puts of each module in a special way to cause the output of the total algorithm to be in proper order [1], [9]. This results in the fastest algorithm, but the program must be written for a specific length. A fourth method is similar, but achieves the unscrambling by properly choosing the mulitplier constants in the modules [10]. A program based on this method could transform many different lengths by using a stored table of appropriate multipliers. The fifth method uses a separate indexing method for the input and output of each module [1], [11]. This results in a slightly slower program than methods three or four, but it is compact and can transform all lengths possible from the combination of modules.

Algorithms for $N = 2^m$

A Cooley–Tukey radix-2 FFT that is calculated in place produces the output in bit-reversed order. The unscrambler usually uses a bit-reverse counter [1], [12] or nested loops [3]. A particularly efficient form has been programmed by Morris [13]. A fast unscrambler has been developed by Evans [14], but it requires a precalculated "seed" table.

The normal in-place Cooley–Tukey FFT does not allow the unscrambling to be done in the modules (butterflies) as is possible with the PFA. That is an intrinsic property of the type of index map used and the fact that the module lengths are not relatively prime. It is possible to design an in-place, in-order radix-2 FFT if the butterflies are calculated two at a time and a special indexing used. This is described in [15], [16] and a structure is shown in [12].

The important observation contained in this correspondence is that a radix-4 or any radix-2^m FFT can be modified so the output is in bit-reversed order. If a normal radix-4 FFT with the output of each butterfly in normal order is used, the output occurs in base-4 reversed order, and similarly for radix-8 and others. However, if for the radix-4 FFT, the short length-4 butterflies are modified to have their outputs in bit-reversed order, the output of the total radix-4 FFT will be in bit-reversed order, not base-4 reversed order. Likewise, if the output of the length-8 butterflies in a radix-8 FFT are placed in bit-reversed order, the output of the total radix-8 FFT will be in bit-reversed order. This allows converting the output order of any radix-2^m FFT to bit-reversed order so that a single bit-reversed counter can be used as an unscrambler.

The structure of the signal flow graph for an FFT with radix-4, 8, 16, $\cdots$, 2^m which is modified for bit-reverse ordered output or for the form of split-radix FFT given in [6] is identical to that for a radix-2 FFT. The only difference in these algorithms is the number and location of the twiddle factors which, of course, change the structure of the program and the speed of execution. The effects of moving the twiddle factors are analyzed in [17] and [18].

Conclusions

We have shown that there are several methods that implement or eliminate the unscrambling process for the PFA, but they each require some compromise. The fastest and simplest method requires that the program be written for a specific length.

For a Cooley–Tukey FFT with a radix which is a power of two, it was shown that a simple modification of the butterflies gives the output in bit-reversed order as a radix-2 does. This can be applied not only to radix-4, radix-8, etc., FFT's, but to programs that mix radix-4 or radix-8 stages to achieve high efficiency with radix-2 stages to allow all power-of-two lengths. The output of the split-radix FFT is also allowed to be in simple bit-reversed order. It should be important in allowing a single special-purpose hardware bit-reversed counter to unscramble the more efficient radix-4, radix-8, and mixed radix FFT's.

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A Note on the Convergence Analysis of LMS Adaptive Filters with Gaussian Data

J. B. FOLEY AND F. M. BOLAND

***Abstract*—Necessary and sufficient conditions for the convergence of LMS adaptive filters with Gaussian data have been established by Horowitz and Senne [3], with the recent support of Feuer and Weinstein [4]. A feature of both of these studies is the necessity to investigate bounds on the roots of rather unwieldy characteristic equations. This note shows how such an investigation can be avoided through the use of a theorem of Gantmacher [5]. In formally applying this theorem, similar results to those of the above studies are obtained in a precise and straightforward manner.**

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The authors are with the Department of Microelectronics and Electrical Engineering, Trinity College, Dublin 2, Ireland.

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I. Introduction

The need to study the stability of LMS adaptive filters through analyzing the convergence properties of a second-moment parameter, such as the mean-square error (MSE), has long been recognized [1], [2]. However, oversimplification in expanding fourth-moment data statistics has meant that these contributions contain rather imprecise expressions for bounds on the maximum step size of the LMS algorithm and the residual excess MSE. The analysis of Horowitz and Senne [3] corrected this imprecision by making the assumption that the reference data have a Gaussian distribution, and then employing the well-known Gaussian fourth-moment expansion. The use of this assumption can be justified on the grounds that in many practical applications, particularly those pertaining to acoustic noise cancellation where the interfering noise may in effect result from a number of weakly correlated point sources, the reference noise statistics will indeed approximate to the Gaussian.

The basic methodology of Horowitz and Senne was to set up a time recursion for the diagonal elements of the weight-error covariance matrix. Cross coupling between this set of equations and an expression for the excess MSE was removed algebraically in the complex frequency domain. A stability criterion was established from the requirement that the roots of the characteristic equation must not lie outside the unit circle. Given the rather unwieldy nature of the characteristic equation in question, the application of this requirement was somewhat involved.

The results of Horowitz and Senne were subsequently confirmed by Feuer and Weinstein [4] who worked entirely with a vector formulation of the recursion for the diagonal elements of the weight-error covariance matrix. In seeking a stability bound, they sought to establish conditions which would ensure that the eigenvalues of the matrix in the vector recursion would be less than unity in magnitude. Again, given the complex nature of the appropriate characteristic polynomial, there ensued some necessary support discussion pertaining to the location of the roots of the polynomial.

The purpose of the present note is to show how the Horowitz–Senne result can be established with clarity and straightforward precision by reference to a theorem of Gantmacher whose definitive work [5] is frequently used by and quoted in the literature of signal processing engineers. The formal proof of the theorem is omitted here, but is definitively set out in the quoted reference. Our general formulation and approach to the problem are similar to those of Feuer and Weinstein.

II. Problem Formulation

The basic problem under consideration is that of an adaptive noise canceller driven by the LMS algorithm of Widrow [6]. From the perspective of convergence studies, the LMS algorithm can be expressed most conveniently as follows (see [3] for a complete derivation):

$$U(j+1) = \left[I - 2\mu Z(j)\, Z(j)^T\right] U(j) + 2\mu e(j)^* Z(j) \quad (1)$$

where

- $U(j)$ is the weight-error vector at time j in the rotated space defined by the eigenvectors of the reference noise autocorrelation matrix,
- $Z(j)$ is the reference noise data vector in the rotated space,
- $e(j)^*$ is the optimal error output, i.e., the error output when $U(j) = 0$,
- I is the identity matrix,
- μ is a positive constant known as the step size.

It may be noted that the reference noise autocorrelation matrix in rotated space is the diagonal matrix

$$\begin{aligned}\Lambda &= \operatorname{diag}\left[\lambda_1 \cdots \lambda_N\right] \\ &= E\left[Z(j)\, Z(j)^T\right] \quad (2)\end{aligned}$$

where the λ's are the eigenvalues of the original noise correlation matrix. Since this latter matrix is positive definite,

$$\lambda_i > 0 \qquad i = 1, \cdots, N. \quad (3)$$

Of principal concern is the asymptotic boundedness of the MSE:

$$\begin{aligned}\xi(j) &= E\left[e(j)^2\right] \\ &= \xi^* + \xi_{ex}(j) \quad (4)\end{aligned}$$

where ξ^* is the optimal MSE, and the excess MSE is given by

$$\begin{aligned}\xi_{ex}(j) &= E\left[U(j)^T \Lambda U(j)\right] \quad (5)\\ &= \operatorname{tr}\left\{\Lambda E\left[U(j)\, U(j)^T\right]\right\} \quad (6)\end{aligned}$$

with $\operatorname{tr}\{\cdot\}$ denoting the trace of the matrix.

Equation (6) shows that the excess MSE is a linear combination of the diagonal elements of the weight error covariance matrix

$$K(j) = E\left[U(j)\, U(j)^T\right] \quad (7)$$

and, consequently, many convergence studies are focused on the asymptotic properties of these elements.

Making the usual assumptions of independent data vectors and of a zero-mean Gaussian noise distribution, Horowitz and Senne employed (1) to derive the following recursion for the diagonal elements of K:

$$\begin{aligned}k_{ii}(j+1) = \left[1 - 4\mu\lambda_i + 8\mu^2\lambda_i^2\right] k_{ii}(j) + 4\mu^2\lambda_i \sum_{p=1}^{N} \lambda_p k_{pp}(j) \\ + 4\mu^2\xi^*\lambda_i \qquad i = 1, \cdots, N. \quad (8)\end{aligned}$$

Now, let

$$S(j)^T = \left[k_{11}(j) \cdots k_{NN}(j)\right] \quad (9)$$

and

$$L^T = \left[\lambda_1 \cdots \lambda_N\right]. \quad (10)$$

The system of equations described by (8) can be expressed in vector form

$$S(j+1) = AS(j) + 4\mu^2\xi^* L \quad (11)$$

where

$$A = I - 4\mu\Lambda + 8\mu^2\Lambda^2 + 4\mu^2 LL^T. \quad (12)$$

Clearly, the stability of the weight-error covariance matrix (and, in turn, the excess MSE) is governed by the requirement that the eigenvalues of the A matrix should be less than unity in magnitude.

III. Stability

Rather than attempt to directly determine the eigenvalues of A (as Feuer and Weinstein [4]), we refer to an appropriate theorem of Gantmacher [5, p. 88] on nonnegative matrices, i.e., matrices all of whose elements are greater than or equal to zero.

Theorem: A necessary and sufficient condition that the real number ρ be greater than the dominant eigenvalue of the nonnegative matrix A is that all the leading principal minors of the characteristic matrix

$$A_\rho = \rho I - A$$

be positive, that is,

$$\rho - a_{11} > 0$$

$$\det \begin{vmatrix} \rho - a_{11} & -a_{12} \\ -a_{21} & \rho - a_{22} \end{vmatrix} > 0$$

$$\cdots$$

$$\det \begin{vmatrix} \rho - a_{11} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \rho - a_{22} & \cdots & -a_{2N} \\ & & \cdots & \\ -a_{N1} & -a_{N2} & \cdots & \rho - a_{NN} \end{vmatrix} > 0. \quad (13)$$

This theorem is an amalgamation of an earlier theorem and a number of subsequent remarks and lemmas, all of which are rigourously proved. It is remarkable in that it provides conditions both necessary and sufficient to bound the dominant eigenvalue.

An off-diagonal term of matrix $\boldsymbol{A}$ as defined by (12)

$$a_{ik} = 4\mu^2\lambda_i\lambda_k \tag{14}$$

is greater than zero from (3), while a diagonal term can be written

$$a_{ii} = (1 - 2\mu\lambda_i)^2 + 8\mu^2\lambda_i^2 \tag{15}$$

which is also greater than zero. Thus, $\boldsymbol{A}$ is a nonnegative matrix. Since the existence of a dominant real eigenvalue is guaranteed by the Perron–Frobenius theorem [5], the conditions for applying the theorem quoted above are fulfilled.

To ensure the stability of (11), we consider the characteristic matrix

$$\boldsymbol{A}_1 = 4\mu\boldsymbol{\Lambda} - 8\mu^2\boldsymbol{\Lambda}^2 - 4\mu^2\boldsymbol{L}\boldsymbol{L}^T. \tag{16}$$

As will become apparent in the sequel, the most critical of the leading principal minors is that of dimension N. Denoting by $\boldsymbol{D}$ the diagonal matrix

$$\begin{aligned} \boldsymbol{D} &= \text{diag}\,[d_1 \cdots d_N] \\ &= 4\mu\boldsymbol{\Lambda} - 8\mu^2\boldsymbol{\Lambda}^2 \end{aligned} \tag{17}$$

we are concerned with the principal minor

$$\Delta_N = \det\,[\boldsymbol{D} - 4\mu^2\boldsymbol{L}\boldsymbol{L}^T]. \tag{18}$$

With $\det\,[\boldsymbol{L}\boldsymbol{L}^T] = 0$ eliminating many cross terms, we find on expansion

$$\begin{aligned} \Delta_N &= (d_1 d_2 \cdots d_N) - 4\mu^2(\lambda_1^2 d_2 d_3 \cdots d_N) \\ &\quad - 4\mu^2(d_1\lambda_2^2 d_3 \cdots d_N) \\ &\quad \cdots \\ &\quad - 4\mu^2(d_1 d_2 \cdots d_{N-1}\lambda_N^2) \qquad (19) \\ &= \det \boldsymbol{D} - 4\mu^2 \det \boldsymbol{D}[\boldsymbol{L}^T\boldsymbol{D}^{-1}\boldsymbol{L}]. \qquad (20) \end{aligned}$$

Making use of (17), we obtain

$$\begin{aligned} \Delta_N &= \prod_{p=1}^{N} [4\mu\lambda_p - 8\mu^2\lambda_p^2] \left\{ 1 - 4\mu^2 \sum_{i=1}^{N} \frac{\lambda_i^2}{4\mu\lambda_i - 8\mu^2\lambda_i^2} \right\} \\ &= (4\mu)^N \prod_{p=1}^{N} \lambda_p[1 - 2\mu\lambda_p] \left\{ 1 - \sum_{i=1}^{N} \frac{\mu\lambda_i}{1 - 2\mu\lambda_i} \right\}. \end{aligned} \tag{21}$$

The LMS algorithm requires a positive step size μ, while (3) ensures the positivity of λ_p. Hence, to ensure the positivity of Δ_N, we require

$$\mu < \frac{1}{2\lambda_i} \qquad i = 1, \cdots, N \tag{22a}$$

$$\sum_{i=1}^{N} \frac{\mu\lambda_i}{1 - 2\mu\lambda_i} < 1 \tag{22b}$$

which are the same conditions as those of [3] and [4].

It is clear that a similar procedure will yield

$$\Delta_{N-1} > 0 \qquad \text{iff} \sum_{i=1}^{N-1} \frac{\mu\lambda_i}{1 - 2\mu\lambda_i} < 1. \tag{23}$$

As a consequence of (22a), condition (22b) is more stringent than condition (23). Hence, (22a) and (22b) constitute the necessary and sufficient conditions for the convergence of the LMS algorithm.

IV. Conclusion

A clear-cut algebraic analysis of LMS-driven adaptive filters with Gaussian input data has been presented. From a similar starting point to that of [3] and [4]—a time recursion for the diagonal elements of the weight-error covariance matrix—necessary and sufficient conditions for the convergence of the excess MSE have been derived making use of a formal theorem of Gantmacher [5]. The results obtained are the same as those of the above studies.

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Comments on "Subband Coding of Images"

C. SANTHOSH KUMAR

In the above,[1] (2) for decimation by 2×2

$$\begin{aligned} Y_{ij}(\omega_1, \omega_2) &= \frac{1}{4} \sum_{k=0}^{1} \sum_{l=0}^{1} H_{ij}\left(\frac{\omega_1 + k\pi}{2}, \frac{\omega_2 + l\pi}{2}\right) \\ &\quad \cdot X\left(\frac{\omega_1 + k\pi}{2}, \frac{\omega_2 + l\pi}{2}\right) \end{aligned} \tag{2}$$

assumes sample period after decimation (i.e., $T' = 2T$ where T' and T are the new and old sample periods, respectively) because ω_1 and ω_2 are also divided by two [1, eq. (4)]. In (2), it appears that the new spectrum contains the old spectrum (before decimation) and the old spectrum shifted by $(\pi/2)$ rad along the ω_1 axis, ω_2 axis, and both ω_1 and ω_2 axes. This is not true for decimation by 2×2; rather it contains the old spectrum and the old spectrum shifted by π rad ($2\pi/M$ where M is the decimation factor) [1]. We feel that (2) has to be rewritten as

$$\begin{aligned} Y_{ij}(\omega_1, \omega_2) &= \frac{1}{4} \sum_{k=0}^{1} \sum_{l=0}^{1} H_{ij}\left(\frac{\omega_1 + 2k\pi}{2}, \frac{\omega_2 + 2l\pi}{2}\right) \\ &\quad \cdot X\left(\frac{\omega_1 + 2k\pi}{2}, \frac{\omega_2 + 2l\pi}{2}\right). \end{aligned} \tag{2'}$$

Using this new equation together with

$$U_{ij}(\omega_1, \omega_2) = Y_{ij}(2\omega_1, 2\omega_2)\, F_{ij}(\omega_1, \omega_2),$$

then only we obtain (3) of the above,[1] which, however, is correct. Thus, the analysis following (3) remains valid.

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The author is with the Digital Communications Group, Transmission R&D, Indian Telephone Industries Ltd., Bangalore 560 016, India.

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