DOMAINS OF POLYHEDRAL TYPE AND BOUNDARY EXTENSIONS OF BIHOLOMORPHISMS

DMITRI ZAITSEV

Abstract. For D, D' analytic polyhedra in $\mathbb{C}^n$, it is proven that a biholomorphic mapping $f\colon D \to D'$ extends holomorphically to a dense boundary subset under certain condition of general position. This result is also extended to a more general class of domains with no smoothness condition on the boundary.

1. Introduction

In 1960 Remmert and Stein [9] proved the following theorem:

Let $D, D' \subset \mathbb{C}^n$ be bounded convex euclidean polyhedra. Suppose that the number of different complex tangent hyperplanes to the faces of D is greater than n and each n of them are linearly independent. Then every biholomorphic (even every proper holomorphic) mapping $f\colon D \to D'$ is an affine isomorphism.

The proof is based on the invariance of the above complex tangent hyperplanes which are called *characteristic decompositions* under proper holomorphic mappings. In fact, Remmert and Stein proved the invariance of characteristic decompositions for a larger class of bounded domains, namely for the so-called *analytic polyhedra*:

A bounded domain $D \subset \mathbb{C}^n$ is called an analytic polyhedron, if there exist a neighborhood $U = U(\overline{D})$ and holomorphic functions $g_i\colon U \to \mathbb{C}$, $i = 1, \dots, s$, such that D is a connected component of the set

$$\{z \in U : |g_1(z)| < 1, \dots, |g_s(z)| < 1\}. \tag{1}$$

We call $\{g_1, \dots, g_s\}$ a set of defining functions. Suppose that this set is minimal, i.e. no function g_i can be removed without changing D. For each $i = 1, \dots, s$, we call the g_i-level set decomposition of D a characteristic decomposition of D.

A trivial consequence of the above theorem is the holomorphic extendibility of f to the boundary of D. One main goal of this paper is to study the extension problem for arbitrary analytic polyhedra. In general, a biholomorphic mapping $f\colon D \to D'$ between analytic polyhedra does not extend holomorphically to the whole boundary. We prove the existence of a holomorphic extension to a dense boundary subset under a conditions which can be seen as a nonlinear version of the above condition of linear independence:

Theorem 1.1. *Let $f\colon D \to D'$ be a biholomorphic mapping between analytic polyhedra in $\mathbb{C}^n$. Suppose that the number of different characteristic decompositions of D is greater than n and that each n of them have linearly independent tangent subspaces at generic points $z \in D$. Then f extends to a biholomorphic mapping between some neighborhoods of dense subsets of ∂D and $\partial D'$ respectively.*

1991 *Mathematics Subject Classification.* 32H40, 32L30, 58F18.

Partially supported by SFB 237 "Unordnung und große Fluktuationen".

We say that $f\colon D \to \mathbb{C}^n$ *extends (bi)holomorphically* to a point $x \in \partial D$, if there exists a neighborhood $V = V(x)$ and a (bi)holomorphic mapping $f_V\colon V \to f_V(V) \subset \mathbb{C}^n$ such that $f = f_V$ on $D \cap V$. We give a characterization of the boundary points, to which a biholomorphic extension exists:

Theorem 1.2. *Let $f\colon D \to D'$ be a biholomorphic mapping between analytic polyhedra in $\mathbb{C}^n$. Suppose that the number of different characteristic decompositions of D is greater than n and that each n of them have linearly independent tangent subspaces at some point $x \in \partial D$, where D is locally connected. Then f extends biholomorphically to x if and only if there exists a sequence $(z_m)_{m\geq 1}$ in D together with $L \in \mathbf{GL}_n(\mathbb{C})$ such that $z_m \to x$ and $d_{z_m} f \to L$ as $m \to \infty$.*

The special properties of analytic polyhedra are their piecewise smoothness and vanishing of the Levi form at smooth boundary points. In this paper we extend the above results to a larger class of bounded domains with no smoothness condition and arbitrary Levi ranks at smooth points.

Definition 1.1. *A bounded domain $D \subset \mathbb{C}^n$ with $D = \overset{\circ}{\overline{D}}$ is called a* domain of polyhedral type*, if there exist a neighborhood $U = U\left(\overline{D}\right) \subset \mathbb{C}^n$, holomorphic maps $g_i\colon U \to \mathbb{C}^{n_i}$ and open subsets $D_i \subset \mathbb{C}^{n_i}$, $i = 1, \dots, s$, such that*

1. *∂D_i contains no positive dimensional analytic set in an open subset of $\mathbb{C}^{n_i}$ for all i;*
2. *D is a connected component of the set*

$$\{z \in U : g_1(z) \in D_1, \dots, g_s(z) \in D_s\}$$

If $n_i = 1$ and $D_i = \{|z| < 1\} \subset \mathbb{C}$, Definition 1.1 gives an arbitrary analytic polyhedron. If $s = 1$ and $g_1 = \mathsf{id}$, it gives an arbitrary bounded domain with $D = \overset{\circ}{\overline{D}}$, whose boundary contains no positive dimensional analytic set in an open set in $\mathbb{C}^n$. In the sequel we say that these domains have *simple boundaries*. In particular, we obtain an arbitrary piecewise smooth strongly pseudoconvex or an arbitrary bounded pseudoconvex real-analytic domain (see [4]). On the other hand, if the n_i's are arbitrary and each D_i is strongly pseudoconvex, Definition 1.1 gives an arbitrary strictly pseudoconvex polyhedra in the sense of [11]. Furthermore, as follows from the definition, the class of domains of polyhedral type is closed under cartesian products and intersections In particular a product of bounded domains with simple boundaries is a domain of polyhedral type.

As above we assume that the set $\{g_1, \dots, g_s\}$ of *defining mappings* is chosen minimal. For each $i = 1, \dots, s$, we call the level set decomposition given by each g_i a *characteristic decomposition* of D. We denote by $G_i(z)$ the level set of g_i through z, i.e. the connected z-component of $g_i^{-1}(g_i(z)) \cap D$.

The theorem of Remmert and Stein (Satz 14 in [9]) on the invariance of characteristic decompositions can be extended to domains of polyhedral type:

Theorem 1.3. *Let $D \subset \mathbb{C}^n$, $D' \subset \mathbb{C}^{n'}$ be domains of polyhedral type and let $f\colon D \to D'$ be a proper holomorphic mapping. Then there exists a function $\varphi\colon \{1, \dots, s\} \to \{1, \dots, s'\}$ together with a proper analytic subset $A \subset D$ such that $f(G_i(z)) \subset G'_{\varphi(i)}(f(z))$ for every $z \in D \setminus A$ and $i = 1, \dots, s$.*

In the cases of analytic polyhedra and of products of domains with simple boundaries the conclusion of Theorem 1.3 is automatically valid for all $z \in D$. In general this is not true.

We call a characteristic decomposition *maximal*, if its fibers are not generically included in the fibers of another characteristic decomposition. For analytic polyhedra and products of bounded domains with simple boundaries, all characteristic decompositions are maximal.

Corollary 1.1. *Let $f\colon D \to D'$ be a biholomorphic mapping between bounded domains of polyhedral type whose maximal characteristic decompositions are $G_1, \dots, G_s$ and $G'_1, \dots, G'_{s'}$ respectively. Then $s' = s$ and there exists a permutation φ of $\{1, \dots, s\}$ such that $\dim_z G_i(z) = \dim_{z'} G'_{\varphi(i)}(z')$ for all generic points $z \in D$ and $z' \in D'$ and for all $i = 1, \dots, s$.*

Theorem 1.3 and Corollary 1.1 can be seen as generalizations of corresponding statements for analytic polyhedra ([9], see also [10] for similar statements involving proper holomorphic correspondences) and at the same time for the products of bounded domains with simple boundaries (see [10], [8], [6], [12], also [3] for automorphisms under no boundary assumption):

Corollary 1.2. *Let $D = D_1 \times \cdots \times D_s \in \mathbb{C}^n$, $D = D'_1 \times \cdots \times D'_{s'} \in \mathbb{C}^n$ be products of bounded domains with simple boundaries and let $f\colon D \to D'$ be a biholomorphic mapping. Then $s = s'$ and there exists a permutation φ of the set $\{1, \dots, s\}$ together with biholomorphic mappings $f_i\colon D_i \to D'_{\varphi(i)}$ such that $f = f_1 \times \cdots \times f_s$.*

The next goal of this paper is to extend Theorem 1.1 to arbitrary domains of polyhedral type. For this we have to reformulate the condition of linear independence of hyperplanes in a way suitable for linear subspaces of arbitrary dimensions. We call this the condition of *general position*. Roughly speaking this means that each two expressions consisting of sums and intersections are in general position, i.e. have the largest possible sum or equivalently the smallest possible intersection (see section 4 for precise definitions).

Define $N(n) := (n+1)(n(n-1)/2+1)$.

Theorem 1.4. *Let $f\colon D \to D'$ be a biholomorphic mapping between domains of polyhedral type in $\mathbb{C}^n$, $n > 1$. Suppose that the number of different characteristic decompositions of D is at least $N(n)$ and that their tangent subspaces are in general position at generic points of D. Then there exists a dense subset $S \subset \partial D$ such that f has a holomorphic extension to every point $x \in S$, where D is locally connected.*

The following result is a criterion for the existence of a biholomorphic extension at a given boundary point $x \in \partial D$.

Theorem 1.5. *Let $f\colon D \to D'$ be a biholomorphic mapping between domains of polyhedral type in $\mathbb{C}^n$ and $x \in \partial D$. Suppose that D is locally connected at x, that the number of different characteristic decompositions of D is at least $N(n)$ and that their tangent subspaces are in general position at x. Then f has a biholomorphic extension to x if an only if there exist a sequence $(z_m)_{m\ge 1}$ in D and $L \in \mathbf{GL}_n(\mathbb{C})$ such that $z_m \to x$ and $d_{z_m} f \to L$ as $m \to \infty$.*

Here we make no restrictions on the dimensions of characteristic decompositions. If the characteristic decompositions of D have special dimensions (e.g. for analytic polyhedra), we give exact estimates $N'(n)$.

Theorem 1.6. *Under the assumptions of Theorem 1.4 suppose that either all characteristic decompositions of D are 1-dimensional or all of them are 1-codimensional.*

Then in Theorem 1.4 *we can replace* $N(n)$ *with* $N'(n) := n+1$. *This estimate is optimal, i.e. if* D *has at most* n *different characteristic decompositions, the statement does not hold in general.*

If D and D' are analytic polyhedra, the boundary regularity problem for biholomorphic mappings $f: D \to D'$ can be reduced in some cases to corresponding problems in one complex variable. In particular, classical results of Caratheodory and of Schwarz can be used to prove the existence of continuous and holomorphic extensions of f respectively (see [5]). For D and D' arbitrary domains of polyhedral type, we have no smoothness condition on the boundaries and this is the reason why the statement of Theorem 1.6 does not hold for $s = n$ in general, e.g. for some domains biholomorphic to polydisks.

Our method is based on Theorem 1.3 and the theory of holomorphic webs (see e.g. [2] for applications of the holomorphic web theory to analytic polyhedra). All results stated here except Theorem 1.3 are proven in section 8. Theorem 1.3 is proved in section 3.

We now mention some applications of the above theorems. We first give conditions on f to be algebraic (i.e. such that each coordinate of f satisfies a nontrivial polynomial equation with polynomial coefficients). Webster [13] proved that *if the Levi form of a real-algebraic hypersurface* $M \subset \mathbb{C}^n$ *is nondegenerate, a biholomorphic mapping sending* M *into another real-algebraic hypersurface* $M' \subset \mathbb{C}^n$ *is always algebraic.* Then Baouendi and Rothschild [1] proved this property for the larger class of *essentially finite* real-algebraic hypersurfaces $M = \{z : \varphi(z, \overline{z}) = 0\}$ (i.e. such that each $x \in M$ is an isolated point of the set $\{z : (\varphi(x, \overline{w}) = 0) \implies (\varphi(z, \overline{w}) = 0)\}$). Applying their result we obtain:

Corollary 1.3. *Under the assumptions of Theorem* 1.4 *suppose that* D *and* D' *are given by real polynomial inequalities and* ∂D *contains an open piece of an essentially finite hypersurface. Then every biholomorphic map* $f: D \to D'$ *is algebraic.*

Furthermore, an application of results from [14] yields an algebraic description of the full group $\mathsf{Aut}(D)$ of biholomorphic automorphisms. Recall that a real Lie group is called an *affine Nash group* if it is diffeomorphic to a (not necessarily closed) submanifold of $\mathbb{R}^m$ given by real polynomial inequalities and if all group operations have graphs of this kind (see [7], [14] for precise definitions).

Corollary 1.4. *Under the assumptions of Theorem* 1.4 *suppose that* D *is given by real-algebraic inequalities and that* ∂D *contains an open piece of a Levi-nondegenerate hypersurface. Then* $\mathsf{Aut}(D)$ *is an affine Nash group and the action* $\mathsf{Aut}(D) \times D \to D$ *is a Nash mapping. In particular, the number of connected components of* $\mathsf{Aut}(D)$ *is finite.*

2. Notation

In the following we use the notation of Definition 1.1. Denote by $G_i(z) \subset D$ the g_i-level set through $z \in D$ and by G_i the decomposition into these level sets. For another domain D' we write (g'_j, D'_j), $j = 1, \dots, s'$, $U' = U(\overline{D}')$ and $G'_j(z')$ for the corresponding data. The system of all maximal characteristic decomposition is called the *characteristic web* of D. All analytic sets are always meant complex-analytic. We use the abbreviation $\mathcal{L} := \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ for the space of linear operators. For $d \leq n$, $G_{n,d}$ denotes the Grassmanian of d-dimensional linear subspaces of $\mathbb{C}^n$. For a collection of linear subspaces $P_1, \dots, P_K \subset \mathbb{C}^n$, denote by $\tilde{P}_k$ the sum of all

P_j with $j \neq k$. We say that a property is satisfied generically or for generic points if it is satisfied for all points in the complement of some proper analytic subset.

3. Invariance of characteristic webs under proper holomorphic mappings

In this section we prove Theorem 1.3.

Lemma 3.1. *For each $i = 1, \dots, s$, there exist a boundary point $x \in \partial D$, coordinate neighborhoods $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$ of x and $\tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2 \subset \mathbb{C}^r \times \mathbb{C}^{n_i - r}$ of $g_i(x)$ and an open subset $D_0 \subset \Omega_1$ with simple boundary such that $D \cap \Omega = D_0 \times \Omega_2$ and $g_i(z_1, z_2) = (z_1, 0)$.*

Proof. Fix some $i = 1, \dots, s$. Since the set of defining mappings $\{g_1, \dots, g_s\}$ is minimal, there exists a point $x \in \partial D$ and a connected neighborhood Ω of x such that

$$\tilde{D} \cap \Omega = \{z \in \Omega : g_i(z) \in D_i\},$$

where $\tilde{D}$ is as in Definition 1.1. Denote by $S \subset \Omega$ the singular locus of g_i, i.e. the set of all points, where g_i has not its maximal rank. Since $D = \overset{0}{\overline{D}}$ and $\Omega \backslash S$ is connected, $\partial D \not\subset S$. Hence, by changing to a smaller Ω, we may assume that $S = \emptyset$. The conclusion of the lemma follows now from the rank theorem and Definition 1.1. □

Let i be fixed and r be the maximal rank of g_i. We apply Lemma 3.1 for D and i and borrow its notation.

Let $z^0 \in \partial D \cap \Omega$ be arbitrary. Our first goal is to prove that for every sequence $(z^m)_{m \geq 1}$ in $D \cap \Omega$ which converges to z^0,

$$\bigotimes_{j=1}^{s'} \frac{\partial (g'_j \circ f)}{\partial z_2}(z^m) \to 0, \quad m \to \infty. \tag{2}$$

Here we mean the tensor product $\mathbb{C}^{n-r} \to \otimes_{j=1}^{s'} \mathbb{C}^{n'_j}$ of corresponding partial derivatives considered as linear maps.

If (2) is not valid, we may assume that

$$\left\| \prod_{j=1}^{s'} \frac{\partial (g'_j \circ f)}{\partial z_2}(z^n) \right\| \geq \varepsilon \tag{3}$$

for some ε and all $m = 1, 2, \dots$. Further we may assume by Montel's theorem that

$$\Phi_m(\xi) := f(z_1^m, z_2^m + \xi) \to \Phi(\xi), \quad m \to \infty \tag{4}$$

uniformly for $\xi \in B$, where $B \subset \mathbb{C}^{n-r}$ is a sufficiently small connected neighborhood of 0. It follows from Lemma 3.1 that $(z_1^m, z_2^m + \xi) \to \partial D$ for all $\xi \in B$.

Since f is proper, $\Phi(B) \subset \partial D'$. It follows from Definition 1.1 that $\partial D'$ is covered by the closed sets $(g'_j)^{-1}(\partial D'_j)$, $j = 1, \dots, s'$. Changing if necessary to a smaller B we may assume that $\Phi(B)$ is contained in $(g'_j)^{-1}(\partial D'_j)$ for some $j = j_0$, i.e. $g'_j(\Phi(B)) \subset \partial D'_j$. Since $\partial D'_j$ contains only zero dimensional analytic sets, $\Phi(B)$ lies in a level set of g'_j. This means that

$$\frac{\partial g'_j \circ \Phi}{\partial \xi} = 0 \tag{5}$$

for $j = j_0$. As a consequence of the uniform convergence we have

$$\frac{\partial(g'_j \circ f)}{\partial z^2}(z^m) \to \frac{\partial(g'_j \circ \Phi)}{\partial \xi}(0), \quad m \to \infty. \tag{6}$$

Together with (5) this contradicts the assumption (3). This shows (2).

Since $z^0 \in \partial D \cap \Omega$ is arbitrary, we can apply Rado's theorem (see e.g. [8]) to the tensor product in (2). It follows that for some $j =: \varphi(i)$,

$$\frac{\partial(g'_j \circ f)}{\partial z_2} = 0 \tag{7}$$

identically on $D \cap \Omega$.

We now write the Jacobian matrices of g_i and $g'_j \circ f$ in the coordinates given by Lemma 3.1:

$$\frac{\partial g_i}{\partial z} = \begin{pmatrix} \mathsf{id} & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial(g'_j \circ f)}{\partial z} = \begin{pmatrix} \partial(g'_j \circ f)/\partial z_1 \\ \partial(g'_j \circ f)/\partial z_2 \end{pmatrix}. \tag{8}$$

We wish to prove that the rank of the matrix

$$\begin{pmatrix} \dfrac{\partial g_i}{\partial z} & \dfrac{\partial g'_j \circ f}{\partial z} \end{pmatrix} \tag{9}$$

is the same as of $\partial g_i/\partial z$, i.e. r. It suffices to show that all $(r+1)\times(r+1)$-minors of (9) which contain r first columns and r first rows are zero. It follows from (8) that such minors are exactly the entries of the matrix $\partial(g'_j \circ f)/\partial z_2$. By (7), they are equal to zero. Therefore the rank of (9) equals r everywhere in D. This implies that the level sets of g_i in its regular locus are included in corresponding level sets of $g'_j \circ f$. Finally, we define A to be the union of singular loci of g_i, $i = 1, \dots, s$. This finishes the proof of Theorem 1.3.

4. Characteristic webs in general position

4.1. **Linear situation.** Let $E_1, E_2 \subset \mathbb{C}^n$ be linear subspaces. Then the pair (E_1, E_2) is in *general position*, if

$$\dim(E_1 + E_2) = \min(\dim E, \dim E_1 + \dim E_2). \tag{10}$$

We wish to discuss this notion for arbitrary systems of linear subspaces $E = (E_1, \dots, E_s)$. Notice that it is not sufficient to require that all sums and all intersections have maximal (minimal) possible dimensions. Indeed, suppose that $E_1, \dots, E_6 \in \mathbb{C}^3$ are 1-dimensional, each 3 subspaces are linearly independent but the intersection

$$(E_1 + E_2) \cap (E_3 + E_4) \cap (E_5 + E_6)$$

is of dimension 1. On the projective level this means that three lines intersect at one point. This tuple is not in general position but this cannot be checked by considering pure sums and pure intersections.

To make the definition precise we introduce (formal) admissible expressions $P(X)$ in variables $X = (X_1, \dots, X_s)$ by the following rules:

1. Each X_i is an admissible expression;
2. If $P(X)$ and $Q(X)$ are admissible expressions, then $P(X)+Q(X)$ and $P(X)\cap Q(X)$ are also admissible.

We call a pair of admissible expressions $(P(X), Q(X))$ *independent*, at each variable X_i appears at most once in $P(X) + Q(X)$. Given a system $E = (E_1, \dots , E_s)$ of linear subspaces of $\mathbb{C}^n$, we denote by $P(E)$ the evaluation of an admissible expression $P(X)$ on E.

Definition 4.1. *A system of linear subspaces $E = (E_1, \dots , E_s)$ of $\mathbb{C}^n$ is said to be in general position, if for every independent pair of admissible expressions $(P(X), Q(X))$, the pair of evaluations $(P(E), Q(E))$ is in general position.*

It follows from the definition that the set of all s-tuples with fixed dimensions $n_1, \dots , n_s$ which are in general position is a Zariski open subset of the product of corresponding Grassmanians G_{n,n_i}.

4.2. **General position for webs.**

Definition 4.2. *We say that a holomorphic web $G = (G_i)_{1\le i\le s}$ is in general position at $z \in U$, if all $G_i(z)$'s are smooth at z and the tangent subspaces $T_zG_i(z) \subset T_zU$ are in general position. A web is in general position if it is in general position at generic points.*

The following statement is straightforward:

Lemma 4.1. *Let G be a holomorphic web in general position. Then there exists a nonzero holomorphic function $\psi : U \to \mathbb{C}$ such that the tangent subspaces $T_zG_i(z) \subset T_zU$ are in general position for all z with $\psi(z) \neq 0$.*

Let ψ' be as in Lemma 4.1 for the web of D'. Applying Rado's theorem to $\psi' \circ f$ we obtain:

Lemma 4.2. *Let $f : D \to D'$ be a proper holomorphic map between domains of polyhedral type in C^n. Suppose that the characteristic webs of D and D' are in general position. Then there exists a dense boundary subset $S \subset \partial D$ with the following property. For every $x \in S$ there exists $x' \in \partial D'$ and a sequence $z_m \to x$ in D such that $f(z_m) \to x'$ and the corresponding webs are in general position at x and at x' respectively.*

5. Rigid tuples of natural numbers

Definition 5.1. *A tuple of natural numbers*

$$(m_1, \dots , m_s) \in \{1, \dots , n - 1\}^s$$

*is called n-*rigid *if for each $i = 1, \dots , s$, there exist systems of formal admissible expressions $P = (P_1, \dots , P_K)$, $Q = (Q_1, \dots , Q_L)$, such that their evaluations on an arbitrary system of linear subspaces $E = (E_1, \dots , E_s)$ of $\mathbb{C}^n$ of dimensions $m_1, \dots , m_s$ in general position satisfy the following properties*

1. *$E_i \subset \tilde{P}_1(E)$;*
2. *$\mathbb{C}^n = P_1(E) \oplus \cdots \oplus P_K(E)$;*
3. *for every $k = 1, \dots , K - 1$, there exists a subset $I_k \subset \{1, \dots , L\}$ such that $\tilde{P}_k(E) + \sum_{l\in I_k} Q_l(E) = \tilde{P}_{k+1}(E) + \sum_{l\in I_k} Q_l(E) = \mathbb{C}^n$ and for all $l \in I_k$, $\tilde{P}_k(E) \cap Q_l(E) = \tilde{P}_{k+1}(E) \cap Q_l(E) = 0$.*

Example 5.1. *The s-tuple* $(1,\dots,1)$ *is n-rigid if and only if* $s>n$. *Indeed, if* $s>n$, *it is sufficient to assume* $i=1$ *and define* $P_k(E):=E_k$, $k=1,\dots,n$, $Q_1(E):=E_{n+1}$ *and* $I_k:=\{1\}$. *On the other hand, if an s-tuple* $(1,\dots,1)$ *is n-rigid, one needs at least n subspaces* E_i *to split* $\mathbb{C}^n$ *in a direct sum. Furthermore, one needs at least one more subspace to obtain* Q_l *as above.*

Example 5.2. *The s-tuple* $(n-1,\dots,n-1)$ *is n-rigid if and only if* $s>n$. *Indeed, if* $s>n$, *it is sufficient to assume* $i=1$ *and define* $P_k(E):=\cap_{1\le j\le n, j\ne k}E_k$ *for* $k=1,\dots,n$, $Q_l(E):=(P_l(E)+P_{l+1}(E))\cap E_{n+1}$, $l=1,\dots,n-1$ *and* $I_k:=\{k\}$. *The necessity of* $n+1$ *subspaces follows as above.*

Proposition 5.1. *In the above notation there exists an integer function* $N(n)$ *such that for each n and* $s\ge N(n)$, *all s-tuples* $(m_1,\dots,m_s)$ *are rigid. One can take* $N(n)=(n+1)(n(n-1)/2+1)$.

Proof. We first prove by induction on n that every system in general position of $s(n):=(n(n-1)/2+1)$ linear subspaces $E_i\subset\mathbb{C}^n$, $i=1,\dots,s(n)$, satisfy conditions 1 and 2 of Definition 5.1 with $K=2$. This is clearly true for $n=2$. Let n be larger than 2. Let $Z\subset\mathbb{C}^n$ be the largest possible direct sum of E_i's. If $Z=\mathbb{C}^n$ we are done. Otherwise Z is smaller and is a sum of at most $(n-1)$ subspaces. There remain at least $s(n)-(n-1)=s(n-1)$ "free" subspaces.

Each remainder subspace E_i is in general position with Z. Since Z is the largest direct sum, the intersection $E_i\cap Z$ is nonzero. We use the induction for Z and the system of subspaces $E_i\cap Z$ (at least $s(n-1)$ of them are nonzero). We conclude that there exist admissible expressions $W_1,W_2\in V$ such that $Z=(Z\cap W_1)\oplus(Z\cap W_2)$ is a nontrivial direct sum. By the conditions of general position we obtain:

$$\dim W_j+\dim Z=n+\dim(Z\cap W_j),\quad j=1,2, \tag{11}$$

and therefore $\dim W_1+\dim W_2=2n-\dim Z$. It follows that

$$\dim(W_1\cap W_2)\ge\dim W_1+\dim W_2-n=n-\dim Z. \tag{12}$$

On the other hand, $W:=W_1\cap W_2$ has zero intersection with Z, i.e. $\dim W\le n-\dim Z$. Together with (12) this implies $V=W\oplus Z$ which is the required direct sum.

Suppose now that we have $(n+1)s(n)$ subspaces in general position. By the first part of the proof, they generate $(n+1)$ independent splittings $V=W_j\oplus U_j$, $\dim W_j\le\dim U_j$, $j=1,\dots,n+1$. Choose W_j with the maximal possible dimension, say W_{n+1}. Then the subspaces $P_1:=W_{n+1}$, $P_2:=U_{n+1}$, $Q_l:=W_l$, $l=1,\dots,n$, satisfy the required properties because of general position. □

The main property of n-rigid tuples which is crucial for our extension results is a certain holomorphic connection between components of linear maps sending one tuple of linear subspaces of general position into another. This is expressed in the following proposition.

Proposition 5.2. *Let* $(m_1,\dots,m_s)$ *be n-rigid, P and Q be systems of formal admissible expressions satisfying conditions of Definition* 5.1. *Furthermore, let* $E_0=(E_{01},\dots,E_{0s})$ *and* $E_0'=(E_{01}',\dots,E_{0s}')$ *be systems of linear subspaces of* $\mathbb{C}^n$ *in general position with* $\dim E_{0i}=\dim E_{0i}'=m_k$. *Then there exist open neighborhoods* $\Omega=\Omega(E_0)$ *and* $\Omega'=\Omega(E_0')$ *in the product of Grassmanians* $G_{n,m_1}\times\cdots\times G_{n,m_s}$ *and* *families of holomorphic maps* $\Phi_k(E,E')\colon \mathcal{L}(P_k(E),P_k(E'))\to\mathcal{L}$*,* $E\in\Omega$*,* $E'\in\Omega'$*,* $k=1,\dots,K$*, such that for all*

$$g=(g_1,\dots,g_K)\in\mathcal{L}(P_1(E)\oplus\cdots\oplus P_K(E),P_1(E')\oplus\cdots\oplus P_K(E'))$$

with $g(Q_l(E))\subset Q_l(E')$*,* $l=1,\dots,L$*, one has* $g=\Phi_k(E,E')(g_k)$*.*

Proof. Let E and E' be close to E_0 and E'_0 respectively. By conditions of Definition 5.1, the canonical projections $\pi_{kl}(E)\colon Q_l(E)\to P_k(E)$, $\pi_{k+1,l}(E)\colon Q_l(E)\to P_{k+1}(E)$, are injective for all $k=1,\dots,K-1$ and $l\in I_k$. Hence the following maps are well-defined and invertible:

$$\varphi_{k,l}(E):=\pi_{k+1,l}\circ\pi_{kl}^{-1}\colon \pi_{kl}(Q_l)\to\pi_{k+1,l}(Q_l). \tag{13}$$

Let $Z_{k,l}(E')\subset P_k(E')$ be a holomorphic family of linear subspaces which satisfies $P_k(E')=\pi_{kl}(Q_l(E'))\oplus Z_{k,l}(E')$. This is possible for E' close E'_0. Then the projection along $Z_{k,l}(E')$, $\psi_{kl}(E')\colon P_k(E')\to\pi_k(Q_l(E'))$ is well-defined.

Fix a holomorphically E-dependent basis $e_1(E),\dots,e_m(E)\in P_{k+1}(E)$ such that $e_j(E)\in\pi_{k+1}(Q_{h(j)}(E))$ for all $j=1,\dots,m$, where $h\colon\{1,\dots,m\}\to I_k$ is an integer function. This is possible by conditions of Definition 5.1.

Define

$$\Phi_{k,k+1}(E,E')(g_k)(e_j(E)):= \left[\varphi_{k,h(j)}(E')\circ\psi_{k,h(j)}(E')\circ g_k\circ(\varphi_{k,h(j)}(E))^{-1}\right](e_j(E)). \tag{14}$$

We obtain a holomorphic family of linear maps

$$\Phi_{k,k+1}(E,E')(g_k)\colon \mathcal{L}(P_k(E),P_k(E'))\to\mathcal{L}(P_{k+1}(E),P_{k+1}(E') \tag{15}$$

such that for every $g=(g_1,\dots,g_K)$ as above, $g_{k+1}=\Phi_{k,k+1}(E,E')(g_k)$. Similarly we construct a family

$$\Phi_{k+1,k}(E,E')\colon \mathcal{L}(P_{k+1}(E),P_{k+1}(E'))\to\mathcal{L}(P_k(E),P_k(E'). \tag{16}$$

Then the required families can be defined as follows:

$$\Phi_{k,r}(E,E'):=\Phi_{r-1,r}(E,E')\circ\cdots\circ\Phi_{k,k+1}(E,E') \tag{17}$$

for $1\le k<r\le K$, similarly for $1\le r<k\le K$ and finally

$$\Phi_k(E,E'):=(\Phi_{k,1}(E,E'),\dots,\Phi_{k,K}(E,E')), \tag{18}$$

where $\Phi_{k,k}(E,E')(g_k):=g_k$. □

6. Differential equations for biholomorphic maps

Our goal here is to construct a system of differential equations satisfied by biholomorphic maps. We say that a domain of polyhedral type $D\subset\mathbb{C}^n$ is rigid, if the tuple of dimensions of maximal characteristic decompositions is n-rigid.

Proposition 6.1. *Let* $f\colon D\to D'$ *be a biholomorphic map between domains of polyhedral type. Suppose that* D *is rigid and the characteristic web of* D *(resp. of* D'*) is in general position at* $x\in\partial D$ *(resp. at* $x'\in\partial D'$*). Then there exist open neighborhoods* $V=V(x)$*,* $V'=V(x')$ *and a holomorphic map* $\Phi\colon V\times V'\times\mathcal{L}\to\mathcal{L}$ *such that for all* $z,w\in D$ *with* $(z,f(z)),(w,f(w))\in V\times V'$ *and* w *sufficiently close to* z*,* $d_wf=\Phi(w,f(w),d_zf)$*.*

Proof. Suppose for simplicity that $G_1, \dots, G_s$ are maximal and $\varphi(i) = i$ in Theorem 1.3. Let P, Q be as in Definition 5.1. Denote by $P(z)$, $Q(z)$ the evaluations on $(T_zG_1(z), \dots, T_zG_s(z))$. Furthermore we evaluate P on the fibers $G_i(z)$ as follows. An intersection in P corresponds to an intersection of fibers and a sum to the composition:

$$G_i(G_j(z)) := \bigcup_{w \in G_j(z)} G_i(w). \tag{19}$$

By our assumptions, the characteristic web of D is in general position at every point $z \in D$ which is close to x. If we restrict all fibers to a small neighborhood $V = V(x)$, this construction yields a non-linear frame $\mathcal{P}(z) = (\mathcal{P}_1(z), \dots, \mathcal{P}_K(z))$. Let $(\tilde{\mathcal{P}}_1(z), \dots, \tilde{\mathcal{P}}_K(z))$ be the complex submanifolds near z which correspond to $\tilde{P}_1, \dots, \tilde{P}_K$ respectively. We define a local biholomorphism

$$\mathcal{P}_1(z) \times \cdots \times \mathcal{P}_K(z) \to U, \quad (w_1, \dots, w_K) \mapsto \tilde{\mathcal{P}}_1(w_1) \cap \cdots \cap \tilde{\mathcal{P}}_K(w_K). \tag{20}$$

Changing to a possibly smaller V we may assume that (20) defines local splittings $V = V_1(z) \times \cdots \times V_K(z)$ for all $z \in V$.

In the similar way we do the same construction for D'. In the sequel we take w sufficiently close to $z \in D$. It follows from Theorem 1.3 that $w \in \mathcal{P}_k(z)$ implies $f(w) \in \mathcal{P}_k(f(z))$. Hence f respects the splitting defined by (20) and can be written in the form

$$f(w_1, \dots, w_K) = (f_1(w_1), \dots, f_K(w_K))$$

with $f_k := f|\mathcal{P}_k(z) \to \mathcal{P}_k(z')$. By Proposition 5.2,

$$d_{w_1} f = \Phi_2(E(z), E(z'))(d_{z_2} f_2) \tag{21}$$

and

$$d_w f = \Phi_1(E(z), E(z'))(d_{w_1} f_1). \tag{22}$$

Putting (21) and (22) together we obtain

$$d_w f = \Phi_1(E(z), E(z'))(\Phi_2(E(z), E(z'))(d_{z_2} f_2)|P_1(z)). \tag{23}$$

This yields the required differential equation, where the right-hand side is defined for all z and z' sufficiently close to x and x' respectively. □

7. Boundedness of *df*

Let φ be as in Theorem 1.3.

Proposition 7.1. *Let $f\colon D \to D'$ be a biholomorphic map between domains of polyhedral type in $\mathbb{C}^n$. Suppose D is rigid and its characteristic web of D is in general position. Then there exists a dense subset $S \subset \partial D$ and for every $x \in S$ a sequence $(z_m)_{m \ge 0}$ in D such that $z_m \to x$, $f(z_m) \to x'$, $m \to \infty$, where $d_{z_m} f$ is bounded and the webs $G_1, \dots, G_s$ (resp. $G'_{\varphi(1)}, \dots, G'_{\varphi(s)}$) is in general position at x (resp. x').*

The proof is given by Lemmata 7.1 and 7.2 below.

Lemma 7.1. *Let $i = 1, \dots, s$ be fixed and $(z_m)_{m \ge 0}$ be a sequence in $G_i(z_0)$ such that $z_m \to x \in \partial D$ and $f(z_m) \to x' \in \partial D'$ as $m \to \infty$. Suppose that D is rigid and the characteristic webs of D and D' are in general position at x and x' respectively. Then $d_{z_m} f$ is bounded.*

Proof. Without loss of generality, $i = 1$. Let $z \in G_1(z_0)$ be close to x. Then there exists a neighborhood $V = V(z) \subset D$ such that $g_1^{-1}(v) \cap V$ are connected for all $v \in \mathbb{C}^{n_1}$. Since g_1 is regular at x, we may assume that $W := g_1(V) \subset \mathbb{C}^{n_1}$ is a submanifold with $y := g_1(x) \in W$. It follows from the connectedness of fibers and condition $f(G_i(z)) \subset G_i'(f(z))$ that there exists a holomorphic map $f_1 \colon W \to \mathbb{C}^{n_1'}$ with $g_1' \circ f = f_1 \circ g_1$ and $f_1(y) = g_1'(x') = y'$. By differentiating we obtain

$$d_{z'} g_1' \circ d_z f = d_y f_1 \circ d_z g_1, \tag{24}$$

where $z' := f(z)$.

Let P and Q be as in Proposition 5.2. We write $E_i(z) := T_z G_i(z)$, $E(z) := (E_1(z), \ldots, E_s(z))$, $P_k(z) := P_k(E(z))$, $\tilde{P}_k(z) := \tilde{P}_k(E(z))$. By the first condition of Definition 5.1, $E_1(z) = T_z G_1(z_0) \subset \tilde{P}_1(z)$. Hence $d_z g_1 | P_1(z)$ is injective and

$$d_z f | P_1(z) = (d_{z'} g_1' | P_1(z'))^{-1} \circ d_y f_1 \circ d_z g_1. \tag{25}$$

If z and z' are close to x and x' respectively, $E(z) \in \Omega$, $E(z') \in \Omega'$ as in Proposition 5.2. Then by Proposition 5.2,

$$d_z f = \Phi_1(E(z), E(z'), d_z f | P_1(z)). \tag{26}$$

Combining (25) and (26) we obtain the boundedness of $d_{z_m} f$. □

Lemma 7.2. *Suppose that D is rigid. For every open subset $W \subset \partial D$ there exist $i = 1, \ldots, s$, and a sequence $(z_m)_{m \geq 0}$ as in Lemma 7.1 with $z_m \to x \in W$.*

Proof. Consider an open subset $\tilde{W} \subset U$ such that $\emptyset \neq \tilde{W} \cap \partial D \subset W$. For brevity we write just U, D, ∂D and g_i for $\tilde{W} \cap U$, $\tilde{W} \cap D$, $\tilde{W} \cap \partial D$ and $g_i | \tilde{W}$ respectively. Without loss of generality, the characteristic web of D is in general position in $\tilde{W}$ and all fibers $g_i^{-1}(v) \cap D$ are biholomorphic to some open sets of appropriate vector spaces. Define $\tilde{D}_i := g_i(D)$.

We first suppose that $g_i(\tilde{D}_i) \cap \partial D \neq \emptyset$ for some $i = 1, \ldots, s$. This means that there exists $z_0 \in D$ such that $\overline{G_i(z_0)} \cap \partial D \neq \emptyset$. Since by our assumption $G_i(z_0)$ is biholomorphic to open subsets of vector spaces, Rado's theorem can be applied to the function ψ from Lemma 4.1 for D'. Hence there exists a sequence $(z_m)_{m \geq 0}$ with $f(z_m) \to x'$, $m \to \infty$, and such that $\psi(x') \neq 0$, i.e. the characteristic web of D' is in general position at x'. This finishes the proof under our assumption.

Now suppose, on the contrary, that $g_i(\tilde{D}_i) \cap \partial D = \emptyset$ for all $i = 1, \ldots, s$. Then $\partial D \subset g_i^{-1}(\partial \tilde{D}_i)$. We fix i and change to a possibly smaller connected subset $\tilde{W}$ of the form $W_1 \times W_2$ such that $g_i(z_1, z_2) = z_1$. In these coordinates $D = \tilde{D}_i \times W_2$ and $\partial D = \partial \tilde{D}_i \times W_2$, where $\partial \tilde{D}_i$ is taken in W_1. This shows that $\tilde{D}$ is locally saturated by the fibers of g_i. However this should be true for all $i = 1, \ldots, s$. It follows from the rigidity of D that ∂D contains open subsets of $\mathbb{C}^n$ which is impossible.

□

8. Proofs of main results

In the following proofs we use notation of corresponding theorems.

Proof of Theorem 1.5. Suppose that the sequence $(z_m)_{m \geq 1}$ exists. By Proposition 5.1, D is rigid. By Theorem 1.3, $d_{z_m}(T_{z_m} G_i(z_m)) = T_{f(z_m)} G'_{\varphi(i)}(f(z_m))$. Passing to the limit as $m \to \infty$ we obtain $L(T_x G_i(x)) = T_{x'} G'_{\varphi(i)}(x')$. Since $L \in \mathbf{GL}_n$, the characteristic web of D' is rigid at x'. Now by Proposition 6.1, f locally satisfies a holomorphic system of partial differential equations $d_w f = \Phi(w, f(w), d_z f)$. By general results, a local solution depends holomorphically on the initial values and parameters. Setting $z = z_m$ for m sufficiently large and using the boundedness of $d_{z_m} f$ we obtain the required extension. The opposite direction is straightforward. □

Proof of Theorem 1.4. By Proposition 7.1, there exists a dense subset $S \subset \partial D$ such the assumptions of Theorem 1.5 are satisfied for every $x \in S$. The required statement follows now from Theorem 1.5. □

Proof of Theorem 1.6. We use the statements of Examples 5.1 and 5.2 istead of Proposition 5.1. Then the proof goes as above. To show the exactness define $D = \delta^n$, $D' = \Omega^n$ with $\delta := \{|z| < 1\} \subset \mathbb{C}$ and $\Omega \subset \mathbb{C}$ a simply connected domain with nowhere real-analytic boundary. By Riemann Mapping Theorem, there exists a biholomorphic mapping $f \colon D \to D'$. However it is not holomorphically extendible to any point $x \in \partial D$. □

Theorems 1.2 and 1.1 follows directly from Theorems 1.5 and 1.4 respectively.

References

[1] M. S. Baouendi and L. P. Rothschild. Mappings of real algebraic hypersurfaces. *J. Amer. Mat. Soc.*, 8:997–1015, 1995.

[2] J. Baumann. Eine gewebetheoretische Methode in der Theorie der holomorphen Abbildungen: Starrheit und Nichtäquivalenz von analytischen Polyedergebieten. *Schriftenreihe des Mathematischen Instituts der Universität Münster*, Heft 24, 1982.

[3] H. Cartan. Sur les fonctions de n variables complexes: le transformations du produit topologique de deux domaines bornés. *Bull. Soc. math. Fr.*, 64:37–48, 1936.

[4] K. Diederich and J. E. Fornæss. Pseudoconvex domains with real-analytic boundaries. *Ann. Math.*, 107:371–384, 1978.

[5] B. L. Fridman. On a class of analytic polyhedra. *Sov. Math. Dokl.*, 19:1258–1261, 1978.

[6] E. Ligocka. On proper holomorphic and biholomorphic mappings between product domains. *Bull. Acad. Pol. Sci*, 28:319–323, 1980.

[7] J. J. Madden and C. M. Stanton. One-dimensional Nash groups. *Pacific Journal of Math.*, 154(2):331–344, 1992.

[8] R. Narasimhan. *Several complex variables*. Chicago Lectures in Mathematics. Univ. of Chicago Press, 1971.

[9] R. Remmert and K. Stein. Eigentliche holomorphe Abbildungen. *Math. Zeitschr.*, 73:159–189, 1960.

[10] H. Rischel. Holomorphe überlagerungskorrespondenzen. *Math. Scand.*, 15:49–63, 1964.

[11] A. G. Sergeev and G.M. Henkin. Uniform estimates for solutions of the $\overline{\partial}$-equation in pseudoconvex polyhedra. *Math. USSR Sb.*, No.4(40), 1981.

[12] Sh. I. Tsyganov. Biholomorphic maps of the direct products of domains. *Math. Notes*, 41:469–472, 1987.

[13] S. Webster. On the mapping problem for algebraic real hypersurfaces. *Inventiones math.*, 43:53–68, 1977.

[14] D. Zaitsev. On the automorphism groups of algebraic bounded domains. *Math. Ann.*, 302:105–129, 1995.

Mathematisches Institut, Universität Tübingen, 72076 Tübingen, Germany, E-mail address: dmitri.zaitsev@uni-tuebingen.de