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Phase space Langevin equation for spin relaxation in a dc magnetic field

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Abstract – A Langevin equation for the quantum Brownian motion of a spin of arbitrary size in a uniform external dc magnetic field is derived from the phase space master equation in the weak coupling and narrowing limits, for the quasiprobability distribution (Wigner) function of spin orientations in the configuration space of polar and azimuthal angles following methods long familiar in quantum optics. The closed system of differential-recurrence equations for the statistical moments describing magnetic relaxation of the spin is obtained as an example of applications of this equation.

Phasespace representations of quantum-mechanical evolution equations \cite{1,2} (generally based on the coherent state representation of the density matrix introduced by Glauber and Sudarshan as widely used in quantum optics \cite{3,4}) when applied to spins (e.g., \cite{5–10}) enable one to study spin relaxation using a master equation. This equation governs the evolution of the \textit{quasiprobability distribution function} \( W(\vartheta, \varphi, t) \) of spin orientations in the relevant phase (here configuration) space \((\vartheta, \varphi)\), where \( \vartheta \) and \( \varphi \) are the polar and azimuthal angles, constituting the canonical variables. Mapping of quantum spin dynamics onto \( \psi \)-number master equations for \( W(\vartheta, \varphi, t) \) exemplifies how these equations reduce to the Fokker-Planck equation governing the rotational Brownian motion of an assembly of classical spins when the spin number \( S \to \infty \) \cite{6,7,10}. Phase space distribution functions for spins originally introduced by Stratonovich \cite{11} for closed systems have been extensively developed for both closed and open spin systems (e.g., \cite{12–15}) and are entirely analogous to the Wigner distribution \( W(x, p, t) \) for the translational motion of a particle in the phase space of positions and momenta \((x, p)\) \cite{1,2}. The Wigner function \( W(\vartheta, \varphi, t) \) analogous to \( W(x, p, t) \) enables the expected value \( \langle \hat{A} \rangle(t) \) of a spin operator \( \hat{A} \) to be determined using the corresponding \( \psi \)-number (or Weyl symbol) \( A(\vartheta, \varphi) \), viz., \cite{12}

\[
\langle \hat{A} \rangle(t) = \frac{2S+1}{4\pi} \int_0^\pi \int_0^{2\pi} A(\vartheta, \varphi) W(\vartheta, \varphi, t) \sin \vartheta \, d\vartheta \, d\varphi
\]

so that quantum-mechanical averages may be calculated just as classical ones \cite{1–4}.

Besides the phase space (generalized coherent state) \cite{5–10,12–15} many other methods for the description of spin dynamics already exist, e.g., the reduced density matrix \cite{16,17}, the stochastic Liouville equation \cite{8,18}, the Langevin equation \cite{19,20}. In general, however, phase space methods are attractive because they map quantum-mechanical evolution equations for the (reduced) density matrix for spins onto a \( \psi \)-number space which has an obvious advantage over the operator equations when one wishes to study the quantum/classical divide. Moreover, in the phase space formalism, only a master equation in configuration space akin to the Fokker-Planck equation for the rotational Brownian motion of classical magnetic dipoles is involved. Hence the formalism is relatively easy to implement for the purpose of practical calculations because the existence of phase space master equations

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enables powerful computational techniques developed for the Fokker-Planck equation \((e.g., \text{continued fractions, mean first passage times, etc. [21,22]) to be seamlessly carried over into the quantum domain [10,23]. The analogy with the Fokker-Planck equation also suggests that the quantum stochastic spin dynamics can be modeled by using the Langlevin equations associated with the phase space master equation [24] rather than the averaged phase space dynamics yielded by that equation alone. We remark that several attempts (see, e.g., [24–26]) to obtain quantum Langevin equations pertaining to stochastic spin dynamics in configuration space have already been made. However, they have been formulated in a manner precluding calculation of the observables. Here, in contrast, besides writing a Langevin equation describing the spin dynamics in phase space directly yielding the stochastic trajectories, we show how this equation also yields the closed system of differential-recurrence relations for the statistical moments (averaged spherical harmonics) governing the spin relaxation.

The dynamics of a spin \(\hat{S}\) in an external dc magnetic field \(\hat{H}_0\) directed along the Z-axis and a random field \(h(t)\) characterizing the collision damping (due to the heat bath) incurred by the precessional motion of the spin may be described by the Hamiltonian [6–9]

\[
\hat{H} = \hat{H}_S + \hat{H}_{SB} + \hat{H}_B,
\]

where \(\hat{H}_S = -\hbar\omega_0\hat{S}_Z\), \(\omega_0 = \gamma \hat{H}_0\) is the precession (Larmor) frequency, \(\gamma\) is the gyromagnetic ratio, the term \(\hat{H}_{SB} = -\gamma \hbar \hat{S}\cdot \hat{S}\) describes the spin-bath interaction, and \(\hat{H}_B\) characterizes the bath. The equation of motion of the density matrix \(\hat{\rho}\) is then

\[
\frac{\partial \hat{\rho}}{\partial t} + i \frac{\hbar}{\omega}[\hat{H}_S, \hat{\rho}] = \hat{Q}(\hat{\rho}),
\]

where \(\hat{Q}(\hat{\rho}) = -(i/\hbar)[\hat{H}_S + \hat{H}_B, \hat{\rho}]\) is the collision kernel operator. The reduced density matrix \(\hat{\sigma} = \text{Tr}_B \hat{\rho}\) (i.e., averaged over the density matrix of the bath) satisfies the equation [4,7,13]

\[
\frac{\partial \hat{\sigma}}{\partial t} - i \omega_0 [\hat{S}_0, \hat{\sigma}] = C[\hat{S}_0, \hat{\sigma}, \hat{S}_0] + C^* [\hat{S}_0, \hat{\sigma}, \hat{S}_0] + B^* [\hat{S}_-, \hat{\sigma}, \hat{S}_+] + B[\hat{S}_-, \hat{\sigma}, \hat{S}_+] + e^{i\hbar \omega_0} \left\{ B^* [\hat{S}_+, \hat{\sigma}, \hat{S}_+] + B[\hat{S}_+, \hat{\sigma}, \hat{S}_+] \right\},
\]

where \(\hat{S}_\pm = \hat{S}_X \pm i \hat{S}_Y\) and \(\hat{S}_0 = \hat{S}_Z\) are the spin operators, \(\beta = (kT)^{-1}\), \(k\) is Boltzmann’s constant, \(T\) is the temperature,

\[
B = \gamma^2 \int_0^\infty \langle h_-(t) h_+(0) \rangle_B e^{-i\omega_0 t} dt,
\]

\[
C = \gamma^2 \int_0^\infty \langle h_0(t) h_0(0) \rangle_B dt,
\]

\[
h_\pm = (h_x \pm ih_y)/2\text{ and } h_0 = h_z, \text{ and the asterisk denotes the complex conjugate. Here the averages are over the equilibrium bath density matrix (assuming axial symmetry about the Z-axis with the averaged field components \(\langle h_\pm(t) \rangle_B = 0\) and \(\langle h_0(t) \rangle_B = 0\). The corresponding evolution equation for \(W(\vartheta, \varphi, t)\) is [13]

\[
\frac{\partial W}{\partial t} - \omega_0 \frac{\partial W}{\partial \varphi} = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left[ \sin \vartheta D_{11} \frac{\partial W}{\partial \vartheta} \right] + D_1 \{ e^{ih\omega_0} - 1 \} \sin^2 \vartheta \frac{\partial W}{\partial \varphi^2},
\]

where \(D_{11}(\vartheta)\) and \(D_{22}(\vartheta)\) are the diffusion coefficients given by

\[
D_{11}(\vartheta) = \frac{D_1}{2} \left( e^{ih\omega_0} + 1 - e^{ih\omega_0} \cos \vartheta \right),
\]

\[
D_{22}(\vartheta) = D_{11}(\vartheta) \sin^2 \vartheta + D_2 = \frac{D_1}{2} \left( e^{ih\omega_0} + 1 \right),
\]

\(D_1 = 2\text{Re}(B)\) and \(D_2 = \text{Re}(C)\) are effective “diffusion” constants related to the random magnetic field imposed by the reservoir on the spin; below, for simplicity, we consider the case \(D_1 = D_2\). Here the Wigner distribution function is given by [12]

\[
W(\vartheta, \varphi, t) = \text{Tr}\{ \hat{\sigma}(t) \hat{\omega}(\vartheta, \varphi) \},
\]

where \(\hat{\omega}(\vartheta, \varphi)\) is the Stratonovich-Wigner operator defined as

\[
\hat{\omega}(\vartheta, \varphi) = \sqrt{\frac{4\pi}{2S + 1}} \sum_{L=0}^S \sum_{\pm} \sum_{M=-L}^L C_{S,S,L,M} Y_L^\pm(\vartheta, \varphi) \hat{T}^{(S)}_{L,M},
\]

such that \(\text{Tr}\{ \hat{\omega} \} = 1\) and

\[
\int_{0}^{2\pi} \int_{-\pi/2}^{\pi/2} \hat{\omega}(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi = I. \tag{11}
\]

Here \(I\) is the identity matrix, \(Y_{L,M}(\vartheta, \varphi)\) are the spherical harmonics [27], \(\hat{T}^{(S)}_{L,M}\) are the irreducible tensor (polarization) operators with matrix elements given by [27]

\[
\left[ \hat{T}^{(S)}_{L,M} \right]_{m',m} = \sqrt{\frac{2L+1}{2S+1}} C_{S,m',L,m} \times C_{S,m,L,M}, \tag{12}
\]

and \(C_{S,S,L,0}^S\) and \(C_{S,S,L,M}^S\) are the Clebsch-Gordan coefficients [27]. Equation (6) is also written for a so-called Q distribution function \(W\) [4,12]. We remark in passing that in general the spin quasiprobability functions belonging to the SU(2) dynamical symmetry group are parametrized by a parameter \(s\). The parameter values \(s = 0\) and \(s = \pm 1\) correspond to the Stratonovich [11] and Berezin [12] contravariant and covariant functions, respectively (the latter are directly related to the \(P\) and \(Q\) symbols.

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appearing naturally in the coherent state representation [3, 4, 12]. Here we consider the \( Q \) distribution function \((s = -1)\) because it alone satisfies the non-negativity condition required of a true probability density function, viz, \( W \geq 0 \).

Equation (6) (which has the form of a Fokker-Planck equation so permitting one to write the corresponding Langevin equation) describes the quantum Brownian motion of spins \( S \) in a dc field \( H_0 \) in the narrowing limit case when the correlation time \( \tau_c \) of the random field (white noise) \( h(t) \) acting on a spin satisfies the condition \( \gamma \hbar \tau_c \ll 1 \), where \( \hbar \) is the averaged amplitude of the random magnetic field. The left-hand side of eq. (6) is the quantum analog of the Liouville equation for spins (now the same as the classical case just as the corresponding result for particles with quadratic Hamiltonians \([28]\)) while the right-hand side (collision kernel) characterizes the spin-bath interaction at temperature \( T \). Essentially, eq. (6) follows from the equation of motion of the reduced density matrix with spin-bath interactions small enough for the weak coupling limit to apply and with bath correlation time so short that the stochastic process originating in it can be regarded as Markovian, i.e., one assumes frequency independent damping. This approximation may be used for relatively high temperatures. In the parameter ranges, where such an approximation is invalid (e.g., throughout the very low-temperature region), a more general form of the phase space master equation with time dependent drift and diffusion coefficients should be used (see, e.g., refs. [8, 14] and [15] where such evolution equations have been derived). We emphasize that the master equation (6) describes the relaxation of \( W(\vartheta, \varphi, t) \) to the equilibrium distribution [9, 10]

\[
W_{eq}(\vartheta) = Z_S^{-1} \left[ \cosh \left( \frac{\beta \hbar \omega_0}{2} \right) + \sinh \left( \frac{\beta \hbar \omega_0}{2} \right) \cos \vartheta \right]^{2S}, \tag{13}
\]

where

\[
Z_S = \left( S + \frac{1}{2} \right) \left[ \cosh \left( \frac{\beta \hbar \omega_0}{2} \right) + \sinh \left( \frac{\beta \hbar \omega_0}{2} \right) \right]^{2S} \sinh \left( \frac{1}{2} \beta \hbar \omega_0 \right), \tag{14}
\]

is the partition function. We further remark that \( W_{eq} \) corresponds to the canonical equilibrium density matrix \( \hat{\rho}_{eq} = e^{\beta \hat{H}_0} Z_S / Z_S \) pertaining to thermal equilibrium without dissipative coupling to the thermal bath. However, in quantum open systems [29], the equilibrium state may deviate from the canonical distribution \( \hat{\rho}_{eq} \) hence \( W_{eq} \) may describe the thermal equilibrium of the open system in the weak coupling and high temperature limits only (for a detailed discussion see Geva et al. [30]).

We have mentioned that the form of the master equation (6) suggests that it may be regarded as stemming from quantum analogs of Langevin equations with multiplicative noise. In order to write these we shall use the Stratonovich interpretation [31] of such equations which constitutes the mathematical idealization of the spin relaxation process [31] so that one can then apply conventional calculus [21]. The Langevin equations for the random variables \( \vartheta \) and \( \varphi \) corresponding to the phase space master equation (6) read in the Stratonovich interpretation as

\[
\dot{\vartheta}(t) = D_1(t) + \gamma \sqrt{\frac{D_{11}(t)}{D_\perp}} \left[ \alpha h_\vartheta(t) - h_\varphi(t) \right] + \frac{D_\perp}{4} (e^{\beta \hbar \omega_0} - 1) \sin \vartheta(t) - \cos \vartheta(t) \sqrt{D_{11}(t)} D_{22}(t), \tag{15}
\]

\[
\dot{\varphi}(t) = D_2 + \gamma \sqrt{\frac{D_{22}(t)}{D_\perp}} \left[ h_\vartheta(t) + \alpha h_\varphi(t) \right], \tag{16}
\]

where \( \alpha = \beta \hbar S D_\perp \) is a dimensionless dissipation parameter, \( D_{11} \) and \( D_{22} \) are the diffusion coefficients defined by eqs. (7) and (8) while \( D_1 \) and \( D_2 \) are the drift coefficients given by

\[
D_1 = \cot \vartheta D_{11} - D_\perp \left( S + \frac{1}{2} \right) (e^{\beta \hbar \omega_0} - 1) \sin \vartheta, \tag{17}
\]

\[
D_2 = -\omega_0. \tag{18}
\]

In writing eqs. (15) and (16) we have neglected all terms of the order of \( \alpha^2 \) because \( \alpha \) is a small parameter. Now the components \( h_\vartheta(t), h_\varphi(t) \) of the random white noise field \( h(t) \) in spherical polar coordinates may also be expressed in terms of Cartesian components \( h_X(t), h_Y(t), h_Z(t) \), as [22]

\[
h_\vartheta = h_X \cos \varphi \cos \vartheta + h_Y \cos \varphi \sin \varphi - h_Z \sin \vartheta, \tag{19}
\]

\[
h_\varphi = -h_X \sin \varphi + h_Y \cos \varphi, \tag{20}
\]

with \( h_i(t) = 0, h_i(t) h_j(t') = 2 \gamma^{-2} D_\perp \delta_{ij} (t - t'), \) \( i, j = X, Y, Z \). Here the overbar means the statistical average over the realizations of \( h(t) \). In order to demonstrate that the Langevin equations (15) and (16) are equivalent to the phase space master equation (6), we recall [21, 22] that for Stratonovich stochastic differential equations involving two random variables \( \{x_1(t) = \vartheta(t), x_2(t) = \varphi(t)\} \), namely,

\[
\dot{x}_1(t) = H_i[x_1(t), x_2(t)] + \sum_{j=1}^{3} G_{ij} [x_1(t), x_2(t)] h_j(t), \tag{19}
\]

the drift \( D_i \) and the diffusion \( D_{ij} \) coefficients are

\[
D_i = \lim_{\tau \to 0} \frac{\bar{x}_i(t) - x_i(t)}{\tau} = H_i + \gamma^{-2} D_\perp \sum_{k=1}^{3} \sum_{j=1}^{3} G_{kj} \frac{\partial}{\partial x_k} G_{ij}, \tag{20}
\]

\[
D_{ij} = \lim_{\tau \to 0} \frac{[x_i(t) - x_i(t)] [x_j(t) - x_j(t)]}{2 \tau} = \gamma^{-2} D_\perp \sum_{k=1}^{3} G_{ik} G_{jk}. \tag{20}
\]
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\( (D_i \text{ and } D_{ij} \text{ are given by eqs. (7), (8), and (17); } \)
\( D_{12} = D_{21} = 0) \). Hence, the corresponding Fokker-Planck equation for the phase space distribution function

\[
\frac{\partial P}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} (D_i P) + \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij} P). \tag{21}
\]

After some algebra eq. (21) reduces to the master equation (6) for \( W(\vartheta, \varphi, t) \). The phase space Langevin equation for spins may also be formulated using the Itô interpretation [21,22] of stochastic differential equations. The Itô Langevin equations also have the form of eqs. (15) and (16), however, the second line in eq. (15) must be omitted since in Itô equations the noise-induced drift is equal to zero [21,22] and the rules of Itô calculus now apply.

Now eqs. (15) and (16) describing the stochastic dynamics of an individual spin in phase space can also be written as a vector Langevin equation, viz.,

\[
\dot{\mathbf{u}} = \gamma \mathbf{u} \times (\mathbf{H}_0 + \dot{\mathbf{D}} h) - \mathbf{u} \times \dot{\mathbf{D}} [\mathbf{u} \times (A \mathbf{H}_0 + \mathbf{q} + \gamma \alpha \mathbf{h})], \tag{22}
\]

where \( \mathbf{u} = \mathbf{S}/|\mathbf{S}| \) is a unit vector specifying in the \( c \)-number space the direction of the spin vector \( \mathbf{S} \) corresponding to the spin operator \( \mathbf{S} \), \( \mathbf{q} \) is a vector perpendicular to \( \mathbf{u} \) which in the spherical coordinate system basis \((\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)\) is

\[
\mathbf{q} = (0, \sqrt{D_{21} D_{11}} \cot \vartheta - \sqrt{D_{22} D_{11}} \cos \vartheta, 0), \tag{23}
\]

\( \dot{\mathbf{D}} \) is a diagonal matrix with the matrix elements defined as

\[
(D)_{11} = (\dot{D})_{22} = \sqrt{\sin^2 \varphi D_{22}^2 + \sin^2 \theta D_{11}^2}, \quad \dot{(D)}_{33} = \frac{D_{11}}{D_{\perp}}, \tag{24}
\]

and

\[
A = \gamma D_{\perp} (S + 1/4) (e^{\beta \omega_0} - 1)/\left(\omega_0 \sqrt{D_{11}/D_{\perp}}\right). \tag{25}
\]

The main features of eq. (22) are that i) \( (\mathbf{u} \cdot \dot{\mathbf{u}}) = 0 \), i.e., the length of the vector \( \mathbf{u} \) does not alter during the motion, and ii) in the classical limit \( (S \to \infty \text{ and } \hbar S = \text{const}) \), eq. (22) reduces to the familiar Landau-Lifshitz equation augmented by a random field term \( \mathbf{h} \) [22], namely,

\[
\dot{\mathbf{u}} = \gamma \mathbf{u} \times (\mathbf{H}_0 + \mathbf{h}) - \alpha \gamma \mathbf{u} \times [\mathbf{u} \times (\mathbf{H}_0 + \mathbf{h})]. \tag{26}
\]

The vector Langevin equation (26) describes the isotropic rotational Brownian motion of a classical spin \( \mathbf{S} \) in a dc field \( \mathbf{H}_0 \). The accompanying Fokker-Planck equation is [6]

\[
\frac{\partial W}{\partial t} = \omega_0 \frac{\partial W}{\partial \varphi} + D_{\perp} \frac{\partial}{\partial \vartheta} \left( \frac{\xi}{\tan^2 \vartheta} \frac{\partial W}{\partial \vartheta} \right) + \frac{\partial}{\partial \varphi} \left( \frac{\sin^2 \vartheta}{\tan \vartheta} \frac{\partial W}{\partial \varphi} \right) + \frac{1}{\sin \vartheta} \frac{\partial^2 W}{\partial \varphi^2}, \tag{27}
\]

where \( \xi = \beta \hbar \omega_0 S \) is a dimensionless parameter. Consequently, the main difference between the quantum and classical Brownian motion of a spin is that in the quantum spin diffusion is always anisotropic \( \sin^2 \vartheta D_{22} \neq D_{11} \) even if \( D_{11} = D_{22} \).

Now having written the Langevin equations (7) and (8) for the random variables \( \vartheta(t) \) and \( \varphi(t) \), one may also write the Langevin equation for an arbitrary function \( f(\vartheta, \varphi) \) as \( \dot{f} = \vartheta(t) \partial_\vartheta f + \varphi(t) \partial_\varphi f \), where \( \vartheta(t) \) and \( \varphi(t) \) are given by eq. (15) and eq. (16), respectively, enabling one to calculate observables. In magnetic relaxation, the relevant observables are statistical averages of the spherical harmonics \( Y_{l,m}(\vartheta, \varphi) \) defined as [27]

\[
Y_{l,m}(\vartheta, \varphi) = \sqrt{(2l+1)(l-m)!/4\pi} P^m_l(\cos \vartheta) e^{im\varphi}, \tag{28}
\]

where \( P^m_l(x) \) are the associated Legendre functions [27]. Hence, by averaging the Langevin equation

\[
\dot{Y}_{l,m} = \vartheta(t) \partial_{\vartheta} Y_{l,m} + \varphi(t) \partial_{\varphi} Y_{l,m}, \tag{29}
\]

over its realizations as described in detail in ref. [22], and subsequently using the recurrence relations [27]

\[
\begin{align*}
\sin \vartheta \partial_{\vartheta} Y_{l,m} &= \frac{1}{l} \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}} Y_{l+1,m} \\
\cos \vartheta \partial_{\vartheta} Y_{l,m} &= -\frac{1}{l+1} \sqrt{\frac{l^2 - m^2}{4l^2 - 1}} Y_{l-1,m},
\end{align*} \tag{30}
\]

we can derive directly (without recourse to the master equation as in [15]) the closed system of differential-recurrence equations for the averaged spherical harmonics \( \langle Y_{l,m}(t) \rangle \), namely,

\[
\frac{d\langle Y_{l,m}(t) \rangle}{dt} = q_{l,m} \langle Y_{l-1,m}(t) \rangle + q_{l,m} \langle Y_{l+1,m}(t) \rangle + q_{l,m}^\pm \langle Y_{l,m}(t) \rangle,
\]

where \( 0 \leq l \leq 2S \),

\[
q_{l,m} = \imath m \omega_0 - \frac{D_{\perp} m^2}{2} \left( 1 + \frac{\imath l}{S} \right) \left[ l(l+1) - m^2 \right],
\]

\[
q_{l,m}^\pm = -D_{\perp} \left( \frac{\imath l}{S} - 1 \right) \left[ l(l+1) - \frac{4 l + 1}{4} \pm S \right] \tag{32}
\]

where \( \xi = \beta \hbar \omega_0 S \) is a dimensionless parameter. Here the number of recurring equations is finite because \( \langle Y_{l,m}(t) \rangle = 0 \) for \( L > 2S \) [5,8,10,14,15] constituting the main difference between this hierarchy and the corresponding classical hierarchy [22], where the number of equations is infinite \( (S \to \infty) \).

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The three-term recurrence eq. (32) can be solved for the average of any spherical harmonic using continued fractions [8–10,14,15]. In particular, noting the correspondence rules of operators $\hat{S}_X$, $\hat{S}_Y$, $\hat{S}_Z$ and Weyl symbols ($c$-numbers) $S_X$, $S_Y$, $S_Z$ in the phase space $(\theta, \varphi)$ [7] and solving the recurrence eq. (32) for $(Y_{1,0})$ and $(Y_{1,\pm 1})$, one can calculate the longitudinal, $\langle \hat{S}_Z \rangle$, and transverse, $\langle \hat{S}_t \rangle = \langle S_X \rangle = i (S_Y)$, components of the magnetization as [32] $\langle \hat{S}_t \rangle = \mp \sqrt{8\pi/3} (S+1)/Y_{1,\pm 1}$ and $\langle \hat{S}_Z \rangle = \sqrt{4\pi/3} (S+1)/(Y_{1,0})$.

For $m = 0$, the recurrence eq. (32) pertains to the averaged Legendre polynomials $\langle P_l \rangle(t) = \sqrt{(2l+1)/4\pi} (Y_{1,0})(t)$ characterizing the longitudinal relaxation. Using the phase space master equation this problem has been treated in [8–10], where various solutions for the linear and non-linear relaxation of the averaged longitudinal component of the spin $\langle \hat{S}_Z \rangle(t)$ as a function of all spin values $S$ in a uniform magnetic field $H_0$ of arbitrary strength have been given. In particular [8–10], the statistical average $\langle \hat{S}_Z \rangle(t)$, comprising $2S$ exponentials, may be accurately approximated by a single exponential with an explicit longitudinal relaxation time $T_1$ which strongly depends on $S$ and the field strength for arbitrary $S$ (the explicit equation for $T_1$ is given in ref. [10]). In other words, even for a giant spin ($S \gg 1$), $\langle \hat{S}_Z \rangle(t)$ still obeys the Bloch equation

$$\frac{d\langle \hat{S}_Z \rangle}{dt} + \frac{\langle \hat{S}_Z \rangle - \langle \hat{S}_Z \rangle_{eq}}{T_1} = 0,$$

where

$$\langle \hat{S}_Z \rangle_{eq} = \left( S + \frac{1}{2} \right) (S+1) \int_0^\pi \cos \theta W_{eq}(\theta) \sin \theta d\theta$$

$$= SB_S(\xi)$$

is the equilibrium average of the operator $\hat{S}_Z$ and

$$B_S(x) = \frac{2S+1}{2S} \coth \left( \frac{2S+1}{2S} x \right) - \frac{1}{2S} \coth \left( \frac{x}{2S} \right)$$

is the Brillouin function. As far as the spin dependence of $T_1$ is concerned, $T_1$ strongly depends on $S$ and in the low-temperature limit ($\xi \gg 1$) is given explicitly by [10]

$$T_1 \sim \left( 2D_\perp S e^{E/S} - 1 \right)^{-1}.$$ 

We remark that Garcés-Palacios and Zueco [33] have treated the same problem using the spin density matrix. Although the results of refs. [10] and [33] have outwardly very different forms; nevertheless numerical calculation shows [10] that both yield exactly the same result establishing an essential corollary between the phase space and spin density matrix formulations.

To conclude, we have demonstrated that the quantum stochastic spin dynamics can be studied via the phase space master equation or the corresponding Langevin equation and subsequent calculation of the quantum-mechanical expectation values in classical fashion. Hence it is possible to reformulate the quantum-mechanical stochastic spin dynamics as an essentially classical problem. We have derived the phase space Langevin equation for a spin of arbitrary size $S$ placed in a uniform magnetic field in the weak spin-bath coupling limit. The form of the Langevin equation is, however, quiet general, hence it also holds for phase space master equations of the form of eq. (6) with drift and diffusion coefficients depending explicitly on time [8,9,14,15]. We emphasize, however, that the Langevin equation is written down from a priori knowledge of the master equation whereas in the classical case the Langevin equations are written down independently of the Fokker-Planck equation and the results of the two methods coincide entirely due to the Gaussian white noise properties of the random field. We reiterate that the density matrix, phase space master equation and Langevin equation treatments are completely equivalent and yield the same results [10]. However, the Langevin equation has, in our opinion, the advantage that it both provides a basis for numerical simulation of the quantum relaxation processes and also allows one to evaluate the observables in the familiar classical manner. We remark that the present problem constitutes the simplest example of the phase space method for spins because the evolution equation is merely the Fokker-Planck equation with time independent drift and diffusion coefficients. This would not be true in general, e.g., for relaxation in mixed magnetocrystalline anisotropy and external field potentials, e.g., for a uniaxial paramagnet subjected to a dc magnetic field. Here, the phase space evolution equation can also be presented in the form of a generalized Fokker-Planck equation, however, it has a much more complicated form, where the drift and diffusion coefficients are differential operators [34,35]. Nevertheless, the corresponding Langevin equation can also be derived in this case just in quantum optics, where the drift and diffusion coefficients of the generalized Fokker-Planck equation of the laser are also differential operators [36]. Furthermore, the Langevin method may also be extended to non-axially symmetric spin systems in order to include spin size effects in important magnetic relaxation problems such as the reversal time of the magnetization, switching and hysteresis curves, etc. Thus, it will be possible to evaluate the temperature dependence of the switching fields and corresponding hysteresis loops via obvious spin size corrected generalizations of the known classical methods used in the analysis of the classical spin dynamics.

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