathsf{c}\,\epsilon_{ab} e^{\alpha\beta}\theta_\beta \frac{\partial}{\partial w_b} \\
\mathbb{C} &= \mathsf{ab}\Big(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\Big)\,, & \mathbb{C}^\dagger &= \mathsf{cd}\Big(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\Big) \\
& \mathbb{H} = (\mathsf{ad}+\mathsf{bc})\Big(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\Big)\,. & & \qquad (3.2)
\end{aligned}
$$

Here we use the conventions

$$
\frac{\partial}{\partial \theta_\alpha}\theta_\beta + \theta_\beta \frac{\partial}{\partial \theta_\alpha} = \delta_\beta^\alpha\,, \qquad \epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta_\alpha^\beta\,, \qquad \epsilon^{\alpha\beta}\epsilon_{\rho\delta} = \delta_\rho^\alpha \delta_\delta^\beta - \delta_\rho^\beta \delta_\delta^\alpha\,,
$$

and the constants $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ satisfy the only condition

$$
\mathsf{ad} - \mathsf{bc} = 1\,,
$$

and are obviously identified with the parameters of the totally symmetric representation we discussed in the previous section.

The representation carried by $\Phi(w,\theta)$ is reducible, and to single out an irreducible component one should restrict oneself to an irreducible superfield $\Phi_M(w,\theta)$ (3.1). Then the basis vectors $|e_{c_1 c_2 \cdots c_M}\rangle$, $|e_{c_1 c_2 \cdots c_{M-1}\gamma}\rangle$ and $|e_{c_1 c_2 \cdots c_{M-2}\gamma_1\gamma_2}\rangle$ of a totally symmetric representation should be identified with the monomials $w_{c_1}\cdots w_{c_M}$, $w_{c_1}\cdots w_{c_{M-1}}\theta_\gamma$, and $w_{c_1}\cdots w_{c_{M-2}}\theta_{\gamma_1}\theta_{\gamma_2}$, respectively.

One can define a natural scalar product on the vector space. One introduces the following basis of monomials

$$
|m,n,\mu,\nu\rangle = N_{mn\mu\nu}\, w_1^m\, w_2^n\, \theta_3^\mu\, \theta_4^\nu\,, \tag{3.3}
$$

where $m,n \geq 0\,,\ \mu,\nu = 0,1\,,\ m+n+\mu+\nu = M$. This basis is assumed to be orthonormal

$$
\langle a,b,\alpha,\beta|m,n,\mu,\nu\rangle = \delta_{am}\delta_{bn}\delta_{\alpha\mu}\delta_{\beta\nu}\,.
$$

The normalization constants $N_{mn\mu\nu}$ can then be determined from the requirement of the unitarity of the representation, see e.g. [50] for the $\mathfrak{su}(2)$ case. In particular the hermiticity condition for the generators $\mathbb{L}_a{}^b$: $\left(\mathbb{L}_a{}^b\right)^\dagger = \mathbb{L}_b{}^a$ leads to the relation [50]

$$
N_{mn\mu\nu} = \left(\frac{1}{m!\,n!}\right)^{1/2} N(m+n)\,.
$$

Further, the condition that the generators $\mathbb{Q}_\alpha{}^a$ and $\mathbb{Q}_a^{\dagger\,\alpha}$ are hermitian conjugate to each other fixes the normalization constants $N(m+n)$ to coincide

$$
N(M-2) = N(M-1) = N(M)\,.
$$

The overall normalization constant can be set to any number, and we choose it so that the normalization constant $N_{M-2,0,1,1}$ be equal to 1. This gives

$$N_{mn\mu\nu} = \left(\frac{(M-2)!}{m!\,n!}\right)^{1/2} .$$

Having defined the scalar product, one can easily check that both the transposed and hermitian conjugate operators are obtained by using the following rules

$$\begin{aligned} (w_a)^\dagger &= \frac{\partial}{\partial w_a}\,, \quad \left(\frac{\partial}{\partial w_a}\right)^\dagger = w_a\,, \quad (\theta_\alpha)^\dagger = \frac{\partial}{\partial\theta_\alpha}\,, \quad \left(\frac{\partial}{\partial\theta_\alpha}\right)^\dagger = \theta_\alpha \\ (w_a)^t &= \frac{\partial}{\partial w_a}\,, \quad \left(\frac{\partial}{\partial w_a}\right)^t = w_a\,, \quad (\theta_\alpha)^t = \frac{\partial}{\partial\theta_\alpha}\,, \quad \left(\frac{\partial}{\partial\theta_\alpha}\right)^t = \theta_\alpha\,, \end{aligned} \tag{3.4}$$

that means that $w_a\,,\ \frac{\partial}{\partial w_a}\,,\ \theta_\alpha\,,\ \frac{\partial}{\partial\theta_\alpha}$ are considered to be real. For the product of several operators the usual rules are applied: $(\mathbb{A}\mathbb{B})^\dagger = \mathbb{B}^\dagger\mathbb{A}^\dagger$ and $(\mathbb{A}\mathbb{B})^t = \mathbb{B}^t\mathbb{A}^t$. In particular, by using these rules one can readily verify that the central element $\mathbb{H}$ is hermitian, and the supersymmetry generators $\mathbb{Q}_\alpha{}^a$ and $\mathbb{Q}^{\dagger}{}_a^{\alpha}$, and the central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ are hermitian conjugate to each other, provided the parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ satisfy the following relations: $\mathsf{d}^* = \mathsf{a}\,,\ \mathsf{c}^* = \mathsf{b}$.

The basis of monomials (3.3) can be also used to find the matrix form of the algebra generators. Denoting the basis vectors of the M-particle bound state representation as $|e_i\rangle$, where $i = 1, \ldots, 4M$, we can define the matrix elements of any differential operator $\mathbb{O}$ acting in the vector space of monomials by the formula

$$\mathbb{O}\cdot|e_i\rangle = O_i^k(z)\,|e_k\rangle, \tag{3.5}$$

and use them to construct the matrix form of $\mathbb{O}$

$$O(z) = \sum_{ik} O_i^k(z)\, E_k{}^i\,, \tag{3.6}$$

where $E_k{}^i \equiv E_{ki}$ are the usual matrix unities.

Let us finally note that the $\mathfrak{sl}(2)$ external automorphism of $\mathfrak{su}(2|2)_{\mathcal{C}}$ just redefines the constants $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ as follows

$$\tilde{\mathsf{a}} = \mathsf{a}\,u_1 - \mathsf{c}\,u_2\,, \quad \tilde{\mathsf{b}} = \mathsf{b}\,u_1 - \mathsf{d}\,u_2\,; \qquad \tilde{\mathsf{c}} = \mathsf{c}\,v_1 - \mathsf{a}\,v_2\,, \quad \tilde{\mathsf{d}} = \mathsf{d}\,v_1 - \mathsf{b}\,v_2\,, \tag{3.7}$$

and can be used to set, e.g. $\mathsf{b} = 0 = \mathsf{c}$. This simplifies the derivation of the S-matrix.

3.2 S-matrix operator

It is clear that the graded tensor product $\mathscr{V}^{M_1}(p_1,\zeta_1)\otimes\mathscr{V}^{M_2}(p_2,\zeta_2)$ of two totally symmetric representations can be identified with a product of two superfields Φ_{M_1}

and Φ_{M_2} depending on different sets of coordinates. An algebra generator acting in the product is given by the sum of two differential operators each acting on its own superfield and depending on its own set of parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ or, equivalently, on p and ζ. Given any two sets of parameters, one obtains a representation of $\mathfrak{su}(2|2)_c$ (generically reducible) with central charges $H, C, \overline{C}$ equal to the sum of central charges of the symmetric representations: $C = C_1 + C_2$, and so on. The same tensor product representation can be obtained from two other symmetric representations $\mathscr{V}^{M_1}(p_1, \tilde{\zeta}_1)\,, \mathscr{V}^{M_2}(p_2, \tilde{\zeta}_2)$ if the parameters $\tilde{\zeta}_i$ satisfy certain shortening conditions [19].

As was extensively discussed in [12], in string theory the Hilbert space of two-particle in-states is identified with the tensor product $\mathscr{V}^{M_1}(p_1, e^{ip_2}) \otimes \mathscr{V}^{M_2}(p_2, 1)$, and the Hilbert space of two-particle out-states is identified with $\mathscr{V}^{M_1}(p_1, 1) \otimes \mathscr{V}^{M_2}(p_2, e^{ip_1})$. The two Hilbert spaces are isomorphic to the product of two superfields Φ_{M_1} and Φ_{M_2}, and the S-matrix $\mathbb{S}$ is an intertwining operator

$$\mathbb{S}(p_1, p_2): \qquad \mathscr{V}^{M_1}(p_1, e^{ip_2}) \otimes \mathscr{V}^{M_2}(p_2, 1) \to \mathscr{V}^{M_1}(p_1, 1) \otimes \mathscr{V}^{M_2}(p_2, e^{ip_1})\,. \tag{3.8}$$

It is realized as a differential operator acting on $\Phi_{M_1}(w_a^1, \theta_\alpha^1)\Phi_{M_2}(w_a^2, \theta_\alpha^2)$, and satisfying the following invariance condition

$$\mathbb{S}(p_1, p_2) \cdot (\mathbb{J}(p_1, e^{ip_2}) + \mathbb{J}(p_2, 1)) = (\mathbb{J}(p_1, 1) + \mathbb{J}(p_2, e^{ip_1})) \cdot \mathbb{S}(p_1, p_2) \tag{3.9}$$

for any of the symmetry generators. Here $\mathbb{J}(p_i, \zeta)$ stands for any of the generators (3.2) realized as a differential operator acting on functions of w_a^i, θ_α^i. The parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ of the operator coefficients are expressed through p_i and ζ by means of formulae (2.8) with η given by eq.(2.11).

Since $\Phi_{M_1}\Phi_{M_2}$ is isomorphic to the graded tensor product of two symmetric representations, the S-matrix operator (3.8) satisfying eq.(3.9) is analogous to the so-called fermionic R-operator, see e.g. [51]. This analogy becomes even more manifest when one considers the action of $\mathbb{S}$ on the tensor product of several symmetric representation. For this reason $\mathbb{S}$ could be also called the fermionic $\mathbb{S}$-operator. In particular, in absence of interactions, i.e. in the limit $g \to \infty$, it must reduce to the identity operator.

It is clear that for bosonic generators $\mathbb{L}_a{}^b, \mathbb{R}_\alpha{}^\beta$ generating the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ subalgebra, the invariance condition (3.9) takes the standard form

$$\left[\, \mathbb{S}(p_1, p_2)\,,\, \mathbb{L}_a{}^b \,\right] = 0 = \left[\, \mathbb{S}(p_1, p_2)\,,\, \mathbb{R}_\alpha{}^\beta \,\right]\,. \tag{3.10}$$

Therefore, the S-matrix is a $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$-invariant differential operator which maps $\Phi_{M_1}(w_a^1, \theta_\alpha^1)\Phi_{M_2}(w_a^2, \theta_\alpha^2)$ into a linear combination of the products of two homogeneous polynomials of degree M_1 and M_2.

Any differential operator acting in $\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2}$ can be viewed as an element of

$$\mathrm{End}(\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2}) \approx \mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2} \,\otimes\, \mathscr{D}_{M_1} \otimes \mathscr{D}_{M_2}\,,$$

where $\mathscr{D}_M$ is the vector space dual to $\mathscr{V}^M$. The dual space is realized as the space of polynomials of degree M in the derivative operators $\frac{\partial}{\partial w_a}$, $\frac{\partial}{\partial \theta_\alpha}$. A natural pairing between $\mathscr{D}_M$ and $\mathscr{V}^M$ is induced by $\frac{\partial}{\partial w_a} w_b = \delta_a^b$, $\frac{\partial}{\partial \theta_\alpha}\theta_\beta = \delta_\alpha^\beta$. Therefore, eq.(3.10) means that the S-matrix is a $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ singlet (invariant) component in the tensor product decomposition of $\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2} \otimes \mathscr{D}_{M_1} \otimes \mathscr{D}_{M_2}$.

Thus, the S-matrix can be naturally represented as

$$\mathbb{S}(p_1,p_2) = \sum_i a_i(p_1,p_2)\, \Lambda_i\,, \tag{3.11}$$

where Λ_i span a complete basis of differential $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$-invariant operators (obviously independent of p_1, p_2), and a_i are coefficients of the S-matrix which could be determined from the remaining invariance conditions (3.9), and some additional requirements such as the YB equation.

The basis of the operators Λ_i can be easily found by first branching the symmetric representations $\mathscr{V}^{M_1}, \mathscr{V}^{M_2}$ into their $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ components, see eq.(3.1)

$$\mathscr{V}^{M_i} = V^{\frac{M_i}{2}} \times V^0_{(0)} + V^{\frac{M_i-1}{2}} \times V^{1/2}_{(1)} + V^{\frac{M_i-2}{2}} \times V^0_{(2)} = \sum_{k=0}^{2} V^{\frac{M_i-k}{2}} \times V^{j_k}_{(k)}\,.$$

Here V^j denotes a spin j representation of $\mathfrak{su}(2)$ realized in the space of homogeneous symmetric polynomials of degree $2j$ of two bosonic variables, while $V^{j_k}_{(k)}$ stands for a spin j_k representation of $\mathfrak{su}(2)$ realized in the space of homogeneous degree k polynomials of two fermionic variables, so that k can take only 3 values: $k = 0, 1, 2$, and $j_0 = 0$, $j_1 = 1/2$, $j_2 = 0$.

The tensor product $\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2}$ is then decomposed into a sum of irreducible $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ components as follows

$$\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2} = \sum_{k_1,k_2=0}^{2} \sum_{j_B=\frac{|M_2-M_1-k_2+k_1|}{2}}^{\frac{M_2+M_1-k_2-k_1}{2}} V^{j_B}_{(M_1-k_1, M_2-k_2)} \times \sum_{j_F=|j_{k_1}-j_{k_2}|}^{j_{k_1}+j_{k_2}} V^{j_F}_{(k_1,k_2)}\,. \tag{3.12}$$

Here $V^{j_B}_{(M,N)}$ denotes a spin j_B representation of $\mathfrak{su}(2)$, which is realized in the space spanned by products of degree M with degree N symmetric polynomials, the first one in two bosonic variables w^1_a and the second in w^2_a. Analogously, $V^{j_F}_{(k_1,k_2)}$ stands for a spin j_F representation of $\mathfrak{su}(2)$ realized in the space of products of degree k_1 polynomials in two fermionic variables θ^1_α with degree k_2 polynomials in two other fermionic variables θ^2_α.

Finally, the dual space $\mathscr{D}_{M_1} \otimes \mathscr{D}_{M_2}$ has a similar decomposition (3.12) in terms of the vector spaces $D^j_{(n_1,n_2)}$ dual to $V^j_{(n_1,n_2)}$. Since $V^j \otimes D^j$ contains the singlet component (the invariant), the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$-invariant differential operators simply correspond to the singlet components of the spaces $V^j \times V^k \otimes D^j \times D^k$ in the tensor product decomposition of $\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2} \otimes \mathscr{D}_{M_1} \otimes \mathscr{D}_{M_2}$.

In principle, it is not difficult to count the number of components in eq.(3.12) as well as the number of invariant operators for arbitrary M_1, M_2. We consider here[7] only the special case of $M_1 = 1\,,\ M_2 = M$ which corresponds to the scattering of a fundamental particle with a M-particle bound state. One has to consider separately three different cases: $M = 1$, $M = 2$ and $M \geq 3$.

For $M = 1$, by using (3.12), one gets

$$\begin{aligned}\mathscr{V}^1 \otimes \mathscr{V}^1 &= V^1_{(1,1)} \times V^0_{(0,0)} + V^0_{(0,0)} \times V^1_{(1,1)} \\ &\quad + V^0_{(1,1)} \times V^0_{(0,0)} + V^0_{(0,0)} \times V^0_{(1,1)} + V^{1/2}_{(1,0)} \times V^{1/2}_{(0,1)} + V^{1/2}_{(0,1)} \times V^{1/2}_{(1,0)}\,.\end{aligned} \tag{3.13}$$

We see that there are six components in this decomposition and they give rise to six *diagonal invariants* of the symbolic form $V^{j_B}_{(N_1,N_2)} \times V^{j_F}_{(k_1,k_2)} \cdot D^{j_B}_{(N_1,N_2)} \times D^{j_F}_{(k_1,k_2)}$. Then the representations $V^0 \times V^0$ and $V^{1/2} \times V^{1/2}$ come with multiplicity 2, and that gives additional four *off-diagonal invariants* of the form $V^{j_B}_{(N_1,N_2)} \times V^{j_F}_{(k_1,k_2)} \cdot D^{j_B}_{(K_1,K_2)} \times D^{j_F}_{(n_1,n_2)}$ with $\{N_i, k_i\} \neq \{K_i, n_i\}$. Thus, the total number of invariant operators in the $M_1 = M_2 = 1$ case is 10.

In the case $M_1 = 1, M_2 = 2$ we get the following decomposition with the nine components

$$\begin{aligned}\mathscr{V}^1 \otimes \mathscr{V}^2 &= V^{3/2}_{(1,2)} \times V^0_{(0,0)} + V^{1/2}_{(0,1)} \times V^1_{(1,1)} + V^1_{(1,1)} \times V^{1/2}_{(0,1)} + V^1_{(0,2)} \times V^{1/2}_{(1,0)} \\ &+ V^0_{(1,1)} \times V^{1/2}_{(0,1)} + V^0_{(0,0)} \times V^{1/2}_{(1,2)} + V^{1/2}_{(1,2)} \times V^0_{(0,0)} + V^{1/2}_{(1,0)} \times V^0_{(0,2)} + V^{1/2}_{(0,1)} \times V^0_{(1,1)}\end{aligned} \tag{3.14}$$

giving rise to 9 diagonal invariants. Representations $V^1 \times V^{1/2}$ and $V^0 \times V^{1/2}$ come with multiplicity 2, and $V^{1/2} \times V^0$ occurs with multiplicity 3, and that gives additional 10 off-diagonal invariants Thus, the total number of invariant operators in the $M_1 = M_2 = 1$ case is 19.

Finally, in the case $M_1 = 1, M_2 \equiv M \geq 3$ we get the following decomposition over the ten components

$$\begin{aligned}\mathscr{V}^1 \otimes \mathscr{V}^M &= V^{\frac{M+1}{2}}_{(1,M)} \times V^0_{(0,0)} + V^{\frac{M-1}{2}}_{(0,M-1)} \times V^1_{(1,1)} + V^{\frac{M-3}{2}}_{(1,M-2)} \times V^0_{(0,2)} \\ &+ V^{\frac{M}{2}}_{(1,M-1)} \times V^{1/2}_{(0,1)} + V^{\frac{M}{2}}_{(0,M)} \times V^{1/2}_{(1,0)} + V^{\frac{M-2}{2}}_{(1,M-1)} \times V^{1/2}_{(0,1)} + V^{\frac{M-2}{2}}_{(0,M-2)} \times V^{1/2}_{(1,2)} \\ &+ V^{\frac{M-1}{2}}_{(1,M)} \times V^0_{(0,0)} + V^{\frac{M-1}{2}}_{(1,M-2)} \times V^0_{(0,2)} + V^{\frac{M-1}{2}}_{(0,M-1)} \times V^0_{(1,1)}\,,\end{aligned} \tag{3.15}$$

which produces 10 diagonal and 10 off-diagonal invariants. So, the total number of invariant operators in the $M_1 = 1\,,\ M_2 \geq 3$ case is 20, and it is independent of M_2.

We will fix the normalization of diagonal invariants such that they provide the orthogonal decomposition of the identity operator

$$\mathbb{I} = \sum \Lambda_i^{\text{diag}}\,.$$

[7] The case $M_1 = M_2 = 2$ will be discussed in detail in section 4.3.

In what follows we will always denote the diagonal invariant operator corresponding to the component $V^{\frac{M_1+M_2}{2}}_{(M_1,M_2)} \times V^0_{(0,0)}$ with the maximum spin $j_B = (M_1 + M_2)/2$ as Λ_1, and we will set the coefficient a_1 of the S-matrix (3.11) to be equal to one.[8]

Having found all invariant operators, one can use the invariance condition (3.9) to determine some of the coefficients a_i. A natural question is how many coefficients will be left undetermined or, in other words, how many different (up to an overall factor) $\mathfrak{su}(2|2)$ invariant S-matrices exist. To answer to this question, one should decompose the tensor product $\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2}$ into the sum of irreducible $\mathfrak{su}(2|2)$ components. Then the number of different $\mathfrak{su}(2|2)$ invariant S-matrices is equal to the number of components in the decomposition. The tensor product decomposition was studied in [19] with the result

$$\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2} = \sum_{n=|M_1-M_2|+2}^{M_1+M_2} \mathscr{W}^n \,, \tag{3.16}$$

where $\mathscr{W}^n$ is a typical (or long) irreducible supermultiplet of dimension $16(n-1)$ denoted as $\{n-2, 0; \vec{C}\}$ in [19].

Assuming that $M_1 \leq M_2$, we find M_1 irreducible components in eq.(3.16), and, therefore, there exists M_1 $\mathfrak{su}(2|2)$ invariant S-matrices solving eq.(3.9). Setting $a_1 = 1$ leaves $M_1 - 1$ coefficients undetermined. They could be fixed by imposing additional equations such as the unitarity condition and the YB equation.

In the simplest case $M_1 = 1\,,\ M_2 \equiv M$ there is only one component in the decomposition (3.16)

$$\mathscr{V}^1 \otimes \mathscr{V}^M = \mathscr{W}^{M+1} \,, \tag{3.17}$$

and, as the consequence, the $\mathfrak{su}(2|2)$ invariant S-matrix is determined uniquely up to an overall factor.

The first nontrivial case corresponds to $M_1 = 2\,,\ M_2 \equiv M \geq 2$, where eq.(3.16) has two components

$$\mathscr{V}^2 \otimes \mathscr{V}^M = \mathscr{W}^{M+2} + \mathscr{W}^M \,. \tag{3.18}$$

A coefficient of the S-matrix which cannot be determined by the invariance condition is fixed by the YB equation as we will demonstrate in section 4.3.

3.3 Matrix form of the S-matrix

The operator formalism provides a very efficient way of solving the invariance conditions (3.9), and checking the YB equation. It is convenient, however, to know

[8]With this choice the string S-matrix can be obtained from (3.11) by multiplying it by the scalar S-matrix of the $\mathfrak{su}(2)$ sector, see section 5 for details.

also the usual matrix form of the S-matrix operator. It can be used to check such properties of the S-matrix as unitarity and crossing symmetry.

To find the matrix form of the S-matrix realized the invariant differential operator, we use the basis of monomials (3.3). Denoting the basis vectors of the M_1- and M_2-particle bound state representations as $|e_i\rangle$ and $|e_I\rangle$, respectively, and the basis vectors of the tensor product of these representations as $|e_{iJ}\rangle$ where $i = 1, \ldots, 4M_1$ and $J = 1, \ldots, 4M_2$, we define the matrix elements of any operator $\mathbb{O}$ acting in the tensor product by the formula

$$\mathbb{O} \cdot |e_{iJ}\rangle = O_{iJ}^{kL}(z_1, z_2)\,|e_{kL}\rangle. \tag{3.19}$$

The elements O_{iJ}^{kL} can be now used to construct various matrix forms of $\mathbb{O}$. In particular, the matrix form of the S-matrix $\mathbb{S}$ which satisfies the *graded* YB equation is given by

$$S_{12}^{g}(z_1, z_2) = \sum_{ikJL} S_{iJ}^{kL}(z_1, z_2)\, E_k{}^i \otimes E_L{}^J\,, \tag{3.20}$$

where $E_k{}^i \equiv E_{ki}$ are the usual matrix unities.

On the other hand, the matrix form of the S-matrix $\mathbb{S}$ which satisfies the *usual* YB equation, see section 4.1, is obtained by multiplying S_{12}^g by the graded identity $I_{12}^g = (-1)^{\epsilon_i\epsilon_J} E_i^i \otimes E_J^J$, where $\epsilon_i\,, \epsilon_J$ are equal to 0 for bosonic and to 1 for fermionic states. It is given by

$$S_{12}(z_1, z_2) = I_{12}^g S_{12}^g = \sum_{ikJL} S_{iJ}^{kL}(z_1, z_2)\,(-1)^{\epsilon_k\epsilon_L}\, E_k{}^i \otimes E_L{}^J\,. \tag{3.21}$$

For reader's convenience we list in appendices the matrix form of the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariant differential operators expressing them as sums over symbols E_{kiLJ}, the latter can be equal to either $E_k{}^i\otimes E_L{}^J$ or to $(-1)^{\epsilon_k\epsilon_L}\, E_k{}^i\otimes E_L{}^J$ (or anything else one wants).

3.4 General properties of S-matrix

In a physical theory the S-matrix should satisfy some general properties.

Physical Unitarity. First of all, for real values of the momenta p_1, p_2 or, equivalently, for the real torus rapidity variables z_1, z_2, the S-matrix must be a unitary operator. Moreover, in a relativistic theory the S-matrix satisfies a more general condition $\mathbb{S}(\theta^*)^\dagger \cdot \mathbb{S}(\theta) = \mathbb{I}$ called the generalized unitarity. For a non-relativistic theory with a S-matrix depending on two parameters the condition takes the following form

$$\mathbb{S}(z_1^*, z_2^*)^\dagger \cdot \mathbb{S}(z_1, z_2) = \mathbb{I}\,. \tag{3.22}$$

To analyze the generalized unitarity condition, we notice that the diagonal invariant operators can be normalized to be hermitian, and it is convenient to normalize and order the off-diagonal invariant operators in such a way that they come in hermitian-conjugate pairs

$$\left(\Lambda_i^{\rm diag}\right)^\dagger = \Lambda_i^{\rm diag}\,, \qquad \left(\Lambda_{2k}^{\rm off-diag}\right)^\dagger = \Lambda_{2k+1}^{\rm off-diag}\,, \tag{3.23}$$

where k takes integer values if the number of diagonal operators is odd, and half-integer values if the number of diagonal operators is even.

It is clear that the generalized unitarity condition imposes severe restrictions on the coefficients a_i of the S-matrix. As was shown in [21], the S-matrix of fundamental particles of the light-cone string theory satisfies the generalized unitarity only for a special choice of the parameters η_i of the fundamental representations. Then, the only degree of arbitrariness left is a constant unitary rotation of the basis of the fundamental representation. We will show in section 4 that the bound state S-matrices also satisfy the generalized unitarity condition.

CPT invariance. In a CPT-invariant relativistic field theory the S-matrix must be a symmetric operator, and we impose the same condition on the scattering matrix for any bound states of the light-cone string theory

$$\mathbb{S}(z_1,z_2)^T = \mathbb{S}(z_1,z_2)\,. \tag{3.24}$$

Taking into account that, due to the definition (3.4), the transposition acts on the invariant operators in the same way as the hermitian conjugation (3.23)

$$\left(\Lambda_i^{\rm diag}\right)^T = \Lambda_i^{\rm diag}\,, \qquad \left(\Lambda_{2k}^{\rm off-diag}\right)^T = \Lambda_{2k+1}^{\rm off-diag}\,,$$

we find that for a symmetric S-matrix no restriction arises on the coefficients a_i corresponding to the diagonal invariant operators, while the off-diagonal coefficients must satisfy the following non-trivial relations

$$a_{2k}^{\rm off-diag} = a_{2k+1}^{\rm off-diag}\,. \tag{3.25}$$

We will see in section 4 that these relations do hold but again only with the choice of η_i as made in [21].

Parity Transformation. The S-matrix is an invertible operator, and one can ask whether the inverse S-matrix is related to the S-matrix itself but not through the inversion procedure. One relation has been already described above, where the inverse S-matrix is just the hermitian conjugate one. Another quite natural relation arises due to the parity transformation $\mathscr{P}$ of the world-sheet coordinate σ, which changes the sign of the torus rapidity variables $z_i \to -z_i$. To describe the action of the parity transformation on the invariant operators, one notices that since they are homogeneous polynomials in fermions θ^i_α and derivatives $\partial/\partial\theta^i_\alpha$, one can introduce the following natural grading on the space of invariant operators

$$\epsilon_\Lambda = \frac{1}{2}\left(n_\theta - n_{\partial/\partial\theta}\right)\,, \tag{3.26}$$

where n_θ and $n_{\partial/\partial\theta}$ are the numbers of θ's and $\partial/\partial\theta$'s occurring in an invariant operator Λ, respectively. It is not difficult to see that the degree of any invariant operator is an integer. Furthermore, the vector space of invariant operators can be supplied with the structure of a graded algebra because

$$\epsilon_{\Lambda_1\cdot\Lambda_2} = \epsilon_{\Lambda_1} + \epsilon_{\Lambda_2}\,.$$

An action of the parity transformation on any invariant operator can be defined in the following way

$$\Lambda^{\mathscr{P}} = (-1)^{\epsilon_\Lambda}\,\Lambda\,. \tag{3.27}$$

Then, the inverse operator $\mathbb{S}^{-1}$ must coincide with the parity-transformed operator

$$\mathbb{S}^{-1}(z_1,z_2) = \mathbb{S}(-z_1,-z_2)^{\mathscr{P}} = \sum_k a_k(-z_1,-z_2)(-1)^{\epsilon_{\Lambda_k}}\,\Lambda_k\,. \tag{3.28}$$

This gives a very simple way of finding the inverse S-matrix. Comparing eqs.(3.22), (3.25) and (3.28), we obtain the following relations between the coefficients of the S-matrix

$$a_k(z_1^*,z_2^*)^* = (-1)^{\epsilon_{\Lambda_k}}\,a_k(-z_1,-z_2)\,. \tag{3.29}$$

Coefficients of the bound state S-matrices we study in section 4 do indeed satisfy these relations.

Unitarity and Hermitian Analyticity. To discuss the next property, we need to distinguish the S-matrix $\mathbb{S}^{MN}$ which describes the scattering of a M-particle bound state with a N-particle one and acts as an intertwiner

$$\mathbb{S}^{MN}(z_1,z_2) = \sum_k a_k(z_1,z_2)\Lambda_k\,: \quad \mathscr{V}^M\otimes\mathscr{V}^N \;\to\; \mathscr{V}^M\otimes\mathscr{V}^N\,, \tag{3.30}$$

from the S-matrix $\mathbb{S}^{NM}$ which describes the scattering of a N-particle bound state with a M-particle one

$$\mathbb{S}^{NM}(z_1,z_2) = \sum_k b_k(z_1,z_2)\Lambda_k^{NM}\,: \quad \mathscr{V}^N\otimes\mathscr{V}^M \;\to\; \mathscr{V}^N\otimes\mathscr{V}^M\,. \tag{3.31}$$

It is clear that the number of invariant operators in $\mathscr{V}^M\otimes\mathscr{V}^N$ and $\mathscr{V}^N\otimes\mathscr{V}^M$ is the same, and, moreover, if Λ_k acts on the products $\Phi_M(w_a^1,\theta_\alpha^1)\Phi_N(w_a^2,\theta_\alpha^2)$ then Λ_k^{NM} acts on the products $\Phi_N(w_a^1,\theta_\alpha^1)\Phi_M(w_a^2,\theta_\alpha^2)$ and it is obtained from Λ_k by means of exchange $w_a^1\leftrightarrow w_a^2$, $\theta_\alpha^1\leftrightarrow\theta_\alpha^2$. Note also that in eq.(3.30) the variable z_1 is the torus rapidity variable of the M-particle bound state, while in eq.(3.31) z_1 is the torus rapidity variable of the N-particle bound state.

The coefficients b_k of the S-matrix $\mathbb{S}^{NM}$ can be used to construct an invariant operator which acts in $\mathscr{V}^M \otimes \mathscr{V}^N$

$$\mathbb{S}_{21}^{NM}(z_2, z_1) = \sum_k b_k(z_2, z_1)\Lambda_k : \quad \mathscr{V}^M \otimes \mathscr{V}^N \quad \to \quad \mathscr{V}^M \otimes \mathscr{V}^N , \tag{3.32}$$

where the subscript 21 indicates that the order of Φ_N and Φ_M was exchanged. We also exchanged z_1 and z_2 to attach z_1 to the M-particle bound state. It is clear that the combined exchange is just the graded permutation of the spaces $\mathscr{V}^N$ and $\mathscr{V}^M$.

Then, the unitarity condition[9] states that the operator $\mathbb{S}_{21}^{NM}(z_2, z_1)$ is the operator inverse to the S-matrix operator $\mathbb{S}^{MN}(z_1, z_2)$

$$\mathbb{S}_{21}^{NM}(z_2, z_1) \cdot \mathbb{S}^{MN}(z_1, z_2) = \mathbb{I} . \tag{3.33}$$

Comparing this formula with eq.(3.28) for the inverse S-matrix operator and eq.(3.29), we get that the coefficients b_k are related to a_k in the following way

$$b_k(z_2, z_1) = (-1)^{\epsilon_{\Lambda_k}} a_k(-z_1, -z_2) = a_k(z_1^*, z_2^*)^* . \tag{3.34}$$

As the consequence of eq.(3.33) and eq.(3.22) one finds another relation

$$\mathbb{S}^{MN}(z_1^*, z_2^*)^\dagger = \mathbb{S}_{21}^{NM}(z_2, z_1) , \tag{3.35}$$

which is the operator form of the hermitian analyticity condition.

In the case of the scattering of M-particle bound states between themselves, that is for $M = N$, there is one more relation between the S-matrix and its inverse. To find it, we notice that for $M = N$ the action of the graded permutation $\mathcal{P}$ on the invariant operators Λ_i can be realized by means of exchange $w_a^1 \leftrightarrow w_a^2$, $\theta_\alpha^1 \leftrightarrow \theta_\alpha^2$. Under this exchange an invariant operator transforms to another invariant operator, the latter, due to $M = N$, acts in the same space

$$\Lambda_k \to \Lambda_k^{\mathcal{P}} .$$

Then, the operator $\mathbb{S}_{21}^{MM}(z_2, z_1) \equiv \mathbb{S}_{21}(z_2, z_1)$ can be obtained from $\mathbb{S}^{MM}(z_1, z_2) \equiv \mathbb{S}(z_1, z_2)$ by exchanging $z_1 \leftrightarrow z_2$ in the coefficients a_k and replacing Λ_k by $\Lambda_k^{\mathcal{P}}$

$$\mathbb{S}_{21}(z_2, z_1) = \mathbb{S}(z_2, z_1)^{\mathcal{P}} = \sum_k a_k(z_2, z_1)\Lambda_k^{\mathcal{P}} . \tag{3.36}$$

Thus, in the operator form the unitarity condition looks as

$$\mathbb{S}(z_2, z_1)^{\mathcal{P}} \cdot \mathbb{S}(z_1, z_2) = \mathbb{I} , \tag{3.37}$$

while the hermitian analyticity condition takes the form

$$\mathbb{S}(z_1^*, z_2^*)^\dagger = \mathbb{S}(z_2, z_1)^{\mathcal{P}} . \tag{3.38}$$

[9] Not to be confused with the generalized physical unitarity (3.22).

The last condition leads to additional non-trivial relations between the coefficients a_k which, as we show in section 4, are satisfied by our bound state S-matrices.

Crossing Symmetry. The S-matrix also satisfies the crossing symmetry relations. We discuss them in detail in section 5.

Yang-Baxter Equation. The $\mathfrak{su}(2|2)$-invariant S-matrix for the full asymptotic spectrum, the latter includes both the fundamental particles and all the bound states, can be schematically represented as the following block-diagonal matrix

$$\mathbb{S} = \begin{pmatrix} \mathbb{S}^{\mathbf{11}} & & & & \\ & \mathbb{S}^{\mathbf{12}} & & & \\ & & \mathbb{S}^{\mathbf{21}} & & \\ & & & \mathbb{S}^{\mathbf{22}} & \\ & & & & \ddots \end{pmatrix} .$$

Here $\mathbb{S}^{\mathbf{11}}$ is the scattering matrix for the fundamental multiplets, $\mathbb{S}^{\mathbf{12}}$ is the scattering matrix for the fundamental multiplet with the two-particle bound state multiplet and so on. One should view $\mathbb{S}$ as an operator acting in the direct sum

$$\mathscr{V} = \mathscr{V}^{\mathbf{1}} \oplus \mathscr{V}^{\mathbf{2}} \oplus \mathscr{V}^{\mathbf{3}} \oplus \ldots ,$$

where $\mathscr{V}^{M}$ is a carrier space for the irreducible representation of $\mathfrak{su}(2|2)_{\mathcal{C}}$ corresponding to the M-particle bound state. The condition of the factorised scattering is equivalent to the Yang-Baxter equation

$$\mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12} = \mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} . \tag{3.39}$$

Each side of the last equality is understood as an operator acting in $\mathscr{V}^{\otimes 3}$.

The YB equation (3.39) is equivalent to an infinite number of equations for the individual S-matrices. They are obtained by restricting $\mathscr{V}^{\otimes 3}$ to the tensor product of three irreducible representations $\mathscr{V}^{M_1} \otimes \mathscr{V}^{M_2} \otimes \mathscr{V}^{M_3}$

$$\mathbb{S}_{23}^{M_2M_3}\mathbb{S}_{13}^{M_1M_3}\mathbb{S}_{12}^{M_1M_2} = \mathbb{S}_{12}^{M_1M_2}\mathbb{S}_{13}^{M_1M_3}\mathbb{S}_{23}^{M_2M_3} . \tag{3.40}$$

In particular, in the next section we will consider a *consistent truncation* of the theory which amounts to keeping the fundamental particles and two-particle bound states only. The corresponding truncated S-matrix is

$$\check{\mathbb{S}} = \begin{pmatrix} \mathbb{S}^{AA} & & & \\ & \mathbb{S}^{AA} & & \\ & & \mathbb{S}^{BA} & \\ & & & \mathbb{S}^{BB} \end{pmatrix} .$$

Here and in what follows for the sake of clarify we identify $\mathbb{S}^{\mathbf{11}} \equiv \mathbb{S}^{AA}$, etc., and we use the notation $\mathscr{V}^1 \equiv \mathscr{V}^A$ for the fundamental representation, and $\mathscr{V}^2 \equiv \mathscr{V}^B$ for the two-particle bound state representation.

Consistency of the truncation means that the scattering of the lower-particle bound states (one- and two- in our present context) between themselves should factorise independently of the presence/absence of the higher-particle bound states. Thus, we require $\check{\mathbb{S}}$ to satisfy the Yang-Baxter equation (3.39) which results in the following set of inequivalent equations for the individual S-matrices

$$\begin{aligned}
\mathbb{S}_{23}^{AA}\mathbb{S}_{13}^{AA}\mathbb{S}_{12}^{AA} &= \mathbb{S}_{12}^{AA}\mathbb{S}_{13}^{AA}\mathbb{S}_{23}^{AA}\,, \\
\mathbb{S}_{23}^{AB}\mathbb{S}_{13}^{AB}\mathbb{S}_{12}^{AA} &= \mathbb{S}_{12}^{AA}\mathbb{S}_{13}^{AB}\mathbb{S}_{23}^{AB}\,, \\
\mathbb{S}_{23}^{BB}\mathbb{S}_{13}^{AB}\mathbb{S}_{12}^{AB} &= \mathbb{S}_{12}^{AB}\mathbb{S}_{13}^{AB}\mathbb{S}_{23}^{BB}\,, \\
\mathbb{S}_{23}^{BB}\mathbb{S}_{13}^{BB}\mathbb{S}_{12}^{BB} &= \mathbb{S}_{12}^{BB}\mathbb{S}_{13}^{BB}\mathbb{S}_{23}^{BB}\,.
\end{aligned}$$

Here the first and the last equations are the standard Yang-Baxter equations for the S-matrices corresponding to scattering of the fundamental and the two-particle bound states respectively. The second and the third equations are defined in the spaces $\mathscr{V}^A \otimes \mathscr{V}^A \otimes \mathscr{V}^B$ and $\mathscr{V}^A \otimes \mathscr{V}^B \otimes \mathscr{V}^B$ correspondingly.

In the matrix language the properties of the S-matrix operator we discussed above take the following form.

Physical Unitarity.

$$S_{12}^g(z_1^*, z_2^*)^\dagger\, S_{12}^g(z_1, z_2) = I \iff S_{12}(z_1^*, z_2^*)^\dagger\, S_{12}(z_1, z_2) = I\,,$$

where S_{12}^g and S_{12} are defined in (3.20) and (3.21).

CPT invariance.

$$S_{12}^g(z_1, z_2)^T = S_{12}^g(z_1, z_2) \iff S_{12}(z_1, z_2)^T = I_{12}^g S_{12}(z_1, z_2) I_{12}^g\,,$$

where the superscript "T" denotes the usual matrix transposition. Thus, the CPT invariance just means that the graded S-matrix is symmetric.

Parity Transformation.

$$S_{12}^g(z_1, z_2)^{-1} = I_{12}^g S_{12}^g(-z_1, -z_2) I_{12}^g \iff S_{12}(z_1, z_2)^{-1} = S_{12}(-z_1, -z_2)\,,$$

where the parity transformation acts on the S-matrix and any invariant operator as $(S_{12}^g)^{\mathscr{P}} = I_{12}^g S_{12}^g I_{12}^g$. This also means that in matrix language the grading of the tensor product of matrix unities ${E_k}^i \otimes {E_L}^J$ is defined to be $(-1)^{\epsilon_k\epsilon_L+\epsilon_i\epsilon_J}$.

Unitarity and Hermitian Analyticity.
In the general case $M \neq N$

$$S_{21}^{g,NM}(z_2, z_1) S_{12}^{g,MN}(z_1, z_2) = I \iff S_{21}^{NM}(z_2, z_1) S_{12}^{MN}(z_1, z_2) = I\,,$$

where

$$S_{21}^{g,NM}(z_2,z_1) = P_{12}^g S_{12}^{g,NM}(z_2,z_1) P_{12}^g = \sum_{IKjl} S^{NM}{}_{Ij}^{Kl}(z_2,z_1)\,(-1)^{\epsilon_K\epsilon_l+\epsilon_I\epsilon_j}\, E_l{}^j \otimes E_K{}^I\,,$$

and P_{12}^g is the graded permutation matrix. The hermitian analyticity condition takes the form

$$S_{12}^{g,MN}(z_1^*,z_2^*)^\dagger = S_{21}^{g,NM}(z_2,z_1) \iff S_{12}^{MN}(z_1^*,z_2^*)^\dagger = S_{21}^{NM}(z_2,z_1)\,.$$

In the case $M=N$, we get $\mathbb{S}^{MM}(z_1,z_2) \to S_{12}(z_1,z_2)$, and the unitarity and hermitian analyticity conditions take the following form

$$\begin{aligned} S_{21}^g(z_2,z_1)\, S_{12}^g(z_1,z_2) = I &\iff S_{21}(z_2,z_1)\, S_{12}(z_1,z_2) = I\,, \\ S_{12}^g(z_1^*,z_2^*)^\dagger = S_{21}^g(z_2,z_1) &\iff S_{12}(z_1^*,z_2^*)^\dagger = S_{21}(z_2,z_1)\,, \end{aligned}$$

where

$$S_{21}^g(z_2,z_1) = P_{12}^g S_{12}^g(z_2,z_1) P_{12}^g = \sum_{ikjl} S_{ij}^{kl}(z_2,z_1)(-1)^{\epsilon_k\epsilon_l+\epsilon_i\epsilon_j}\, E_l{}^j \otimes E_k{}^i\,,$$

and this means that the action of the graded permutation operator $\mathcal{P}$ is just given by the conjugation by the graded permutation matrix.

Yang-Baxter Equation. The operator YB equation (3.39) has the same matrix form only for the S-matrix S_{12} defined by (3.21):

$$S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}\,,$$

where to simplify the notation we omit the explicit dependence of the S-matrices on z_i and M_i. To show this, we notice that the matrix form of the operators $\mathbb{S}_{12}, \mathbb{S}_{13}$ and $\mathbb{S}_{23}$ is

$$S_{12}^{\rm matr} = S_{12}^g\,, \quad S_{13}^{\rm matr} = I_{23}^g S_{13}^g I_{23}^g\,, \quad S_{23}^{\rm matr} = I_{12}^g I_{13}^g S_{23}^g I_{13}^g I_{12}^g\,,$$

where S_{ij}^g denotes the usual embedding of the matrix S^g into the product of three spaces. We see that the graded form of the S-matrix satisfies the following graded YB equation

$$I_{12}^g I_{13}^g S_{23}^g I_{13}^g I_{12}^g I_{23}^g S_{13}^g I_{23}^g S_{12}^g = S_{12}^g I_{23}^g S_{13}^g I_{23}^g I_{12}^g I_{13}^g S_{23}^g I_{13}^g I_{12}^g\,.$$

In terms of the S-matrix $S_{12} = I_{12}^g S_{12}^G$ the previous equation takes the form

$$I_{12}^g I_{13}^g I_{23}^g S_{23} I_{12}^g I_{23}^g S_{13} I_{23}^g I_{12}^g S_{12} = I_{12}^g S_{12} I_{23}^g I_{13}^g S_{13} I_{12}^g I_{13}^g S_{23} I_{13}^g I_{12}^g\,.$$

Taking into account the identities

$$I_{12}^g I_{23}^g S_{13} = S_{13} I_{12}^g I_{23}^g\,, \quad I_{12}^g I_{13}^g S_{23} = S_{23} I_{13}^g I_{12}^g\,,$$

we conclude that S_{12} satisfies the usual YB equation. This completes our discussion of the properties of the S-matrix.

4. S-matrices

In this section we discuss the explicit construction and properties of the three S-matrices describing the scattering of fundamental particles and two-particle bound states.

4.1 The S-matrix $\mathbb{S}^{AA}$

As the warm-up exercise, here we will explain how to obtain the known scattering matrix $\mathbb{S}^{AA}$ for fundamental particles [8, 12] in the framework of our new superspace approach.

According to our general discussion in section 3, the scattering matrix $\mathbb{S}^{AA}$ is an $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant differential operator which acts on the tensor product of two fundamental multiplets

$$\mathbb{S}^{AA} : \qquad \mathscr{V}^A \otimes \mathscr{V}^A \quad \rightarrow \quad \mathscr{V}^A \otimes \mathscr{V}^A \,.$$

Recall that

$$\mathscr{V}^A \otimes \mathscr{V}^A = \mathscr{W}^2 \,,$$

where $\mathscr{W}^2$ is a long irreducible supermultiplet of dimension 16 [19]. Under the action of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ this supermultiplet branches as follows

$$\begin{aligned} \mathscr{W}^2 = V^0 \times V^0 + V^0 \times V^0 + V^1 \times V^0 + \\ + V^0 \times V^1 + V^{1/2} \times V^{1/2} + V^{1/2} \times V^{1/2}. \end{aligned} \tag{4.1}$$

The 16-dimensional basis of the tensor product space $\mathscr{V}^A \otimes \mathscr{V}^A$

$$\mathscr{V}^A \otimes \mathscr{V}^A = \text{Span}\Big\{ w_a^1 w_b^2 \,, w_a^1 \theta_\alpha^2 \,, w_a^1 \theta_\alpha^2 \,, \theta_\alpha^1 \theta_\beta^2 \Big\}, \qquad a = 1,2; \quad \alpha = 3,4$$

can be easily adopted to the decomposition (4.1):

$$\begin{aligned} w_a^1 w_b^2 &= \frac{1}{2}(w_a^1 w_b^2 - w_b^1 w_a^2) + \frac{1}{2}(w_a^1 w_b^2 + w_b^1 w_a^2) \quad \rightarrow \quad (V^0 + V^1) \times V^0 \\ w_a^1 \theta_\alpha^2 \quad &\rightarrow \quad V^{1/2} \times V^{1/2} \\ w_a^2 \theta_\alpha^1 \quad &\rightarrow \quad V^{1/2} \times V^{1/2} \\ \theta_\alpha^1 \theta_\beta^2 &= \frac{1}{2}(\theta_\alpha^1 \theta_\beta^2 - \theta_\beta^1 \theta_\alpha^2) + \frac{1}{2}(\theta_\alpha^1 \theta_\beta^2 + \theta_\beta^1 \theta_\alpha^2) \quad \rightarrow \quad V^0 \times (V^0 + V^1) \end{aligned} \tag{4.2}$$

The space $\mathscr{D}_2$ of the second order differential operators dual to $\mathscr{W}^2$ branches under $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in the same way:

$$\begin{aligned} \mathscr{D}_2 = D^0 \times D^0 + D^0 \times D^0 + D^1 \times D^0 + \\ + D^0 \times D^1 + D^{1/2} \times D^{1/2} + D^{1/2} \times D^{1/2}. \end{aligned} \tag{4.3}$$

A basis adopted to this decomposition is formed by the following differential operators

$$
\begin{aligned}
\epsilon^{cd}\frac{\partial}{\partial w_1^c}\frac{\partial}{\partial w_2^d} &\quad\rightarrow\quad D^0\times D^0\,, \quad \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_1^\alpha}\frac{\partial}{\partial\theta_2^\beta} \quad\rightarrow\quad D^0\times D^0\\
\frac{\partial}{\partial w_a^2}\frac{\partial}{\partial\theta_\alpha^1} &\quad\rightarrow\quad D^{1/2}\times D^{1/2}\,, \quad \frac{\partial}{\partial w_a^1}\frac{\partial}{\partial\theta_\alpha^2} \quad\rightarrow\quad D^{1/2}\times D^{1/2}\\
\frac{\partial}{\partial w_1^a}\frac{\partial}{\partial w_2^b}+\frac{\partial}{\partial w_1^b}\frac{\partial}{\partial w_2^a} &\quad\rightarrow\quad D^1\times D^0\,, \quad \frac{\partial}{\partial\theta_1^\alpha}\frac{\partial}{\partial\theta_2^\beta}+\frac{\partial}{\partial\theta_1^\beta}\frac{\partial}{\partial\theta_2^\alpha} \quad\rightarrow\quad D^0\times D^1
\end{aligned}
\tag{4.4}
$$

The S-matrix we are looking for is an element of the space $\mathscr{W}^2\otimes\mathscr{D}_2$ which is invariant under the action of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. We obviously have

$$
\begin{aligned}
\mathscr{W}^2\otimes\mathscr{D}_2 = &\,(V^0\times V^0+V^0\times V^0+V^1\times V^0+V^0\times V^1+V^{1/2}\times V^{1/2}+V^{1/2}\times V^{1/2})\\
&\otimes(D^0\times D^0+D^0\times D^0+D^1\times D^0+D^0\times D^1+D^{\frac{1}{2}}\times D^{1/2}+D^{1/2}\times D^{1/2})
\end{aligned}
$$

The invariants of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ arising in this tensor product decomposition are easy to count. Indeed, since $V^j\otimes D^j$ contains the singlet component (invariant), the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariants simply correspond to the singlet components of the spaces $V^j\otimes D^j\times V^k\otimes D^k$ in the tensor product decomposition of $\mathscr{W}^2\otimes\mathscr{D}_2$. Thus, the invariant subspace is generated by

$$
\begin{aligned}
\mathrm{Inv}(\mathscr{W}^2\otimes\mathscr{D}_2) = &\,4(V^0\otimes D^0)\times(V^0\otimes D^0)+4(V^{1/2}\otimes D^{1/2})\times(V^{1/2}\otimes D^{1/2})+\\
&+(V^1\otimes D^1)\times(V^0\otimes D^0)+(V^0\otimes D^0)\times(V^1\otimes D^1),
\end{aligned}
$$

where integers in front of the spaces in the r.h.s of the last formulas indicate the corresponding multiplicities. Therefore, in the present case we find 10 invariant elements Λ_k in the space $\mathscr{W}^2\otimes\mathscr{D}_2$. Their explicit form can be easily found by using the bases (4.2) and (4.4):

$$
\begin{aligned}
\Lambda_1 &= \frac{1}{2}(w_a^1w_b^2+w_b^1w_a^2)\frac{\partial^2}{\partial w_a^1\partial w_b^2}\,, & \Lambda_7 &= \epsilon^{ab}w_a^1w_b^2\epsilon_{\alpha\beta}\frac{\partial^2}{\partial\theta_\beta^2\partial\theta_\alpha^1}\\
\Lambda_2 &= \frac{1}{2}(w_a^1w_b^2-w_b^1w_a^2)\frac{\partial^2}{\partial w_a^1\partial w_b^2}\,, & \Lambda_8 &= \epsilon^{\alpha\beta}\theta_\alpha^1\theta_\beta^2\epsilon_{ab}\frac{\partial^2}{\partial w_a^1\partial w_b^2}\\
\Lambda_3 &= \frac{1}{2}(\theta_\alpha^1\theta_\beta^2+\theta_\beta^1\theta_\alpha^2)\frac{\partial^2}{\partial\theta_\beta^2\partial\theta_\alpha^1}\,, & \Lambda_9 &= w_a^1\theta_\alpha^2\frac{\partial^2}{\partial w_a^2\partial\theta_\alpha^1}\\
\Lambda_4 &= \frac{1}{2}(\theta_\alpha^1\theta_\beta^2-\theta_\beta^1\theta_\alpha^2)\frac{\partial^2}{\partial\theta_\beta^2\partial\theta_\alpha^1}\,, & \Lambda_{10} &= w_a^2\theta_\alpha^1\frac{\partial^2}{\partial w_a^1\partial\theta_\alpha^2}\\
\Lambda_5 &= w_a^1\theta_\alpha^2\frac{\partial^2}{\partial w_a^1\partial\theta_\alpha^2}\,, & &\\
\Lambda_6 &= w_a^2\theta_\alpha^1\frac{\partial^2}{\partial w_a^2\partial\theta_\alpha^1}\,. & &
\end{aligned}
\tag{4.5}
$$

Thus, the S-matrix is the following differential operator in the space $\mathscr{V}^A \otimes \mathscr{V}^A$:

$$\mathbb{S}^{AA}(z_1, z_2) = \sum_{k=1}^{10} a_k \Lambda_k$$

The differential operators $\Lambda_1, \ldots \Lambda_6$ correspond to the diagonal invariants in $\mathscr{W}^2 \otimes \mathscr{D}_2$. Their normalization has been chosen in such a way that for $a_1 = a_2 = \ldots = a_6 = 1$ and $a_7 = a_8 = a_9 = a_{10} = 0$ the operator $\mathbb{S}^{AA}$ coincides with an identity operator.

The unknown coefficients a_k can be now determined from the permutation relations of $\mathbb{S}^{AA}$ with the supersymmetry generators. We find

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 2\,\frac{(x_1^+ - x_2^+)(x_1^- x_2^+ - 1)x_2^-}{(x_1^+ - x_2^-)(x_1^- x_2^- - 1)x_2^+} - 1\,, \\
a_3 &= \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2}, \\
a_4 &= \frac{(x_1^- - x_2^+)}{(x_2^- - x_1^+)}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2} - 2\,\frac{(x_2^- x_1^+ - 1)(x_1^+ - x_2^+)x_1^-}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)x_1^+}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2}, \\
a_5 &= \frac{x_2^- - x_1^-}{x_2^- - x_1^+}\frac{\tilde{\eta}_2}{\eta_2}, \\
a_6 &= \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-}\frac{\tilde{\eta}_1}{\eta_1}, \\
a_7 &= -\frac{i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)}\frac{1}{\eta_1\eta_2}, \\
a_8 &= \frac{i x_1^- x_2^- (x_1^+ - x_2^+)}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)x_1^+ x_2^+}\tilde{\eta}_1\tilde{\eta}_2, \\
a_9 &= \frac{x_1^+ - x_1^-}{x_1^+ - x_2^-}\frac{\tilde{\eta}_2}{\eta_1}, \\
a_{10} &= \frac{x_2^- - x_2^+}{x_2^- - x_1^+}\frac{\tilde{\eta}_1}{\eta_2}\,,
\end{aligned}$$

The coefficients a_k are determined up to an overall scaling factor, and we normalize them in a canonical way by setting $a_1 = 1$. The parameters η_k are not fixed by the invariance condition. They are determined by imposing the generalized unitarity condition and the YB equation [21], and are given by the following formulas

$$\eta_1 = e^{ip_2/2}\eta(z_1, 1)\,, \quad \eta_2 = \eta(z_2, 1)\,, \quad \tilde{\eta}_1 = \eta(z_1, 1)\,, \quad \tilde{\eta}_2 = e^{ip_1/2}\eta(z_2, 1)\,, \quad (4.6)$$

where $\eta(z, M)$ is defined by (2.15).

The S-matrix satisfies all the properties we discussed in the previous section. First, the physical unitarity condition $\mathbb{S}^\dagger \cdot \mathbb{S} = \mathbb{I}$ can be easily checked by using the explicit form of the coefficients a_i, and the hermitian conjugation conditions

$$(\Lambda_i)^\dagger = \Lambda_i\,, \quad i = 1, \ldots, 6\,; \qquad (\Lambda_7)^\dagger = \Lambda_8\,, \;\; (\Lambda_9)^\dagger = \Lambda_{10}\,.$$

Moreover, with the choice of η_i (4.6) made in [21], the S-matrix also satisfies the generalized unitarity condition $\mathbb{S}(z_1^*, z_2^*)^\dagger \cdot \mathbb{S}(z_1, z_2) = \mathbb{I}\,.$

Second, the S-matrix is a symmetric operator $\mathbb{S}(z_1, z_2)^T = \mathbb{S}(z_1, z_2)$, and the coefficients a_i satisfy the following relations

$$a_7(z_1, z_2) = a_8(z_1, z_2)\,, \quad a_9(z_1, z_2) = a_{10}(z_1, z_2)\,, \tag{4.7}$$

if one uses η_i given by (4.6).

Third, by using the action of the parity transformation on an invariant operator defined in (3.27), we find

$$\Lambda_i^{\mathscr{P}} = \Lambda_i\,,\; i = 1, \ldots, 6; 9, 10\,; \quad \Lambda_7^{\mathscr{P}} = -\Lambda_7\,,\; \Lambda_8^{\mathscr{P}} = -\Lambda_8\,,$$

and check that the inverse operator $\mathbb{S}^{-1}$ is equal to the parity-transformed S-matrix operator $\mathbb{S}^{-1}(z_1, z_2) = \mathbb{S}(-z_1, -z_2)^{\mathscr{P}}$, and the relations (3.29) between on the coefficients of the S-matrix are satisfied.

Fourth, as was discussed in the previous section, in the operator formalism the action of the graded permutation on the invariant operators Λ_i can be realized by means of the exchange $w_a^1 \leftrightarrow w_a^2$, $\theta_\alpha^1 \leftrightarrow \theta_\alpha^2$. By using the explicit form of Λ_i one finds

$$\Lambda_i^{\mathcal{P}} = \Lambda_i\,,\; i = 1, 2, 3, 4\,; \quad \Lambda_5^{\mathcal{P}} = \Lambda_6\,,\; \Lambda_7^{\mathcal{P}} = -\Lambda_7\,,\; \Lambda_8^{\mathcal{P}} = -\Lambda_8\,,\; \Lambda_9^{\mathcal{P}} = \Lambda_{10}\,, \tag{4.8}$$

and verifies the unitarity condition $\mathbb{S}(z_2, z_1)^{\mathcal{P}} \cdot \mathbb{S}(z_1, z_2) = \mathbb{I}$, and the hermitian analyticity condition $\mathbb{S}(z_1^*, z_2^*)^\dagger = \mathbb{S}(z_2, z_1)^{\mathcal{P}}$, which implies the following relations between the coefficients a_i of the S-matrix

$$\begin{aligned} &a_i(z_2, z_1)^* = a_i(z_1^*, z_2^*)\,,\; i = 1, 2, 3, 4\,; \quad a_5(z_2, z_1)^* = a_6(z_1^*, z_2^*)\,, \\ &a_7(z_2, z_1)^* = -a_8(z_1^*, z_2^*)\,, \;\; a_9(z_2, z_1)^* = a_9(z_1^*, z_2^*)\,,\; a_{10}(z_2, z_1)^* = a_{10}(z_1^*, z_2^*)\,. \end{aligned} \tag{4.9}$$

Comparing relations (4.9) and (4.7), we conclude that the coefficients a_i also satisfy the following relations

$$\begin{aligned} &a_7(z_2, z_1)^* = -a_7(z_1^*, z_2^*)\,,\; a_8(z_2, z_1)^* = -a_8(z_1^*, z_2^*)\,, \\ &a_9(z_2, z_1)^* = a_{10}(z_1^*, z_2^*)\,,\; a_{10}(z_2, z_1)^* = a_9(z_1^*, z_2^*)\,. \end{aligned}$$

Finally, given the η's, one can check the fulfilment of the Yang-Baxter equation

$$\mathbb{S}_{23}^{AA}\mathbb{S}_{13}^{AA}\mathbb{S}_{12}^{AA} = \mathbb{S}_{12}^{AA}\mathbb{S}_{13}^{AA}\mathbb{S}_{23}^{AA}\,, \tag{4.10}$$

where each side of the last equality is understood as an operator acting in $\mathscr{V}^{A\otimes 3}$, the latter being identified with the product of three superfields $\Phi_1(w_a^1, \theta_\alpha^1)$, $\Phi_2(w_a^2, \theta_\alpha^2)$ and $\Phi_3(w_a^3, \theta_\alpha^3)$. This completes the construction of the graded fermionic $\mathbb{S}$-operator which describes the scattering of two fundamental multiplets.

By using the general procedure outlined in section 3, we can also construct the matrix representation for the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant differential operators acting in the tensor product of fundamental particle representations. For reader's convenience we present them below

$$\Lambda_1 = E_{1111} + \frac{1}{2}E_{1122} + \frac{1}{2}E_{1221} + \frac{1}{2}E_{2112} + \frac{1}{2}E_{2211} + E_{2222}$$

$$\Lambda_2 = \frac{1}{2}E_{1122} - \frac{1}{2}E_{1221} - \frac{1}{2}E_{2112} + \frac{1}{2}E_{2211}$$

$$\Lambda_3 = E_{3333} + \frac{1}{2}E_{3344} + \frac{1}{2}E_{3443} + \frac{1}{2}E_{4334} + \frac{1}{2}E_{4433} + E_{4444}$$

$$\Lambda_4 = \frac{1}{2}E_{3344} - \frac{1}{2}E_{3443} - \frac{1}{2}E_{4334} + \frac{1}{2}E_{4433}$$

$$\Lambda_5 = E_{1133} + E_{1144} + E_{2233} + E_{2244}$$

$$\Lambda_6 = E_{3311} + E_{3322} + E_{4411} + E_{4422}$$

$$\Lambda_7 = E_{1324} - E_{1423} - E_{2314} + E_{2413}$$

$$\Lambda_8 = E_{3142} - E_{3241} - E_{4132} + E_{4231}$$

$$\Lambda_9 = E_{1331} + E_{1441} + E_{2332} + E_{2442}$$

$$\Lambda_{10} = E_{3113} + E_{3223} + E_{4114} + E_{4224}$$

Here the symbols E_{kilj} can be equal to either $E_k{}^i \otimes E_l{}^j$ or to $(-1)^{\epsilon_k \epsilon_l} E_k{}^i \otimes E_l{}^j$, where $E_k{}^i \equiv E_{ki}$ are the standard 4×4 matrix unities. Computing the matrix representation for $\mathbb{S}^{AA}$ by using $E_{kilj} = (-1)^{\epsilon_k \epsilon_l} E_k{}^i \otimes E_l{}^j$ we find that it precisely coincides with the $\mathfrak{su}(2|2)$-invariant S-matrix found in [12, 21].

In the matrix language the above-described properties of the S-matrix operator take the form discussed in the previous section.

4.2 The S-matrix $\mathbb{S}^{AB}$

The S-matrices $\mathbb{S}^{AB}$ and $\mathbb{S}^{BA}$ for scattering of a fundamental particle A with the two-particle bound state B are $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariant third-order differential operators which act as intertwiners

$$\begin{array}{llll} \mathbb{S}^{AB}: & \mathscr{V}^A\otimes\mathscr{V}^B & \to & \mathscr{V}^A\otimes\mathscr{V}^B \\ \mathbb{S}^{BA}: & \mathscr{V}^B\otimes\mathscr{V}^A & \to & \mathscr{V}^B\otimes\mathscr{V}^A \end{array}$$

Below we discuss the S-matrix $\mathbb{S}^{AB}$ in detail and comment on its relation to $\mathbb{S}^{BA}$.

According to section 3.2, the tensor product $\mathscr{V}^A\otimes\mathscr{V}^B$ is isomorphic to a long supermultiplet $\mathscr{W}^3$ of dimension 32. The total number of invariant differential operators Λ_k for this case is 19 and, therefore,

$$\mathbb{S}^{AB}=\sum_{k=1}^{19}a_k\Lambda_k\,. \tag{4.11}$$

Representations for Λ_k by third-order differential operators and by rank 32 matrices are listed in appendix 6.1.1. The invariants $\Lambda_1,\ldots,\Lambda_9$ are diagonal and they are normalized to provide the orthogonal decomposition of unity.

Since the tensor product $\mathscr{V}^A\otimes\mathscr{V}^B$ is irreducible, all the coefficients a_k in (4.11) can be determined, up to an overall scaling factor and parameters η_k, from the invariance conditions involving the generators $\mathbb{Q}$ and $\mathbb{Q}^\dagger$. The corresponding solution for the coefficients a_k is presented in appendix 6.1.2.

Now we are ready to discuss the properties of $\mathbb{S}^{AB}$. First, under the hermitian conjugation the operators Λ_k transform as follows

$$(\Lambda_i)^\dagger=\Lambda_i,\quad i=1,\ldots,9;\qquad (\Lambda_{10+2k})^\dagger=\Lambda_{11+2k}\,,\qquad k=0,\ldots,4.$$

These transformation rules allow one to check the generalized unitarity condition. Second, with our choice of η's the operator $\mathbb{S}^{AB}$ is symmetric, i.e. $(\mathbb{S}^{AB})^{T}=\mathbb{S}^{AB}$. Consequently, the coefficients a_k satisfy the following relations

$$a_{10+2k}(z_1,z_2)=a_{11+2k}(z_1,z_2)$$

for $k=0,\ldots,4$.

Third, the parity transformation (3.27) acts on the invariant operators as follows

$$\Lambda_i^{\mathscr{P}}=\Lambda_i\,,\ i=1,\ldots,9;14,15,18,19\,;\quad \Lambda_k^{\mathscr{P}}=-\Lambda_k\,,\ k=10,11,12,13,16,17\,,$$

and the inverse S-matrix is equal to the parity-transformed S-matrix, and the relations (3.29) between on the coefficients of the S-matrix are satisfied.

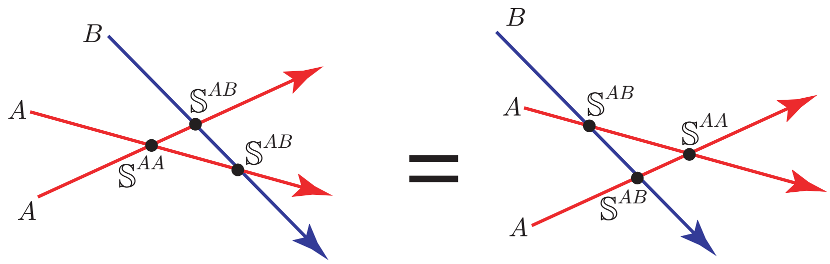

Figure 1: Factorisation of the three-particle S-matrix for the scattering process involving two fundamental particles and one two-particle bound state.

Fourth, the unitarity condition (3.33) $\mathbb{S}^{BA}_{21}(z_2,z_1)\cdot\mathbb{S}^{AB}(z_1,z_2)=\mathbb{I}$ allows us to find the coefficients of the S-matrix S^{BA} by using the relations (3.34)

$$b_k(z_1,z_2)=(-1)^{\epsilon_{\Lambda_k}}\,a_k(-z_2,-z_1)=a_k(z_2^*,z_1^*)^*\,,$$

and check that the resulting S-matrix S^{BA} satisfies the invariance conditions (3.9).

Having found the S-matrices $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$ one can check the factorization property of the three-particle S-matrix involving two fundamental particles and one two-particle bound state, see Fig. 1. The corresponding Yang-Baxter equation

$$\mathbb{S}^{AB}_{23}\mathbb{S}^{AB}_{13}\mathbb{S}^{AA}_{12}=\mathbb{S}^{AA}_{12}\mathbb{S}^{AB}_{13}\mathbb{S}^{AB}_{23}$$

is an operator identity on the triple tensor product $\mathscr{V}^A\otimes\mathscr{V}^A\otimes\mathscr{V}^B$, which we have shown to hold.

Finally, one can check that on solutions of the equation

$$x_1^-=y_2^+ \tag{4.12}$$

the coefficients $a_4,a_7,a_9,a_{16},a_{17}$ vanish. As the consequence, the S-matrix $\mathbb{S}^{AB}$ degenerates and has rank 12. This precisely corresponds to forming a three-particle bound state of dimension 12. Under the condition (4.12) the 32-dim tensor product representation $\mathscr{V}^A\otimes\mathscr{V}^B=\mathscr{W}^3$ becomes reducible [19]. However, as was discussed in the Introduction, this representation is indecomposable. The invariant subspace is a 20-dim short representation, while the three-particle bound state representation should be understood as a factor representation.

4.3 The S-matrix $\mathbb{S}^{BB}$

The S-matrix $\mathbb{S}^{BB}$ is the following differential operator in the space $\mathscr{V}^B\otimes\mathscr{V}^B$:

$$\mathbb{S}^{BB}(z_1,z_2)=\sum_{k=1}^{48}a_k\Lambda_k\,,$$

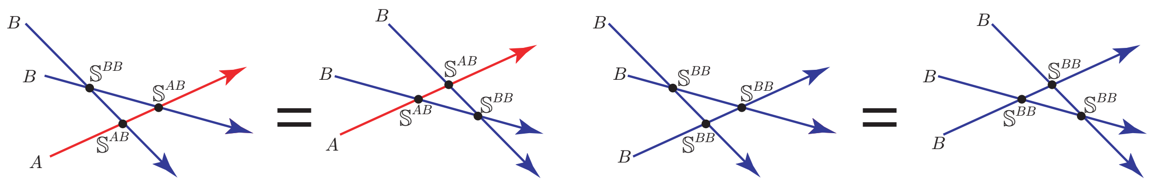

Figure 2: The Yang-Baxter equation corresponding to the scattering process of a fundamental particle A with two two-particle bound states B.

where 48 invariant operators Λ_k and the corresponding matrices are listed in the appendix 6.2. In particular, $\Lambda_1, \ldots \Lambda_{16}$ correspond to the diagonal invariants. Their normalization has been chosen in such a way that for $a_1 = a_2 = \ldots = a_{16} = 1$ and $a_{17} = \ldots = 0$ the operator $\mathbb{S}^{BB}$ coincides with the identity operator.

The operator $\mathbb{S}^{BB}$ provides the first example of an invariant differential operator for which the coefficients a_k cannot be fully determined from the commutation relations with supersymmetry generators. As was already mentioned in section 3.2, the tensor product decomposition $\mathscr{V}^B \otimes \mathscr{V}^B$ comprises two long multiplets $\mathscr{W}^2$ and $\mathscr{W}^4$. As a consequence, a solution of the invariance condition is a two-parametric family

$$a_k \equiv a_k(a_1, a_2; \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}), \qquad k > 2,$$

where we have chosen the independent coefficients a_1 and a_2 to parametrize our solution. In addition, a_k depend in a very complicated way on the parameters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ which describe four representations of the centrally extended $\mathfrak{su}(2|2)$ involved in the invariance condition. It appears convenient to explicitly distinguish between two independent solutions by introducing

$$\begin{aligned} a_1^f &= 1, \quad a_2^f = 0, \quad a_k^f \equiv a_k(1, 0; \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}), \quad k > 2, \\ a_1^s &= 0, \quad a_2^s = 1, \quad a_k^s \equiv a_s(0, 1; \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}), \quad k > 2, \end{aligned}$$

so that a general solution is a linear combination of these two. Consequently, we introduce two differential operators

$$\mathbb{S}_1^{BB} = \sum_{k=1}^{48} a_k^f \Lambda_k,$$

$$\mathbb{S}_2^{BB} = \sum_{k=1}^{48} a_k^s \Lambda_k.$$

Thus, the S-matrix $\mathbb{S}^{BB}$ with the canonical normalization $a_1 = a_1^f = 1$ can be now written as

$$\mathbb{S}^{BB} = \mathbb{S}_1^{BB} + q\, \mathbb{S}_2^{BB}, \tag{4.13}$$

where $q \equiv a_2$ is a single parameter which cannot be determined from the invariance condition. Below we will find it by solving the YB equations involving $\mathbb{S}^{AB}$ and $\mathbb{S}^{BB}$.

First we consider the YB equation involving one fundamental particle and two bound states

$$\mathbb{S}_{23}^{BB}\mathbb{S}_{13}^{AB}\mathbb{S}_{12}^{AB} = \mathbb{S}_{12}^{AB}\mathbb{S}_{13}^{AB}\mathbb{S}_{23}^{BB}\,, \tag{4.14}$$

where $\mathbb{S}^{AB}$ is given by eq.(4.13). Evaluated on basis elements of the tensor product $\mathscr{V}^A \otimes \mathscr{V}^B \otimes V^B$ this operator equation is equivalent to 256 relations between the coefficients of the S-matrices involved. Analysis of these relations allows one to extract the following form of the unknown coefficient q

$$q = -\frac{3a_{11}^{AB}(a_1^f + a_3^f - 2a_{13}^f) - 8(2a_1^{AB} + a_2^{AB} - 3a_6^{AB})a_{24}^f}{3a_{11}^{AB}(b_1^s + b_3^s - 2b_{13}^s) - 8(2a_1^{AB} + a_2^{AB} - 3a_6^{AB})b_{24}^s}\,, \tag{4.15}$$

where by a_k^{AB} we denoted the coefficients of $\mathbb{S}^{AB}$ to distinguish them from a_k of $\mathbb{S}^{BB}$.

This result for q might seem rather odd. Indeed, the coefficient q must also guarantee the fulfilment of another YB equation, which involves the bound states only

$$\mathbb{S}_{23}^{BB}\mathbb{S}_{13}^{BB}\mathbb{S}_{12}^{BB} = \mathbb{S}_{12}^{BB}\mathbb{S}_{13}^{BB}\mathbb{S}_{23}^{BB}\,. \tag{4.16}$$

This relation implies that q should be a function of the variables z_1, z_2 living on the rapidity tori associated to two bound states, while solution (4.15) involves the rapidity torus of a fundamental particles. This issue can be resolved by noting that the following important identity takes place

$$2a_1^{AB} + a_2^{AB} - 3a_6^{AB} = 3\frac{(1-y_2^+y_2^-)}{\sqrt{y_2^+y_2^-}}\frac{(y_2^+ + y_2^-)}{(y_2^+ - y_2^-)}a_{11}^{AB}\,. \tag{4.17}$$

Due to this identity the coefficients a_k^{AB} drop out from the formula for q and we find

$$q = -\frac{a_1^f + a_3^f - 2a_{13}^f - 8\frac{(1-y_2^+y_2^-)}{\sqrt{y_2^+y_2^-}}\frac{(y_2^+ + y_2^-)}{(y_2^+ - y_2^-)}a_{24}^f}{b_1^s + b_3^s - 2b_{13}^s - 8\frac{(1-y_2^+y_2^-)}{\sqrt{y_2^+y_2^-}}\frac{(y_2^+ + y_2^-)}{(y_2^+ - y_2^-)}b_{24}^s}\,.$$

Thus, q depends on the coefficients of the operator $\mathbb{S}^{BB}$ only.

Having found the coefficients a_k^f, a_k^s and q, we can now reconstruct all the coefficients $a_k = a_k^f + q a_k^s$. The coefficients $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ of the bound state representations are parametrized in terms of the variables $y_1^\pm$ and $y_2^\pm$ satisfying the constraints

$$y_1^+ + \frac{1}{y_1^+} - y_1^- - \frac{1}{y_1^-} = \frac{4i}{g}\,, \qquad y_2^+ + \frac{1}{y_2^+} - y_2^- - \frac{1}{y_2^-} = \frac{4i}{g}\,. \tag{4.18}$$

In practice, the formulae arising for a_k as the functions of $y_1^\pm, y_2^\pm$ are very involved, but they can be drastically simplified by using constraints (4.18). Most efficiently, this can be done by using the external $\mathfrak{sl}(2)$ automorphism of the centrally extended algebra $\mathfrak{su}(2|2)$ discussed in section 2.1. Indeed, in the invariance condition (3.9) the supersymmetry generators $\mathbb{Q}$ and $\mathbb{Q}^\dagger$ can be replaced by $\tilde{\mathbb{Q}}$ and $\tilde{\mathbb{Q}}^\dagger$ given by (2.3). As the result, we will find

$$\begin{aligned} a_1^f &= 1\,, \quad a_2^f = 0\,, \quad a_k^f \equiv a_k(1,0;\tilde{\mathsf{a}},\tilde{\mathsf{b}},\tilde{\mathsf{c}},\tilde{\mathsf{d}})\,, \quad k>2,\\ a_1^s &= 0\,, \quad a_2^s = 1\,, \quad a_k^s \equiv a_s(0,1;\tilde{\mathsf{a}},\tilde{\mathsf{b}},\tilde{\mathsf{c}},\tilde{\mathsf{d}})\,, \quad k>2, \end{aligned}$$

where the $\mathfrak{sl}(2)$-transformed representation coefficients are given by eqs.(3.7) and they depend on arbitrary parameters u_1, u_2 and v_1, v_2 modulo the constraint $u_1v_1 - u_2v_2 = 1$. On the other hand, the coefficients a_k^s and a_k^f are $\mathfrak{sl}(2)$ invariant and they must be independent of u and v. By picking up various values of these parameters one can achieve enormous simplification of a_k. In this way we find

$$\begin{aligned} q = {}& \frac{y_1^+ y_2^-(y_1^- - y_2^+)(-1 + y_1^- y_2^+)}{y_1^- y_2^+(-1 + y_1^- y_2^-)(y_2^- - y_1^+)} \times \\ & \times \frac{-3y_1^+ y_2^- + 2y_2^+ y_2^- + y_1^-(y_2^-)^2 y_2^+ + y_1^+ y_2^+ + 2y_1^- y_2^- y_1^+ y_2^+ - 3y_1^- y_2^-(y_2^+)^2}{y_1^+ y_2^- - 2y_2^- y_2^+ + y_1^+ y_2^+ + y_1^+ y_2^+(y_2^-)^2 - 2y_2^+ y_2^-(y_1^+)^2 + y_1^+ y_2^-(y_2^+)^2}\,. \end{aligned}$$

The final form of the coefficients a_k is given in appendix 6.2.2. Quite remarkably, we find that $a_{45} = a_{46} = a_{47} = a_{48} = 0$. Finally, one can check that with these coefficients a_k the YB equations (4.14) and (4.16) are satisfied. Thus, solving the invariance condition together with the YB equations we found the unique, up to an overall scale, S-matrix describing the scattering of two-particle bound states.

The found S-matrix $\mathbb{S}^{BB}$ enjoys all the properties discussed in section 3.4. Below we outline the transformation rules for invariants Λ_k under the hermitian conjugation, graded permutation and parity.

First, by using the hermitian conjugation rules (3.4) we find the following relations

$$(\Lambda_i)^\dagger = \Lambda_i\,, \quad i = 1,\dots,16\,; \qquad (\Lambda_{17+2k})^\dagger = \Lambda_{18+2k}\,, \quad k = 0,\dots,15\,.$$

Second, the S-matrix is a symmetric operator, c.f. (3.24), and, as a consequence, the coefficients a_i satisfy the following relations

$$a_{17+2k}(z_1,z_2) = a_{18+2k}(z_1,z_2)\,, \quad k = 0,\dots,15\,. \tag{4.19}$$

It is worth emphasizing that these relations hold only for the choice of η_i as in eq.(6.3).

Third, the parity transformation $\mathscr{P}$ acts on the invariants Λ_k as follows

$$\begin{aligned} \Lambda_k^{\mathscr{P}} &= \Lambda_k \quad \text{for} \quad k = 1,\dots,18 \quad \text{and} \quad k = 25,\dots,36;\\ \Lambda_k^{\mathscr{P}} &= -\Lambda_k \quad \text{for} \quad k = 19,\dots,24 \quad \text{and} \quad k = 37,\dots,48. \end{aligned}$$

Finally, the graded permutation acts on Λ_i in the following way

$$\Lambda_i \leftrightarrow \Lambda_i\,,\ i=1,\ldots,8\,;\quad \Lambda_{9+2k} \leftrightarrow \Lambda_{10+2k}\,,\ k=0,1,2,3\,;\quad \Lambda_i \leftrightarrow \Lambda_i\,,\ i=17,18\,;$$
$$\Lambda_i \leftrightarrow -\Lambda_i\,,\ i=19,\ldots,24\,;\quad \Lambda_{25+2k} \leftrightarrow \Lambda_{26+2k}\,,\ k=0,1,2,3\,;$$
$$\Lambda_{33+k} \leftrightarrow \Lambda_{35+k}\,,\ k=0,1\,;\quad \Lambda_{37+k+4m} \leftrightarrow -\Lambda_{39+k+4m}\,,\ k=0,1\,,\ m=0,1,2\,,$$

leading to the unitarity condition $\mathbb{S}(z_2,z_1)^{\mathcal{P}} \cdot \mathbb{S}(z_1,z_2) = \mathbb{I}$, and to the hermitian analyticity condition $\mathbb{S}(z_1^*,z_2^*)^\dagger = \mathbb{S}(z_2,z_1)^{\mathcal{P}}$, which together with (4.19) implies the following relations between the coefficients a_i of the S-matrix

$$a_i(z_2,z_1)^* = a_i(z_1^*,z_2^*)\,,\ i=1,\ldots,8\,;\quad a_{9+2k}(z_2,z_1)^* = a_{10+2k}(z_1^*,z_2^*)\,,\ k=0,1,2,3\,;$$
$$a_{17}(z_2,z_1)^* = a_{18}(z_1^*,z_2^*)\,;\quad a_{19+k}(z_2,z_1)^* = -a_{19+k}(z_1^*,z_2^*)\,,\ k=0,\ldots,5\,;$$
$$a_{25+k}(z_2,z_1)^* = a_{25+k}(z_1^*,z_2^*)\,,\ k=0,\ldots,7\,;\quad a_{33+k}(z_2,z_1)^* = a_{35+k}(z_1^*,z_2^*)\,,\ k=0,1\,;$$
$$a_{37+k+4m}(z_2,z_1)^* = -a_{39+k+4m}(z_1^*,z_2^*)\,,\ k=0,1\,,\ m=0,1,2\,.$$

One can check that the relations do hold.

Further, one can see that on solutions of the equation

$$y_1^- = y_2^+ \tag{4.20}$$

the coefficients $a_2, a_4, a_6, a_7, a_8, a_{11}, a_{12}, a_{15}, \ldots, a_{22}, a_{27}, \ldots, a_{30}, a_{41}, \ldots, a_{48}$ vanish. As a result, the S-matrix $\mathbb{S}^{BB}$ degenerates and has rank 16. This precisely corresponds to forming a 16-dim four-particle bound state multiplet. Under the condition (4.20) the 48-dim component $\mathscr{W}^4$ of $\mathscr{V}^B$ becomes reducible but indecomposable. The invariant subspace is a 32-dim short multiplet, while the four-particle bound state representation is the factor representation.

5. Crossing symmetry

In this section we introduce the S-matrices which describe the bound state scattering in the light-cone string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, and discuss the conditions imposed on them by crossing symmetry.

5.1 String S-matrix

The canonical $\mathfrak{su}(2|2)$ invariant S-matrices can be used to find the corresponding $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ invariant string S-matrices which describe the scattering of bound states in the light-cone string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. To this end, one should multiply the tensor product of two copies of the canonical $\mathfrak{su}(2|2)$ S-matrix by a scalar factor so that the resulting matrix would satisfy the crossing symmetry relations. Thus, the string S-matrix describing the scattering of M-particle with N-particle bound states is

$$\mathcal{S}^{MN}(z_1,z_2) = S_0^{MN}(z_1,z_2)\,\mathbb{S}^{MN}(z_1,z_2) \otimes \mathbb{S}^{MN}(z_1,z_2)\,. \tag{5.1}$$

The scalar factor S_0^{MN} can be easily identified by noting that, since Λ_1 is the diagonal invariant operator corresponding to the component $V_{(M,N)}^{\frac{M+N}{2}} \times V_{(0,0)}^{0}$ with the maximum spin $j_B = (M+N)/2$, the term

$$S_0^{MN}(z_1, z_2) a_1(z_1, z_2)^2 \, \Lambda_1 \otimes \Lambda_1$$

describes the bound state scattering in the $\mathfrak{su}(2)$ sector of the theory, the latter contains bound states with the single charge non-vanishing[10]. The S-matrix $S_{\mathfrak{su}(2)}^{MN}$ corresponding to this sector is just a scalar function equal to the coefficient in front of $\Lambda_1 \otimes \Lambda_1$. Since we have set the coefficient a_1 equal to unity, we see that the scalar factor S_0 in eq.(5.1) must be equal to the S-matrix of the $\mathfrak{su}(2)$ sector

$$S_0^{MN}(z_1, z_2) = S_{\mathfrak{su}(2)}^{MN}(z_1, z_2) \, .$$

In its turn, the S-matrix in the $\mathfrak{su}(2)$ sector can be determined either by using the fusion procedure [52, 53] or simply by evoking the $\mathfrak{su}(2)$ sector of the asymptotic Bethe equations [4] for fundamental particles.[11] The only subtlety which escaped from considerations in [52, 53] is that the $\mathfrak{su}(2)$-sector S-matrix depends on extra phases which come from the parameters η_k [12]. Taking these phases into account and using the results by [52, 53], we get the following $\mathfrak{su}(2)$-sector S-matrix

$$\begin{aligned} S_{\mathfrak{su}(2)}^{MN}(z_1, z_2) = e^{ia(p_1\epsilon_2 - p_2\epsilon_1)} \left(\frac{y_1^-}{y_1^+}\right)^N \left(\frac{y_2^+}{y_2^-}\right)^M \sigma(y_1^\pm, y_2^\pm)^2 \, \times \\ \times \, G(N-M) G(N+M) \prod_{k=1}^{M-1} G(N-M+2k)^2 \, , \end{aligned} \qquad (5.2)$$

where we assumed $M \leq N$ and introduced the function

$$G(\ell) = \frac{u_1 - u_2 + \frac{i\ell}{g}}{u_1 - u_2 - \frac{i\ell}{g}} \, .$$

In eq.(5.2) the parameters $y_k^\pm$ satisfy the following relations

$$y_k^+ + \frac{1}{y_k^+} - y_k^- - \frac{1}{y_k^-} = M_k \frac{2i}{g} \, , \quad M_1 = M \, , \; M_2 = N$$

and the spectral parameters u_k are expressed in terms of $y_k^\pm$ as follows

$$u_k = \frac{1}{2}\left(y_k^+ + \frac{1}{y_k^+} + y_k^- + \frac{1}{y_k^-}\right) \, .$$

[10]Recall that in the light-cone gauge the particles do not carry the charge $J_1 \equiv J$ that is taken to infinity in the decompactification limit, see [54, 55, 11] for detail.

[11]The $\mathfrak{su}(2)$-sector S-matrix exhibits an intricate structure of poles and zeroes which has been shown in [56, 57] to have a natural interpretation in terms of on-shell intermediate bound states.

The first factor in eq.(5.2) depends on the parameter a which is the parameter of the generalized light-cone gauge [36], and $\epsilon_i = H(p_i)$ is the energy of a bound state. The gauge-dependent factor solves the homogeneous crossing equation [58]. In what follows we set $a = 0$.

The gauge-independent dressing factor is $\sigma(y_1^\pm, y_2^\pm) = e^{i\theta(y_1^\pm, y_2^\pm)}$. Here the dressing phase [4]

$$\theta(y_1^+, y_1^-, y_2^+, y_2^-) = \sum_{r=2}^{\infty}\sum_{n=0}^{\infty} c_{r,r+1+2n}(g)\Big[q_r(y_1^\pm)q_{r+1+2n}(y_2^\pm) - q_r(y_2^\pm)q_{r+1+2n}(y_1^\pm)\Big]$$

is a two-form on the vector space of conserved charges $q_r(y^\pm)$

$$q_r(y_k^-, y_k^+) = \frac{i}{r-1}\left[\left(\frac{1}{y_k^+}\right)^{r-1} - \left(\frac{1}{y_k^-}\right)^{r-1}\right].$$

An important observation made in [52, 53] is that as a function of *four* parameters $y_1^+, y_1^-, y_2^+, y_2^-$ the dressing factor is universal for scattering of any bound states. We point out, however, that as a function of the torus rapidity variables z_1, z_2, the dressing factor depends on M and N, and this results in different crossing symmetry equations satisfied by each of the dressing factors.

The general crossing equations can be derived by combining the fusion procedure with the known crossing equations for the dressing factor of the fundamental S-matrix [14]. In fact, the simplest form of these equations is obtained for a variable Σ which differs from σ by inclusion of the extra phases in eq.(5.2)

$$\Sigma^{MN}(z_1, z_2) = \left(\frac{y_1^-}{y_1^+}\right)^{\frac{N}{2}} \left(\frac{y_2^+}{y_2^-}\right)^{\frac{M}{2}} \sigma(y_1^\pm, y_2^\pm)\,, \tag{5.3}$$

where on the r.h.s. the variables $y_i^\pm$ should be expressed through z_i by using eq.(2.14).

Then, it is not difficult to show that Σ^{MN} should satisfy the following crossing symmetry equations[12]

$$\begin{aligned} &\Sigma^{MN}(z_1, z_2)\,\Sigma^{MN}(z_1 + \omega_2^{(M)}, z_2) = h(y_1^\pm, y_2^\pm)\prod_{k=0}^{M-1} G(M-N-2k)\,, \\ &\Sigma^{MN}(z_1, z_2)\,\Sigma^{MN}(z_1, z_2 - \omega_2^{(N)}) = h(y_1^\pm, y_2^\pm)\, G(M-N)\prod_{k=0}^{N-1} G(M-N+2k)\,, \end{aligned} \tag{5.4}$$

where the function $h(y_1^\pm, y_2^\pm)$ is given by

$$h(y_1^\pm, y_2^\pm) = \frac{(y_1^- - y_2^+)\Big(1 - \frac{1}{y_1^- y_2^-}\Big)}{(y_1^+ - y_2^+)\Big(1 - \frac{1}{y_1^+ y_2^-}\Big)}\,.$$

[12]The second equation in (5.4) follows from the first one by using the unitarity condition $\Sigma^{MN}(z_1, z_2)\Sigma^{NM}(z_2, z_1) = 1$.

Quite interesting, for $M=1$ the right hand sides of two crossing relations in eq.(5.4) are the same

$$\begin{aligned} \Sigma^{1N}(z_1,z_2)\,\Sigma^{1N}(z_1+\omega_2^{(1)},z_2) &= h(y_1^\pm,y_2^\pm)G(1-N)\,,\\ \Sigma^{1N}(z_1,z_2)\,\Sigma^{1N}(z_1,z_2-\omega_2^{(N)}) &= h(y_1^\pm,y_2^\pm)G(1-N)\,. \end{aligned} \tag{5.5}$$

Although the function $\sigma(y_1^+,y_1^-,y_2^+,y_2^-)$ does not have an explicit dependence on M and N, it satisfies the crossing equations which do depend on these numbers, the reason for this lies in the fact that the crossing equations hold on the constraint surfaces of $y_i^\pm$ only and the latter depend on M and N.

5.2 Crossing symmetry equations for the S-matrices

In this section we show that the canonical $\mathfrak{su}(2|2)$ invariant S-matrices we discussed in the previous section are compatible with the crossing equations (5.4). To simplify the notation, we consider only one copy of $\mathfrak{su}(2|2)$.

Fundamental S-matrix

We start with the canonical S-matrix $\mathbb{S}^{AA}\equiv\mathbb{S}$ and multiply it with the square root of the $\mathfrak{su}(2)$-sector S-matrix (5.2)

$$\mathcal{S}(z_1,z_2)=\Sigma(z_1,z_2)\,G(2)^{1/2}\,\mathbb{S}(z_1,z_2),$$

where $\Sigma(z_1,z_2)\equiv\Sigma^{11}(z_1,z_2)$ is the dressing phase (5.3).

The simplest way to formulate the crossing symmetry equations for $\mathcal{S}$ is to use its matrix form (3.21). Then, as was shown in [12, 21], requiring the dressing factor Σ to satisfy eqs.(5.4)

$$\Sigma(z_1,z_2)\,\Sigma(z_1+\omega_2,z_2)=h(x_1^\pm,x_2^\pm)\,,\quad \Sigma(z_1,z_2)\,\Sigma(z_1,z_2-\omega_2)=h(x_1^\pm,x_2^\pm)\,,$$

one finds that the string S-matrix exhibits the following crossing symmetry relations

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(z_1,z_2)\mathscr{C}_1\mathcal{S}_{12}(z_1+\omega_2,z_2)=I\,,\quad \mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(z_1,z_2)\mathscr{C}_1\mathcal{S}_{12}(z_1,z_2-\omega_2)=I\,.$$

Here t_1 denotes transposition in the first matrix space and $\mathscr{C}$ is a constant charge conjugation matrix

$$\mathscr{C}=\begin{pmatrix}\sigma_2 & 0\\ 0 & i\,\sigma_2\end{pmatrix}\,, \tag{5.6}$$

where σ_2 is the Pauli matrix.

It is of interest to understand how the crossing relations are formulated in the operator language. To this end, we need to find out how the operator corresponding to the charge-conjugated and transposed in the first space S-matrix $\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}\mathscr{C}_1$ can be obtained from the invariant operators Λ_k. The corresponding transformation rule for Λ_k appears to be simple and natural. Denoting the transformed operator as $\Lambda_k^{c_1}$, we find that the latter can be obtained from Λ_k by means of the following substitution

$$w_a^1 \to \epsilon_{ab} \frac{\partial}{\partial w_b^1}\,, \quad \frac{\partial}{\partial w_a^1} \to \epsilon^{ab} w_b^1\,, \quad \theta_\alpha^1 \to i\epsilon_{\alpha\beta} \frac{\partial}{\partial \theta_\beta^1}\,, \quad \frac{\partial}{\partial \theta_\alpha^1} \to i\epsilon^{\alpha\beta} \theta_\beta^1\,, \tag{5.7}$$

and in the product of several variables entering Λ_k we should also change their order according to the rule $(\mathbb{A}\mathbb{B})^{c_1} = (-1)^{\epsilon_{\mathbb{A}}\epsilon_{\mathbb{B}}} \mathbb{B}\mathbb{A}$. With these rules we find that the invariant operators transform under the crossing transformation as follows

$$\Lambda_1^{c_1} = \frac{1}{2}\Lambda_1 + \frac{3}{2}\Lambda_2\,,\ \Lambda_2^{c_1} = \frac{1}{2}\Lambda_1 - \frac{1}{2}\Lambda_2\,,\ \Lambda_3^{c_1} = \frac{1}{2}\Lambda_3 + \frac{3}{2}\Lambda_4\,,\ \Lambda_4^{c_1} = \frac{1}{2}\Lambda_3 - \frac{1}{2}\Lambda_4\,,$$
$$\Lambda_5^{c_1} = \Lambda_5\,,\ \Lambda_6^{c_1} = \Lambda_6\,,\ \Lambda_7^{c_1} = -i\Lambda_{10}\,,\ \Lambda_8^{c_1} = -i\Lambda_9\,,\ \Lambda_9^{c_1} = -i\Lambda_8\,,\ \Lambda_{10}^{c_1} = -i\Lambda_7\,.$$

Notice that the crossing transformation does not square to the identity because the off-diagonal operators change their sign.

Then the crossing symmetry relations take the following operator form

$$\mathcal{S}^{c_1}(z_1, z_2) \cdot \mathcal{S}(z_1 + \omega_2, z_2) = \mathbb{I}\,,$$

or, by using the unitarity condition (3.33), we can write them as

$$\mathcal{S}^{c_1}(z_1, z_2) = \mathcal{S}^{\mathcal{P}}(z_1 + \omega_2, z_2)\,.$$

The last equation implies a number of non-trivial relations between the S-matrix coefficients which can be shown to hold.

The fact that the operators $\Lambda_1, \ldots, \Lambda_4$ transform non-trivially under crossing indicates that the basis of invariant operators we use is not particularly suitable for exhibiting the crossing symmetry. It is not difficult to see that a better basis can be obtained by regarding a differential operator acting in the space $\mathscr{V}^M \otimes \mathscr{V}^N$ not as an element of $(\mathscr{V}^M \otimes \mathscr{V}^N) \otimes (\mathscr{D}_M \otimes \mathscr{D}_N)$, but as an element of $(\mathscr{V}^M \otimes \mathscr{D}_M) \otimes (\mathscr{V}^N \otimes \mathscr{D}_N)$. The relation between two bases is a linear transformation arising due to the reordering of the factors in the tensor product. The brackets in the tensor products above indicate the order of the tensor product decomposition. In the new basis the crossing transformation would act in the first factor $\mathscr{V}^M \otimes \mathscr{D}_M$ only leading to a simpler transformation of the invariant operators. A disadvantage of this crossing-adjusted basis is that the corresponding coefficients of the S-matrix will be given by some linear combinations of our simple coefficients a_k, and, for this reason, they will not be simple anymore. Moreover, it would be difficult to find a common normalization for all S-matrices similar to the condition $a_1 = 1$ we imposed.

S-matrix $\mathbb{S}^{AB}$

Multiplying the canonical S-matrix $\mathbb{S}^{AB}$ with the square root of the $\mathfrak{su}(2)$-sector S-matrix (5.2) taken for $M = 1, N = 2$, we get

$$\mathcal{S}^{AB}(z_1, z_2) = \Sigma^{AB}(z_1, z_2)\ (G(1)G(3))^{1/2}\ \mathbb{S}^{AB}(z_1, z_2),$$

where $\Sigma^{AB} \equiv \Sigma^{12}$ is the dressing phase (5.3).

We use again the matrix form (3.21) of $\mathcal{S}^{AB}$ to formulate the crossing symmetry equations. One can show that if the dressing factor Σ^{AB} satisfies eq.(5.4)

$$\Sigma^{AB}(z_1, z_2)\, \Sigma^{AB}(z_1 + \omega_2, z_2) = h(x_1^\pm, y_2^\pm)G(-1) = h(x_1^\pm, y_2^\pm)\frac{u_1 - u_2 - \frac{i}{g}}{u_1 - u_2 + \frac{i}{g}}\,,$$

then the crossing equation corresponding to the shift of the torus rapidity variable z_1 of the fundamental particle takes the same form as eq(5.6)

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{AB,t_1}(z_1, z_2)\mathscr{C}_1\mathcal{S}_{12}^{AB}(z_1 + \omega_2, z_2) = I\,,$$

where $\omega_2 \equiv \omega_2^{(1)}$ is the imaginary half-period of the fundamental particle rapidity torus, and $\mathscr{C}$ is defined in eq.(5.6).

A matrix equation corresponding to the shift of the second rapidity z_2 of the two-particle bound state takes its simplest form not for the S-matrix $\mathcal{S}_{12}^{AB}(z_1, z_2)$ acting in the space $\mathscr{V}^A \otimes \mathscr{V}^B$ but for the S-matrix $\mathcal{S}_{12}^{BA}(z_2, z_1)$ acting in $\mathscr{V}^B \otimes \mathscr{V}^A$. The dressing factor Σ^{BA} of $\mathcal{S}^{BA}$ is related to Σ^{AB} through the unitarity condition $\Sigma^{AB}(z_1, z_2)\Sigma^{BA}(z_2, z_1) = 1$ and it satisfies the crossing equation

$$\Sigma^{BA}(z_2, z_1)\, \Sigma^{BA}(z_2 + \omega_2^{(2)}, z_1) = h(y_2^\pm, x_1^\pm)\,. \tag{5.8}$$

With this equation for the dressing factor, the crossing symmetry relation corresponding to the shift of the two-particle bound state rapidity z_2 acquires the form similar to eq.(5.8)

$$\mathscr{C}_{B,1}^{-1}\mathcal{S}_{12}^{BA,t_1}(z_2, z_1)\mathscr{C}_{B,1}\mathcal{S}_{12}^{BA}(z_2 + \omega_2^{(2)}, z_1) = I\,, \tag{5.9}$$

where $\mathscr{C}_B$ is an 8×8 charge conjugation matrix acting in the two-particle bound state representation

$$\mathscr{C}_B = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

We can formulate the crossing transformations in the operator language too. By using (5.7), we find that the invariant operators transform under the crossing transformation of the fundamental particle space as follows

$$\Lambda_1^{c_1} = \frac{1}{3}\Lambda_1 + \frac{4}{3}\Lambda_2\,, \quad \Lambda_2^{c_1} = \frac{2}{3}\Lambda_1 - \frac{1}{3}\Lambda_2\,, \quad \Lambda_3^{c_1} = \frac{1}{2}\Lambda_3 + \frac{3}{2}\Lambda_4\,, \quad \Lambda_4^{c_1} = \frac{1}{2}\Lambda_3 - \frac{1}{2}\Lambda_4\,,$$
$$\Lambda_5^{c_1} = \Lambda_5\,, \quad \Lambda_6^{c_1} = \Lambda_6\,, \quad \Lambda_7^{c_1} = \frac{1}{2}\Lambda_7 + \frac{3}{2}\Lambda_8\,, \quad \Lambda_8^{c_1} = \frac{1}{2}\Lambda_7 - \frac{1}{2}\Lambda_8\,, \quad \Lambda_9^{c_1} = \Lambda_9\,,$$
$$\Lambda_{10}^{c_1} = -\Lambda_{10}\,, \quad \Lambda_{11}^{c_1} = -\Lambda_{11}\,, \quad \Lambda_{12}^{c_1} = -i\Lambda_{19}\,, \quad \Lambda_{13}^{c_1} = -i\Lambda_{18}\,, \quad \Lambda_{14}^{c_1} = -i\Lambda_{16}\,,$$
$$\Lambda_{15}^{c_1} = -i\Lambda_{17}\,, \quad \Lambda_{16}^{c_1} = -i\Lambda_{14}\,, \quad \Lambda_{17}^{c_1} = -i\Lambda_{15}\,, \quad \Lambda_{18}^{c_1} = -i\Lambda_{13}\,, \quad \Lambda_{19}^{c_1} = -i\Lambda_{12}\,.$$

With these transformation rules, the first crossing symmetry equation acquires the following operator form

$$\mathcal{S}^{AB,c_1}(z_1,z_2)\cdot\mathcal{S}^{AB}(z_1+\omega_2,z_2) = \mathbb{I}\,.$$

The second crossing symmetry relation is found by transforming the invariant operators under the crossing transformation of the two-particle bound space by means of the following substitution

$$w_a^2 \to \epsilon_{ab}\frac{\partial}{\partial w_b^2}\,, \quad \frac{\partial}{\partial w_a^2} \to \epsilon^{ab} w_b^2\,, \quad \theta_\alpha^2 \to -i\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta^2}\,, \quad \frac{\partial}{\partial\theta_\alpha^2} \to -i\epsilon^{\alpha\beta}\theta_\beta^2\,. \tag{5.11}$$

Notice the opposite sign in the transformation of fermions in comparison to eq.(5.7). We find

$$\Lambda_1^{c_2} = \frac{1}{3}\Lambda_1 + \frac{4}{3}\Lambda_2\,, \quad \Lambda_2^{c_2} = \frac{2}{3}\Lambda_1 - \frac{1}{3}\Lambda_2\,, \quad \Lambda_3^{c_2} = \frac{1}{2}\Lambda_3 + \frac{3}{2}\Lambda_4\,, \quad \Lambda_4^{c_2} = \frac{1}{2}\Lambda_3 - \frac{1}{2}\Lambda_4\,,$$
$$\Lambda_5^{c_2} = \Lambda_5\,, \quad \Lambda_6^{c_2} = \Lambda_6\,, \quad \Lambda_7^{c_2} = \frac{1}{2}\Lambda_7 + \frac{3}{2}\Lambda_8\,, \quad \Lambda_8^{c_2} = \frac{1}{2}\Lambda_7 - \frac{1}{2}\Lambda_8\,, \quad \Lambda_9^{c_2} = \Lambda_9\,,$$
$$\Lambda_{10}^{c_2} = -\Lambda_{11}\,, \quad \Lambda_{11}^{c_2} = -\Lambda_{10}\,, \quad \Lambda_{12}^{c_2} = -i\Lambda_{18}\,, \quad \Lambda_{13}^{c_2} = -i\Lambda_{19}\,, \quad \Lambda_{14}^{c_2} = -i\Lambda_{17}\,,$$
$$\Lambda_{15}^{c_2} = -i\Lambda_{16}\,, \quad \Lambda_{16}^{c_2} = -i\Lambda_{15}\,, \quad \Lambda_{17}^{c_2} = -i\Lambda_{14}\,, \quad \Lambda_{18}^{c_2} = -i\Lambda_{12}\,, \quad \Lambda_{19}^{c_2} = -i\Lambda_{13}\,.$$

Then the second crossing symmetry equation acquires the following operator form

$$\mathcal{S}^{AB,c_2}(z_1,z_2)\cdot\mathcal{S}^{AB}(z_1,z_2-\omega_2^{(2)}) = \mathbb{I}\,.$$

The last equation implies a number of non-trivial relations between the S-matrix coefficients which can be shown to hold.

S-matrix $\mathbb{S}^{BB}$

The crossing equations for the string S-matrix $\mathcal{S}^{BB}$ are basically the same as for $\mathcal{S}^{AA}$. All one needs is to replace $\mathscr{C}$ by $\mathscr{C}_B$ in eqs.(5.6).

We multiply the canonical S-matrix $\mathbb{S}^{BB} \equiv \mathbb{S}^{22}$ with the square root of the $\mathfrak{su}(2)$-sector S-matrix (5.2)

$$\mathcal{S}^{BB}(z_1, z_2) = \Sigma^{BB}(z_1, z_2)\,\left(G(2)^2 G(4)\right)^{1/2}\,\mathbb{S}^{BB}(z_1, z_2),$$

where $\Sigma^{BB}(z_1, z_2) \equiv \Sigma^{22}(z_1, z_2)$ is the dressing phase (5.3) satisfying the equations

$$\begin{aligned}
\Sigma^{BB}(z_1, z_2)\,\Sigma^{BB}(z_1 + \omega_2^{(2)}, z_2) &= h(y_1^\pm, y_2^\pm)G(-2) = h(y_1^\pm, y_2^\pm)\frac{u_1 - u_2 - \frac{2i}{g}}{u_1 - u_2 + \frac{2i}{g}}\,, \\
\Sigma^{BB}(z_1, z_2)\,\Sigma^{BB}(z_1, z_2 - \omega_2^{(2)}) &= h(y_1^\pm, y_2^\pm)G(2) = h(y_1^\pm, y_2^\pm)\frac{u_1 - u_2 + \frac{2i}{g}}{u_1 - u_2 - \frac{2i}{g}}\,.
\end{aligned} \tag{5.12}$$

With these equations for the dressing phase, the string S-matrix exhibits the following crossing symmetry relations

$$\begin{aligned}
\mathscr{C}_{B,1}^{-1}\mathcal{S}_{12}^{BB,t_1}(z_1, z_2)\mathscr{C}_{B,1}\mathcal{S}_{12}^{BB}(z_1 + \omega_2^{(2)}, z_2) &= I\,, \\
\mathscr{C}_{B,1}^{-1}\mathcal{S}_{12}^{BB,t_1}(z_1, z_2)\mathscr{C}_{B,1}\mathcal{S}_{12}^{BB}(z_1, z_2 - \omega_2^{(2)}) &= I\,.
\end{aligned} \tag{5.13}$$

To formulate the crossing transformations in the operator language we use (5.7) to find how the invariant operators transform under the crossing transformation

$$\begin{aligned}
&\Lambda_1^{c_1} = \frac{1}{6}\Lambda_1 + \frac{5}{3}\Lambda_2 + \frac{5}{6}\Lambda_3\,, \quad \Lambda_2^{c_1} = \frac{1}{3}\Lambda_1 + \frac{1}{3}\Lambda_2 - \frac{1}{3}\Lambda_3\,, \quad \Lambda_3^{c_1} = \frac{1}{2}\Lambda_1 - \Lambda_2 + \frac{1}{2}\Lambda_3\,, \\
&\Lambda_4^{c_1} = \frac{1}{4}\Lambda_4 + \frac{3}{4}\Lambda_5 + \frac{3}{4}\Lambda_6 + \frac{9}{4}\Lambda_7\,, \quad \Lambda_5^{c_1} = \frac{1}{4}\Lambda_4 - \frac{1}{4}\Lambda_5 + \frac{3}{4}\Lambda_6 - \frac{3}{4}\Lambda_7\,, \\
&\Lambda_6^{c_1} = \frac{1}{4}\Lambda_4 + \frac{3}{4}\Lambda_5 - \frac{1}{4}\Lambda_6 - \frac{3}{4}\Lambda_7\,, \quad \Lambda_7^{c_1} = \frac{1}{4}\Lambda_4 - \frac{1}{4}\Lambda_5 - \frac{1}{4}\Lambda_6 + \frac{1}{4}\Lambda_7\,,
\end{aligned}$$

$$\begin{aligned}
&\Lambda_8^{c_1} = \Lambda_8\,, \quad \Lambda_9^{c_1} = \frac{1}{3}\Lambda_9 + \frac{4}{3}\Lambda_{11}\,, \quad \Lambda_{11}^{c_1} = \frac{2}{3}\Lambda_9 - \frac{1}{3}\Lambda_{11}\,, \\
&\Lambda_{10}^{c_1} = \frac{1}{3}\Lambda_{10} + \frac{4}{3}\Lambda_{12}\,, \quad \Lambda_{12}^{c_1} = \frac{2}{3}\Lambda_{10} - \frac{1}{3}\Lambda_{12}\,, \quad \Lambda_k^{c_1} = \Lambda_k\,, \quad k = 13, 14, 15, 16\,, \\
&\Lambda_{17}^{c_1} = \Lambda_{26}\,, \quad \Lambda_{26}^{c_1} = \Lambda_{17}\,, \quad \Lambda_{18}^{c_1} = \Lambda_{25}\,, \quad \Lambda_{25}^{c_1} = \Lambda_{18}\,,
\end{aligned}$$

$$\begin{aligned}
&\Lambda_{19}^{c_1} = i\Lambda_{28} - i\Lambda_{32}\,,\ \Lambda_{23}^{c_1} = -2i\Lambda_{28} - i\Lambda_{32}\,,\ \Lambda_{28}^{c_1} = \frac{i}{3}\Lambda_{19} - \frac{i}{3}\Lambda_{23}\,, \\
&\Lambda_{20}^{c_1} = i\Lambda_{27} - i\Lambda_{31}\,,\ \Lambda_{24}^{c_1} = -2i\Lambda_{27} - i\Lambda_{31}\,,\ \Lambda_{27}^{c_1} = \frac{i}{3}\Lambda_{20} - \frac{i}{3}\Lambda_{24}\,, \\
&\Lambda_{31}^{c_1} = -\frac{2i}{3}\Lambda_{20} - \frac{i}{3}\Lambda_{24}\,, \qquad \Lambda_{32}^{c_1} = -\frac{2i}{3}\Lambda_{19} - \frac{i}{3}\Lambda_{23}\,, \\
&\Lambda_{21}^{c_1} = i\Lambda_{29}\,, \quad \Lambda_{29}^{c_1} = i\Lambda_{21}\,, \quad \Lambda_{22}^{c_1} = i\Lambda_{30}\,, \quad \Lambda_{30}^{c_1} = i\Lambda_{22}\,,
\end{aligned}$$

$$\begin{aligned}
&\Lambda_{33}^{c_1} = -i\Lambda_{42}\,, \quad \Lambda_{42}^{c_1} = -i\Lambda_{33}\,, \quad \Lambda_{34}^{c_1} = -i\Lambda_{41}\,, \quad \Lambda_{41}^{c_1} = -i\Lambda_{34}\,, \\
&\Lambda_{35}^{c_1} = -i\Lambda_{43}\,, \quad \Lambda_{43}^{c_1} = -i\Lambda_{35}\,, \quad \Lambda_{36}^{c_1} = -i\Lambda_{44}\,, \quad \Lambda_{44}^{c_1} = -i\Lambda_{36}\,, \\
&\Lambda_{37}^{c_1} = -\Lambda_{37}\,, \quad \Lambda_{38}^{c_1} = -\Lambda_{38}\,, \quad \Lambda_{39}^{c_1} = -\Lambda_{40}\,, \quad \Lambda_{40}^{c_1} = -\Lambda_{39}\,.
\end{aligned}$$

With this formulae the first crossing symmetry equation acquires the following operator form

$$\mathcal{S}^{BB,c_1}(z_1,z_2)\cdot\mathcal{S}^{BB}(z_1+\omega_2^{(2)},z_2)=\mathbb{I}\,,$$

and leads to non-trivial relations between the S-matrix coefficients. For instance, one finds

$$\frac{1}{6}+\frac{1}{3}a_2+\frac{1}{2}a_3=\frac{G(-2)G(-4)}{h(y_1^\pm,y_2^\pm)}\,,$$

where a_i are the coefficients of the canonical S-matrix, see Appendix 6.2.2. We have checked that all the relations between the S-matrix coefficients are satisfied. This concludes our discussion of the crossing symmetry for the string S-matrices.

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6. Appendices

6.1 The S-matrix $\mathbb{S}^{AB}$

6.1.1 Invariant differential operators for $\mathbb{S}^{AB}$

To construct a corresponding basis of invariant differential operators Λ_k, we have to figure out all $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ singlet components in the tensor product decomposition of $\mathscr{W}^3\otimes\mathscr{D}_3$. The basis of $\mathscr{W}^3$:

$$\mathscr{W}^3=\text{Span}\Big\{w_a^1w_c^2w_d^2,\,w_a^1w_b^2\theta_\alpha^2,\,\theta_\alpha^1w_a^2w_b^2,\,\theta_\alpha^1w_a^2\theta_\beta^2,\,w_a^1\theta_\alpha^2\theta_\beta^2,\,\theta_\alpha^1\theta_\beta^2\theta_\gamma^2\Big\}\,,$$

as well as the basis of the dual space $\mathscr{D}_3$, can be adopted to the decomposition (3.14). For instance,

$$w_a^1w_c^2w_d^2=\frac{1}{3}(w_a^1w_c^2w_d^2+w_a^2w_c^1w_d^2+w_a^2w_c^2w_d^1)+\frac{1}{3}(\epsilon_{ab}w_c^2+\epsilon_{ac}w_b^2)\epsilon^{kl}w_k^1w_l^2\,,$$

where two terms in the r.h.s. provide the bases for the spaces $V^{3/2}_{(1,2)}\times V^0_{(0,0)}$ and $V^{1/2}_{(1,2)}\times V^0_{(0,0)}$, respectively. Similarly,

$$\begin{aligned}
w_a^1w_b^2\theta_\alpha^2 &= \frac{1}{2}(w_a^1w_b^2+w_a^2w_b^1)\theta_\alpha^2+\frac{1}{2}\epsilon_{ab}\epsilon^{kl}w_k^1w_l^2\theta_\alpha^2\to(V^1_{(1,1)}+V^0_{(1,1)})\times V^{1/2}_{(0,1)}\\
\theta_\alpha^1w_a^2\theta_\beta^2 &= \frac{1}{2}w_a^2(\theta_\alpha^1\theta_\beta^2+\theta_\beta^1\theta_\alpha^2)+\frac{1}{2}w_a^2(\theta_\alpha^1\theta_\beta^2-\theta_\beta^1\theta_\alpha^2)\to V^{1/2}_{(0,1)}\times(V^1_{(1,1)}+V^0_{(1,1)})\\
\theta_\alpha^1w_a^2w_b^2 &\to V^1_{(0,2)}\times V^{1/2}_{(1,0)}\,,\quad w_a^1\theta_\alpha^2\theta_\beta^2\to V^{1/2}_{(1,0)}\times V^0_{(1,1)}\,,\quad \theta_\alpha^1\theta_\beta^2\theta_\gamma^2\to V^0_{(0,0)}\times V^{1/2}_{(1,2)}
\end{aligned}$$

Projecting on the singlet components of the tensor product decomposition $\mathscr{V}^3 \otimes \mathscr{D}_3$, one obtains 19 invariant differential operators of the third order whose explicit form is given below.

$$\Lambda_1 = \frac{1}{6}(w^1_a w^2_b w^2_c + w^2_a w^1_b w^2_c + w^2_a w^2_b w^1_c)\ \frac{\partial^3}{\partial w^2_c \partial w^2_b \partial w^1_a}$$
$$\Lambda_2 = \frac{1}{6}(\epsilon_{ab} w^2_c + \epsilon_{ac} w^2_b)\epsilon^{kl} w^1_k w^2_l\ \frac{\partial^3}{\partial w^2_c \partial w^2_b \partial w^1_a}$$
$$\Lambda_3 = \frac{1}{2}(w^1_a w^2_b + w^2_a w^1_b)\theta^2_\alpha\ \frac{\partial^3}{\partial \theta^2_\alpha \partial w^2_b \partial w^1_a}$$
$$\Lambda_4 = \frac{1}{2}(w^1_a w^2_b - w^2_a w^1_b)\theta^2_\alpha\ \frac{\partial^3}{\partial \theta^2_\alpha \partial w^2_b \partial w^1_a}$$
$$\Lambda_5 = \frac{1}{2} w^2_a w^2_b \theta^1_\alpha\ \frac{\partial^3}{\partial w^2_b \partial w^2_a \partial \theta^1_\alpha}$$
$$\Lambda_6 = \frac{1}{2} w^1_a \theta^2_\alpha \theta^2_\beta\ \frac{\partial^3}{\partial \theta^2_\beta \partial \theta^2_\alpha \partial w^1_a}$$
$$\Lambda_7 = \frac{1}{2} w^2_a (\theta^1_\alpha \theta^2_\beta + \theta^1_\beta \theta^2_\alpha)\ \frac{\partial^3}{\partial w^2_a \partial \theta^2_\beta \partial \theta^1_\alpha}$$
$$\Lambda_8 = \frac{1}{2} w^2_a (\theta^1_\alpha \theta^2_\beta - \theta^1_\beta \theta^2_\alpha)\ \frac{\partial^3}{\partial w^2_a \partial \theta^2_\beta \partial \theta^1_\alpha}$$
$$\Lambda_9 = \frac{1}{2} \theta^1_\alpha \theta^2_\beta \theta^2_\gamma\ \frac{\partial^3}{\partial \theta^2_\gamma \partial \theta^2_\beta \partial \theta^1_\alpha}$$
$$\Lambda_{10} = \frac{1}{2} \epsilon^{kl} w^1_k w^2_l w^2_a \epsilon_{\alpha\beta}\ \frac{\partial^3}{\partial \theta^2_\beta \partial \theta^2_\alpha \partial w^1_a}$$
$$\Lambda_{11} = \frac{1}{2} \epsilon^{\alpha\beta} \theta^2_\alpha \theta^2_\beta \epsilon_{ab} w^1_c\ \frac{\partial^3}{\partial w^2_c \partial w^2_b \partial w^1_a}$$
$$\Lambda_{12} = \epsilon^{kl} w^1_k w^2_l \epsilon_{\alpha\beta} w^2_a\ \frac{\partial^3}{\partial w^2_a \partial \theta^2_\beta \partial \theta^1_\alpha}$$
$$\Lambda_{13} = \epsilon^{\alpha\beta} \theta^1_\alpha \theta^2_\beta \epsilon_{ab} w^2_c\ \frac{\partial^3}{\partial w^2_c \partial w^2_b \partial w^1_a}$$
$$\Lambda_{14} = \frac{1}{2} \epsilon^{\alpha\beta} \theta^2_\alpha \theta^2_\beta \epsilon_{\gamma\delta} w^1_a\ \frac{\partial^3}{\partial w^2_a \partial \theta^2_\delta \partial \theta^1_\gamma}$$
$$\Lambda_{15} = \frac{1}{2} \epsilon^{\alpha\beta} \theta^1_\alpha \theta^2_\beta \epsilon_{\gamma\delta} w^2_a\ \frac{\partial^3}{\partial \theta^2_\delta \partial \theta^2_\gamma \partial w^1_a}$$
$$\Lambda_{16} = \epsilon_{ab} \epsilon^{\alpha\beta} \theta^1_\alpha \theta^2_\beta \theta^2_\gamma\ \frac{\partial^3}{\partial \theta^2_\gamma \partial w^2_b \partial w^1_a}$$
$$\Lambda_{17} = \epsilon^{kl} w^1_k w^2_l \epsilon_{\alpha\beta} \theta^2_\gamma\ \frac{\partial^3}{\partial \theta^2_\gamma \partial \theta^2_\beta \partial \theta^1_\alpha}$$
$$\Lambda_{18} = w^1_a w^2_b \theta^2_\alpha\ \frac{\partial^3}{\partial w^2_b \partial w^2_a \partial \theta^1_\alpha}$$
$$\Lambda_{19} = w^2_a w^2_b \theta^1_\alpha\ \frac{\partial^3}{\partial \theta^2_\alpha \partial w^2_b \partial w^1_a}$$

6.1.2 Coefficients of $\mathbb{S}^{AB}$

Here $x_1^\pm$ are the parameters of the fundamental particle with momentum $p_1 = 2\,\mathrm{am} z_1$, and $y_2^\pm$ are the parameters of the two-particle bound state with momentum $p_2 = 2\,\mathrm{am} z_2$. The parameters η_k are given by

$$\eta_1 = e^{ip_2/2}\eta(z_1,1)\,, \quad \eta_2 = \eta(z_2,2)\,, \quad \tilde\eta_1 = \eta(z_1,1)\,, \quad \tilde\eta_2 = e^{ip_1/2}\eta(z_2,2)\,, \tag{6.1}$$

where $\eta(z,M)$ is defined by (2.15). The coefficients are meromorphic functions of z_1 and z_2 defined on the product of two tori. Note, however, that the tori have different moduli.

$$a_1 = 1$$

$$a_2 = -\frac{1}{2} - \frac{3}{2}\frac{(1-x_1^-y_2^+)(x_1^+-y_2^+)y_2^-}{(1-x_1^-y_2^-)(y_2^--x_1^+)y_2^+}$$

$$a_3 = -\frac{x_1^- - y_2^-}{y_2^- - x_1^+}\frac{\tilde\eta_2}{\eta_2}$$

$$a_4 = -\frac{(1-x_1^-y_2^+)(x_1^--y_2^+)y_2^-}{(1-x_1^-y_2^-)(y_2^--x_1^+)y_2^+}\frac{\tilde\eta_2}{\eta_2}$$

$$a_5 = -\frac{x_1^+ - y_2^+}{y_2^- - x_1^+}\frac{\tilde\eta_1}{\eta_1}$$

$$a_6 = \frac{x_1^-\left(2y_2^+y_2^- - x_1^-y_2^- - x_1^-y_2^+ - x_1^+y_2^+{y_2^-}^2 - x_1^+{y_2^+}^2y_2^- + 2x_1^-x_1^+y_2^+y_2^-\right)}{2x_1^+(x_1^+-y_2^-)(x_1^-y_2^--1)y_2^+}\frac{\tilde\eta_2^2}{\eta_2^2}$$

$$a_7 = -\frac{x_1^- - y_2^+}{y_2^- - x_1^+}\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2}$$

$$a_8 = \frac{x_1^- - y_2^+}{y_2^- - x_1^+}\left(1 - 2\frac{(1-y_2^-x_1^+)(x_1^+-y_2^+)x_1^-}{(1-x_1^-y_2^-)(x_1^--y_2^+)x_1^+}\right)\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2}$$

$$a_9 = -\frac{(1-y_2^-x_1^+)(x_1^--y_2^+)}{(1-x_1^-y_2^-)(y_2^--x_1^+)}\frac{\tilde\eta_1}{\eta_1}$$

$$a_{10} = -\frac{i}{2}\frac{(x_1^--x_1^+)(y_2^--y_2^+)^2}{(1-x_1^-y_2^-)(y_2^--x_1^+)}\frac{1}{\eta_2^2}$$

$$a_{11} = \frac{i}{2}\frac{x_1^-y_2^-}{x_1^+y_2^+}\frac{(x_1^--x_1^+)\,\tilde\eta_2^2}{(1-x_1^-y_2^-)(y_2^--x_1^+)}$$

$$a_{12} = \frac{i}{\sqrt{2}}\frac{(x_1^--x_1^+)(x_1^+-y_2^+)(y_2^--y_2^+)}{(1-x_1^-y_2^-)(y_2^--x_1^+)}\frac{1}{\eta_1\eta_2}$$

$$a_{13} = -\frac{i}{\sqrt{2}}\frac{x_1^- y_2^-}{x_1^+ y_2^+}\frac{(x_1^- - y_2^+)\,\tilde\eta_1\tilde\eta_2}{(1 - x_1^- y_2^-)(y_2^- - x_1^+)}$$

$$a_{14} = \frac{1}{\sqrt{2}}\frac{x_1^-}{x_1^+}\frac{(1 - y_2^- x_1^-)(x_1^- - x_1^+)}{(1 - x_1^- y_2^-)(y_2^- - x_1^+)}\frac{\tilde\eta_2^2}{\eta_1\eta_2}$$

$$a_{15} = \frac{1}{\sqrt{2}}\frac{x_1^-}{x_1^+}\frac{(1 - y_2^- x_1^+)(y_2^- - y_2^+)}{(1 - x_1^- y_2^-)(y_2^- - x_1^+)}\frac{\tilde\eta_1\tilde\eta_2}{\eta_2^2}$$

$$a_{16} = -\frac{i}{\sqrt{2}}\frac{x_1^- y_2^-}{x_1^+ y_2^+}\frac{(x_1^- - y_2^+)}{(1 - x_1^- y_2^-)(y_2^- - x_1^+)}\frac{\tilde\eta_1\tilde\eta_2^2}{\eta_2}$$

$$a_{17} = \frac{i}{\sqrt{2}}\frac{(x_1^- - x_1^+)(x_1^- - y_2^+)(y_2^- - y_2^+)}{(1 - x_1^- y_2^-)(y_2^- - x_1^+)}\frac{\tilde\eta_2}{\eta_1\eta_2^2}$$

$$a_{18} = \frac{1}{\sqrt{2}}\frac{x_1^- - x_1^+}{y_2^- - x_1^+}\frac{\tilde\eta_2}{\eta_1}$$

$$a_{19} = \frac{1}{\sqrt{2}}\frac{y_2^- - y_2^+}{y_2^- - x_1^+}\frac{\tilde\eta_1}{\eta_2}$$

6.1.3 Matrix form of invariant differential operators Λ_k

We use the basis of monomials (3.3). The basis vectors $|e_i\rangle$ of the fundamental representation are $|e_a\rangle = w_a^1$, $|e_\alpha\rangle = \theta_\alpha^1$, and the basis vectors $|e_J\rangle$ of the two-particle bound state representation are

$$\begin{aligned}
|e_1\rangle &= \frac{w_1^2 w_1^2}{\sqrt{2}}\,, \quad |e_2\rangle = w_1^2 w_2^2\,, \quad |e_3\rangle = \frac{w_2^2 w_2^2}{\sqrt{2}}\,, \quad |e_4\rangle = \theta_3^2\theta_4^2\,, \\
|e_5\rangle &= w_1^2\theta_3^2\,, \quad |e_6\rangle = w_1^2\theta_4^2\,, \quad |e_7\rangle = w_2^2\theta_3^2\,, \quad |e_8\rangle = w_2^2\theta_4^2\,.
\end{aligned} \tag{6.2}$$

A $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariant differential operator is represented in the matrix form as a sum over symbols E_{kiLJ} which can be equal to either ${E_k}^i\otimes {E_L}^J$ or to $(-1)^{\epsilon_k\epsilon_L}\,{E_k}^i\otimes {E_L}^J$ (or anything else one wants).

$$\begin{aligned}
\Lambda_1 =\, & E_{1111} + \frac{2}{3}E_{1122} + \frac{1}{3}E_{1133} + \frac{1}{3}\sqrt{2}E_{1221} + \frac{1}{3}\sqrt{2}E_{1232} + \frac{1}{3}\sqrt{2}E_{2112} + \frac{1}{3}\sqrt{2}E_{2123} \\
& + \frac{1}{3}E_{2211} + \frac{2}{3}E_{2222} + E_{2233}
\end{aligned}$$

$$\Lambda_2 = \frac{1}{3}\Big(E_{1122} + 2E_{1133} - \sqrt{2}E_{1221} - \sqrt{2}E_{1232} - \sqrt{2}E_{2112} - \sqrt{2}E_{2123} + 2E_{2211} + E_{2222}\Big)$$

$$\begin{aligned}
\Lambda_3 =\, & E_{1155} + E_{1166} + \frac{1}{2}E_{1177} + \frac{1}{2}E_{1188} + \frac{1}{2}E_{1275} + \frac{1}{2}E_{1286} + \frac{1}{2}E_{2157} + \frac{1}{2}E_{2168} + \frac{1}{2}E_{2255} \\
& + \frac{1}{2}E_{2266} + E_{2277} + E_{2288}
\end{aligned}$$

$$\Lambda_4 = \frac{1}{2}\left(E_{1177} + E_{1188} - E_{1275} - E_{1286} - E_{2157} - E_{2168} + E_{2255} + E_{2266}\right)$$

$$\Lambda_5 = E_{3311} + E_{3322} + E_{3333} + E_{4411} + E_{4422} + E_{4433}$$

$$\Lambda_6 = E_{1144} + E_{2244}$$

$$\Lambda_7 = E_{3355} + \frac{1}{2}E_{3366} + E_{3377} + \frac{1}{2}E_{3388} + \frac{1}{2}E_{3465} + \frac{1}{2}E_{3487} + \frac{1}{2}E_{4356} + \frac{1}{2}E_{4378} + \frac{1}{2}E_{4455} + E_{4466} + \frac{1}{2}E_{4477} + E_{4488}$$

$$\Lambda_8 = \frac{1}{2}\left(E_{3366} + E_{3388} - E_{3465} - E_{3487} - E_{4356} - E_{4378} + E_{4455} + E_{4477}\right)$$

$$\Lambda_9 = E_{3344} + E_{4444}$$

$$\Lambda_{10} = E_{1124} + \sqrt{2}E_{1234} - \sqrt{2}E_{2114} - E_{2224}$$

$$\Lambda_{11} = E_{1142} - \sqrt{2}E_{1241} + \sqrt{2}E_{2143} - E_{2242}$$

$$\Lambda_{12} = E_{1326} + \sqrt{2}E_{1338} - E_{1425} - \sqrt{2}E_{1437} - \sqrt{2}E_{2316} - E_{2328} + \sqrt{2}E_{2415} + E_{2427}$$

$$\Lambda_{13} = E_{3162} + \sqrt{2}E_{3183} - \sqrt{2}E_{3261} - E_{3282} - E_{4152} - \sqrt{2}E_{4173} + \sqrt{2}E_{4251} + E_{4272}$$

$$\Lambda_{14} = E_{1346} - E_{1445} + E_{2348} - E_{2447}$$

$$\Lambda_{15} = E_{3164} + E_{3284} - E_{4154} - E_{4274}$$

$$\Lambda_{16} = -E_{3147} + E_{3245} - E_{4148} + E_{4246}$$

$$\Lambda_{17} = -E_{1374} - E_{1484} + E_{2354} + E_{2464}$$

$$\Lambda_{18} = \sqrt{2}E_{1351} + E_{1372} + \sqrt{2}E_{1461} + E_{1482} + E_{2352} + \sqrt{2}E_{2373} + E_{2462} + \sqrt{2}E_{2483}$$

$$\Lambda_{19} = \sqrt{2}E_{3115} + E_{3127} + E_{3225} + \sqrt{2}E_{3237} + \sqrt{2}E_{4116} + E_{4128} + E_{4226} + \sqrt{2}E_{4238}$$

6.2 The S-matrix $\mathbb{S}^{BB}$

6.2.1 Invariant differential operators for $\mathbb{S}^{BB}$

Recall that $\mathscr{V}^B \otimes \mathscr{V}^B = \mathscr{W}^2 + \mathscr{W}^4$, where $\mathscr{W}^2$ and $\mathscr{W}^4$ are long supermultiplets of dimension 16 and 48, respectively. The branching rule for $\mathscr{W}^2$ under the action of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is given by eq.(4.1). As to $\mathscr{W}^4$, it branches as follows

$$\mathscr{W}^4 = V^0 \times V^0 + 3V^1 \times V^0 + V^2 \times V^0 + V^1 \times V^1 + 2V^{3/2} \times V^{1/2} + 2V^{1/2} \times V^{1/2},$$

where the integers in the r.h.s. denote the multiplicities of the corresponding representations. Thus, the sum $\mathscr{W}^2 + \mathscr{W}^4$ contains $6 + 10 = 16$ $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ multiplets. The basis vectors of $\mathscr{V}^B \otimes \mathscr{V}^B$ give rise to the corresponding basis vectors adopted to the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ decomposition. Schematically,

$$\begin{array}{ll}
w^1_a w^1_b w^2_c w^2_d \to (V^0 + V^1 + V^2) \times V^0 & w^1_a \theta^1_\alpha \theta^2_\beta \theta^2_\gamma \to V^{1/2} \times V^{1/2} \\
w^1_a w^1_b w^2_c \theta^2_\alpha \to (V^{3/2} + V^{1/2}) \times V^{1/2} & w^2_a w^2_b \theta^1_\alpha \theta^1_\beta \to V^1 \times V^0 \\
w^1_a w^1_b \theta^2_\alpha \theta^2_\beta \to V^1 \times V^0 & w^2_a \theta^1_\alpha \theta^1_\beta \theta^2_\gamma \to V^{1/2} \times V^{1/2} \\
w^1_a w^2_b w^2_c \theta^1_\alpha \to (V^{3/2} + V^{1/2}) \times V^{1/2} & \theta^1_\alpha \theta^1_\beta \theta^2_\gamma \theta^2_\delta \to V^0 \times V^0 \\
w^1_a w^2_b \theta^1_\alpha \theta^2_\beta \to (V^0 + V^1) \times (V^0 + V^1) &
\end{array}$$

Here the basis elements of the spaces $V^k \times V^m$ are obtained from those of $\mathscr{V}^B \otimes \mathscr{V}^B$ by proper (anti)symmetrization of the $\mathfrak{su}(2)$ indices. Analogously, one can construct the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ basis of the dual space $\mathscr{D}_2 + \mathscr{D}_4$. Finally, by analyzing the tensor product $(\mathscr{W}^2 + \mathscr{W}^4) \otimes (\mathscr{D}_2 + \mathscr{D}_4)$, one finds that it has 48 singlet components. Thus, there are 48 independent invariant differential operators Λ_k which we present below.

$$\Lambda_1 = \frac{1}{24}\Big[w^2_a w^2_b w^1_c w^1_d + 4 w^1_a w^2_b w^1_c w^2_d + w^1_a w^1_b w^2_c w^2_d\Big] \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial w^2_d}$$

$$\Lambda_2 = \frac{1}{12}\epsilon_{ac}\epsilon_{bd}\epsilon^{kl}\epsilon^{mn} w^1_k w^2_l w^1_m w^2_n \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial w^2_d}$$

$$\Lambda_3 = \frac{1}{8}\epsilon_{ad}(w^1_b w^2_c + w^2_b w^1_c)\epsilon^{kl} w^1_k w^2_l \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial w^2_d}$$

$$\Lambda_4 = \frac{1}{4}(w^1_a w^2_b + w^2_a w^1_b)(\theta^1_\alpha \theta^2_\beta + \theta^1_\beta \theta^2_\alpha) \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_5 = \frac{1}{4}(w^1_a w^2_b + w^2_a w^1_b)(\theta^1_\alpha \theta^2_\beta - \theta^1_\beta \theta^2_\alpha) \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_6 = \frac{1}{4}\epsilon_{ab}\epsilon^{kl} w^1_k w^2_l (\theta^1_\alpha \theta^2_\beta + \theta^1_\beta \theta^2_\alpha) \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_7 = \frac{1}{4}\epsilon_{ab}\epsilon^{kl} w^1_k w^2_l (\theta^1_\alpha \theta^2_\beta - \theta^1_\beta \theta^2_\alpha) \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_8 = \frac{1}{4}\theta^1_\alpha\theta^1_\beta\theta^2_\gamma\theta^2_\delta \; \frac{\partial^4}{\partial\theta^2_\delta\partial\theta^2_\gamma\partial\theta^1_\beta\partial\theta^1_\alpha}$$

$$\Lambda_9 = \frac{1}{6}(w^1_a w^1_b w^2_c + 2w^1_a w^2_b w^1_c)\theta^2_\alpha \; \frac{\partial^4}{\partial w^1_a\partial w^1_b\partial w^2_c\partial\theta^2_\alpha}$$

$$\Lambda_{10} = \frac{1}{6}(2w^2_a w^2_b w^1_c + w^1_a w^2_b w^2_c)\theta^1_\alpha \; \frac{\partial^4}{\partial w^1_a\partial w^2_b\partial w^2_c\partial\theta^1_\alpha}$$

$$\Lambda_{11} = \frac{1}{3}\epsilon_{bc} w^1_a \epsilon^{kl} w^1_k w^2_l \theta^2_\alpha \; \frac{\partial^4}{\partial w^1_a\partial w^1_b\partial w^2_c\partial\theta^2_\alpha}$$

$$\Lambda_{12} = \frac{1}{3}\epsilon_{ab} w^2_c \epsilon^{kl} w^1_k w^2_l \theta^1_\alpha \; \frac{\partial^4}{\partial w^1_a\partial w^2_b\partial w^2_c\partial\theta^1_\alpha}$$

$$\Lambda_{13} = \frac{1}{4} w^1_a w^1_b \theta^2_\alpha\theta^2_\beta \; \frac{\partial^4}{\partial w^1_a\partial w^1_b\partial\theta^2_\beta\partial\theta^2_\alpha}$$

$$\Lambda_{14} = \frac{1}{4} w^2_a w^2_b \theta^1_\alpha\theta^1_\beta \; \frac{\partial^4}{\partial w^2_a\partial w^2_b\partial\theta^1_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{15} = w^2_a \epsilon_{\beta\gamma}\theta^1_\alpha \epsilon^{\rho\delta}\theta^1_\rho\theta^2_\delta \; \frac{\partial^4}{\partial w^2_a\partial\theta^2_\gamma\partial\theta^1_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{16} = w^1_a \epsilon_{\alpha\beta}\theta^2_\gamma \epsilon^{\rho\delta}\theta^1_\rho\theta^2_\delta \; \frac{\partial^4}{\partial w^1_a\partial\theta^2_\gamma\partial\theta^2_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{17} = \frac{1}{4}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\epsilon^{ab} w^1_a w^2_b \epsilon^{cd} w^1_c w^2_d \; \frac{\partial^4}{\partial\theta^2_\delta\partial\theta^2_\gamma\partial\theta^1_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{18} = \frac{1}{4}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\theta^1_\alpha\theta^1_\beta\theta^2_\gamma\theta^2_\delta \; \epsilon_{ac}\epsilon_{bd} \; \frac{\partial^4}{\partial w^1_a\partial w^1_b\partial w^2_c\partial w^2_d}$$

$$\Lambda_{19} = \frac{1}{2}\epsilon^{kl} w^1_k w^2_l \epsilon^{cd} w^1_c w^2_d \epsilon_{ab}\epsilon_{\alpha\beta} \; \frac{\partial^4}{\partial w^1_a\partial w^2_b\partial\theta^2_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{20} = \frac{1}{2}\epsilon^{kl} w^1_k w^2_l \epsilon^{\alpha\beta}\theta^1_\alpha\theta^2_\beta \epsilon_{ac}\epsilon_{bd} \; \frac{\partial^4}{\partial w^1_a\partial w^1_b\partial w^2_c\partial w^2_d}$$

$$\Lambda_{21} = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{ab}\epsilon^{\gamma\delta}\epsilon^{\rho\lambda}\theta^1_\gamma\theta^1_\rho\theta^2_\delta\theta^2_\lambda \; \frac{\partial^4}{\partial w^1_a\partial w^2_b\partial\theta^2_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{22} = \frac{1}{2}\epsilon^{kl} w^1_k w^2_l \epsilon^{\mu\rho}\theta^1_\mu\theta^2_\rho \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \; \frac{\partial^4}{\partial\theta^2_\delta\partial\theta^2_\gamma\partial\theta^1_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{23} = \frac{1}{2}(w^1_b w^2_c + w^2_b w^1_c)\epsilon^{kl} w^1_k w^2_l \epsilon_{\alpha\beta} \; \frac{\partial^4}{\partial w^1_b\partial w^2_c\partial\theta^2_\beta\partial\theta^1_\alpha}$$

$$\Lambda_{24} = \frac{1}{2}\epsilon^{\alpha\beta}\theta^1_\alpha\theta^2_\beta\,\epsilon_{ad}(w^1_b w^2_c + w^1_c w^2_b)\ \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial w^2_d}$$

$$\Lambda_{25} = \frac{1}{4} w^1_a w^1_b \theta^2_\alpha \theta^2_\beta\ \frac{\partial^4}{\partial w^2_a \partial w^2_b \partial \theta^1_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{26} = \frac{1}{4} w^2_a w^2_b \theta^1_\alpha \theta^1_\beta\ \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial \theta^2_\beta \partial \theta^2_\alpha}$$

$$\Lambda_{27} = \frac{1}{3}\epsilon_{ac} w^1_b \theta^2_\alpha \epsilon^{kl} w^1_k w^2_l\ \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial w^2_c \partial \theta^1_\alpha}$$

$$\Lambda_{28} = \frac{1}{3}\epsilon_{ac} w^2_b \theta^1_\alpha \epsilon^{kl} w^1_k w^2_l\ \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial \theta^2_\alpha}$$

$$\Lambda_{29} = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2_\gamma \epsilon^{\rho\delta}\theta^1_\rho \theta^2_\delta w^1_a\ \frac{\partial^4}{\partial w^2_a \partial \theta^2_\gamma \partial \theta^1_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{30} = \frac{1}{2}\epsilon_{\beta\gamma}\theta^1_\alpha \epsilon^{\rho\delta}\theta^1_\rho \theta^2_\delta w^2_a\ \frac{\partial^4}{\partial w^1_a \partial \theta^2_\gamma \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{31} = \frac{1}{6}(2w^1_a w^1_b w^2_c + w^2_a w^1_b w^1_c)\theta^2_\alpha\ \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial w^2_c \partial \theta^1_\alpha}$$

$$\Lambda_{32} = \frac{1}{6}(2w^1_a w^2_b w^2_c + w^2_a w^2_b w^1_c)\theta^1_\alpha\ \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial \theta^2_\alpha}$$

$$\Lambda_{33} = w^1_a w^1_b \theta^2_\alpha \theta^2_\beta\ \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{34} = w^1_a w^2_b \theta^1_\alpha \theta^2_\beta\ \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial \theta^2_\beta \partial \theta^2_\alpha}$$

$$\Lambda_{35} = w^2_a w^2_b \theta^1_\alpha \theta^1_\beta\ \frac{\partial^4}{\partial w^1_a \partial w^2_b \partial \theta^2_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{36} = w^1_a w^2_b \theta^1_\alpha \theta^2_\beta\ \frac{\partial^4}{\partial w^2_a \partial w^2_b \partial \theta^1_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{37} = \frac{1}{4} w^1_b w^2_c \epsilon^{kl} w^1_k w^2_l \epsilon_{\alpha\beta}\ \frac{\partial^4}{\partial w^1_b \partial w^1_c \partial \theta^2_\beta \partial \theta^2_\alpha}$$

$$\Lambda_{38} = \frac{1}{4}\epsilon_{ad} w^1_b w^1_c \epsilon^{\alpha\beta}\theta^2_\alpha \theta^2_\beta\ \frac{\partial^4}{\partial w^1_a \partial w^1_b \partial w^2_c \partial w^2_d}$$

$$\Lambda_{39} = \frac{1}{4} w^1_b w^2_c \epsilon^{kl} w^1_k w^2_l \epsilon_{\alpha\beta}\ \frac{\partial^4}{\partial w^2_b \partial w^2_c \partial \theta^1_\beta \partial \theta^1_\alpha}$$

$$\Lambda_{40} = \frac{1}{4}\epsilon_{ad}w_b^2 w_c^2 \epsilon^{\alpha\beta}\theta_\alpha^1\theta_\beta^1 \, \frac{\partial^4}{\partial w_a^1 \partial w_b^1 \partial w_c^2 \partial w_d^2}$$

$$\Lambda_{41} = w_a^1 \epsilon^{kl} w_k^1 w_l^2 \epsilon_{\alpha\beta}\theta_\gamma^2 \, \frac{\partial^4}{\partial w_a^1 \partial \theta_\gamma^2 \partial \theta_\beta^2 \partial \theta_\alpha^1}$$

$$\Lambda_{42} = \theta_\alpha^2 \epsilon^{\rho\delta}\theta_\rho^1\theta_\delta^2 \epsilon_{bc} w_a^1 \, \frac{\partial^4}{\partial w_a^1 \partial w_b^1 \partial w_c^2 \partial \theta_\alpha^2}$$

$$\Lambda_{43} = w_c^2 \epsilon^{kl} w_k^1 w_l^2 \epsilon_{\beta\gamma}\theta_\alpha^1 \, \frac{\partial^4}{\partial w_c^2 \partial \theta_\gamma^2 \partial \theta_\beta^1 \partial \theta_\alpha^1}$$

$$\Lambda_{44} = \theta_\alpha^1 \epsilon^{\rho\delta}\theta_\rho^1\theta_\delta^2 \epsilon_{ab} w_c^2 \, \frac{\partial^4}{\partial w_a^1 \partial w_b^2 \partial w_c^2 \partial \theta_\alpha^1}$$

$$\Lambda_{45} = w_a^1 \epsilon^{kl} w_k^1 w_l^2 \epsilon_{\beta\gamma}\theta_\alpha^2 \, \frac{\partial^4}{\partial w_a^2 \partial \theta_\gamma^2 \partial \theta_\beta^1 \partial \theta_\alpha^1}$$

$$\Lambda_{46} = \theta_\alpha^1 \epsilon^{\rho\delta}\theta_\rho^1\theta_\delta^2 \epsilon_{bc} w_a^2 \, \frac{\partial^4}{\partial w_a^1 \partial w_b^1 \partial w_c^2 \partial \theta_\alpha^2}$$

$$\Lambda_{47} = w_c^2 \epsilon^{kl} w_k^1 w_l^2 \epsilon_{\alpha\beta}\theta_\gamma^1 \, \frac{\partial^4}{\partial w_c^1 \partial \theta_\gamma^2 \partial \theta_\beta^2 \partial \theta_\alpha^1}$$

$$\Lambda_{48} = \theta_\alpha^2 \epsilon^{\rho\delta}\theta_\rho^1\theta_\delta^2 \epsilon_{ab} w_c^1 \, \frac{\partial^4}{\partial w_a^1 \partial w_b^2 \partial w_c^2 \partial \theta_\alpha^1}$$

6.2.2 Coefficients of the S-matrix $\mathbb{S}^{BB}$

Below $y_1^\pm$ and $y_2^\pm$ parameterize the two-particle bound state with momenta $p_1 = 2\,\mathrm{am} z_1$ and $p_2 = 2\,\mathrm{am} z_2$, respectively, and they are constrained by eqs.(4.18). The parameters η_k are given by

$$\eta_1 = e^{ip_2/2}\eta(z_1,2)\,, \quad \eta_2 = \eta(z_2,2)\,, \quad \tilde{\eta}_1 = \eta(z_1,2)\,, \quad \tilde{\eta}_2 = e^{ip_1/2}\eta(z_2,2)\,, \quad (6.3)$$

where $\eta(z,M)$ is defined by eq.(2.15). The coefficients a_k are meromorphic functions of z_1 and z_2 defined on the product of two equivalent tori. We also use the variables u_1 and u_2

$$u_1 = \frac{1}{2}\Big(y_1^+ + \frac{1}{y_1^+} + y_1^- + \frac{1}{y_1^-}\Big)\,, \qquad u_2 = \frac{1}{2}\Big(y_2^+ + \frac{1}{y_2^+} + y_2^- + \frac{1}{y_2^-}\Big)\,.$$

With this notation the coefficients a_k have the form

$$a_1 = 1$$

$$a_2 = -\frac{(u_1 - u_2 - \frac{2i}{g})(y_1^- - y_2^+)(-1 + y_1^- y_2^+) y_2^- y_1^+}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-) y_1^- y_2^+} - \frac{(y_1^- - y_2^+)(y_1^- - y_1^+)(-1 + y_1^- y_2^+) y_2^-}{2(u_1 - u_2 + \frac{2i}{g})(-1 + y_1^- y_2^-) y_1^- y_2^+}$$
$$- \frac{3(y_1^- - y_2^+)(y_1^- - y_1^+)(y_1^+ - y_2^+)(-1 + y_1^- y_2^+) y_2^-}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-) y_1^- y_2^+}$$

$$a_3 = -\frac{(y_1^- - y_1^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)}{(u_1 - u_2 + \frac{2i}{g})(-1 + y_1^- y_2^-) y_1^+ y_2^+}$$
$$+ \frac{(y_2^- - y_1^+)(-y_2^- + y_2^+) + y_2^-(y_1^+ - y_2^+)(1 - y_1^- y_2^+) - y_2^-(-y_2^- + y_2^+)(1 - y_1^- y_2^+)}{(-1 + y_1^- y_2^-)(y_2^- - y_1^+) y_2^+}$$

$$a_4 = -\frac{y_1^- - y_2^+}{y_2^- - y_1^+} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}$$

$$a_5 = -\frac{2(y_1^- - y_2^-)(y_1^+ - y_2^+)(-1 + y_2^- y_1^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_1^+ y_2^-} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2} + \frac{y_1^- - y_2^+}{y_2^- - y_1^+} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}$$

$$a_6 = -\frac{(u_1 - u_2 - \frac{2i}{g})(y_1^- - y_2^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}$$

$$a_7 = -\frac{(y_1^- - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(-1 + y_1^- y_2^-)(y_2^- - y_1^+) y_1^+ y_2^+ y_2^-} \left[y_2^- y_1^+ + 3 y_1^- (y_2^-)^2 y_1^+ - 4 (y_2^-)^2 (y_1^+)^2 \right.$$
$$-2 y_2^- y_2^+ - 2 y_1^- (y_2^-)^2 y_2^+ + y_1^+ y_2^+ - 5 y_1^- y_2^- y_1^+ y_2^+ + 5 (y_2^-)^2 y_1^+ y_2^+ - y_1^- (y_2^-)^3 y_1^+ y_2^+ + 2 y_2^- (y_1^+)^2 y_2^+$$
$$\left. +2 y_1^- (y_2^-)^2 (y_1^+)^2 y_2^+ + 4 y_1^- y_2^- (y_2^+)^2 - 3 y_2^- y_1^+ (y_2^+)^2 - y_1^- (y_2^-)^2 y_1^+ (y_2^+)^2 \right] \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}$$

$$a_8 = -\frac{(y_1^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_2^- y_1^+)^2 y_1^-}{2(u_1 - u_2 + \frac{2i}{g})(-1 + y_1^- y_2^-)(y_2^- - y_1^+)(y_1^+)^2 y_2^-} \frac{\tilde{\eta}_1^2 \tilde{\eta}_2^2}{\eta_1^2 \eta_2^2} - \frac{(y_1^- - y_2^+)(y_1^- - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_1^+ y_2^-} \frac{\tilde{\eta}_1^2 \tilde{\eta}_2^2}{\eta_1^2 \eta_2^2}$$

$$a_9 = -\frac{y_1^- - y_2^-}{y_2^- - y_1^+} \frac{\tilde{\eta}_2}{\eta_2}$$

$$a_{10} = -\frac{y_1^+ - y_2^+}{y_2^- - y_1^+} \frac{\tilde{\eta}_1}{\eta_1}$$

$$a_{11} = -\frac{(y_1^- - y_2^-)(y_1^- - y_2^+)(-1 + y_1^- y_2^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_1^- y_2^+} \frac{\tilde{\eta}_2}{\eta_2}$$

$$a_{12} = \frac{(y_1^- - y_2^+)(-y_1^+ + y_2^+)(-1 + y_1^- y_2^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_1^- y_2^+} \frac{\tilde{\eta}_1}{\eta_1}$$

$$a_{13} = \frac{y_2^- - y_1^-}{y_2^- - y_1^+} \frac{\tilde{\eta}_2^2}{\eta_2^2} - \frac{(y_1^- - y_2^-)(y_1^- - y_1^+)(y_2^+(-1 + y_1^+ y_2^-) + y_2^-(-1 + y_1^+ y_2^+))}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_1^+ y_2^+ y_2^-} \frac{\tilde{\eta}_2^2}{\eta_2^2}$$

$$a_{14} = -\frac{(u_1 - u_2 - \frac{2i}{g})(y_1^+ - y_2^+) y_2^+}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_2^-} \frac{\tilde{\eta}_1^2}{\eta_1^2} - \frac{(y_1^- + y_1^+ - 2 y_2^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+) y_2^-} \frac{\tilde{\eta}_1^2}{\eta_1^2}$$

$$a_{15} = \frac{(y_1^- - y_2^+)(y_1^+ - y_2^+)(1 - y_2^- y_1^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1^2\tilde{\eta}_2}{\eta_1^2\eta_2}$$

$$a_{16} = -\frac{(y_1^- - y_2^-)(y_1^- - y_2^+)(-1 + y_2^- y_1^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1\tilde{\eta}_2^2}{\eta_1\eta_2^2}$$

$$a_{17} = -\frac{(y_1^- - y_1^+)^2(y_1^- - y_2^+)(y_2^- - y_2^+)^2(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(-1 + y_1^- y_2^-)(y_2^- - y_1^+)y_1^- y_2^-}\frac{1}{\eta_1^2\eta_2^2}$$

$$a_{18} = -\frac{y_1^- y_2^-(y_1^- - y_2^+)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-)(y_1^+)^2(y_2^+)^2}\tilde{\eta}_1^2\tilde{\eta}_2^2$$

$$a_{19} = -\frac{i(y_1^- - y_1^+)(y_1^- - y_2^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_1^- y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-)y_1^- y_2^+}\frac{1}{\eta_1\eta_2}$$

$$a_{20} = \frac{i(y_1^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_1^- y_2^+)y_2^-}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-)y_1^+ {y_2^+}^2}\tilde{\eta}_1\tilde{\eta}_2$$

$$a_{21} = -\frac{i(y_1^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_2^- y_1^+)y_1^-}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-)(y_1^+)^2 y_2^+}\frac{\tilde{\eta}_1^2\tilde{\eta}_2^2}{\eta_1\eta_2}$$

$$a_{22} = \frac{i(y_1^- - y_1^+)(y_1^- - y_2^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)(-1 + y_1^- y_2^-)y_1^+ y_2^-}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1^2\eta_2^2}$$

$$a_{23} = -\frac{i(y_1^- - y_2^-)(y_1^- - y_1^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^- y_2^-}\frac{1}{\eta_1\eta_2}$$

$$a_{24} = \frac{i(y_1^- - y_2^-)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^+}\tilde{\eta}_1\tilde{\eta}_2$$

$$a_{25} = -\frac{(y_1^- - y_1^+)^2(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_2^2}{\eta_1^2}$$

$$a_{26} = -\frac{(y_2^- - y_2^+)^2(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1^2}{\eta_2^2}$$

$$a_{27} = \frac{(y_1^- - y_1^+)(y_1^- - y_2^+)(-1 + y_1^- y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^- y_2^+}\frac{\tilde{\eta}_2}{\eta_1}$$

$$a_{28} = \frac{(u_1 - u_2 - \frac{2i}{g})(y_2^- - y_2^+)}{(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)}\frac{\tilde{\eta}_1}{\eta_2} - \frac{(y_2^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_1^+ y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^+}\frac{\tilde{\eta}_1}{\eta_2}$$

$$a_{29} = \frac{(y_1^- - y_1^+)(y_1^- - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1\tilde{\eta}_2^2}{\eta_1^2\eta_2}$$

$$a_{30} = \frac{(y_1^- - y_2^+)(y_2^- - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1^2\tilde{\eta}_2}{\eta_1\eta_2^2}$$

$$a_{31} = \frac{y_1^- - y_1^+}{y_2^- - y_1^+}\frac{\tilde{\eta}_2}{\eta_1}$$

$$a_{32} = \frac{y_2^- - y_2^+}{y_2^- - y_1^+}\frac{\tilde{\eta}_1}{\eta_2}$$

$$a_{33} = \frac{(y_1^- - y_2^-)(y_1^- - y_1^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_2^2}{\eta_1\eta_2}$$

$$a_{34} = \frac{(y_1^- - y_2^-)(y_2^- - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_2^2}$$

$$a_{35} = \frac{(y_2^- - y_2^+)(y_1^+ - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1^2}{\eta_1\eta_2}$$

$$a_{36} = \frac{(y_1^- - y_1^+)(y_1^+ - y_2^+)(-1 + y_2^- y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^-}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1^2}$$

$$a_{37} = \frac{i(y_1^- - y_2^-)(y_1^- - y_1^+)(y_2^- - y_2^+)^2}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^- y_2^-}\frac{1}{\eta_2^2}$$

$$a_{38} = -\frac{i(y_1^- - y_2^-)(y_1^- - y_1^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^+}\tilde{\eta}_2^2$$

$$a_{39} = \frac{i(y_1^- - y_1^+)(y_1^- - y_1^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^- y_2^-}\frac{1}{\eta_1^2}$$

$$a_{40} = -\frac{i(y_1^+ - y_2^+)(y_2^- - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^+}\tilde{\eta}_1^2$$

$$a_{41} = -\frac{i(y_1^- - y_2^-)(y_1^- - y_1^+)(y_1^- - y_2^+)(y_2^- - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^- y_2^-}\frac{\tilde{\eta}_2^2}{\eta_1\eta_2^2}$$

$$a_{42} = \frac{i(y_1^- - y_2^-)(y_1^- - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^+}\frac{\tilde{\eta}_1\tilde{\eta}_2^2}{\eta_2}$$

$$a_{43} = -\frac{i(y_1^- - y_1^+)(y_1^- - y_2^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})y_1^- y_2^-(y_2^- - y_1^+)}\frac{\tilde{\eta}_1}{\eta_1^2\eta_2}$$

$$a_{44} = \frac{i(y_1^- - y_2^+)(y_1^+ - y_2^+)}{2(u_1 - u_2 + \frac{2i}{g})(y_2^- - y_1^+)y_1^+ y_2^+}\frac{\tilde{\eta}_1^2\tilde{\eta}_2}{\eta_1}$$

$$a_{45} = a_{46} = a_{47} = a_{48} = 0$$

6.2.3 Matrix form of invariant differential operators Λ_k

We use the basis of the two-particle bound state representation (6.2). A $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ invariant differential operator is represented in the matrix form as a sum over symbols E_{KILJ} which can be equal to either $E_K{}^I \otimes E_L{}^J$ or to $(-1)^{\epsilon_K \epsilon_L}\, E_K{}^I \otimes E_L{}^J$ (or anything else one wants).

$$\begin{aligned}\Lambda_1 =\ & E_{1111} + \frac{1}{2}E_{1122} + \frac{1}{6}E_{1133} + \frac{1}{2}E_{1221} + \frac{1}{3}E_{1232} + \frac{1}{6}E_{1331} + \frac{1}{2}E_{2112} + \frac{1}{3}E_{2123} \\ & + \frac{1}{2}E_{2211} + \frac{2}{3}E_{2222} + \frac{1}{2}E_{2233} + \frac{1}{3}E_{2321} + \frac{1}{2}E_{2332} + \frac{1}{6}E_{3113} + \frac{1}{3}E_{3212} + \frac{1}{2}E_{3223} \\ & + \frac{1}{6}E_{3311} + \frac{1}{2}E_{3322} + E_{3333}\end{aligned}$$

$$\Lambda_2 = \frac{1}{3}\left(E_{1133} - E_{1232} + E_{1331} - E_{2123} + E_{2222} - E_{2321} + E_{3113} - E_{3212} + E_{3311}\right)$$

$$\begin{aligned}\Lambda_3 = \frac{1}{2}\big(&E_{1122} + E_{1133} - E_{1221} - E_{1331} - E_{2112} + E_{2211} + E_{2233} - E_{2332} \\ & - E_{3113} - E_{3223} + E_{3311} + E_{3322}\big)\end{aligned}$$

$$\begin{aligned}\Lambda_4 =\ & E_{5555} + \frac{1}{2}E_{5566} + \frac{1}{2}E_{5577} + \frac{1}{4}E_{5588} + \frac{1}{2}E_{5665} + \frac{1}{4}E_{5687} + \frac{1}{2}E_{5775} + \frac{1}{4}E_{5786} \\ & + \frac{1}{4}E_{5885} + \frac{1}{2}E_{6556} + \frac{1}{4}E_{6578} + \frac{1}{2}E_{6655} + E_{6666} + \frac{1}{4}E_{6677} + \frac{1}{2}E_{6688} + \frac{1}{4}E_{6776} \\ & + \frac{1}{4}E_{6875} + \frac{1}{2}E_{6886} + \frac{1}{2}E_{7557} + \frac{1}{4}E_{7568} + \frac{1}{4}E_{7667} + \frac{1}{2}E_{7755} + \frac{1}{4}E_{7766} + E_{7777} \\ & + \frac{1}{2}E_{7788} + \frac{1}{4}E_{7865} + \frac{1}{2}E_{7887} + \frac{1}{4}E_{8558} + \frac{1}{4}E_{8657} + \frac{1}{2}E_{8668} + \frac{1}{4}E_{8756} \\ & + \frac{1}{2}E_{8778} + \frac{1}{4}E_{8855} + \frac{1}{2}E_{8866} + \frac{1}{2}E_{8877} + E_{8888}\end{aligned}$$

$$\begin{aligned}\Lambda_5 =\ & \frac{1}{4}\big(2E_{5566} + E_{5588} - 2E_{5665} - E_{5687} + E_{5786} - E_{5885} - 2E_{6556} - E_{6578} + 2E_{6655} \\ & + E_{6677} - E_{6776} + E_{6875} + E_{7568} - E_{7667} + E_{7766} + 2E_{7788} - E_{7865} - 2E_{7887} \\ & - E_{8558} + E_{8657} - E_{8756} - 2E_{8778} + E_{8855} + 2E_{8877}\big)\end{aligned}$$

$$\begin{aligned}\Lambda_6 =\ & \frac{1}{4}\big(2E_{5577} + E_{5588} + E_{5687} - 2E_{5775} - E_{5786} - E_{5885} + E_{6578} + E_{6677} + 2E_{6688} \\ & - E_{6776} - E_{6875} - 2E_{6886} - 2E_{7557} - E_{7568} - E_{7667} + 2E_{7755} + E_{7766} + E_{7865} \\ & - E_{8558} - E_{8657} - 2E_{8668} + E_{8756} + E_{8855} + 2E_{8866}\big)\end{aligned}$$

$$\begin{aligned}\Lambda_7 =\ & \frac{1}{4}\big(E_{5588} - E_{5687} - E_{5786} + E_{5885} - E_{6578} + E_{6677} + E_{6776} - E_{6875} - E_{7568} \\ & + E_{7667} + E_{7766} - E_{7865} + E_{8558} - E_{8657} - E_{8756} + E_{8855}\big)\end{aligned}$$

$$\Lambda_8 = E_{4444}$$

$$
\begin{aligned}
\Lambda_9 =\ & E_{1155} + E_{1166} + \frac{1}{3}E_{1177} + \frac{1}{3}E_{1188} + \frac{\sqrt{2}}{3}E_{1275} + \frac{\sqrt{2}}{3}E_{1286} + \frac{\sqrt{2}}{3}E_{2157} + \frac{\sqrt{2}}{3}E_{2168} \\
& + \frac{2}{3}E_{2255} + \frac{2}{3}E_{2266} + \frac{2}{3}E_{2277} + \frac{2}{3}E_{2288} + \frac{\sqrt{2}}{3}E_{2375} + \frac{\sqrt{2}}{3}E_{2386} + \frac{\sqrt{2}}{3}E_{3257} \\
& + \frac{\sqrt{2}}{3}E_{3268} + \frac{1}{3}E_{3355} + \frac{1}{3}E_{3366} + E_{3377} + E_{3388}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{10} =\ & E_{5511} + \frac{2}{3}E_{5522} + \frac{1}{3}E_{5533} + \frac{1}{3}\sqrt{2}E_{5721} + \frac{1}{3}\sqrt{2}E_{5732} + E_{6611} + \frac{2}{3}E_{6622} \\
& + \frac{1}{3}E_{6633} + \frac{1}{3}\sqrt{2}E_{6821} + \frac{1}{3}\sqrt{2}E_{6832} + \frac{1}{3}\sqrt{2}E_{7512} + \frac{1}{3}\sqrt{2}E_{7523} + \frac{1}{3}E_{7711} \\
& + \frac{2}{3}E_{7722} + E_{7733} + \frac{1}{3}\sqrt{2}E_{8612} + \frac{1}{3}\sqrt{2}E_{8623} + \frac{1}{3}E_{8811} + \frac{2}{3}E_{8822} + E_{8833}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{11} =\ & \frac{1}{3}\Big(2E_{1177} + 2E_{1188} - \sqrt{2}E_{1275} - \sqrt{2}E_{1286} - \sqrt{2}E_{2157} - \sqrt{2}E_{2168} + E_{2255} + E_{2266} \\
& + E_{2277} + E_{2288} - \sqrt{2}E_{2375} - \sqrt{2}E_{2386} - \sqrt{2}E_{3257} - \sqrt{2}E_{3268} + 2E_{3355} + 2E_{3366}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{12} =\ & \frac{1}{3}\Big(E_{5522} + 2E_{5533} - \sqrt{2}E_{5721} - \sqrt{2}E_{5732} + E_{6622} + 2E_{6633} - \sqrt{2}E_{6821} - \sqrt{2}E_{6832} \\
& - \sqrt{2}E_{7512} - \sqrt{2}E_{7523} + 2E_{7711} + E_{7722} - \sqrt{2}E_{8612} - \sqrt{2}E_{8623} + 2E_{8811} + E_{8822}\Big)
\end{aligned}
$$

$$
\Lambda_{13} = E_{1144} + E_{2244} + E_{3344}
$$

$$
\Lambda_{14} = E_{4411} + E_{4422} + E_{4433}
$$

$$
\Lambda_{15} = E_{4455} + E_{4466} + E_{4477} + E_{4488}
$$

$$
\Lambda_{16} = E_{5544} + E_{6644} + E_{7744} + E_{8844}
$$

$$
\Lambda_{17} = E_{1434} - E_{2424} + E_{3414}
$$

$$
\Lambda_{18} = E_{4143} - E_{4242} + E_{4341}
$$

$$
\begin{aligned}
\Lambda_{19} =\ & E_{1538} - E_{1637} - E_{1736} + E_{1835} - E_{2528} + E_{2627} + E_{2726} - E_{2825} \\
& + E_{3518} - E_{3617} - E_{3716} + E_{3815}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{20} =\ & E_{5183} - E_{5282} + E_{5381} - E_{6173} + E_{6272} - E_{6371} - E_{7163} + E_{7262} \\
& - E_{7361} + E_{8153} - E_{8252} + E_{8351}
\end{aligned}
$$

$$\Lambda_{21} = E_{4548} - E_{4647} - E_{4746} + E_{4845}$$

$$\Lambda_{22} = E_{5484} - E_{6474} - E_{7464} + E_{8454}$$

$$\begin{aligned}\Lambda_{23} = &\sqrt{2}E_{1526} + E_{1538} - \sqrt{2}E_{1625} - E_{1637} + E_{1736} - E_{1835} - \sqrt{2}E_{2516} + \sqrt{2}E_{2615}\\ &+ \sqrt{2}E_{2738} - \sqrt{2}E_{2837} - E_{3518} + E_{3617} - E_{3716} - \sqrt{2}E_{3728} + E_{3815} + \sqrt{2}E_{3827}\end{aligned}$$

$$\begin{aligned}\Lambda_{24} = &\sqrt{2}E_{5162} + E_{5183} - \sqrt{2}E_{5261} - E_{5381} - \sqrt{2}E_{6152} - E_{6173} + \sqrt{2}E_{6251} + E_{6371}\\ &+ E_{7163} + \sqrt{2}E_{7283} - E_{7361} - \sqrt{2}E_{7382} - E_{8153} - \sqrt{2}E_{8273} + E_{8351} + \sqrt{2}E_{8372}\end{aligned}$$

$$\Lambda_{25} = E_{1441} + E_{2442} + E_{3443}$$

$$\Lambda_{26} = E_{4114} + E_{4224} + E_{4334}$$

$$\begin{aligned}\Lambda_{27} = &\frac{1}{3}\Big(\sqrt{2}E_{1572} + \sqrt{2}E_{1682} - 2E_{1771} - 2E_{1881} - E_{2552} + \sqrt{2}E_{2573} - E_{2662} + \sqrt{2}E_{2683}\\ &+ \sqrt{2}E_{2751} - E_{2772} + \sqrt{2}E_{2861} - E_{2882} - 2E_{3553} - 2E_{3663} + \sqrt{2}E_{3752} + \sqrt{2}E_{3862}\Big)\end{aligned}$$

$$\begin{aligned}\Lambda_{28} = &\frac{1}{3}\Big(\sqrt{2}E_{5127} - E_{5225} + \sqrt{2}E_{5237} - 2E_{5335} + \sqrt{2}E_{6128} - E_{6226} + \sqrt{2}E_{6238} - 2E_{6336}\\ &- 2E_{7117} + \sqrt{2}E_{7215} - E_{7227} + \sqrt{2}E_{7325} - 2E_{8118} + \sqrt{2}E_{8216} - E_{8228} + \sqrt{2}E_{8326}\Big)\end{aligned}$$

$$\Lambda_{29} = -E_{5445} - E_{6446} - E_{7447} - E_{8448}$$

$$\Lambda_{30} = -E_{4554} - E_{4664} - E_{4774} - E_{4884}$$

$$\begin{aligned}\Lambda_{31} = &E_{1551} + \frac{\sqrt{2}}{3}E_{1572} + E_{1661} + \frac{\sqrt{2}}{3}E_{1682} + \frac{1}{3}E_{1771} + \frac{1}{3}E_{1881} + \frac{2}{3}E_{2552} + \frac{\sqrt{2}}{3}E_{2573}\\ &+ \frac{2}{3}E_{2662} + \frac{\sqrt{2}}{3}E_{2683} + \frac{\sqrt{2}}{3}E_{2751} + \frac{2}{3}E_{2772} + \frac{\sqrt{2}}{3}E_{2861} + \frac{2}{3}E_{2882} + \frac{1}{3}E_{3553}\\ &+ \frac{1}{3}E_{3663} + \frac{\sqrt{2}}{3}E_{3752} + E_{3773} + \frac{\sqrt{2}}{3}E_{3862} + E_{3883}\end{aligned}$$

$$\begin{aligned}\Lambda_{32} = &E_{5115} + \frac{\sqrt{2}}{3}E_{5127} + \frac{2}{3}E_{5225} + \frac{\sqrt{2}}{3}E_{5237} + \frac{1}{3}E_{5335} + E_{6116} + \frac{\sqrt{2}}{3}E_{6128} + \frac{2}{3}E_{6226}\\ &+ \frac{\sqrt{2}}{3}E_{6238} + \frac{1}{3}E_{6336} + \frac{1}{3}E_{7117} + \frac{\sqrt{2}}{3}E_{7215} + \frac{2}{3}E_{7227} + \frac{\sqrt{2}}{3}E_{7325} + E_{7337} + \frac{1}{3}E_{8118}\\ &+ \frac{\sqrt{2}}{3}E_{8216} + \frac{2}{3}E_{8228} + \frac{\sqrt{2}}{3}E_{8326} + E_{8338}\end{aligned}$$

$$\Lambda_{33} = \sqrt{2}E_{1546} - \sqrt{2}E_{1645} + E_{2548} - E_{2647} + E_{2746} - E_{2845} + \sqrt{2}E_{3748} - \sqrt{2}E_{3847}$$

$$\Lambda_{34} = \sqrt{2}E_{5164} + E_{5284} - \sqrt{2}E_{6154} - E_{6274} + E_{7264} + \sqrt{2}E_{7384} - E_{8254} - \sqrt{2}E_{8374}$$

$$\Lambda_{35} = \sqrt{2}E_{4516} + E_{4528} - \sqrt{2}E_{4615} - E_{4627} + E_{4726} + \sqrt{2}E_{4738} - E_{4825} - \sqrt{2}E_{4837}$$

$$\Lambda_{36} = \sqrt{2}E_{5461} + E_{5482} - \sqrt{2}E_{6451} - E_{6472} + E_{7462} + \sqrt{2}E_{7483} - E_{8452} - \sqrt{2}E_{8473}$$

$$\Lambda_{37} = E_{1124} + E_{1234} - E_{2114} + E_{2334} - E_{3214} - E_{3324}$$

$$\Lambda_{38} = E_{1142} - E_{1241} + E_{2143} - E_{2341} + E_{3243} - E_{3342}$$

$$\Lambda_{39} = E_{1421} + E_{1432} - E_{2411} + E_{2433} - E_{3412} - E_{3423}$$

$$\Lambda_{40} = E_{4112} + E_{4123} - E_{4211} + E_{4233} - E_{4321} - E_{4332}$$

$$\Lambda_{41} = -\sqrt{2}E_{1574} - \sqrt{2}E_{1684} + E_{2554} + E_{2664} - E_{2774} - E_{2884} + \sqrt{2}E_{3754} + \sqrt{2}E_{3864}$$

$$\Lambda_{42} = -\sqrt{2}E_{5147} + E_{5245} - \sqrt{2}E_{6148} + E_{6246} - E_{7247} + \sqrt{2}E_{7345} - E_{8248} + \sqrt{2}E_{8346}$$

$$\Lambda_{43} = -E_{5425} - \sqrt{2}E_{5437} - E_{6426} - \sqrt{2}E_{6438} + \sqrt{2}E_{7415} + E_{7427} + \sqrt{2}E_{8416} + E_{8428}$$

$$\Lambda_{44} = -E_{4552} - \sqrt{2}E_{4573} - E_{4662} - \sqrt{2}E_{4683} + \sqrt{2}E_{4751} + E_{4772} + \sqrt{2}E_{4861} + E_{4882}$$

$$\Lambda_{45} = -\sqrt{2}E_{1475} - \sqrt{2}E_{1486} + E_{2455} + E_{2466} - E_{2477} - E_{2488} + \sqrt{2}E_{3457} + \sqrt{2}E_{3468}$$

$$\Lambda_{46} = -\sqrt{2}E_{4157} - \sqrt{2}E_{4168} + E_{4255} + E_{4266} - E_{4277} - E_{4288} + \sqrt{2}E_{4375} + \sqrt{2}E_{4386}$$

$$\Lambda_{47} = -E_{5524} - \sqrt{2}E_{5734} - E_{6624} - \sqrt{2}E_{6834} + \sqrt{2}E_{7514} + E_{7724} + \sqrt{2}E_{8614} + E_{8824}$$

$$\Lambda_{48} = -E_{5542} + \sqrt{2}E_{5741} - E_{6642} + \sqrt{2}E_{6841} - \sqrt{2}E_{7543} + E_{7742} - \sqrt{2}E_{8643} + E_{8842}$$

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