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Effective-eigenvalue approach to the nonlinear Langevin equation for the Brownian motion in a tilted periodic potential. II. Application to the ring-laser gyroscope

W. T. Coffey*
Department of Microelectronics and Electrical Engineering, Trinity College, Dublin 2, Ireland

Yu. P. Kalmykov
Institute of Radio Engineering and Electronics, Russian Academy of Sciences, Vvedenskii Sq. 1, Fryazino, Moscow Region 141120, Russia

E. S. Massawe
Department of Microelectronics and Electrical Engineering, Trinity College, Dublin 2, Ireland
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The effective-eigenvalue method is used to obtain an approximate solution for the mean beat-signal spectrum for the ring-laser gyroscope in the presence of quantum noise. The accuracy of the effective-eigenvalue method is demonstrated by comparing the exact and approximate calculations. It shows clearly that the effective-eigenvalue method yields a simple and concise analytical description of the solution of the problem under consideration.

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I. INTRODUCTION

One of the most remarkable devices which laser light has made possible is the ring-laser gyroscope. The most important advantage of the ring-laser gyroscope as a rotation sensor over its mechanical counterpart is that it has no moving parts [1] so that it has a longer lifetime. However, its accuracy and sensitivity suffer from the well-known lock-in effect [2], i.e., at low rotation rates, the backscattering couples the counterpropagating waves and the beat note disappears. In recent times the problem of the lock-in effect has been avoided by adding an external controlled rotation rate [2] so that for the most part the laser operates outside the dead band. Thus virtually the only limitation which remains on the sensitivity of the ring-laser gyroscope is the quantum fluctuations [2,3]. It is therefore important to have a clear understanding of the effects of such noise on the action of the gyroscope.

Several investigators have studied the effects of the quantum noise on the mean beat frequency and spectrum [2-5]. Cresser, *et al.* [4] suggested a numerical algorithm for the exact solution of the spectrum of the beat signal in terms of an infinite continued fraction by solving the Fokker-Planck equation for the probability density function of the phase variable. However, this numerical approach to the problem has the disadvantage that it does not yield a closed-form solution. Thus the qualitative behavior of the system is not at all obvious.

A number of investigators (see, e.g., [3,5]) have attempted to overcome this problem by deriving approximate analytical expressions for various ranges of the gyroscope parameters. However, a general equation which would be valid for all the parameter ranges of interest has not yet been derived. It is obviously valuable to have a general analytical expression for the spectrum. Such an expression will be obtained in the present paper.

As is well known, the underlying Langevin equation for the ring-laser gyroscope is similar to that for the Brownian motion of a particle in a tilted periodic potential [1]. Such a Langevin equation also arises in a number of other physical situations: Josephson junctions [6], self-locking in a laser [7], the laser with injected signal [8], and the theory of phase-locking techniques in radio engineering [9], etc. In Ref. [10] the Josephson junction has been considered as an example of the Brownian motion in a tilted-cosine potential. We have shown [10] that the effective-eigenvalue method (described in detail in Ref. [11]) allows us to obtain a simple analytical expression for the impedance of the junction for an externally applied small-signal alternating current in the presence of noise. However, this approach can also be applied to the ring-laser gyroscope [1,3–5], where we remark that there the quantity of interest is the beat-signal spectrum.

It is the purpose of this paper to show how the effective-eigenvalue method can also be applied to the ring-laser gyroscope to obtain a simple and concise analytical description of the beat-signal spectrum. The method constitutes a truncation procedure which allows one to obtain a closed-form approximation to the solution of the infinite hierarchy of differential-difference equations obtained directly from the nonlinear Langevin equation without recourse to the Fokker-Planck equation. These equations govern the time behavior of the statistical averages characterizing the dynamics of the ring-laser gyroscope operating in the presence of quantum noise. We show that the effective-eigenvalue method is a valuable and extremely powerful tool for the purpose of obtaining a simple analytical solution for the beat-signal spectrum. The solution obtained from the effective-eigenvalue method is shown to agree closely with the exact solution for a wide range of frequencies as demonstrated by the plots of the mean beat signal as a function of frequency in all regimes of interest. We remark that our form of the exact solution for the beat-signal spectrum also has the merit of being considerably more convenient for numerical calculations while constituting a simpler algorithm than that obtained in [4]. Our exact results agree with those of Cresser *et al.* [4].

II. THE AVERAGED LANGEVIN EQUATION FOR THE RING-LASER GYROSCOPE

The dynamical behavior of the ring-laser gyroscope operating in the steady state is described by the Langevin equation [1,3]

$$\dot{\phi}(t)-b\sin\phi(t)=a+\Gamma(t)\ , \qquad (1)$$

where ϕ is the relative phase between the clockwise and counterclockwise modes of the laser, the gyroscope rotation rate coefficient a is given by

$$a=-\frac{8\pi}{\lambda L}(\mathbf{A}\cdot\mathbf{\Omega})\ , \qquad (2)$$

$|\mathbf{A}|$ is the area covered by the optical path of length L along the ring cavity, $\mathbf{\Omega}$ is the rotation rate of the gyroscope, and λ is the wavelength of the laser; b is the backscattering coefficient, $\Gamma(t)$ is the noise source which is assumed to be Gaussian with

$$\overline{\Gamma(t)}=0,\quad \overline{\Gamma(0)\Gamma(t)}=2D\delta(t)\ , \qquad (3)$$

where $D=(\nu/2Q\bar{n})$ is the diffusion coefficient due to random noise (ν is the frequency of the light field, $\bar{n}$ is the averaged number of photons in the field at steady state, Q is the quality factor of the cavity, and the overbar denotes "statistical average of").

In order to proceed we change the variable in the Langevin equation (1) by means of the transformation

$$r^n=e^{-in\phi}\quad (n=\ldots,-1,0,1,\ldots)$$

so that

$$\frac{d}{dt}r^n(t)=\frac{nb}{2}[r^{n+1}(t)-r^{n-1}(t)]-inr^n(t)[a+\Gamma(t)]\ . \qquad (4)$$

The multiplicative noise term $r^n(t)\Gamma(t)$ in Eq. (4) contributes a noise-induced drift term to the average [12]. This term poses an interpretation problem in averaging Eq. (4). We recall that, taking the Langevin equation for a stochastic variable $\xi(t)$ as [12]

$$\frac{d}{dt}\xi(t)=h(\xi(t),t)+g(\xi(t),t)\Gamma(t)\ , \qquad (5)$$

with

$$\overline{\Gamma(t)}=0,\quad \overline{\Gamma(t)\Gamma(t')}=2\delta(t-t')\ ,$$

and interpreting it as a Stratonovich stochastic equation [12], we have

$$\dot{x}=\lim_{\tau\to 0}\left.\left\{\frac{1}{\tau}\overline{[\xi(t+\tau)-x]}\right\}\right|_{\xi(t)=x}$$
$$=h(x,t)+g(x,t)\frac{\partial}{\partial x}g(x,t)\ , \qquad (6)$$

where $\xi(t+\tau)$, $\tau>0$ is a solution of Eq. (5) which at time t has the sharp value $\xi(t)=x$. It should be noted that the quantity x in Eq. (6) is itself a random variable with probability density function $W(x,t)$ defined such that $W(x,t)dx$ is the probability of finding x in the interval $(x,x+dx)$. Thus on averaging Eq. (6) over $W(x,t)$ we obtain

$$\frac{d}{dt}\langle x\rangle=\langle h(x,t)\rangle+\left\langle g(x,t)\frac{\partial}{\partial x}g(x,t)\right\rangle\ , \qquad (7)$$

where the angular brackets mean the relevant quantity averaged over $W(x,t)$.

We may use the above results to evaluate the average of the multiplicative noise term in Eq. (4). We have

$$g(r^n)=-inr^n\ ,$$
$$g(r^n)\frac{\partial}{\partial r^n}g(r^n)=-n^2r^n\ , \qquad (8)$$

and

$$\frac{d}{dt}r^n=\frac{nb}{2}(r^{n+1}-r^{n-1})-inar^n-Dn^2r^n\ . \qquad (9)$$

Thus we obtain the hierarchy of differential-difference equations for averages

$$\frac{d}{dt}\langle r^n\rangle+\frac{1}{\tau_0}\left[n^2+\frac{inx\gamma}{2}\right]\langle r^n\rangle$$
$$=\frac{n\gamma}{4\tau_0}(\langle r^{n-1}\rangle-\langle r^{n+1}\rangle]\ , \qquad (10)$$

where

$$x=-\frac{a}{b} \qquad (11)$$

is the ratio of the rotation rate to the backscattering coefficient,

$$\gamma=-\frac{2b}{D} \qquad (12)$$

is the ratio of the backscattering coefficient to the diffusion coefficient, and

$$\tau_0=\frac{1}{D} \qquad (13)$$

is the characteristic relaxation time.

We remark that $r^n(t)$ in Eq. (4) and r^n in Eqs. (9) and (10) have different meanings. Namely, $r^n(t)$ in Eq. (4) is a stochastic variable while in Eqs. (9) and (10) r^n is the sharp (definite) value $r^n(t)=r^n$ at time t. (Instead of using different symbols for the two quantities we have distinguished the sharp values at time t from stochastic variables by deleting the time argument as in Ref. [12].) The quantity r^n above is itself a random variable which must be averaged over an ensemble of gyroscopes. The symbol $\langle\ \rangle$ means such an ensemble average.

It should be noted that Eq. (10) is a well-known result which may be obtained from the relevant Fokker-Planck equation [1–5].

III. CALCULATION OF THE BEAT-SIGNAL SPECTRUM OF THE RING-LASER GYROSCOPE

For the ring-laser gyroscope operating at steady state the quantity of interest is the spectrum of the beat signal [4] defined as

$$\alpha(\omega)=\int_{-\infty}^{\infty}\langle\cos\phi(0)\cos\phi(t)\rangle_0 e^{i\omega t}dt =2\,\mathrm{Re}\left\{\int_0^{\infty}C(t)e^{i\omega t}dt\right\}, \tag{14}$$

where

$$C(t)=\langle\cos\phi(0)\cos\phi(t)\rangle_0 \tag{15}$$

is the stationary beat-signal autocorrelation function $\cos\phi(t)$ which is a measure of the total detected intensity [1,3]. The appearance of the cosine term in Eq. (14) arises from the heterodyning of the counter-rotating waves [3].

A numerical method for the exact calculation of $\alpha(\omega)$ has been given in Ref. [4]. Another representation of the exact solution can be obtained in the manner described below.

On multiplying both sides of Eq. (9) by $\cos\phi(0)$ and averaging over the stationary distribution function [12]

$$W_0(\phi(0))=c_0e^{-U(\phi(0))/kT}\left[1-\frac{(1-e^{-U(\phi(0))/kT})\int_0^{\phi(0)}e^{U(\phi')/kT}d\phi'}{\int_0^{2\pi}e^{-U(\phi')/kT}d\phi'}\right],\quad U(\phi)=-\frac{\gamma kT}{2}(\cos\phi+x\phi) \tag{16}$$

we obtain just as in Eq. (10) the differential-difference equation for the stationary averages:

$$\frac{d}{dt}\psi_n(t)+\frac{1}{\tau_0}\left[n^2+\frac{inx\gamma}{2}\right]\psi_n(t) =\frac{n\gamma}{4\tau_0}[\psi_{n-1}(t)-\psi_{n+1}(t)], \tag{17}$$

where

$$\psi_n(t)=\langle\cos\phi(0)r^n(t)\rangle_0 . \tag{18}$$

It is obvious that $C(t)$ from Eq. (15) is related to $\psi_1(t)$ by

$$C(t)=\mathrm{Re}\{\psi_1(t)\} . \tag{19}$$

Let us introduce instead of $\psi_n(t)$ the true correlation functions $C_n(t)$ defined as

$$C_n(t)=\psi_n(t)-\psi_n(\infty) =\langle\cos\phi(0)r^n(t)\rangle_0-\langle\cos\phi(0)\rangle_0\langle r^n(0)\rangle_0 . \tag{20}$$

Then we obtain from Eq. (17)

$$\left[n^2+\frac{inx\gamma}{2}\right]\psi_n(\infty)=\frac{n\gamma}{4}[\psi_{n-1}(\infty)-\psi_{n+1}(\infty)] \tag{21}$$

and

$$\frac{d}{dt}C_n(t)+\frac{1}{\tau_0}\left[n^2+\frac{inx\gamma}{2}\right]C_n(t) =\frac{n\gamma}{4\tau_0}[C_{n-1}(t)-C_{n+1}(t)] . \tag{22}$$

With the help of the one-sided Fourier transform

$$\tilde{C}_n(\omega)=\int_0^{\infty}e^{-i\omega t}C_n(t)dt \tag{23}$$

we may now obtain from Eq. (22) the usual three-term recurrence relation

$$\left[-i\omega\tau_0+n^2+\frac{inx\gamma}{2}\right]\tilde{C}_n(\omega) =\frac{\gamma n}{4}[\tilde{C}_{n-1}(\omega)-\tilde{C}_{n+1}(\omega)]+\tau_0C_n(0) . \tag{24}$$

Equation (24) has the same form as Eq. (21) of Ref. [10] and hence has a solution of similar form, viz.,

$$\begin{aligned}\tilde{C}_1(\omega)&=\frac{4\tau}{\gamma_0}\Big(\tilde{S}_1(\omega)C_1(0)-\tfrac{1}{2}\tilde{S}_1(\omega)\tilde{S}_2(\omega)C_2(0)+\tfrac{1}{3}\tilde{S}_1(\omega)\tilde{S}_2(\omega)\tilde{S}_3(\omega)C_3(0)+\cdots+\frac{1}{n}(-1)^{n+1}C_n(0)\prod_{i=1}^{n}\tilde{S}_i(\omega)+\cdots\Big)\\&=\frac{4\tau_0}{\gamma}\tilde{S}_1(\omega)C_1(0)\left\{1-\frac{\tilde{S}_2(\omega)C_2(0)}{2C_1(0)}\left[1-\frac{2\tilde{S}_3(\omega)C_3(0)}{3C_2(0)}\left[1-\frac{3\tilde{S}_4(\omega)C_4(0)}{4C_3(0)}(1-\cdots)\right]\right]\right\},\end{aligned} \tag{25}$$

where the continued fraction $\tilde{S}_n(\omega)$ is defined as

$$\tilde{S}_n(\omega)=\frac{0.5}{-2i\omega\tau_0/\gamma n+2n/\gamma+ix+0.5\tilde{S}_{n+1}(\omega)} .$$

The quantity $C_n(0)/C_{n-1}(0)$ appearing in Eq. (25) can be expressed in terms of $S_1=\tilde{S}_1(0)$ and $S_n=\tilde{S}_n(0)$ as follows. We have from Eq. (20)

$$\begin{aligned}C_n(0)&=\langle\cos\phi(0)r^n(0)\rangle_0-\langle\cos\phi(0)_0\rangle\langle r^n(0)\rangle_0\\&=\tfrac{1}{2}\langle r^{n+1}(0)+r^{n-1}(0)\rangle_0-\mathrm{Re}(S_1)\langle r^n(0)\rangle_0\\&=\tfrac{1}{2}S_1S_2\cdots S_{n-1}(1+S_nS_{n+1})\\&\quad-\mathrm{Re}(S_1)S_1S_2\cdots S_n .\end{aligned} \tag{26}$$

Hence we have for $n\geq 2$

$$\frac{C_n(0)}{C_{n-1}(0)}=\frac{S_{n-1}(1+S_nS_{n+1})-2S_{n-1}S_n\mathrm{Re}(S_1)}{1+S_{n-1}S_n-2S_{n-1}\mathrm{Re}(S_1)} . \tag{27}$$

Taking account of the stationary solution of Eq. (4), i.e., [12],

$$S_p\left[p^2+\frac{ix\gamma p}{2}\right]=\frac{p\gamma}{4}(1-S_pS_{p+1}) , \tag{28}$$

with

$$S_p=\frac{\langle r^p\rangle_0}{\langle r^{p-1}\rangle_0} ,$$

we have

$$\frac{C_n(0)}{C_{n-1}(0)}=\frac{1-S_n(ix+2n/\gamma)-S_n\mathrm{Re}(S_1)}{S_n+ix+2(n-1)/\gamma-\mathrm{Re}(S_1)},\quad n\geq 2 . \tag{29}$$

We also note that

$$\begin{aligned}C_1(0)&=\tfrac{1}{2}(S_2S_1+1)-S_1\mathrm{Re}(S_1)\\&=1-S_1(ix+2/\gamma)-S_1\mathrm{Re}(S_1) .\end{aligned} \tag{30}$$

We can show in the same way that

$$\begin{aligned}\tilde{C}_{-1}(\omega)&=\tilde{C}_1^*(-\omega)\\&=\frac{4\tau_0}{\gamma}\tilde{S}_1^*(-\omega)C_1^*(0)\left\{1-\frac{\tilde{S}_2^*(-\omega)C_2^*(0)}{2C_1^*(0)}\left[1-\frac{2\tilde{S}_3^*(-\omega)C_3^*(0)}{3C_2^*(0)}\left[1-\frac{3\tilde{S}_4^*(-\omega)C_4^*(0)}{4C_3^*(0)}(1-\cdots)\right]\right]\right\} ,\end{aligned} \tag{31}$$

where the symbol $*$ means the complex conjugate. Thus on using Eqs. (19), (20), (25), and (31) we obtain the beat-signal spectrum as

$$\begin{aligned}\alpha(\omega)&=2\pi\psi_1(\infty)\delta(\omega)+\mathrm{Re}\{\tilde{C}_1(\omega)+\tilde{C}_{-1}(\omega)\}\\&=2\pi\,\mathrm{Re}^2(S_1)\delta(\omega)+\mathrm{Re}\{\tilde{C}_1(\omega)+\tilde{C}_1^*(-\omega)\} .\end{aligned} \tag{32}$$

Equations (25) and (29–32) are very convenient for numerical calculations and constitute a simpler algorithm than that used in Ref. [4] in order to evaluate $\alpha(\omega)$. They allow us to calculate $\alpha(\omega)$ exactly.

We shall now show how the effective-eigenvalue method [11] can be applied in order to evaluate $\alpha(\omega)$. In order to implement it and to determine the effective eigenvalue for the quantity of interest we need to keep the equations for $C_1(t)$ and $C_2(t)$ of Eq. (22). Thus the effective-eigenvalue method requires that $C_1(t)$ and $C_2(t)$ obey the coupled equations

$$\frac{d}{dt}C_1(t)+\lambda_1C_1(t)+\frac{\gamma}{4\tau_0}C_2(t)=0 , \tag{33}$$

$$\frac{d}{dt}C_2(t)+\lambda_2^{\mathrm{ef}}C_2(t)-\frac{\gamma}{2\tau_0}C_1(t)=0 , \tag{34}$$

where

$$\lambda_1=1+ix\gamma/2 \tag{35}$$

and λ_2^{ef} is the effective eigenvalue to be determined.

According to the effective-eigenvalue approach [11], λ_2^{ef} is determined as

$$\lambda_2^{\mathrm{ef}}=-\frac{\dot{C}_2(0)-(\gamma/2\tau_0)C_1(0)}{C_2(0)} , \tag{36}$$

where $\dot{C}_2(0)$ is determined by Eq. (22) for $n=2$, namely,

$$\dot{C}_2(0)=-\frac{1}{\tau_0}(4+ix\gamma)C_2(0)-\frac{\gamma}{2\tau_0}[C_3(0)-C_1(0)] . \tag{37}$$

$C_2(0)$ and $C_3(0)$ are given by Eq. (26).

Substitution of Eq. (37) into Eq. (36) yields

$$\begin{aligned}\lambda_2^{\mathrm{ef}}&=\frac{(1/\tau_0)(4+ix\gamma)C_2(0)+(\gamma/2\tau_0)C_3(0)}{C_2(0)}\\&=\frac{1}{\tau_0}(4+ix\gamma)+\frac{\gamma}{2\tau_0}\frac{C_3(0)}{C_2(0)} .\end{aligned} \tag{38}$$

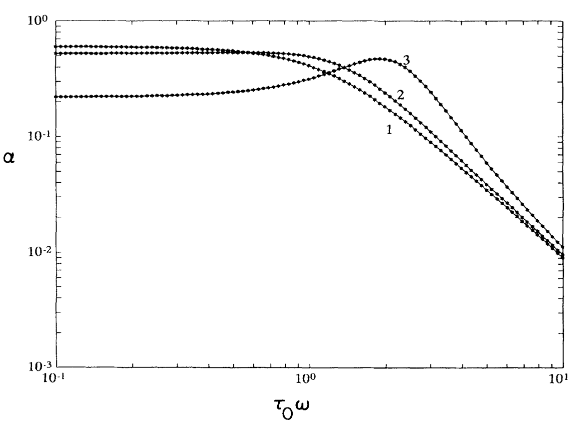

FIG. 1. Comparison of the exact (solid lines) and approximate (stars) solutions for "incoherent" part of the beat-signal spectrum α vs $\tau_0\omega$ at $\gamma=-2$. Curves 1: $x=-0.5$ (the locked region), 2: $x=-0.9(|a|\leq b)$ and 3: $x=-2.0$ (the unlocked region).

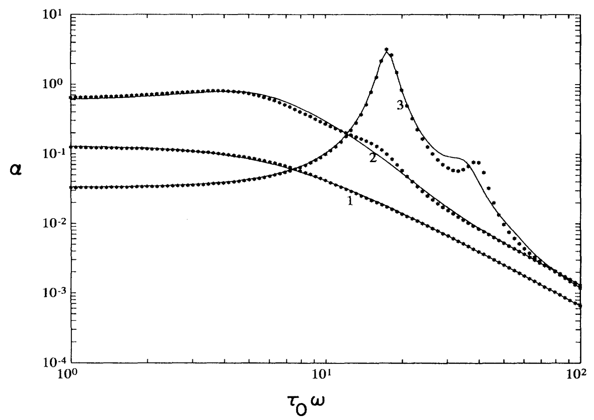

FIG. 2. Same as Fig. 1 for $\gamma=-20$.

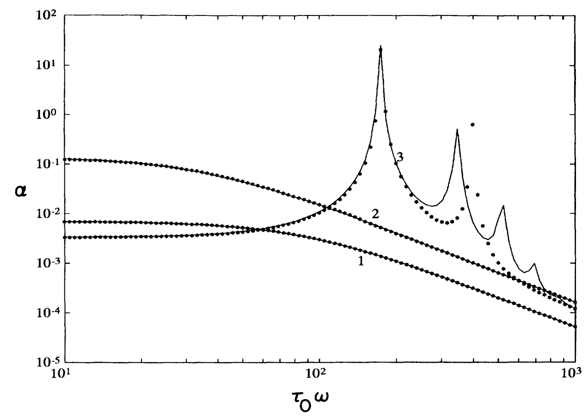

FIG. 3. Same as Fig. 1 for $\gamma=-200$ (the noise-free limit).

On substituting $C_3(0)/C_2(0)$ from Eq. (27) for $n=3$ into Eq. (38) we thus obtain

$$\lambda_2^{\text{ef}}=\frac{1}{\tau_0}\left[4+ix\gamma+\frac{\gamma}{2}\left[\frac{1-S_3(ix+6/\gamma)-S_3\text{Re}(S_1)}{S_3+ix+4/\gamma-\text{Re}(S_1)}\right]\right] . \tag{39}$$

The solution of Eqs. (33) and (34) is easily found using the one-sided Fourier transform. The result is (where we emphasize that λ_1 and λ_2^{ef} are complex)

$$\tilde{C}_1(\omega)=\frac{C_1(0)(\lambda_2^{\text{ef}}-i\omega)-\gamma C_2(0)/4\tau_0}{(\lambda_1-i\omega)(\lambda_2^{\text{ef}}-i\omega)+\gamma^2/8\tau_0^2} , \tag{40}$$

where

$$\tilde{C}_1(\omega)=\int_0^{\infty}e^{-i\omega t}C_1(t)dt . \tag{41}$$

A similar calculation for $\tilde{C}_{-1}(\omega)$ yields

$$\tilde{C}_{-1}(\omega)=\frac{C_1^*(0)(\lambda_2^{\text{ef}*}-i\omega)-\gamma C_2^*(0)/4\tau_0}{(\lambda_1^*-i\omega)(\lambda_2^{\text{ef}*}-i\omega)+\gamma^2/8\tau_0^2} . \tag{42}$$

Thus we may now calculate from Eqs. (14), (15), (40), and (42) the beat-signal spectrum in the simple analytic form

$$\alpha(\omega)=2\pi\,\text{Re}^2(S_1)\delta(\omega)+\text{Re}\left\{\frac{C_1(0)(\lambda_2^{\text{ef}}-i\omega)-\gamma C_2(0)/4\tau_0}{(\lambda_1-i\omega)(\lambda_2^{\text{ef}}-i\omega)+\gamma^2/8\tau_0^2}+\frac{C_1^*(0)(\lambda_2^{\text{ef}*}-i\omega)-\gamma C_2^*(0)/4\tau_0}{(\lambda_1^*-i\omega)(\lambda_2^{\text{ef}*}-i\omega)+\gamma^2/8\tau_0^2}\right\} , \tag{43}$$

where λ_1 and λ_2^{ef} are given by Eqs. (35) and (39), respectively.

IV. RESULTS AND DISCUSSION

As seen from Eqs. (32) and (43) the spectrum $\alpha(\omega)$ contains two parts, a "coherent" δ-function spectrum and an "incoherent" broad spectrum [4]. The "incoherent" part of the laser-gyroscope beat-signal spectrum is presented in Figs. 1–3 for different values of the parameters a/b and γ/b. Our exact results coincide with those given in Ref. [4]. One can see by inspection of these figures that the effective-eigenvalue method gives a good quantitative description of the main features of the spectrum $\alpha(\omega)$ in all regions of interest [1,3–5]: $|a|<b$ (the so-called locked region), $|a|\lesssim b$ and $|a|>b$ (the unlocked region). As expected at the noise-free (or strong-backscattering) limit ($|\gamma|\gg 1$), the approximate solution in the unlocked region gives only a qualitative description of the second-harmonic contribution to $\alpha(\omega)$ and does not describe higher harmonics at all (see Fig. 3). This is due to our truncation of the hierarchy (22) at the second level, which leads to the appearance of two harmonics in the spectrum only while the noise-free spectrum of the gyroscope consists of an infinite number of harmonics [1,3]. The effective-eigenvalue method may, however, be applied in this instance as well if we truncate the hierarchy (22) at a higher level. In particular, in order to give a quantitative description of the second-harmonic band and explain qualitatively the third harmonic, one would have to truncate the hierarchy at the third level and calculate the effective eigenvalue λ_3^{ef} for $C_3(t)$.

V. CONCLUSIONS

The purpose of this paper is to apply the effective-eigenvalue method to the ring-laser gyroscope. We find that it yields an analytical expression (43) which describes the main features of the beat-signal spectrum in all regions of interest (with the exception of the second- and higher-harmonic contributions to the spectrum for large γ) and agrees closely with the exact solution. We also present an alternative form of the exact solution of the problem which is much simpler for numerical calculations in comparison with the previously reported algorithm [4].

*Author to whom correspondence should be addressed.

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