Bulletin de la Société Royale des Sciences de Liège, 60e année, 6, 1991, pp. 401-404

ON A PROBLEM OF H. BOHR

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Abstract

An exponent of convergance, for convergent power series in infinitely many variables, is established using methods developed in the study of the relationship between neclearity and the existence of an absolute basis for holomorphic function in locally convex spaces.

In his investigation of Dirichlet series, H. Bohr [1] employed power series expressions in infinitely many variables and posed a certain convergence problem. In this article we provide a solution of this problem using methods which arose in our investigation of the relationship between absolute monomial expansions of holomorphic functions and nuclearity in locally convex spaces [3] and which were motivated by a theorem of Bohr [2].

Let $N^{(N)}$ denote the set of non-negative integers which are eventually zero. We identify, in the usual way, N^n, the n-tuples of non-negative integers, with a subset of $N^{(N)}$ and, with this identification, $N^n \subset N^{n+1}$ and $\bigcup_{n=1}^{\infty} N^n = N^{(N)}$. If $z = (z_n)_{n=1}^{\infty}$ is sequence of complex numbers and $m = (m_n)_{n=1}^{\infty} \in N^{(N)}$ we let $z^m = \prod_{n=1}^{\infty} z_n^{m_n}$ (we let $0^0 = 1$) and, by abuse of language, we also let z^m denote the monomial $z \to z^m$. For each positive integer n let

$$\Delta^n = \{(z_i)_{i=1}^n \in C^n : \quad |z_i| \leq 1 \text{ for } i = 1, \ldots, n\}.$$

AMS Classification: Primary 46G20, Secondary 46E10

Key words and phrases, Absolute Monomial expansion, holomorphic function

Présenté par J. Schmets le 20 juin 1991

We identify Δ^n with a subset of Δ^{n+1} in the usual way and let $\Delta^\infty = \bigcup_{n=1}^{\infty} \Delta^n$.

Bohr [1] considers the formal monomial expansions $\sum_{m \in N^{(N)}} a_m z^m$ with the following properties;

(a)
$$\sum_{m \in N^n} |a_m| < \infty \quad \text{for all } n$$

(b)
$$\sup_n \left\{ \sup \left\{ \left| \sum_{m \in N^n} a_m z^m \right|; \quad |z_i| \le 1, \quad i = 1, \ldots, n \right\} \right\} < \infty$$

We let B denote the set of power series satisfying (a) and (b). Using Taylor series expansions it is easily seen that there is a one to one correspondence between elements of B and functions f on Δ^∞ with the following properties;

(c) the Taylor series expansion of $f|_{\Delta^n}$ converges absolutely on Δ^n,

(d)
$$\sup_n \|f\|_{\Delta^n} = \|f\|_{\Delta^\infty} < \infty.$$

Let

$$C = \left\{ (\epsilon_n)_{n=1}^{\infty}; \quad 0 < \epsilon_n < 1 \text{ all } n, \sum_{m \in N^{(N)}} |a_m| \epsilon^m < \infty \text{ for all } \sum_{m \in N^{(N)}} a_m z^m \epsilon B \right\}$$

Bohr [1] wished to find the largest real number S such that if $0 < \epsilon_n < 1$ all n and $\sum_{n=1}^{\infty} \epsilon_n^S < \infty$ then $(\epsilon_n)_{n=1}^{\infty} \in C$. He showed that $S \ge 2$ and Toeplitz [4] showed $S \le 4$. In this article we show that $S = 2$. We found it convenient to consider a smaller set of functions.

Let δ denote a fixed real number greater than 1. For each positive integer n let

$$D^n = \{(z_i)_{i=1}^n \in \mathbf{C}^n; \quad |z_i| < \delta, \quad i = 1, \ldots, n\}.$$

and let $D^\infty = \bigcup_{n=1}^{\infty} D^n$.

We let $\tilde{B}$ denote the set of all functions f on D^∞ such that

(e) $f|_{D^n}$ is a holomorphic function (of n complex variables) for all n,

(f) $\|f\|_{D^\infty} < \infty$.

Since $\Delta^\infty \subset D^\infty$ and Δ^n is a compact polydisc in D^n for all n it follows that (e) $\Longrightarrow$ (c) $\Longleftrightarrow$ (a) and (f) $\Longrightarrow$ (d) $\Longleftrightarrow$ (b). Hence $\tilde{B} \subset B$. The identifiction of $f \in \tilde{B}$ is with its monomial expansion $\sum_{m \in N^{(N)}} a_m z^m \in B$.

Endowed with the norm $\|.\|_{D^\infty}$, $\tilde{B}$ is a Banach space. Now suppose $(\epsilon)_{n=1}^{\infty}$ belongs to C. Then

$$\sum_{m\in N^{(N)}} |a_m|\epsilon^m < \infty \text{ for all } f \sim \sum_{m\in N^{(N)}} a_m z^m \in \tilde{B}.$$

Since the mapping

$$f \sim \sum_{m\in N^{(N)}} a_m z^m \longrightarrow |a_m|$$

is continuous, by the Cauchy integral formula, for each m it follows that the set

$$\left\{ \sum_{m\in N^{(N)}} a_m z^m \in \tilde{B}; \quad \sum_{m\in N^{(N)}} |a_m|\epsilon^m \le 1 \right\}$$

is a closed convex balanced absorbing subset of $(\tilde{B}, \|.\|_{D^\infty})$ and hence, since Banach spaces are barrelled, there exists a positive constant α such that

$$\sum_{m\in N^{(N)}} |a_m|\epsilon^m \le \alpha \|f\|_{D^\infty} \qquad (*)$$

for all $f \sim \sum_{m\in N^{(N)}} a_m z^m \in \tilde{B}$.

Theorem. $S = 2$.

Proof. Let $(\epsilon_n)_{n=1}^{\infty} \in C$. Clearly B and C are invariant under any permutation of coordinates. By the proof of theorem 3.1 in [3] we may take $\alpha = 1$ in $(*)$ and it follows that

$$\sum_{j=1}^{n} \epsilon_{l(j)} \le 2\sqrt{n} \qquad (**)$$

for any sequence of positive integers $l(1), \ldots, l(n)$. Hence, $(\epsilon_n)_{n=1}^{\infty}$ is convergent to 0 in R. If $(\epsilon'_n)_n$ is the nondecreasing rearrangement of $(\epsilon_n)_n$ then

$$n\epsilon'_n \le \sum_{j=1}^{n} \epsilon'_n \le 2\sqrt{n}$$

and, for any $\delta > 0$,

$$\sum_{n=1}^{\infty} \epsilon_n^{2+\delta} = \sum_{n=1}^{\infty} (\epsilon'_n)^{2+\delta} \le 2^{2+\delta} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{1+(\delta/2)}} < \infty.$$

This completes the proof.

Using $(**)$, it is also easy to construct a sequence $(\epsilon_n)_n$, $0 < \epsilon_n < 1$, such that $\sum \epsilon_n^{2+\delta} < \infty$ for $\delta > 0$ but $(\epsilon_n)_n \notin$ C.

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