THE DUNFORD–PETTIS PROPERTY FOR CERTAIN PLANAR UNIFORM ALGEBRAS

Joseph A. Cima and Richard M. Timoney

We show that if K is a compact subset of the complex plane $\mathbf{C}$ and if A is a T-invariant uniform algebra on K (e.g., $A=R(K)$ or $A=A(K)$), then both A and A^* have the Dunford–Pettis property.

We deduce our results from a recent sufficient condition of Bourgain [1] for subspaces X of $C(L)$ (L a compact Hausdorff space) to have the Dunford–Pettis property (DPP). We actually convert his condition into a definition of a "Bourgain algebra" X_B associated with every subspace X of $C(L)$. Bourgain's condition is that X_B equals $C(L)$.

Wojtaszczyk [12; 13] studied the Dunford–Pettis property for planar uniform algebras, and his results were improved by Delbaen [5]. The DPP for the disc algebra was shown earlier by Chaumat [3], Cnop and Delbaen [4], and Kisljakov [10]. Bourgain [1] used his condition to show that the ball and polydisc algebras have the DPP; Bourgain has also shown in [2] that H^∞ of the unit disc has the DPP. In all of these cases the dual of the algebra is shown to have the DPP, which implies that the algebra itself has the DPP.

No necessary conditions for the DPP of a uniform algebra A seem to be known. However, a result of Milne [11] easily implies that there are uniform algebras A without the DPP. He shows that there are uniform algebras A with (norm one) complemented infinite-dimensional reflexive closed subspaces Y. (For A one can take the uniform algebra on the closed unit ball of the dual space Y^* (equipped with the weak topology) generated by the constants and the functions in $Y=Y^{**}$.) Such uniform algebras do not have the DPP since reflexive infinite dimensional Banach spaces always fail to have the DPP.

For a survey of the Dunford–Pettis property (for Banach spaces) we refer to Diestel [6]. One of several equivalent definitions of the DPP is as follows.

DEFINITION 1. A Banach space X has the DPP if, whenever $(x_n)_n$ is a sequence in X tending weakly to 0 and $(x_n^*)_n$ is a sequence in X^* tending weakly to 0, then

$$\lim_{n\to\infty}\langle x_n, x_n^*\rangle = 0.$$

For any compact Hausdorff space L, we use $C(L)$ to denote the Banach space of continuous complex-valued functions on L with the supremum norm. For K a compact subset of the complex plane, we use $R(K)$ to denote the closure in $C(K)$ of the rational functions with poles off K, and $A(K)$ for those functions in $C(K)$ which are analytic on the interior K^0 of K.

Let X be a subspace of $C(L)$ for L any compact Hausdorff space. Notice that X^{**} may be identified with a subspace $X^{\perp\perp}$ of $C(L)^{**}$ by standard duality theory.

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If $\phi \in C(L)$, we can consider the multiplication operator $x \to \phi x$ on $C(L)$ as extended (by its double transpose) to an operator on $C(L)^{**}$. We write ϕx^{**} for the action of this double transpose on $x^{**} \in C(L)^{**}$.

We now define two "Bourgain algebras" for a subspace X of $C(L)$.

DEFINITION 2. Let X be a subspace of $C(L)$. Then X_b denotes the space of all functions ϕ in $C(L)$ satisfying the condition

(b) if $(x_n)_n$ is a weakly null sequence in X, then

$$\lim_{n\to\infty} \operatorname{dist}(\phi x_n, X) = 0.$$

(Here the distance is measured in the norm of $C(L)$.)

We define X_B to be those $\phi \in C(L)$ satisfying

(B) if $(x_n^{**})_n$ is a weakly null sequence in X^{**}, then

$$\lim_{n\to\infty} \operatorname{dist}(\phi x_n^{**}, X^{**}) = 0.$$

(Now the distance is measured in $C(L)^{**}$.)

The following is a remarkable result of Bourgain [1] which we will use.

THEOREM 3. *Let X be a closed subspace of $C(L)$ (L a compact Hausdorff space).*

(i) *If $X_B = C(L)$, then X and X^* both have the Dunford–Pettis property.*

(ii) *If $X_b = C(L)$, then X has the Dunford–Pettis property.*

Proof. (i) follows directly from Proposition 2 of Bourgain [1] (as in the proof of Theorem 1 of [1] for the case of the ball algebra). (ii) follows by repeating the proof of Proposition 2 of [1] deleting all uses of the principle of local reflexivity.

In fact, Bourgain carries out his argument in the somewhat more general setting of functions with values in a finite-dimensional space E. We could define X_B and $X_b \subset C(L)$ when $X \subset C(L, E)$. However we avoid this additional generality.

PROPOSITION 4. *Let X be a subspace of $C(L)$, L a compact Hausdorff space.*

(i) *If $\bar{X}$ is the norm closure of X in $C(L)$, then $(\bar{X})_B = X_B$, $(\bar{X})_b = X_b$.*

(ii) $X_B \subset X_b$.

(iii) *Both X_b and X_B are closed subalgebras of $C(L)$ and contain the constant functions.*

(iv) *If X is an algebra, then $X \subset X_B \subset X_b$.*

(v) *If X is a uniform algebra on a compact subset K of $\mathbf{C}$ which contains the function $f(z) = z$, then $X_B = C(K)$ if and only if the function $\phi(z) = \bar{z}$ is in X_B. (The same statement holds for X_b.)*

Proof. (i) is not important to us, but it is easy to check. We can assume that X is closed for the rest of the proof.

(ii) Since weakly null sequences in X are also weakly null in X^{**}, we only need to know that, for any $\psi \in C(L)$, $\operatorname{dist}(\psi, X) = \operatorname{dist}(\psi, X^{**})$ (where the first distance is in $C(L)$ and the second in $C(L)^{**}$). To see this recall that the second dual of $C(L)/X$ is canonically isometric to $C(L)^{**}/X^{\perp\perp} = C(L)^{**}/X^{**}$.

(iii) It is not at all difficult to see that X_b and X_B contain the constants. To show that X_B (for example) is an algebra we need only show that if $\phi_1, \phi_2 \in X_B$ then $\phi_1+\phi_2 \in X_B$ and $\phi_1\phi_2 \in X_B$. Since it is marginally more difficult we check that $\phi_1\phi_2 \in X_B$.

Let $(x_n^{**})_n$ be a weakly null sequence in X^{**}. Choose $y_n^{**} \in X^{**}$ with

$$\|\phi_1 x_n^{**} - y_n^{**}\| \le \operatorname{dist}(\phi_1 x_n^{**}, X^{**}) + 1/n.$$

Since $(\phi_1 x_n^{**})_n$ is weakly null in $C(L)^{**}$, we easily deduce from condition (B) that $(y_n^{**})_n$ must be weakly null (in X^{**}). Then choose $z_n^{**} \in X^{**}$ with

$$\|\phi_2 y_n^{**} - z_n^{**}\| \le \operatorname{dist}(\phi_2 y_n^{**}, X^{**}) + 1/n.$$

Now, by the triangle inequality,

$$\|\phi_1\phi_2 x_n^{**} - z_n^{**}\| \le \|\phi_2\| \, \|\phi_1 x_n^{**} - y_n^{**}\| + \|\phi_2 y_n^{**} - z_n^{**}\|.$$

Since the right-hand side tends to zero as n approaches ∞, we conclude that $\phi_1\phi_2 \in X_B$.

The proof that X_B (or X_b) is closed is equally straightforward. Suppose $\phi = \lim_{k\to\infty} \phi_k$ and $\phi_k \in X_B$ for all B. Let $(x_n^{**})_n$ be a weakly null sequence in X^{**} and let ϵ be any positive number. Let M be $\sup\|x_n^{**}\|$ and note that M is finite. Choose k so that

$$\|\phi - \phi_k\|_\infty < \epsilon/2M.$$

Then choose N so that

$$\operatorname{dist}(\phi_k x_n^{**}, X^{**}) < \epsilon/2$$

for all $n \ge N$. The triangle inequality gives

$$\begin{aligned}\operatorname{dist}(\phi x_n^{**}, X^{**}) &\le \|\phi - \phi_k\| \, \|x_n^{**}\| + \operatorname{dist}(\phi_k x_n^{**}, X^{**}) \\ &< M(\epsilon/2M) + \epsilon/2 = \epsilon\end{aligned}$$

for $n \ge N$, which shows that $\phi \in X_B$.

(iv) is clear because if X is an algebra, $\phi \in X$, and $x^{**} \in X^{**}$, then $\phi x^{**} \in X^{**}$.

(v) One implication is trivial. On the other hand, if $\bar{z} \in X_B$ and $z \in X \subset X_B$, it follows from (iii) and the Stone–Weierstrass theorem that X_B must be $C(K)$.

EXAMPLE. The disc algebra $A(D)$ has the DPP. To check this (using the above) we need only show that $\bar{z} \in A(D)_b$. Suppose $(x_n)_n$ is a weakly null sequence in $A(D)$. Then $x_n(0) \to 0$ and $y_n(z) = (x_n(z) - x_n(0))/z$ is in $A(D)$. For $z = e^{i\theta}$, $|\bar{z}x_n(z) - y_n(z)| = |x_n(0)|$. Hence $\|\bar{z}x_n(z) - y_n(z)\|_\infty \to 0$ as $n \to \infty$. Thus $\bar{z} \in A(D)_b$ and we are done. □

DEFINITION 5. A subalgebra A of $C(K)$ ($K \subset \mathbf{C}$ compact) is called T-invariant (see Gamelin [9]) if, whenever ϕ is a smooth function with compact support and f is a bounded Borel function on $\mathbf{C}$ with $f|_K \in A$, then $G|_K \in A$ where

$$G(w) = \phi(w)f(w) + \frac{1}{\pi} \iint \frac{f(z)}{z-w} \frac{\partial \phi}{\partial \bar{z}} \, dx\, dy.$$

We recall that T-invariant algebras A contain all the polynomials (in fact, $R(K) \subset A$). Extending functions in A to be zero on $\mathbf{C} \setminus K$, the definition implies that

$$T_\phi f(w) = \phi(w) f(w) + \frac{1}{\pi} \iint_K \frac{f(z)}{z-w} \frac{\partial \phi}{\partial \bar{z}} \, dx \, dy$$

is in A whenever $f \in A$ and ϕ is smooth on $\mathbf{C}$. (Compact support of ϕ is not relevant since the integral is now taken over K.) Finally, we recall that $A(K)$ and $R(K)$ are T-invariant on K. (Note that $P(K)$, the closure of the polynomials in $C(K)$, is T-invariant on the polynomial hull $\hat{K}$ of K, since $P(K) = R(\hat{K})$.)

LEMMA 6. *For $K \subset \mathbf{C}$ compact we denote Lebesgue area measure on K by λ. Then the measures*

$$\left\{ d\lambda_w(z) = \frac{1}{z-w} \, d\lambda(z) : w \in K \right\}$$

form a norm compact subset of $M(K)$, the space of all regular Borel measures on K ($M(K)$ has the variation norm).

Proof. We first check that $\{1/|z-w| : w \in K\}$ is uniformly integrable with respect to $d\lambda$. Let d denote the diameter of K and let μ be Lebesgue area measure on $\{|z| \leq d\}$. Let $d\mu_0(z) = 1/|z| \, d\mu(z)$.

Since μ_0 is clearly absolutely continous with respect to μ, if $\epsilon > 0$ is given we can find $\delta > 0$ so that $\mu(E) < \delta$ implies $\mu_0(E) < \epsilon$ (for E a measurable subset of $\{|z| \leq d\}$).

Now, if $E \subset K$ and $\lambda(E) < \delta$ then, for $w \in K$, we have

$$|\lambda_w|(E) = \iint_E \frac{1}{|z-w|} \, dx \, dy = \mu_0(E-w),$$

where $E-w$ denotes the translate of E by $-w$. Also $\lambda(E) = \mu(E-w) < \delta$ and hence $|\lambda_w|(E) = \mu_0(E-w) < \epsilon$ for all $w \in K$. This gives uniform integrability. An application of the Vitali convergence theorem (see [8, p. 150]) shows that the map $w \to \lambda_w : K \to M(K)$ is continuous and the norm compactness of $\{\lambda_w : w \in K\}$ follows. □

THEOREM 7. *If A is a T-invariant algebra on $K \subset \mathbf{C}$, then both A and its dual A^* have the Dunford–Pettis property.*

Proof. By Theorem 3 and Proposition 4(v) we need only check that the function $\phi(z) = \bar{z}$ must be in A_B.

Let $(x_n^{**})_n$ be a weakly null sequence in A^{**}. For convenience we assume that $\|x_n^{**}\| < 1$ for all n. Let $\epsilon > 0$ be given.

From Lemma 6 we deduce that, for n large enough,

$$\sup_{w \in K} |\langle \lambda_w, x_n^{**} \rangle| < \epsilon/2.$$

Fix such an n. Then there is a net $(x_\alpha)_\alpha$ in A converging weak* to x_n^{**} and satisfying $\|x_\alpha\| \leq 1$.

By Lemma 6 we may also assume that

$$(*) \qquad \sup_{w\in K} |\langle \lambda_w, x_\alpha \rangle| < \epsilon \text{ (for all } \alpha).$$

With $\phi(z)=\bar{z}$, let

$$\begin{aligned} f_\alpha(w) &= (T_\phi x_\alpha)(w) \\ &= \phi(w)x_\alpha(w) + \frac{1}{\pi}\int_K x_\alpha(z)\, d\lambda_w(z). \end{aligned}$$

Using (*) we deduce that $\|f_\alpha - \phi x_\alpha\|_\infty \leq \epsilon/\pi < \epsilon$.

Now because balls in A^{**} are weak* compact, we can find a weak* limit point $f^{**}\in A^{**}$ of $(f_\alpha)_\alpha$. Since ϕx_α converges in the weak* topology to ϕx_n^{**} and the norm of the limit of a weak* convergent net cannot exceed the limit superior of the norms, we conclude that $\|\phi x_n^{**} - f^{**}\| \leq \epsilon$. Since this is true of all large n, we have the desired conclusion that

$$\lim_{n\to\infty} \operatorname{dist}(\phi x_n^{**}, A^{**}) = 0$$

and thus that $\phi(z)=\bar{z}$ is in A_B.

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Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina 27514