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**A Forward Advancing Wave Expansion Method
for Numerical Solution of Large-scale Sound
Propagation Problems**

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A thesis submitted to the University of Dublin in partial fulfilment of the
requirements for the degree of

Doctor of Philosophy

Department of Mechanical and Manufacturing Engineering,
Trinity College Dublin,
February, 2007.

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Leandro Miguel Barrera Rolla,
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Summary

The study of atmospheric sound propagation has become an important subject since noise pollution problems emerged as a highly relevant matter in several areas such as sociology, economics, regulations and standards.

Modelling sound propagation over large domains represents a major challenge for current numerical tools due the large computational resources required to obtain accurate solutions.

In this thesis a “one-way” wave based field discretization method for solving the Helmholtz equation in large-scale problems is proposed and is referred to as the Forward Wave Expansion Method (FWEM). The FWEM is derived from a highly efficient discretization procedure based on interpolation of wave functions known as the Wave Expansion Method (WEM).

The Wave Expansion Method (WEM) is a very flexible, efficient full field method for solving the Helmholtz equation, which uses mesh densities as low as 3 nodes per wavelength and can model complicated ground topography, ground impedance inhomogeneities and inhomogeneous moving media.

The FWEM computes the propagated sound field by means of an exclusively forward advancing solution. This technique does not require the inversion of large system matrices and thus permits the solution of large scale acoustic problems where backscatter is not of interest. A computationally light model is thus proposed which retains many formulation advantages of WEM. At ranges where the propagating wave is nearly planar, the mesh spacing needed to obtain accurate results can be significantly reduced which further improves the overall computational efficiency.

The accuracy of the two dimensional implementation of the FWEM was tested for a number of benchmarking problems. Diffraction and refraction effects were included in the analysis. Comparisons to data computed with the standard WEM and with theoretical solutions (when available) show the forward approximation to be highly accurate using nodal spacings as low as 0.2 nodes per wavelength in the propagation direction and 3 nodes per wavelength in the quadrature (vertical) direction.

For the three dimensional implementation only the solution for the free propagation problem was compared with theoretical results. The method was successfully implemented to model diffraction effects in two propagating models. These models included continuous and finite barriers. Accurate results were obtained for a limited part of the computational domain. Further refinements, however, are required to make the FWEM a viable modelling tool in three dimensions.

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Leandro Miguel Barrera Rolla

Nomenclature

A	damping constant for the energy absorbing boundary layer
c	speed of sound $[m/s]$
$\mathbf{d}_n$	unit propagation direction vector of the n^{th} plane wave
f	frequency $[Hz]$
$\mathbf{f}$	forcing vector (vector of driving forces) $[Pa]$
G	Green's kernel function
$\mathbf{h}$	array of plane waves functions
$\mathbf{H}$	matrix of plane wave functions
$H_0^{(1)}$	zero order Hankel function of the first kind
$H_0^{(2)}$	zero order Hankel function of the second kind
j	imaginary unit
k	acoustic wavenumber $[1/m]$
$\mathbf{n}$	normal direction vector
M	number of neighbour points
N	number of plane waves directions used for the discretization
p	acoustic pressure $[Pa]$
$\mathbf{p}$	array of acoustic pressure values $[Pa]$
$\mathbf{r}$	general position vector $[m]$
r	scalar distance between two points $[m]$

$\mathbf{S}$	system's dynamic stiffness matrix
v	fluid velocity vector $[m/s]$
Z	specific acoustic impedance $[kg/m^2s]$
δ	Kronecker delta
γ	amplitude of a plane wave
λ	wavelength $[m]$
ρ_0	mass density of the air $[kg/m^3]$
ω	circular frequency $[rad/s]$
∂	partial derivative operator
∇^2	Laplace operator
$+$	generalised pseudo-inverse

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Chapter 1

Introduction

1.1 Relevance of modelling long-range atmospheric noise propagation

Noise pollution is a serious growing problem for many people worldwide, which causes annoyance, lowers quality of life and productivity. In severe cases it can lead to health problems and cognitive difficulties. Growing road, rail and air traffic tends to lead to strong adverse reactions from the citizens affected by it. Reliable evaluation and prediction of noise maps is therefore required in order to assess objectively the exact size of the noise problem and to monitor the efficiency of plans to control and reduce these problems. Consequently, assessment and prediction of outdoor sound propagation is important for decisions concerning environmental noise abatement plans and transport/industrial development.

1.2 Introduction to atmospheric acoustics

Sound propagation in the atmosphere is a complex phenomenon that is dependent on geometrical spreading, molecular absorption, turbulence and scattering. Some other mechanisms occur for waves near the ground, where a major factor is the fact that different ground surfaces differ greatly in their acoustical properties.

Wind and temperature profiles all affect how sound propagates through the atmosphere. Variations in sound speed across the medium cause refraction of the sound waves either upwards or downwards, which can either decrease or increase sound pressure levels significantly. If the medium is not stationary sound waves are convected by the mean motion of the field.

1.3 Scope of the thesis

Propagation of sound above a plane and heterogeneous ground in an homogeneous or stratified atmosphere has been extensively studied analytically, numerically and experimentally. The influence of turbulence and wind gradients has been investigated considerably for the case of sound propagation over plane unobstructed ground. However in complex environments, the modelling of outdoor sound propagation must take into account the mixed influence of the terrain topography and the meteorological conditions.

The two factors that determine the usefulness of a computational method as a predictive tool for atmospheric sound propagation are accuracy and computational cost. The accuracy will determine the reliability of the method, the computational cost will determine the viability of the numerical scheme.

There are a number of approaches available for exterior atmospheric noise propagation modelling. These can be grouped into two classes.

1. Semi-Empirical Methods
2. Physical Methods

Semi-empirical methods usually require relatively low computational resources and can be used to generate approximate noise maps over large areas. However, as they ignore phase information and rely on a number of approximations they are not accurate in many situations particularly those involving multiple reflections and diffractions and complex ground effects and geometries. In addition their frequency dependency is crude and they can only model gross atmospheric effects. However as sources are often crudely modelled these methods may be adequate for urban planning. On the other hand physical methods are much more powerful from a modelling perspective but they require significant computational resources. There is a continual need to improve this computational efficiency and Wave-based methods are promising in this regard.

The discretization method proposed in the present work, the Forward Wave Expansion Method (FWEM), is a development of a novel wave-based method, the Wave Expansion Method (WEM) that considerably reduces the computational resources required to obtain accurate results. The FWEM may be a significant step toward a more useful prediction method for long range outdoor sound propagation.

1.4 Structure of the thesis

Chapter 2 gives a review of the most widely used numerical methods for long-range problems. A general description of present semi-empirical models followed by a brief description of the Fast Field Program (FFP) and Ray trace techniques. Special consideration is given to the most commonly accepted techniques for this kind of sound propagation problems, the Boundary Element Method (BEM) and the Parabolic Equation Methods (PEM). A complete description of the Wave Expansion Method (WEM) is also included since it constitutes the base of the method introduced in the present work, the Forward Wave Expansion Method (FWEM).

The implementation of the FWEM for two-dimensional problems is described in chapter 3. A detailed explanation of the modelling issues encountered during the development of the technique is presented. A number of benchmark sound propagation problems are presented to evaluate the accuracy and flexibility of the FWEM.

In chapter 4, the implementation of the FWEM is extended to three-dimensional propagation problems. The results for some propagation problems are discussed.

Chapter 5 presents the general conclusions. Some issues for future development in the technique are proposed.

Chapter 2

Review of numerical methods for long-range wave propagation problems

2.1 Introduction

There is a continuous search for accurate methods to model outdoor sound propagation. These methods should take into account geometric divergence, refraction of sound, diffraction by obstacles, wind and turbulence effects, complicated ground topography and impedance discontinuities. Different source distributions and realistic source directivity should also be considered in the models.

Different methods have been developed to compute sound propagation under some of these conditions. The resulting model is always an *approximation*, a simplified version of the real world. The accuracy of the result will depend on the approximations used in the derivation of each modelling method and

different applications will require different levels of accuracy. The accuracy of the result is highly dependent on the accuracy of the input data.

Many numerical methods have been, and are still being under development in a search for an accurate, reliable and efficient solution to sound propagation in complex media, i.e. Ray Trace techniques, Fast Field programs, Boundary Element methods, Finite Element techniques, Parabolic Equation Methods, etc.

A brief description of the most widely used methods for modelling noise propagation in the atmosphere is given in the following sections. Descriptions will be limited to outlines of basic formulations. Special consideration is given to the Boundary Element Method and the Parabolic Equation Methods as these are the most commonly accepted techniques for this kind of sound propagation problems. A complete description of the Wave Expansion Method is also included since it constitutes the base of the method introduced in the present work, the Forward Wave Expansion Method.

2.2 Semi-empirical Models

Due to the limitations of physically based numerical methods, semi-empirical models based on geometrical acoustics, are normally used for assessment and management of environmental noise. Most of these methods consist in point to point calculations. That is, at certain receiver position, the sound pressure (or the equivalent SPL) produced by a source is calculated by incoherent summation over the contributions from a number of propagation paths. To account for ground reflections, an approximation for the spherical wave reflection at an impedance plane is usually applied.

The sources are usually represented as a number of incoherent point sources. In the case of road and rail noise propagation problems, each section of the road (or rail) is modelled as a line of point sources. A different approach is to consider the actual source as a line source. The line source may then be broken into a number of sections, each treated as individual line sources. Another technique is to model each source as a finite set of incoherent point sources, allowing the method to include some simplified directivity pattern of the source.

A number of “correction” terms are usually applied to the propagation paths to include effects of the atmosphere, the ground surface, and diffracting obstacles into the models. The surface of the ground between sources and receivers is usually assumed to be uniformly hard or soft, or fibrous.

Some examples of these methods are the ISO9613-2 (International Organisation for Standardisation) [5], the Nord2000 (Nordic Region’s Standard) [6], the XP S 31-133 (French National Standard), the CRTN (UK National Standard), the FHWA (USA), the RLS90 (Germany), ASJ Method - 1993 (Japan), etc. Steele [7] presented an interesting review of some of these methods. These calculation schemes differ with respect to reliability and accuracy. The need for an “standardised” empirical modelling method comes from the fact that application of different empirical methods return different results. The selection of the “best” method to use is now an issue in itself. An example of this was the *HARMONOISE* project [8], to find standardised methods for noise mapping in all EU Member States. The main purpose of this project (Aug 2001 - Jan 2005) was to develop “unified”

prediction methods for road and railway noise. In this project empirical modelling methods are considered as “engineering methods”. These methods are used together with more “physical” methods (Parabolic Equation methods coupled with Boundary Element and Ray Tracing Methods) as reference models for “calibration” purposes. The physical methods are used only for calibration purposes due to the fact that their applicability is restricted to highly idealised problems and/or the computational resources required are still prohibitive for many of the conditions encountered in actual problems.

2.3 Fast Field Program

The fast field program (FFP) technique was developed for the prediction of underwater sound propagation and was later adapted to propagation in the atmosphere [9–11]. The FFP is based on a Fourier transformation of the wave equation from the spatial domain to the wave number domain. The transformed wave equation is solved numerically and the result is transformed back to the spatial domain.

FFP methods are conditioned to range-independent horizontally stratified atmosphere problems. The ground boundary has to be represented by a flat ground with range-independent properties. The effective sound speed can be an arbitrary function of height above the ground.

The FFP works numerically from an exact integral representation of the sound field within each layer in terms of coefficients that may be determined from boundary conditions. This method is based on the Hankel transform of the Helmholtz wave equation. The solution of the transformed equations is

referred to as the height-dependent pressure. The inverse Hankel transform for the physical sound pressure is approximated by two Fourier integrals. The method gets its name from the fast Fourier transform used to evaluate these integrals.

The FFP is a powerful technique for sound propagation modelling in a range-independent environment. If the ground is uneven as, for example, in the presence of a hill or a sound barrier; or if the atmospheric conditions are range-dependent the FFP methods fail.

2.4 Ray trace techniques

Ray tracing is a well known algorithm in acoustic modelling, specially in the area of room acoustics [12–16]. Ray trace techniques are based on geometrical acoustics, where the sound is supposed to act like rays. This assumption is valid when the wavelength of the propagated sound is small compared to the area of surfaces and large compared to the roughness of surfaces.

Ray model techniques consist on the calculation of the paths of all sound rays between the source and the receiver, and the computation of the sound pressure by summation of the contributions of these sound rays.

There are several variations of the algorithm. In the basic algorithm the sound source emits sound rays, which are then reflected at surfaces according to their specular reflection and the sound pressure at the receiver is computed as an incoherent sum of the rays that intersect the receiver. The receivers are typically spheres of proper radius, and the detection mechanism make it possible to compute the sound energy density inside the re-

ceiver volume. The way sound rays are emitted can either be predefined or randomised. A typical choice is to have a uniform distribution of rays over a sphere. With the use of a predefined distribution of rays an accurate solution can be achieved with fewer rays.

The main drawback of ray tracing methods is that they may fail to find all significant propagation paths containing wedge diffractions [17]. The receiver position and diffracting edges are often approximated by volumes of space (in order to enable intersections with infinitely thin rays), which can lead to false hits and paths counted multiple times [18]. Moreover, important propagation paths may be missed by all samples. In order to minimise the likelihood of large errors, ray tracing systems often generate a large number of samples, which requires a large amount of computation [19].

Similar to the ray tracing technique are the beam tracing methods [12, 13, 20] such as cone beam tracing and pyramid beam tracing. These methods find propagation paths from a source by tracing beams through the medium. For each beam, obstacles are tested for intersection with the beam in front-to-back visibility order. As intersecting obstacles are detected, the original beam is truncated to remove the shadow region and a new beam is created in a similar fashion to ray-tracing.

The primary advantage of beam tracing is that it leverages geometric coherence, since each beam represents an infinite number of potential ray paths emanating from the source location. The disadvantage is that the geometric operations required to trace beams through a 3D model (i.e., intersection and clipping) are relatively complex, as each beam may be reflected and/or

obstructed by several surfaces [19]. As for ray tracing, diffraction contribution modelling is a problem for beam tracing methods [20]. Though the beam tracing methods may represent an attractive alternative to ray tracing techniques for some problems, they are very time consuming, and in many cases the pure ray-tracing method is still the best choice [14].

Ray tracing has been used to predict outdoor sound propagation [17, 20–27]. Speed of sound gradients can be included in the models even though the mathematical conditions for ray solutions are seldom met in the real life problem [24]. In most cases the sound speed and wind speed are assumed to vary linearly with height. Therefore, the ray models may not be very useful for accurate computations of the sound pressure field, particularly in situations with irregular sound speed profiles.

2.5 Boundary Element Methods

The Boundary Element Method (BEM) is a numerical method for field problems based on boundary integral equations. The main advantage of the BEM over other methods like the widely known finite element methods (FEM) or the finite differences (FD) techniques, is that it requires discretization of the surface instead of a discretization of the whole domain. This considerably reduces the computational requirements, especially in the field of aeroacoustical modelling, that usually deals with problems with a high ratio of volume to surface area. For large scale outdoor sound propagation modelling, methods like the FEM and the FD are inapplicable while the BEM remains an attractive solution for some problems. The BEM can easily accommodate complex boundary geometries in the solution.

Another important characteristic that makes BEM a very useful numerical tool for outdoor sound propagation modelling is that radiation conditions can be naturally imposed. The use of BEM in acoustic applications include, for example, traffic noise control, interior car and aircraft cabin insulation and turbofan noise simulation. The BEM has been widely used for noise barrier performance assessment [28–35].

There are two kinds of boundary element methods, the direct boundary element method, based on the classical Helmholtz integral equation, and the variational indirect BEM. The indirect approach solves the Helmholtz equation in both the interior and the exterior domain simultaneously. An extended analysis of boundary element methods in acoustics can be found in references [36–38]. The basic formulation and implementation of the method is described in the Appendix A.

The BEM can be used successfully for the prediction of acoustic propagation and scattering in homogeneous media. Using Sommerfeld radiation condition, free field boundary conditions are satisfied exactly. This means simple and accurate modelling of problems involving infinite and semi-infinite domains.

There is no extra difficulty introduced in the method by considering impedance discontinuities, complicated ground topography, sound barriers of different shapes and materials, etc. On the other hand, difficulties arise when characteristics like inhomogeneous atmospheres have to be introduced in the model. Premant and Gabillet [39] developed a Green's function tech-

nique for predicting sound propagation over a sound barrier in the presence of downward refraction (Meteo-BEM) by including meteorological effects in the Green's function. The choice of this particular high dependent Green's function allows only media with a positive constant sound speed gradient to be studied. This approach increases the computational cost of the method. Iterative techniques, like GMRES, can be applied to alleviate the matrix storage requirements. These methods are sometimes difficult to implement, this depends on the problem matrix characteristics.

Another extension of the BEM, which accounts for a refracting atmosphere as well as a non-uniform boundary was introduced by Taherzade *et al.* [40]. This time the Greens function is evaluated using the fast field program method. This method is known as BIE/FFP and represents considerable improvements on previous applications of BEM to barriers with a refracting atmosphere. The BEM has also been applied to transport noise problems in three dimensions, however this increases the computational times significantly.

It is important to note that the computational advantage over full field methods only applies when direct solutions of the system matrix equation is performed. When iterative techniques are used comparable efficiencies may be restored when fast multipole methods are integrated into the BEM formulation.

A very efficient boundary method in which the local basis are constructed using wave functions has been presented by Desmet *et al.* [41–43] for vibro-acoustic modelling, called the Wave-Based prediction Technique (WBT).

The dynamic response variables are described by a set of wave functions which exactly satisfy the governing differential equations. This allows a reduced nodal density to be used for the boundary discretization which could potentially be an attractive alternative for large scale problems. To the best of the authors knowledge it has yet to be applied to model outdoor sound propagation.

In general BEM calculations are still very computationally demanding as the system matrices are fully populated and not always symmetric.

2.6 Parabolic Equation Method

One of the most accepted and widely used physically based method for large scale wave propagation problems is the Parabolic Equation method (PE). In acoustics, the PE approximates the elliptic Helmholtz equation with a parabolic equation by neglecting the backscattered field. The primary motivation for the PE generally has been to treat wave propagation in a complicated inhomogeneous medium. The advantage of the PE approximations for long-range problems is that a parabolic differential equation can be solved marching in range whereas the elliptic equation must be numerically solved in the entire range-depth region simultaneously.

The PE approximation was introduced in the 1940's by Leontovich and Fock to solve electromagnetic long-range wave propagation problems. Thereafter, the PE has been applied and developed to solve wave propagation problems in numerous scientific areas. In the early 1970's, Tappert introduced the PE approximation method to the ocean acoustic community. Since then,

a profuse number of developments have been carried out to improve the accuracy and efficiency of the approximation [2, 44–53], and to expand its applicability to more realistic conditions [54–61]. The parabolic equation was originally limited to narrow-angle propagation; the energy outside the angle of propagation is neglected. The narrow-angle PE was later improved to handle wide-angle propagation problems. An ample account of the historical development of the method for underwater acoustics has been presented by Lee *et al.* [62].

The following is the basic derivation of the parabolic equation.

2.6.1 Parabolic approximation

The Helmholtz equation for the acoustic pressure, p , is

$$\nabla^2 p + k^2 p = 0 \quad (2.1)$$

where $k_{(r,z)} = \omega/c_{(r,z)}$ is the wave number, ω is the circular frequency and $c_{(r,z)}$ is the sound speed.

Equation 2.1 can be rewritten in cylindrical coordinates as:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0$$

For a problem with azimuthal symmetry the above equation becomes:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0 \quad (2.2)$$

Assuming a solution of the form:

$$p = \varphi_{(r,z)} \cdot \nu_{(r)}$$

Equation 2.2 becomes

$$\left[\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} \right] \varphi + \left[\frac{\partial^2 \varphi}{\partial r^2} + \left(\frac{1}{r} + \frac{2}{\nu} \frac{\partial \nu}{\partial r} \right) \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + k^2 \varphi \right] \nu = 0$$

Through the choice of a constant reference wavenumber, k_0 , the elliptic wave equation can be decomposed into two equations:

$$\left[\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} \right] = -\nu k_0^2 \quad (2.3)$$

and

$$\left[\frac{\partial^2 \varphi}{\partial r^2} + \left(\frac{1}{r} + \frac{2}{\nu} \frac{\partial \nu}{\partial r} \right) \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + k_0^2 n^2 \varphi \right] = \varphi k_0^2, \quad (2.4)$$

k_0 is related to $k_{(r,z)}$ by $k_{(r,z)} = k_0 n_{(r,z)}$. $n_{(r,z)}$ is the index of refraction defined as $n_{(r,z)} = c_0/c_{(r,z)}$ where $c_{(r,z)}$ is the sound speed and c_0 is the reference sound speed.

Rearranging terms, the equations become:

$$\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r} \frac{\partial \nu}{\partial r} + k_0^2 \nu = 0 \quad (2.5)$$

and

$$\frac{\partial^2 \varphi}{\partial r^2} + \left(\frac{1}{r} + \frac{2}{\nu} \frac{\partial \nu}{\partial r} \right) \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + k_0^2 n^2 \varphi - k_0^2 \varphi = 0 \quad (2.6)$$

Equation 2.5 can be regarded as a second-order ordinary differential equation in the range variable r [62]. It has a solution conformed by two waves, an outgoing wave involving the zeroth-order Hankel function of the first kind, $H_0^{(1)}(k_0 r)$, and an incoming wave involving the zeroth-order Hankel function

of the second kind, $H_0^{(2)}(k_0 r)$. Neglecting the incoming wave, $\nu(r)$ has the solution

$$\nu \approx H_0^{(1)}(k_0 r)$$

For $k_0 r \gg 1$ (far-field approximation), the solution for $\nu(r)$ can be written as:

$$\nu \approx \sqrt{\frac{2}{\pi k_0 r}} \cdot e^{j(k_0 r - \pi/4)} \quad (2.7)$$

which is the asymptotic expansion for large arguments [63]. If we substitute equation 2.7 into 2.6 we get

$$\frac{\partial^2 \varphi}{\partial r^2} + 2jk_0 \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + k_0^2 (n^2 - 1) \varphi = 0 \quad (2.8)$$

The resulting form is still an elliptic equation and is only valid in the far-field. It is from this far-field elliptic equation that the parabolic wave equation is obtained.

Assuming commutation of operators (small variations of φ in r and z):

$$\frac{\partial}{\partial r} \frac{\partial}{\partial z} \varphi = \frac{\partial}{\partial z} \frac{\partial}{\partial r} \varphi$$

Equation 2.8 can be factored as:

$$\left(\frac{\partial}{\partial r} + jk_0 - jk_0 \sqrt{1 + Q} \right) \left(\frac{\partial}{\partial r} + jk_0 + jk_0 \sqrt{1 + Q} \right) \varphi = 0$$

where

$$Q = \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + (n^2 - 1) \quad (2.9)$$

By considering only the outgoing wave, equation 2.8 reduces to:

$$\frac{\partial \varphi}{\partial r} = jk_0 \left(\sqrt{1 + Q} - 1 \right) \varphi \quad (2.10)$$

The square-root pseudo-differential operator requires further approximation. Different approximate solution methods exist that derive different forms of parabolic equations.

2.6.2 Standard PE

Tappert derived the parabolic equation assuming that the first term of equation 2.8 can be neglected as $\varphi_{(r,z)}$ usually varies slowly with r (paraxial approximation). Equation 2.8 then reduces to

$$2jk_0 \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + k_0^2 (n^2 - 1) \varphi = 0 \quad (2.11)$$

which is the narrow-angle parabolic wave equation.

An alternative derivation can be obtained approximating the square-root operator by the following expansion

$$\sqrt{1+Q} \approx 1 + \frac{Q}{2} - \frac{Q^2}{8} + \dots$$

If only the first two terms of the expansion are used for the approximation equation 2.10 becomes

$$\frac{\partial \varphi}{\partial r} = jk_0 \frac{Q}{2} \varphi \quad (2.12)$$

that is:

$$\frac{\partial \varphi}{\partial r} = \frac{jk_0}{2} \left(\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + (n^2 - 1) \right) \varphi$$

Rearranging terms we get:

$$\frac{\partial^2 \varphi}{\partial z^2} + 2jk_0 \frac{\partial \varphi}{\partial r} + k_0^2 (n^2 - 1) \varphi = 0$$

which is exactly the same as equation 2.11.

Equation 2.11 is the original PE introduced to the ocean acoustics community by Tappert and is recognised as the Standard PE. The solution of the Standard (narrow-angle) PE is accurate up to elevation angles of about $\leq 10^\circ$. This restriction is imposed basically by the paraxial assumption.

2.6.3 Wide-angle PE

The narrow-angle PE was derived from equation 2.10 using the first two terms of the polynomial approximation to the square-root operator. For many propagation problems, the narrow-angle PE is not sufficiently accurate. A wide-angle PE [44] can be obtained if the quadratic term of the polynomial is also included in the approximation to the square-root operator. This is

$$\sqrt{1+Q} \approx 1 + \frac{Q}{2} - \frac{Q^2}{8} + \dots$$

Equation 2.10 becomes

$$\frac{\partial \varphi}{\partial r} = jk_0 \left(\frac{Q}{2} - \frac{Q^2}{8} \right) \varphi \quad (2.13)$$

Equation 2.13 allow propagation up to elevation angles of about 40° .

Alternative wide-angle PE equations can be obtained if the square-root operator is approximated by the rational function

$$\sqrt{1+Q} = \frac{a_0 + a_1 Q}{b_0 + b_1 Q} \quad (2.14)$$

where a_0 , a_1 , b_0 and b_1 are arbitrary constants. Different choices of this co-

efficients result in different pseudo-partial differential equations of the form

$$\frac{\partial \varphi}{\partial r} = jk_0 \left(\frac{a_0 + a_1 Q}{b_0 + b_1 Q} - 1 \right) \varphi \quad (2.15)$$

Using $a_0 = 1$, $a_1 = 0.5$, $b_0 = 1$ and $b_1 = 0$, equation 2.15 is equivalent to the narrow-angle PE (Eqn. 2.11).

Claerbout [64] introduced the following rational approximation ($a_0 = 1$, $a_1 = 0.75$, $b_0 = 1$ and $b_1 = 0.25$) for the approximation of seismic waves:

$$\sqrt{1+Q} \approx \frac{1 + \frac{3}{4}Q}{1 + \frac{1}{4}Q}$$

Expanded in a power series in Q , this rational approximation agrees with the expansion of $\sqrt{1+Q}$ through the quadratic term. Equation 2.10 becomes

$$\left(1 + \frac{Q}{4} \right) \frac{\partial \varphi}{\partial r} = ik_0 \frac{Q}{2} \varphi \quad (2.16)$$

This is Claerbout's wide-angle parabolic equation. The main reason for using the rational approximation instead of the polynomial one is that the first one leads to a tridiagonal matrix equation for a PE step, while the second one would lead to a matrix equation with more nonzero elements.

Gilbert and White [65–67] introduced an implementation of Claerbout's wide-angle PE for atmospheric acoustics. Their method is based on an implicit Crank-Nicholson extrapolation of the sound field, and therefore is referred to as the Crank-Nicholson parabolic equation method (CNPE). The horizontal and vertical grid spacings Δr and Δz in the CNPE method are limited to a maximum value of about $\lambda/10$, where λ is the wavelenght of the

sound wave [65, 68]. Two numerical approaches are commonly applied for the solution of the CNPE, a finite-difference approach [65, 66] and a finite-element discretization [67]. In the finite-element approach it is possible to use a non-uniform vertical grid spacing, therefore, small scale vertical variations in both density and sound speed can be more accurately treated.

Gilbert and Di [69] developed a “fast PE method” for computing sound propagation in the atmosphere, the Green’s function parabolic equation method (GFPE). The GFPE method is more efficient than the Crank-Nicholson PE method. This method is derived from the split-step Fourier parabolic equation method (SSPE) developed for underwater propagation problems [46]. The SSPE method is more efficient than the CNPE method because larger range discretization steps can be used. However, the SSPE give inaccurate results when large impedance discontinuities occur. In atmospheric acoustic problems the impedance discontinuity at the air-ground interface is much larger than any encountered in ocean acoustics, hence, the standard SSPE method cannot be used directly [69]. The GFPE method incorporates the boundary condition at the air-ground interface into the solution. The approach used is a split-step Fourier method derived from a general Green’s function solution to the one-way wave equation. In the GFPE method, the vertical grid must be of equal spacing. This mesh spacing is limited to about $\lambda/10$. The horizontal grid spacing may be chosen considerably larger, usually of about 10λ [68]. The derivation by Gilbert and Di is based on the spectral theorem of functional analysis.

The basic equations for the GFPE method are:

$$\psi_{(r+\Delta r, z)} = e^{\frac{j\Delta r \delta k^2}{2k_r}} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{\psi}_{(r, k')} + R_{(k')} \tilde{\psi}_{(r, -k')} \right) e^{j\Delta r \sqrt{k_r^2 - k'^2} - k_r} e^{jk'z} dk' + 2j\beta \tilde{\psi}_{(r, \beta)} e^{j\Delta r \sqrt{k_r^2 - \beta^2} - k_r} e^{-j\beta z} \right) \quad (2.17)$$

where

$$\tilde{\psi}_{(r, k')} = \int_0^\infty e^{jk'z'} \psi_{(r, z')} dz'$$

is the spatial Fourier transform of $\psi_{(r, z)}$. In the equations, $\psi = \sqrt{r}p$, Δr is the finite-difference range step, $R_{(k')} = (k'Z_g - k_0)/(k'Z_g + k_0)$ the plane-wave reflection coefficient, with Z_g the normalised ground impedance and k_0 the wave number at zero height, and $\beta = k_0/Z_g$ is the surface-wave pole in the reflection coefficient. The squared wavenumber k^2 is split into a constant contribution (range dependent but constant at each range step), k_r^2 , and a contribution $\delta k_{(z)}^2$ that varies with height:

$$k_{(z)}^2 = k_r^2 + \delta k_{(z)}^2$$

The right hand side of equation 2.17 is the product for an exponential refraction factor and the sum of three terms. The first term represents the direct sound wave, the second term represents the sound wave reflected by the ground, and the third term represents the surface wave. The fast Fourier transform (FFT) is used to implement equation 2.17.

The splitting method used by Gilbert and Di can cause some phase errors in the wide angle components of the field. An improved Green's function parabolic equation method was presented by Salomons [70]. Salomons reported that different numerical implementations of the GFPE method de-

rived in considerably different numerical accuracies. The accuracy of the GFPE method, at fixed values of the range step and vertical spacing, can be improved by using an improved approximation for the forward and inverse Fourier integrals based on the midpoint rule of numerical integration. Salomons reported that the maximum acceptable range step required by the improved GFPE varies roughly between 5λ and 50λ for refracting atmosphere problems (typically 10λ [68]). The vertical mesh spacing should not exceed about $\lambda/10$ [68]. These results were obtained for regular meshes in rectangular domains. Salomons stated that the accuracy of the standard approximation for the Fourier integrals can always be improved by using a smaller vertical grid spacing, hence, in a sense, the improved approximation improves the efficiency rather than the accuracy of the method. Salomons derived equation 2.17 from a Kirchhoff-Helmholtz integral equation. A different exponential refraction factor that improve the accuracy of the method was implemented in equation 2.17:

$$e^{\frac{j\Delta r \delta k^2}{2k_r}} \rightarrow e^{j\Delta r \delta k_{(z)}}$$

It should be noted that the GFPE method yields a solution for the sound field that may be considered as a solution of the narrow-angle PE. The GFPE is less accurate than the CNPE method in situations with wide-angle propagation and large sound speed gradients.

An alternate parabolic equation method with large extrapolation steps was presented by Sack and West [71], the Lagrange Padé parabolic equation (LPPE). The LPPE is based on a Lagrange series expansion of implicit functions. The power series are converted to Padé approximants. The method

can be interpreted as a sequence of CN-like steps with varying parameters. These parameters have to be determined numerically, and they depend on Δr . The implementation of the field solution is best performed using a set of sub-ranges with increasing order of the Padé approximation and increase in horizontal step size. The vertical discretization, Δz , remains the same as in the standard CNPE.

2.6.4 Numerical solution techniques

The numerical solutions of the parabolic equations derived in the previous sections can be obtained by the following numerical techniques: the Split-Step Fourier algorithm, the finite-difference technique [47, 65, 66], the Ordinary-differential-equation approach [45], the Finite-element discretization method [67], and Padé approximations [2–4, 51, 52, 61, 72].

The first numerical technique used to solve the parabolic equation was the split-step Fourier algorithm [73] introduced by Tappert. The split-step algorithm is an efficient method for solving initial-value problems, but it lacks flexibility for boundary modelling. It was already mentioned that this approach fails to give accurate results when large impedance discontinuities occur. Thus, a more general purpose solution is required. For this reason, implicit finite-difference and finite-element schemes, and Padé approximations methods have been developed to solve the PE. The three methods are described in Appendix B.

Recent modelling developments on PE methods have been directed at refining and expanding the capabilities of these existing techniques.

2.6.5 Upper attenuating stratum

The numerical grid is truncated at height $z = z_M$. To simulate radiation boundary conditions at infinity, reflections from the top of the grid must be prevented. Such boundary conditions can be approximated by adding an artificial sound-absorbing layer at the top part of the sound-speed profile. To implement this, the impedance of the top layer ($z = z_M$) has to be equal to ρc (ρ is the density of the air), which implies that the plane-wave reflection coefficient is equal to zero for normal incidence. Below the top layer, at grid points within the absorbing layer, a gradually increasing attenuation is introduced by adding an imaginary term $jA [(z - z_t) / (z_M - z_t)]^2$ to the wave number, k , where A is a constant, and z_t is the height of the bottom of the absorbing layer. It is most important that the damping is not applied too quickly as this could produce reflections back into the lower part of the atmosphere. The configuration of the artificial absorbing layer so that the reflection back into the computational region are minimised can not be determined by any straightforward analytical procedure [66]. An important parameter for the absorbing layer is the thickness of the layer, namely, $z_t - z_M$. As a general rule, the minimum thickness of the absorbing layer should be of around 30 wavelengths. The attenuation absorbs sound reaching the top of the numerical grid so that the amplitudes of waves reflecting off the top end of the grid are greatly reduced.

The sound field should not be influenced by the absorbing layer, so the height z_t of the bottom of the layer should not be chosen too small. The optimum choice depends on the geometry, the frequency, and the sound speed profile [68].

2.6.6 Source specification

As an initial value problem in range, the parabolic equation approximation requires an initial value for the solution at range zero, $r = 0$. The starting field should represent a monopole source. The exact expression for the field $p_{(r,z)}$ of a monopole source in an unbounded non-refracting atmosphere is

$$p_{(r,z)} = \frac{e^{jk\mathbf{d}}}{\mathbf{d}}$$

where $\mathbf{d}$ is the radial distance from the source. This expression cannot be used for the starting field because it diverges at the source position, and sound waves with large elevation angles are produced which generate numerical errors in a PE computation.

Gauss function for the source

The initial input must be normalised to approximate a point source in the far field. This can be represented by a Gauss function of the vertical coordinate, centred on $z = z_s$ [66]:

$$\varphi_{(r_s,z)} = S\sqrt{k_0} \cdot e^{\frac{-k_0^2}{2}(z-z_s)^2} \quad (2.18)$$

where r_s and z_s indicate the position of the source and S represents the source strength. The sharper the function is in the z domain, the more distributed is the transform becomes in the k_z domain.

This function is essentially for a cylindrical source. In order to make the results apply to a spherical source, we simply multiply the output $\varphi_{(r,z)}$ by $\exp(ik_0r)/\sqrt{r}$ at each range step.

A similar approach that takes into account the effect the ground in the starting field is given by [69, 70, 74]:

$$\varphi(r_s, z) = S\sqrt{k_0} \left(e^{-\frac{k_0^2}{2}(z-z_s)^2} + \frac{Z_g - 1}{Z_g + 1} e^{-\frac{k_0^2}{2}(z+z_s)^2} \right)$$

where Z_g is the normalised ground impedance.

2.6.7 Non-flat terrain profiles and noise barriers

The methods described above compute the propagated sound over plane unobstructed ground for 2D problems, or 3D problems considering azimuthal symmetry. The computational domain has to be rectangular and discretised with regular meshes.

A parabolic equation approach for the computation of sound diffraction by an infinitely long, thin screen on an absorbing ground was presented by Salomons [74]. This approach uses the CNPE for the calculations and a finite-difference extrapolation technique [65]. To include the screen in the model, the position of the screen has to be chosen in such a way that it covers a number of grid points on a vertical line segment. The effect of the screen on the sound pressure field is approximated by setting the sound pressure equal to zero at the grid points covered by the screen. To evaluate the accuracy of the method the source was positioned at 2 m from the ground level and at 30 m and 100 m from the screen. In those situations the PE results were in good agreement with an analytical solution (Pierce [75]).

The same approach, to set the sound pressure equal to zero at the grid points covered by the screen, was used by Rasmussen and Arranz [76]. Gauvreau

et al. [4] applied the same principle to a split-step Padé PE method. Both papers reported that the approach was reasonably accurate for thin barrier diffraction problems.

Different methods were developed to model sound propagation over non-flat terrain profiles. Sack and West [77] presented a PE method for sound propagation over “smooth” terrain profiles, the generalised terrain parabolic equation method (GTPE). The spatial transformation used in the GTPE method follows the terrain profile and is incorporated in a matrix vector parabolic equation with range dependent elements. The GTPE is much more complicated and less efficient than the CNPE. A function $H_{(r)}$, defining the terrain profile height as a function of range r must be obtained to compute the GFPE solution. Large values of $H_{(r)}$ and $dH_{(r)}/dr$ are not permitted. Blanc-Benon and Juvé [78] stated that the GTPE “*seems difficult to apply for a parabolic equation including wind terms*”. They proposed a different approach with a parabolic equation including wind terms. The non-flat ground is treated as a succession of flat domains. After each flat domain the coordinate system is rotated so that the r axis stays parallel to the ground. The values of the pressure field of the domain N is used as the initial solution for the domain $N + 1$. Reasonably accurate results were reported. Errors were attributed to variations of the local impedance and fluctuations of the meteorological parameters.

2.6.8 3D PE techniques

A number of PE techniques have been developed for 3D problems, specially for ocean acoustics applications [54, 56, 57, 60, 79–81]. The standard method consists in splitting the 3D PE into an $N \times 2D$ part and an azimuthal part.

This splitting method offers the possibility to update any 2D PE code to handle azimuthal coupling, which explains its popularity. However, solving this azimuthal part (which characterises full 3D computations) increases the CPU time. Whereas the computation times are still acceptable at short ranges and low frequencies (in comparison with any 2D run), 3D computations can become unreasonably intensive as the range increases, even at very low frequency. Indeed, as the desired maximum range increases, the required number of azimuthal points also increases and hence the computational demand can become very large [81]. Fawcett [80] introduced a different azimuthally coupled PE method. Fast Fourier transforms were used to solve the PE azimuthal operators. Fawcett compared his method with a $N \times 2D$ PE method. He reported that the 3D coupled method could properly model “qualitatively” the 3D refractive effects of modal propagation in a range-dependent wave guide while the $N \times 2D$ method was unable to do that. He also reported that in order to correctly model the 3D refractive effects of the waveguide, a large number of angular Fourier components were required. The required number of components increase approximately linearly with range, this limits the applicability of the method for very large domains.

The use of a higher-order finite-difference scheme in azimuth was introduced by Stum and Fawcett [81] allowing to reduce the required number of points in the azimuthal direction while still obtaining accurate solutions. The authors reported that the efficiency of this method was comparable to the one of the FFT-based methods, but that it could be use for PE operator expansions that can not be solved using FFT. The higher order schemes were derived using Taylor’s expansion and thus implicitly assume high lev-

els of regularity of the acoustic field. If the required regularity is absent, the advantages of using such higher-order schemes could eventually break down.

The propagation angles for the methods discussed above are “wide-angle” in the vertical plane and “narrow-angle” in the azimuthal plane. Chen *et al.* [82] introduced a three-dimensional azimuthal wide angle model for the PE. The authors demonstrated with an example, the importance of the capability of handling the azimuth wide-angle. The numerical approach used in reference [82] to solve the azimuthal wide-angle equation applies a combination of Finite-difference schemes, Ordinary-differential equation method, and Rational function approximations. As a result the method is called FOR3D. An example presented in that work used mesh spacings of $\Delta r = \lambda/6$, $\Delta z = \lambda/30$ and $\Delta\theta = 1^\circ$.

2.7 Wave Expansion Method

Full domain discretization methods have the advantage over BEM that inhomogeneities in the propagation medium can be easily introduced in the model. The big disadvantage of techniques like the well known finite difference (FD) and finite element methods (FE) is that they require prohibitive computational resources for computations on large domains. The computational cost is closely related with the number of points per wavelength needed to accurately discretise each dimension of the domain under consideration. Existing standard FD and FE methods require a mesh spacing of about 6 - 10 points per wavelength to obtain acceptable accuracy in the solution of the Helmholtz equation [83].

In order to alleviate some of these limitations a new physically based method,

the *Wave Expansion Method* (WEM) [83–85], was developed. This method was derived from the so called *Green Function Discretization Method* (GFD) [83] originally introduced by Caruthers *et al.*. The WEM solves the Helmholtz equation by domain discretization, relating the value of the unknown at each discrete point of the domain to the values at a selected set of neighbouring points. That is, given some point $\mathbf{r}_0$, it determines $p(\mathbf{r}_0)$ in terms of the value of $p(\mathbf{r}_m)$ at a discrete set of neighbouring points, $\mathbf{r}_m$; $m = 1, 2, \dots, M$. As $\mathbf{r}_0$ is arbitrary, the WEM thus represents a local interpolation formula to obtain the pressure field in the domain. This discretization methods offer an enormous improvement in computational efficiency by reducing the number of points per wavelength required to obtain accurate solutions and generate sparse system matrices which have a high bandwidth especially in 3D. The discretization is perhaps optimal in the sense that it does not seem reasonable to expect that anything less than two points per wavelength, the classic Nyquist limit in DSP, is feasible [83]. It was shown [83, 85] that the GFD and WED method fail precisely at a point corresponding to this limit in two dimensions for square grids.

2.7.1 Green's Function Discretization

As it was mentioned above, the method consists of determining the acoustic pressure p at a point $\mathbf{r}_0$, given the amplitude and phase data of a simple harmonic acoustic field at M neighbouring points, $\mathbf{r}_m$ (see Fig. 2.1).

Figure 2.1: Illustration of the basic concept.

For each point in the domain, the pressure may be approximated by the superposition of the fields generated by N hypothetical simple sources of strength γ_n which may be located, just as a matter of convenience, at a constant distance $R = |\mathbf{r}_n - \mathbf{r}_0|$ from the subject point at $\mathbf{r}_0$.

Then

$$p(\mathbf{r}_0) = \sum_{n=1}^N \gamma_n G(\mathbf{r}_0; \mathbf{r}_n), \quad \mathbf{r}_0 \in V_M, \quad (2.19)$$

where

$$G(\mathbf{r}_0; \mathbf{r}_n) = \frac{j}{4} H_0^{(2)}(k |\mathbf{r}_0 - \mathbf{r}_n|), \quad \text{in } 2D,$$

and

$$G(\mathbf{r}_0; \mathbf{r}_n) = \frac{e^{-jk|\mathbf{r}_0 - \mathbf{r}_n|}}{4\pi} |\mathbf{r}_0 - \mathbf{r}_n|, \quad \text{in } 3D.$$

Following the same procedure at the neighbouring points, we obtain

$$p(\mathbf{r}_m) = \sum_{n=1}^N \gamma_n G(\mathbf{r}_m; \mathbf{r}_n), \quad m = 1, 2, \dots, M. \quad (2.20)$$

In matrix notation,

$$\mathbf{p} = \mathbf{G} \cdot \boldsymbol{\gamma}, \quad (2.21)$$

where $\mathbf{P}$ and $\boldsymbol{\gamma}$ are $M \times 1$ and $N \times 1$ matrices, respectively, and $\mathbf{G}$ is an $M \times N$ matrix. Solving equation 2.21 for the coefficients $\{\gamma_n\}_{n=1}^N$ we obtain the approximate pressure at $\mathbf{r}_0$ from equation 2.19.

Minimum norm solution

By choosing $N > M$, there exist an infinite set of solutions $\boldsymbol{\gamma}$ to equation 2.21, all producing pressure vectors $\mathbf{p}$ which satisfy the Helmholtz equation exactly in V_M and match the M given values at $\mathbf{r}_m$ [84, 85]. The matrix $\mathbf{G}$ can be “pseudo-inverted” (Moore-Penrose pseudo-inverse) in order to obtain, among the infinite possible solutions for $\boldsymbol{\gamma}$, the *optimal* one for which $\sum_{n=1}^N \gamma_n^2$ is a minimum. The pseudo-inversion then ensures that all the constraints are satisfied exactly, and that the least “energetic” solution otherwise results [83, 85].

Once the pseudo-inverse has been calculated, the solution for the coefficients $\boldsymbol{\gamma}$ equation 2.21 yields

$$\boldsymbol{\gamma} = \mathbf{G}^+ \mathbf{p} \quad (2.22)$$

Letting $\mathbf{g}$ represent the column vector with elements $g_n(\mathbf{r}_0) = G(\mathbf{r}_0, \mathbf{r}_n)$, then

$$p(\mathbf{r}_0) = \mathbf{g}^T \mathbf{G}^+ \mathbf{p} \quad (2.23)$$

Equation 2.23 is referred to as the Green Function Interpolation (GFI). Repeated for each point in a grid network, it also represents a discretization for numerical solution of the Helmholtz equation and will be referred to

simply as Green Function Discretization method (GFD) [83].

Boundary conditions

Boundary conditions may be implemented by augmenting the row count of the matrix $\mathbf{G}$ in equation 2.21, including the possibility of multiple restraints which may occur at mesh corners [84, 85]. Boundary conditions related to the normal derivatives of the pressure (i.e. Neumann or radiation) at cell points $\{\mathbf{r}_m\}_{m=0}^M$ can be readily computed by taking the normal derivative of the constraint equation 2.21

$$\frac{\partial p(\mathbf{r}_m)}{\partial \mathbf{n}} = \frac{\partial G(\mathbf{r}_m; \mathbf{r}_n)}{\partial \mathbf{n}} \cdot \boldsymbol{\gamma}, \quad (2.24)$$

Equation 2.21 takes the form

$$\left\{ \begin{array}{c} \mathbf{p} \\ \partial p(\mathbf{r}_m)/\partial n \end{array} \right\} = \left[\begin{array}{c} \mathbf{G} \\ \partial G(\mathbf{r}_m; \mathbf{r}_n)/\partial n \end{array} \right] \cdot \boldsymbol{\gamma} = \mathbf{G}_{\text{aug}} \cdot \boldsymbol{\gamma} \quad (2.25)$$

The interpolating formula so obtained is not only an exact solution of the continuum equation at all neighbouring points, but also satisfies the boundary condition at up to $N - M$ points along the boundary in the neighbourhood of the subject point. After pseudo-inversion, the augmented matrix $\mathbf{G}_{\text{aug}}$ may be partitioned into $(\mathbf{G}_{\text{aug}}^+)_L$ and $(\mathbf{G}_{\text{aug}}^+)_R$. Then, a modified template (Eqn. 2.23) with a forcing right hand side may be established as

$$p(\mathbf{r}_0) - g^T (\mathbf{G}_{\text{aug}}^+)_L \mathbf{p} = g^T (\mathbf{G}_{\text{aug}}^+)_R \left\{ \frac{\partial p(\mathbf{r}_m)}{\partial \mathbf{n}} \right\} \quad (2.26)$$

In addition, Dirichlet constraints may be directly imposed on the appropriate degree of freedom in the overall assembled stiffness matrix.

Once the templates are formed, an overall sparse non-symmetric complex equation system

$$\mathbf{K} \cdot \mathbf{p} = \mathbf{f} \quad (2.27)$$

may be assembled with each template independently contributing a row.

2.7.2 Plane Wave Discretization

In theory, any complete solution of the governing equation may be used to interpolate the desired field variable [85]. A disadvantage associated to the use of Green's functions, for acoustic problems in which the coefficients of the governing equation have an spacial dependency, is that it is not possible to determine a single universally valid free space Green's function [85]. To avoid this problem, the infinite radial approximation of the Green's function, plane waves, may be used. The formulation is completely analogous to that outlined above for the GFD method. The pressure at the central point in the cell, may thus be approximated as a combination of plane waves

$$p(\mathbf{r}_0) = \sum_{n=1}^N \gamma_n e^{-jk(\mathbf{d}_n \cdot \mathbf{r}_0)} \quad (2.28)$$

where $\mathbf{d}_n$ is the unit propagation direction vector of the n^{th} plane wave with complex amplitude γ_n .

Using matrix notation equation 2.28 may be written as

$$p(\mathbf{r}_0) = \mathbf{h}(\mathbf{r}_0) \cdot \boldsymbol{\gamma} \quad (2.29)$$

where $\mathbf{h}(\mathbf{r}_0)$ is a $1 \times N$ row vector of plane waves functions evaluated at $\mathbf{r}_0$ and $\boldsymbol{\gamma}$ is a column vector of the wave strengths.

If we apply the same approximation to the other nodal positions in the cell we can write

$$\mathbf{p} = \mathbf{H} \cdot \boldsymbol{\gamma} \quad (2.30)$$

where $\mathbf{p}$ is a $M \times 1$ vector of the pressures at each surrounding node $\{\mathbf{r}_m\}_{m=1}^M$, and

$$H_{mn} = e^{-jk(\mathbf{d}_n \cdot \mathbf{r}_m)}$$

A computational template may then be formed by combining equation 2.29 and 2.30 to give

$$p(\mathbf{r}_0) = \mathbf{h}_{(\mathbf{r}_0)} \mathbf{H}^+ \mathbf{p} \quad (2.31)$$

Equation 2.31 is referred to as the Wave Expansion Discretization (WED) method [85, 86].

2.7.3 Boundary conditions

Because of the flexibility of the basic formulation, there is more than one method to impose the boundary conditions. In this section, a method proposed by Caruthers *et al.* [84] will be explained.

Dirichlet

For points lying adjacent to a boundary, Dirichlet boundary conditions are directly imposed by the partitioning

$$\mathbf{G}^+ \mathbf{p} = [\mathbf{G}_F^+ | \mathbf{G}_B^+] \begin{Bmatrix} \mathbf{p}_F \\ \mathbf{p}_B \end{Bmatrix} = \mathbf{G}_F^+ \mathbf{p}_F + \mathbf{G}_B^+ \mathbf{p}_B$$

where $\mathbf{p}_F$ is the column vector of field points whose component values are unknown and $\mathbf{p}_B$ are imposed at up to $N - M$ arbitrary distributed points

along the boundary. These conditions are implemented simply constraining the appropriate degrees of freedom in the assembled matrix equation 2.27.

Neumann

When the subject point, $\mathbf{r}_0$, is part of a boundary, Neumann boundary conditions may be imposed following the procedure described in Section 2.7.1. This is, at the subject point and each neighbour node that is also in the boundary, $\mathbf{r}_m^b$, a normal velocity condition is applied

$$\frac{\partial p(\mathbf{r}_m^b)}{\partial \mathbf{n}_m} = v_m \quad (2.32)$$

to augment the constraint equation 2.30

$$\begin{Bmatrix} \mathbf{p} \\ v_m \\ \vdots \end{Bmatrix} = \begin{bmatrix} \mathbf{H} \\ \partial h(\mathbf{r}_m^b) / \partial \mathbf{n}_m \\ \vdots \end{bmatrix} \boldsymbol{\gamma} = \mathbf{H}_{\text{aug}} \cdot \boldsymbol{\gamma} \quad (2.33)$$

where $\mathbf{n}_m$ is the direction of the outward normal at the boundary point $\mathbf{r}_m^b$.

Applying the pseudo-inversion to $\mathbf{H}_{\text{aug}}$, substituting and left-right partitioning the resulting equation, the expression of equation 2.29 yields

$$p(r_0) = \mathbf{h}_{(r_0)} (\mathbf{H}_{\text{aug}}^+)_L \mathbf{p} + \mathbf{h}_{(r_0)} (\mathbf{H}_{\text{aug}}^+)_R \cdot \mathbf{v} \quad (2.34)$$

where $\mathbf{v}$ is a vector of the corresponding normal velocities.

This is not a “natural” boundary condition as would be the case in standard finite element methods [85].

Radiation

Caruthers et al. [84], proposed a “naturally” imposed radiation condition at a point $\mathbf{r}_0^b$ laying in a radiating boundary, by selecting the directions of the fundamental solutions $\left\{e^{-jk\mathbf{d}_n \cdot \mathbf{r}^b}\right\}_{n=1}^N$ “travelling” outwards the computational domain, as shown in figure 3.4.

Figure 2.2: “Natural” radiation boundary condition implementation.

A special “radiating” template equation is then constructed including only outward propagating directions,

$$\mathbf{p} = \mathbf{H}_{\text{rad}} \cdot \boldsymbol{\gamma}$$

The pressure at the subject radiating point, is then given by

$$p(r_0) = \mathbf{h}_{(r_0)} \mathbf{H}_{\text{rad}}^+ \mathbf{p} \quad (2.35)$$

An alternative [85], “non-natural” approach, is to use the specific acoustic impedance of freely propagating plane waves ($\rho_0 c$) defined as

$$\frac{\partial p(\mathbf{r}^b)}{\partial \mathbf{n}} = -j \frac{\rho_0 \omega}{Z} p(\mathbf{r}^b) \quad (2.36)$$

where $\mathbf{r}^b$ is any point lying on the radiating boundary, $\mathbf{n}$ is the radiating normal outward boundary, and ρ_0 is the air density.

The implementation of these boundary condition follows a similar procedure as described for Neumann condition implementation. In this case, equation 2.30 is augmented to

$$\begin{Bmatrix} \mathbf{p} \\ 0 \\ \vdots \end{Bmatrix} = \begin{bmatrix} \mathbf{H} \\ \partial h(\mathbf{r}_m^b) / \partial \mathbf{n}_m + j (\rho_0 \omega / Z) h(\mathbf{r}_m^b) \\ \vdots \end{bmatrix} \gamma = \mathbf{H}_{\text{aug}} \cdot \gamma \quad (2.37)$$

After pseudo-inversion and partitioning, the template reduces to

$$p_{(r_0)} = \mathbf{h}_{(r_0)} (\mathbf{H}_{\text{aug}}^+)_L \mathbf{p}$$

Combination/Corners

It can happen that more than one restraint might need to be imposed at the boundary points, such as may occur at “corner” points. In this cases, the Neumann and radiation boundary condition procedures outlined in the previous sections can be readily combined. If an impedance approach is used for the radiation boundary condition, then equations 2.33 and 2.37 may be used in conjunction to augment the template matrix $\mathbf{H}$ according to the normal directions. For the natural radiation boundary approach, first a ma-

trix radiating template $\mathbf{H}_{\text{rad}}$ is constructed from the correspondent outward directions only, and then augmented using the relationship 2.33.

As it was discussed in previous sections, the WEM represents a very important improvement in numerical modelling over the traditional methods (FEM, FD, etc.) since it minimise the number of points per wavelength needed to discretise the domain in study, from 10 to less than 3. For very large problems, the computational cost of solving a problem with this method can still be prohibitive. Because of this, further improvement on this matter is required. A possible answer to scattering problems in very large acoustical domains is to consider sound waves travelling in only one direction. The basic idea of this approach is to calculate the pressure values for every point in the acoustic domain by means of a “forward” advancing solution, a concept analogous to the PE.

2.8 Discussion

In the previous sections, a general account of the most popular methods for outdoor sound propagation modelling was presented.

The most widely used methods for assessment and management of environmental noise are the so called semi-empirical methods. The reason for this, is that these methods require less computational resources than the physically based methods. Many approximations and simplifications are required. This affects their accuracy and application of different empirical methods may return different results. The HARMONOISE model appears to address this issue. Another big drawback of semi-empirical methods is that they require large input data sets.

Physically based methods have been subjected to a profuse body of research and developments in a quest for a more accurate solution to outdoor sound propagation modelling. The main challenge resides in modelling sound propagation through arbitrary complex meteorological conditions and irregular terrain profiles with varying characteristics. Most of the techniques are accurate for some defined conditions but fail in others.

The FFP method is conditioned to range-independent problems.

Ray racing techniques have frequency limitations given by the relationship between the wavenumber and the obstacle's surfaces sizes and roughness. For situations with irregular sound speed profiles Ray techniques are not very useful.

Boundary element methods represent a powerful tool for modelling sound propagation over complicated land profiles with varying surface's conditions. The use of BEM is limited when inhomogeneous atmospheres have to be introduced in the model. Computational limits prevent the use of BEM over large ranges.

The PE method represents one of the most powerful alternatives for large scale sound propagation modelling. Range-dependent inhomogeneities in the atmosphere can be introduced in the model in a simple way. Some difficulties arise when modelling sound propagation over complicated terrain profiles. The PE method seem to offer one of the best flexibility-efficiency relation of all the physically based methods reviewed in this chapter. However, arbitrary terrain profiles limit the applicability of this techniques. For 3D problems, the efficiency of the method decrease very rapidly with the length (range) of the computational domain.

The WEM method is a remarkably flexible method. Arbitrary shaped ground and range-dependent inhomogeneous atmosphere can be simply modelled with this technique. The WEM method is highly efficient as the required node spacing is reduced to less than three nodes per wavelength. Nevertheless, the computational requirements for large problems remain prohibitive.

In the following chapters a development of the WEM method is presented, the Forward Wave Expansion Method (FWEM). The main goal for the FWEM is to extend the efficiency of the WEM whilst maintaining the flexibility and applicability of the method for long range sound propagation. The FWEM computes the propagated sound field by means of an exclusively forward advancing solution, neglecting the backscattered field. The concept is thus analogous to the (one way) Parabolic Equation Method (usually discretised using standard finite difference or finite element methods). These techniques do not require the inversion of large system matrices and thus enable the solution of large scale acoustic problems where backscatter is not of interest.

Chapter 3

Forward-advancing Wave Expansion Method (FWEM)

3.1 Introduction

The FWEM computes the propagated sound field by means of an exclusively forward advancing solution, neglecting the backscattered field, a concept analogous to methods such as the (one way) Parabolic Equation Method (PEM) (usually discretised using standard finite difference or finite element methods). A computationally light model is then formulated whilst retaining many other formulation advantages of WEM.

At ranges where the propagating wave is nearly planar, the mesh spacing in the propagating direction needed to obtain accurate results can be significantly reduced, further improving the overall computational efficiency. In the following section a detailed description of the FWEM implementation is presented.

3.2 Implementation

To compute the sound pressure distribution of the domain shown in figure 3.1a with FWEM, a “local mesh” is formed by using a few “columns” of nodes (see Fig. 3.1b). A minimum of 3 columns is required for each sub-domain. If the solution is advancing to the right, a free radiating condition is imposed on the right-hand side boundary of the local mesh. Neumann, Dirichlet, impedance or radiation boundary conditions may be also applied to the upper and lower faces as required by the model.

Figure 3.1: Domain representation.

The implementation of the general wave discretization method utilised within each subdomain now follows essentially the standard WEM formulations [85, 86] and is repeated here for completeness.

For each point $\mathbf{r}_0$ in the subdomain, the pressure can be approximated by the superposition of N plane waves of strength γ_n :

$$p(\mathbf{r}_0) = \sum_{n=1}^N \gamma_n e^{-jk(\mathbf{d}_n \cdot \mathbf{r}_0)} \quad (3.1)$$

where $\mathbf{d}_n$ is the unit propagation direction vector of the n^{th} plane wave (see Fig. 3.2).

Figure 3.2: Example of a WEM computational cell.

It is important to note that the set of propagation directions of the fundamental solutions shown in figure 3.2 is the one used for the standard WEM. It will be shown later in this chapter that a different set of propagation directions may be applied for the FWEM.

Using matrix notation, equation 3.1 may be written as

$$p(\mathbf{r}_0) = \mathbf{h}(\mathbf{r}_0) \cdot \boldsymbol{\gamma} \quad (3.2)$$

where $\mathbf{h}(\mathbf{r}_0)$ is a $1 \times N$ row vector of plane waves functions evaluated at $\mathbf{r}_0$ and $\boldsymbol{\gamma}$ is a column vector of the wave strengths, γ_n .

If the same approximation is applied to the other nodal positions in the cell we can write

$$\mathbf{p}_m = \mathbf{H} \cdot \boldsymbol{\gamma} \quad (3.3)$$

where $\mathbf{p}_m$ is a $M \times 1$ vector of the pressure values at each surrounding node $\{\mathbf{r}_m\}_{m=1}^M$, and $\mathbf{H}$ is an $M \times N$ matrix with values:

$$H_{mn} = e^{-jk(\mathbf{d}_n \cdot \mathbf{r}_m)}.$$

If the number of plane waves used in the approximation is equal to the number of neighbours in the computational cell, this is $N = M$, then there is a unique solution for γ , given by

$$\gamma = \mathbf{H}^{-1} \mathbf{p}_m. \quad (3.4)$$

However, the solution for vector γ may be composed of high amplitude and phase variations, and may approximate $\mathbf{p}_m$ by means of a large amount of destructive interference. This could lead to an artificially irregular behaviour in the computed field.

If the system is under-constrained, $N > M$, an infinite set of solutions for γ would satisfy equation 3.4. This allows the selection of the best approximation for γ from all the possible solutions. The solution with minimum norm would, for example, exclude the results formed by large amplitude and phase variations, and thus return an optimal result. The solution for which $\sum_{n=1}^N \gamma_n^2$ is a minimum norm can be obtained using a Moore-Penrose pseudo-inversion of matrix $\mathbf{H}$. Another advantage of under-constraining the system, $N > M$, is that in principle, it is possible to accommodate up to $N - M$ additional constraints on the equations. Boundary condition constraints can then be conveniently added to the interpolating formula (see Sec. 3.2.1).

Ruiz [85] studied the behaviour of the WEM solution with different values for N and different distributions of the directions of propagation of the fundamental solutions. A set of $N = 20$ evenly distributed fundamental solutions (plane waves) was then proposed. This approach is used in the present work.

A computational template may then be formed by combining equations 3.2 and 3.3 to give

$$p_{(\mathbf{r}_0)} - \mathbf{h}_{(\mathbf{r}_0)} \mathbf{H}^+ \mathbf{p}_m = 0, \quad (3.5)$$

where the $\mathbf{H}^+$ is a pseudo-inverse of matrix $\mathbf{H}$.

The template from equation 3.5 may be then used to form individual rows of the system's dynamic stiffness matrix, $\mathbf{S}$. The row corresponding to point $\mathbf{r}_0$ is

$$\mathbf{S}_{(p_0, p_m)} = -\mathbf{h}_{(r_0)} \mathbf{H}^+, \quad (3.6)$$

$$\mathbf{S}_{(p_0, p_0)} = 1,$$

where p_m stand for the corresponding m^{th} neighbour point, p_0 is the subject point (central point of the cell).

The complete equation system is then given by

$$\mathbf{S} \cdot \mathbf{p} = \mathbf{f} \quad (3.7)$$

where $\mathbf{p}$ is the solution vector of pressure values and $\mathbf{f}$ is the forcing vector defined by the initial and boundary conditions of the problem.

After the sound pressure is calculated for the first subdomain, the solution may be advanced in space. A new subdomain is assembled. This new subdomain is formed by the last column of nodes of the previous subdomain (with driving pressure values already computed) and a number of the following neighbour columns. As a "staring" field, Dirichlet condition is imposed on the first column of the new subdomain using precalculated values. Im-

pedance, velocity condition or volumetric loading could also be used. When possible, Dirichlet condition was used in the present work because it is the most straight forward implementation. It will be shown later in this chapter that a volumetric loading might be needed in some cases.

A free radiating condition is imposed on the opposite boundary of the local mesh. Neumann, Dirichlet, Impedance or Radiation boundary conditions may then be applied to the upper and lower faces as required by the model. Figure 3.3 shows the restraint scheme applicable to each local mesh.

Figure 3.3: FWEM restraint scheme.

Once the subdomain is assembled and the “new” boundary conditions (true and artificial) are applied, the local equation system can be solved for acoustic pressure at these nodes. A neighbouring subdomain can then be formulated and the solution advanced (Fig. 3.1 a and b).

3.2.1 Boundary condition implementation

When the subject point, $\mathbf{r}_0$, is part of a boundary surface the corresponding condition for each boundary node in the cell has to be introduced into the

system of equations. As was mentioned earlier, in an under-constrained system it is possible to accommodate up to $N - M$ additional constraints on the equations. Boundary condition constraints can then be conveniently added to the interpolating formula.

Dirichlet boundary condition

This boundary condition is directly implemented constraining the appropriate degrees of freedom in the assembled stiffness matrix. This is:

$$\begin{aligned} \mathbf{S}_{(p_d, \cdot)} &= 0, \\ \mathbf{S}_{(p_d, p_d)} &= 1, \end{aligned} \tag{3.8}$$

where points p_d are lying adjacent to a boundary with Dirichlet condition. Equation 3.7 is now:

$$\mathbf{S} \cdot \mathbf{p} = \mathbf{f} \tag{3.9}$$

where the corresponding elements $f_{(\mathbf{r}_d)}$, in the forcing vector $\mathbf{f}$, are given by the corresponding Dirichlet pressure values at $\mathbf{r}_d$.

Neumann boundary condition

At each of the cell points lying on the surface with the Neumann boundary condition, $\mathbf{r}_b$, the normal velocity condition is given by

$$\frac{\partial p_{(\mathbf{r}_b)}}{\partial \mathbf{n}_b} = v_b \tag{3.10}$$

where $\mathbf{n}_b$ is the corresponding outward normal direction of the surface for the node, $\mathbf{r}_b$.

Boundary conditions related to the normal derivatives of the pressure can be computed by taking the normal derivative of the constraint equation 3.3, this is

$$\frac{\partial p(\mathbf{r})}{\partial \mathbf{n}} = \frac{\partial \mathbf{h}(\mathbf{r})}{\partial \mathbf{n}} \gamma.$$

Equation 3.3 may then be directly augmented to

$$\begin{Bmatrix} \mathbf{p} \\ v_b \\ \vdots \end{Bmatrix} = \begin{bmatrix} \mathbf{H} \\ \partial \mathbf{h}_{(\mathbf{r}_b)} / \partial \mathbf{n}_b \\ \vdots \end{bmatrix} \gamma = \mathbf{H}_{\text{aug}} \gamma. \quad (3.11)$$

Equation 3.11 is not only an exact solution of the continuum equation at all neighbouring points, but also satisfies the boundary conditions at up to $N - M$ discrete points along the boundary in the neighbourhood of the subject point.

Applying the pseudo-inversion to $\mathbf{H}_{\text{aug}}$, substituting and left-right partitioning the resulting equation, the expression of equation 3.2 then yields

$$p_{(\mathbf{r}_0)} - \mathbf{h}_{(\mathbf{r}_0)} (\mathbf{H}_{\text{aug}}^+)_L \mathbf{p} = \mathbf{h}_{(\mathbf{r}_0)} (\mathbf{H}_{\text{aug}}^+)_R \mathbf{q} \quad (3.12)$$

where $(\mathbf{H}_{\text{aug}}^+)_L$ corresponds to the first m columns of $\mathbf{H}_{\text{aug}}^+$ and $(\mathbf{H}_{\text{aug}}^+)_R$ is conformed by the B remaining columns where B is the total number of cell points in the Neumann boundary.

The equation system 3.7 is then modified by a forcing right hand side:

$$\mathbf{S} \cdot \mathbf{p} = \mathbf{f} \quad (3.13)$$

where $\mathbf{f}$ corresponds to the right hand side of equation 3.12.

Impedance boundary condition

Impedance boundary condition implementation follows a similar procedure as the one described above. If the condition for points $\mathbf{r}_i$, is given by

$$\frac{\partial p(\mathbf{r}_i)}{\partial \mathbf{n}_i} = -j \frac{\rho_0 \omega}{Z_i} p(\mathbf{r}_i) \quad (3.14)$$

where $\mathbf{r}_i$ is any point laying on the boundary, $\mathbf{n}_i$ is the direction of the outward normal at the corresponding node, ω is the angular frequency, Z_i is the specific acoustic impedance of the surface at that point, and ρ_0 is the density of air.

Equation 3.3 is then augmented to

$$\begin{Bmatrix} \mathbf{p} \\ 0 \\ \vdots \end{Bmatrix} = \begin{bmatrix} \mathbf{H} \\ \partial \mathbf{h}_{(\mathbf{r}_i)} / \partial \mathbf{n}_i + j (\rho_0 \omega / Z) \mathbf{h}_{(\mathbf{r}_i)} \\ \vdots \end{bmatrix} \boldsymbol{\gamma} = \mathbf{H}_{\text{aug}} \boldsymbol{\gamma}. \quad (3.15)$$

After pseudo-inversion and partitioning, the template reduces to

$$p(\mathbf{r}_i) - \mathbf{h}_{(\mathbf{r}_i)} (\mathbf{H}_{\text{aug}}^+)_L \mathbf{p} = 0. \quad (3.16)$$

The template of equation 3.16 determine the corresponding row of the system equation (Eqn. 3.7).

Natural radiation boundary condition

A simple method to implement a natural radiation boundary condition [87] is to restrict the propagation directions of the fundamental basis solutions to a selection of those propagating outwards from the computational domain as shown in figure 3.4.

Figure 3.4: “Natural” radiation boundary condition implementation.

A special “radiating” template equation may then be constructed,

$$\mathbf{p}_m = \mathbf{H}_{\text{rad}} \cdot \boldsymbol{\gamma}$$

and the template for the radiation point given by

$$p(\mathbf{r}_0) - \mathbf{h}(\mathbf{r}_0) \mathbf{H}_{\text{rad}}^+ \mathbf{p}_m = 0. \quad (3.17)$$

In the same way as the previously described templates, equation 3.17 determine the corresponding row of the system equation (Eqn. 3.7).

3.3 Specific modelling issues

The actual implementation of the FWEM for long range propagation problems requires further modification to the standard WEM presented in the previous section. The main points that need to be considered in the adaptation of the standard WEM into the forward advancing one are:

- Free radiation boundary conditions.
- Modelling of the source.
- Selection of the propagation directions of the fundamental solutions.

In order to present these modifications to the reader, a benchmark propagation problem consisting in a point source located in the corner of a rectangular computational domain is used (see Fig. 3.5).

Figure 3.5: Geometry of the benchmark problem.

The operating frequency used for the calculations in this section was 1000 Hz. The subdomains used in the examples were 1 metre long. A perfectly reflecting ground surface was assumed (however no reflection is expected as the source is located exactly at ground level).

To assess the accuracy of the sound pressure field computed with the FWEM, the results were compared with the theoretical solution for free field propa-

gation, the Hankel function

$$P_{theoretical} = -\frac{j}{4} H_0^{(2)}(k\mathbf{r}).$$

where $H_0^{(2)}$ is a Hankel function of second kind and zero order.

To illustrate the importance of the modifications the sound pressure field was computed using the standard WEM discretization for the each subdomain. The sound pressure level (SPL) was used as the parameter for comparison

$$\text{SPL} = -20 \log_{10} \left(\frac{p}{p_0} \right)$$

where $p_0 = 1$ [Pa] was used as reference pressure.

The SPL computed at a line 3 metres above the ground level is presented in figure 3.6. The result is compared with the theoretical solution.

Figure 3.6: Sound pressure level (SPL) computed at a line 3 metres above the ground level with the standard WEM implementation for each subdomain.

— Hankel Function, — FWEM.

The SPL field computed with the standard WEM implementation at each subdomain is shown in figure 3.7. The theoretical solution is presented in figure 3.8 for reference.

Figure 3.7: Sound pressure level (SPL) field computed with the standard WEM implementation for each subdomain.

Figure 3.8: Theoretical solution for the sound pressure level (SPL) field.

The SPL field computed using the FWEM with the modified implementation is shown in figure 3.9.

Figure 3.9: Sound pressure level (SPL) field computed with the modified FWEM implementation (compare with Fig. 3.7).

Figure 3.10 shows the SPL computed at 3 metres above the ground level.

A detailed description of the different modifications of the standard WEM implementation for the FWEM is given in the following sections.

As mentioned before, the modifications mainly concern the source model, the free radiation boundary condition implementation and the selection of the propagation directions of the fundamental solutions.

Figure 3.10: Sound pressure level (SPL) computed at 3 metres above the ground level with the modified FWEM. — Hankel Function, - - FWEM, · · · FWEM using the standard WEM for each subdomain.

3.3.1 Free radiation boundary conditions

Upper attenuating boundary layer

A major problem of using domain-based techniques for sound propagation in unbounded domains is that the computational domain has to be truncated. This truncation produces “fictitious” boundaries where perfect radiation condition should be imposed to prevent “artificial” reflections back to the domain.

The performance of the radiation condition developed for the original WEM (see Sec. 3.2.1) is poor when the angle of incidence of the sound waves on the boundary is very shallow. This happens in the upper boundary of the computational domain when the length/height relationship is large. This type of computational domain is common in long range sound propagation

problems. Figure 3.11 shows a typical example of this type of domain.

Figure 3.11: Typical length/height proportion in a long range propagation problem.

In these cases, if the FWEM is implemented without any “special” treatment in the upper boundary (with radiation boundary condition), the reflections introduced in the computational domain may be unacceptable. An example of the computed sound pressure field produced by a point source of 1 KHz in a computational domain 200 metres long and 30 metres high, using the FWEM with standard radiation boundary condition in the upper boundary, is presented in figure 3.12.

Figure 3.12: Sound pressure level (SPL) field computed with the FWEM without treatment in the upper boundary.

The computed pressure field is highly contaminated with reflections “artificially” generated by the upper boundary.

A plot of the SPL at a high of 3 metres above the ground level is shown in figure 3.13. The results are compared with the theoretical solution.

Figure 3.13: Sound pressure level (SPL) computed at 3 metres above the ground level with standard radiation condition in the upper boundary. — Hankel Function, — FWEM.

In the previous example, the length/height relationship is not as big as the one of figure 3.11, however, the spurious reflections introduced in the computational domain are already excessive. To reduce spurious reflections from the upper boundary back into the computational domain, additional treatment of the radiation boundary condition may be applied by introducing an artificial absorbing boundary layer [74]. In this layer, an artificial attenuation is gradually increased with height, starting with zero attenuation at $z = z_t$ (where z_t is the height of the bottom of the absorbing layer). This attenuation is achieved by adding an imaginary term to the wave number

$k(z)$ according to [66,68],

$$k = k_0 - jA [(z - z_t) / (z_M - z_t)]^2 \quad (3.18)$$

where k_0 is the wave number in the operational part of the domain, A is the damping constant, and z_M is the z coordinate of the top boundary (see Fig. 3.14). The artificial damping constant, A , should not be too large as this will produce reflections from the bottom of the absorbing layer back into the lower part of the atmosphere. For this reason a quadratic variation of the wave number k is chosen over, for example, a linear variation. Nevertheless, it is not clear how to optimise the selection of A in the present context [66]. For the computations in the present work, the value used for the parameter A was chosen with a trial and error method.

Figure 3.14: Diagram of the implementation of the artificial absorbing layer.

For this example, after approximately 5 trials, a damping constant of $A = 1.5$ was chosen and the thickness of the boundary layer was $30 \times \lambda$ deep. This means that, as the operating frequency for was 1 KHz, a layer of approximately 10 metres had to be added to the computational domain which results in a computational domain 40 metres high in the calculations.

The absorption effect of the layer can be seen in figure 3.15.

Figure 3.15: Sound pressure level (SPL) field computed with upper boundary absorbing layer.

The computed SPL for the operational part of the domain is shown in figure 3.16.

Figure 3.16: Sound pressure level (SPL) field computed with upper boundary absorbing layer, operational part of the domain.

It is clear from the figures how the reflections “artificially” generated in the upper boundary (see Fig. 3.12) are almost completely eliminated by the absorbing layer.

A plot of the SPL at a high of 3 metres above the ground level, computed using the absorbing layer, is shown in figure 3.17. The result is compared with the theoretical solution (Hankel function).

Figure 3.17: Sound pressure level (SPL) computed at 3 metres above the ground level with absorbing layer. — Hankel Function, - - - FWEM, . . . FWEM without absorbing layer in the upper boundary.

Figure 3.17 shows the accuracy of the solution.

Propagation directions for the natural radiation boundary condition

The standard implementation of the radiation condition consists in restricting the propagation directions of the fundamental basis solutions to a selection of those propagating outwards from the computational domain, *i.e.* propagation angle of $8\pi/9$ (see Fig. 3.18). For free radiating boundaries the central angle of the “cone” of directions (0°) is parallel to the surface’s normal at the object point (see Fig. 3.18).

Figure 3.18: Directions of the fundamental solutions for free radiating boundaries (160°).

This implementation is sufficiently effective for the standard WEM. In the FWEM, radiation condition is imposed to the vertical boundaries at the end of every subdomain. Any “small” error introduced by these boundaries implementation would build up as the solution advance and the computed sound field at long range could become completely inaccurate. Therefore, the radiation condition implemented must be very rigorous.

If the implemented radiation condition is the same as for the standard WEM (like the one shown in figure 3.18) the sound field computed with the FWEM is highly distorted and contaminated with spurious reflections. Figure 3.19 shows the SPL field computed using an angle of $8\pi/9$ for the radiation condition implementation.

Figure 3.19: Sound pressure level (SPL) field computed with an angle of $8\pi/9$.

It is apparent from the figure that the computed field is inaccurate.

If the propagation angle is restricted further, *i.e.* to an angle of $\pi/2$ (see Fig. 3.20) the accuracy of the computed sound field is clearly improved. This result is shown in figure 3.21.

Figure 3.20: Directions of the fundamental solutions for free radiating boundaries (90°).

Figure 3.21: Sound pressure level (SPL) field computed with an angle of $\pi/2$.

The computed SPL field was noticeably improved but it is still inaccurate.

Reducing the propagation angle even further, *i.e.* to $\pi/12$ (see Fig. 3.22), the accuracy is yet improved, as shown in figure 3.23.

Figure 3.22: Directions of the fundamental solutions for free radiating boundaries (30°).

Figure 3.23: Sound pressure level (SPL) field computed with with an angle of $\pi/12$.

Figure 3.24 shows the solution for the SPL computed at a line 3 metres from the ground level with the three different sets of directions.

Figure 3.24: SPL computed at 3 metres above the ground level with different propagation angles for the radiation condition implementation. — Hankel Function, — FWEM ($8\pi/9$), — FWEM ($\pi/2$), — FWEM ($\pi/12$).

The result for the last 50 metres of the domain are shown in figure 3.24.

Figure 3.25: SPL at 3 metres above the ground level with different propagation angles for the radiation condition implementation, last 50 metres of the domain. — Hankel Function, — FWEM ($8\pi/9$), — FWEM ($\pi/2$), — FWEM ($\pi/12$).

It is clear from the figures that not even in the best case are the reflections from the vertical boundaries with “artificial” radiation boundary condition removed completely. However, figures 3.24 and 3.25 show that for the last case, the remaining reflections do not greatly affect the accuracy of the computed sound field in the far field.

Improved natural radiation boundary condition

Although the treatment of radiation boundaries presented above, in combination with an absorbing upper boundary layer, gives good results for the calculations in the far field, some spurious reflections may be introduced and are noticeable in the “near” field.

An improved solution is obtained if “more information” of the radiation condition is introduced in the system of equations. A way of introducing this “extra” radiation information in the system is by considering as radiating boundary points not only the nodes that lay in a boundary with radiation condition, but also their neighbours.

Figure 3.26 shows the “normal” radiation boundary’s disposition (3.26 a) and the improved one (3.26 b).

Figure 3.26: Radiation condition for vertical "artificial" radiation boundaries.

An example of the computed sound field produced by a point source in the left lower corner of a $100\text{ m} \times 30\text{ m}$ domain is used to illustrate the difference in the solutions. Figures 3.27 and 3.28 show the SPL fields computed using the normal cell disposition and the improved one respectively.

Figure 3.27: Sound pressure level (SPL) field computed with standard radiation boundary condition.

Figure 3.28: Sound pressure level (SPL) field computed with the improved radiation boundary condition.

The SPL computed with the improved radiation boundary condition at a line 3 metres above the ground level is shown in figure 3.29.

Figure 3.29: SPL computed at a line 3 metres above the ground level with the improved radiation boundary condition. — FWEM, — Hankel Function.

The improvement in the solution for the far field is apparent from the figures. Although spurious reflections are still present in both cases, the distortion of the wave present in figure 3.27 is removed in the second case (Fig. 3.28).

3.3.2 Source modelling

As in the PEM, the FWEM will present some problems in modelling the sound sources and computing the pressure values in regions near sources or in locations where the curvature of the wavefronts in the sound field is high. Results show that the pressure values obtained in the acoustic field remote from these will not be influenced significantly by these local inaccuracies.

Using a larger number of columns for the first subdomain, or any part of the domain where the curvature of the wave front could be high, can reduce the inaccuracies in the results.

Source calibration

The wave expansion method, as implemented up to here, can be used to model sound propagation only in a qualitative way. To obtain accurate quantitative results, a source calibration has to be carried out. A calibration method was developed for the WEM by Birgersson [88]. This works as follows, if we consider two contiguous cells containing the excitation source, in node 0, the actual pressure at that point is unknown (see Fig. 3.30).

Figure 3.30: Contiguous cells containing the excitation source in node 0.

The produced pressures at nodes 1 to 13 are given by the exact response to a cylindrical source at the corresponding distance.

From the formulation of the WEM, we have that:

$$p_{(0)} = \mathbf{hH}^+ \begin{Bmatrix} p_{(5)} \\ p_{(6)} \\ p_{(9)} \\ p_{(8)} \\ p_{(10)} \\ p_{(11)} \\ p_{(12)} \\ p_{(13)} \end{Bmatrix} + f_0, \quad p_{(5)} = \mathbf{hH}^+ \begin{Bmatrix} p_{(1)} \\ p_{(2)} \\ p_{(3)} \\ p_{(4)} \\ p_{(6)} \\ p_{(7)} \\ p_{(8)} \\ p_{(0)} \end{Bmatrix} \quad (3.19)$$

From this relationships we have

$$p_{(0)} = \left(p_{(5)} - \mathbf{hH}^+_{(1:7)} \cdot \left\{ p_{(1)} \ p_{(2)} \ p_{(3)} \ p_{(4)} \ p_{(6)} \ p_{(7)} \ p_{(8)} \right\}' \right) / \mathbf{hH}^+_{(8)}$$

Using the above relationship, $p_{(0)}$ can be eliminated from equation 3.19 and we get the following expression for f_0 :

$$f_0 = (p_{(5)} - \mathbf{hH}^+_{(1:7)} \cdot \mathbf{p1}) / \mathbf{hH}^+_{(8)} - \mathbf{hH}^+ \begin{Bmatrix} p_{(5)} \\ p_{(6)} \\ p_{(9)} \\ p_{(8)} \\ p_{(10)} \\ p_{(11)} \\ p_{(12)} \\ p_{(13)} \end{Bmatrix} \quad (3.20)$$

where $\mathbf{p1} = \{p_{(5)}p_{(6)}p_{(9)}p_{(8)}p_{(10)}p_{(11)}p_{(12)}p_{(13)}\}'$.

Using the above formulation, excitation sources can be introduced into the model applying the computed f_i in the corresponding row of the force vector, $\mathbf{f}$.

3.4 Numerical issues of the FWEM

A very important matter concerning the stability of the method, is the condition number of the equation matrices. A poor conditioning will encourage the propagation of error particularly over large distances.

To study the behaviour of the condition number for the equation stiffness matrix $\mathbf{S}$, the behaviour of a nine node square computational cell of dimension $2dr \times 2dr$ as shown in figure 3.31 was considered.

Figure 3.31: Nine node computational cell.

The position of the cell is determined by the general coordinates of point 1.

Figure 3.31 shows the resulting condition number for stiffness matrix as a function of the position of the cell for a propagation problem in homogeneous media. The coordinates of the cell range from $\mathbf{r} = (0, 0)$ to $\mathbf{r} = (0, 1000)$.

Figure 3.32: Condition number behaviour for homogeneous media.

For propagation problems in homogeneous media, the condition number of the stiffness matrix remained constant.

To study the behaviour of the condition number for inhomogeneous media, a change in the wave number k for points 7, 8 and 9 of the cell was introduced. The variation in the wave number was produced using equation 3.18. Two different values of A , 0.01 and 0.05, were used.

Figure 3.33 shows that the condition number varies with the position of the cell.

Figure 3.33: Condition number behaviour for inhomogeneous media.

It is clear from the plots that the system becomes very unstable with distance when dealing with inhomogeneous media. Fortunately, this problem is very easily resolved by always using local coordinates for each cell position. That is, at any cell in the computational domain a new coordinate system is defined to compute the local stiffness matrix. The localised coordinates for the cell points will then be: $\mathbf{r} = (0, 0)$ for the point in position 1, $\mathbf{r} = (dr, 0)$ for position 2, $\mathbf{r} = (dr, dr)$ for position 5, etc.

The resulting behaviour of the condition number is shown in figure 3.34.

Figure 3.34: Condition number behaviour for inhomogeneous media with local coordinate system.

Applying this simple change in the coordinate system, the equation system becomes completely stable.

3.5 Explicit and semi-explicit solution

In a large number of practical problems a regular meshing scheme can be applied to a considerable section of the computational domain. In every section where a regular mesh is used, the advancing method may be made

completely explicit if there are no wavenumber parameter changes in the marching direction of the solution, *i.e.* range-independent atmospheric and boundary conditions.

For example, a regular mesh allows the use of symmetrical subdomains (see Fig. 3.35).

Figure 3.35: Division of a regular mesh into symmetric subdomains.

The system equation stiffness matrix can be assembled for the first subdomain using the procedure explained in section 4.2. From equation 3.6 we have

$$\mathbf{S}_{(p_i, p_m)} = -\mathbf{h}_{(r_i)} \mathbf{H}^+$$

$$\mathbf{S}_{(p_i, p_i)} = 1$$

where

$$\mathbf{h}_{(r_i)} = e^{-jk_i(\mathbf{d}_n \cdot \mathbf{r}_i)}$$

and $\mathbf{H}^+$ is the pseudo-inverse of

$$\mathbf{H}_{mn} = e^{-jk_m(\mathbf{d}_n \cdot \mathbf{r}_m)}$$

As pointed out in section 3.4, a local coordinate system is used for each computational cell. Because of this, the local coordinates for each point of the two symmetric subdomains are exactly the same. If the wavenumber k is the same for every symmetrically corresponding point (i.e. $k_1 = k_3$, $k_{17} = k_{19}$, etc. where $k_i = k_{(r_i)}$) matrix $\mathbf{H}$ and vector $\mathbf{h}$ are equivalent for every template, therefore the resulting stiffness matrix is identical. This allows the solver to compute the stiffness matrix for the first symmetric subdomain and apply it to compute the solution for the next ones. This considerably reduces the computing time required.

If the wavenumber parameters are constant across a computational front for the field nodes but there is a change in the boundary conditions a semi-explicit scheme may be applied. Changes in the values on a boundary with Dirichlet condition have no influence in the stiffness matrix. The new restraint simply affects the corresponding value in the forcing vector $\mathbf{f}$ (Eqn. 3.13). In case of Neumann or Impedance boundary conditions, the constraint equation 3.3 is augmented when the row of the stiffness matrix corresponding to the subject point is computed. Therefore, only these particular rows have to be re-assembled with the new boundary conditions/values, and the rest of the pre-calculated stiffness matrix can be used. The ratio of the boundary points (Dirichlet, Neumann and Impedance) to the field nodes is considerably small, consequently the semi-explicit technique still offers an important saving of computing time.

Examples

To demonstrate the advantages quantitatively, the CPU time required to compute the sound pressure field was assessed for two examples. The geometries used consisted of two rectangular domains. The first one was 50 metres long and 40 metres high (200,901 nodes) and the second one was 100 meters long and 40 metres high (401,401 nodes). The sound pressure field was computed with the WEM, the FWEM, and the explicit-FWEM. The numerical implementations were programmed in `Matlab` Version 6.5, and the computations performed on an IBM, Pentium-IV machine with 882 MB of RAM and a clock speed of 2.2 Ghz. The system matrix was solved using Gaussian elimination (backslash operator in `Matlab`).

The CPU time (in seconds) required by each method to compute the solution is shown in the following table:

	50 × 40 [m]	100 × 40 [m]
WEM	5095	18777
FWEM	723	1403
explicit-FWEM	133	135

It can be seen that the FWEM requires less computational time. The advantage of the explicit solution is evident from the results, enhancing the speed of the method. This technique was used to compute the sound field in the examples presented in section 4.4.

3.6 Segmentation of the computational domain

Because of the flexibility of the FWEM a segmentation of the domain can be simply implemented. This could prove to be a very important feature when solving the sound field for very large problems. In a normal environmental application the computational domain would easily have dimensions in the order of $100,000\text{ m}^2$. The solution for a frequency of 1KHz would require more than 10,000,000 nodes using a mesh spacing of 3 nodes per wave length (this mesh spacing is only necessary for the first part of the domain, once the wave front is sufficiently plane it can be considerably coarsened). A mesh with this number of nodes could be difficult to manage from the point of view of meshing and storage.

A much easier way of handle this is to divide the domain into a number of smaller segments. These segments are then much easier to mesh and manipulate. Figures 3.36 and 3.37 show an example of the segmentation. A large computational domain of dimensions 1000×50 metres is thus subdivided into 5 smaller segments of dimensions 200×50 metres.

Figure 3.36: Complete computational domain.

Figure 3.37: Computational domain divided into segments.

The FWEM can compute the sound field for the first segment independently from the rest. Using only the last column of nodes of this first segment the solution for the next one can be computed. The same procedure can be followed to compute the sound field for the rest of the segments. The resulting sound fields presented in the following section were obtained using this procedure.

3.7 Numerical results

The advancing method was tested for four benchmark problems. The importance of a correct selection of the directions of propagation of the fundamental solutions (plane waves) used for the discretization was analysed. A new advantage of the FWEM then emerged. When the propagating wave front is sufficiently plane, the mesh spacing in the direction of propagation can be considerably increased improving the efficiency of the method (this would be analogous to the signal demodulation used in the Parabolic Equation Method).

Firstly, the FWEM is tested for a free field propagation problem over a perfectly reflecting ground. The results were compared with a theoretical solution, a Hankel function of the second kind and zero order,

$$P_{theoretical} = -\frac{j}{4} H_0^{(2)}(k\mathbf{r}).$$

Secondly, the accuracy of the method for modelling diffraction of sound is evaluated in two diffraction problems, sound diffraction by a simple wedge and by a thin sound barrier. In both cases the ground and the obstacles were considered to be perfectly reflective ($\partial p / \partial \mathbf{n} = 0$).

The insertion loss (IL) was the parameter chosen to assess the accuracy of the solution for the diffraction problems. This parameter is defined by the following relationship:

$$\text{IL} = 20 \log_{10} \left(\frac{P_w}{P_o} \right)$$

where P_w is the sound pressure in the presence of the obstacle and P_o is the sound pressure that would be produced without the obstacle.

The insertion loss computed with the FWEM was compared with an analytical solution, the Fresnel Integration Method [89] and with the standard WEM for the sound barrier problem. For the wedge problem only the standard WEM was used for validation.

Finally, the capability of the FWEM for modelling sound propagation in a refracting atmosphere was tested. The sound field was computed under downwind and upwind refraction conditions. The results were compared with the standard WEM.

For the following examples, a representative operational frequency of 1000 Hz was used. The dimensions and “shapes” of the geometries used in the examples were defined arbitrarily. There is no “special” relationship between the dimensions and angles used in the geometries and the sound wavelength. Similar results were obtained in tests carried out by the author for different operational frequencies and different geometries.

3.7.1 Free field propagation

The geometry of the first problem is shown in Fig. 3.38.

Figure 3.38: Geometry of the problem.

The source was positioned at point $(x,y) = (0,0)$. The operating frequency was 1000 Hz. The domain was 200 metres long and 40 metres high. An absorbing layer $30 \times \lambda$ deep was applied. This setup leaves an operational altitude of 30 metres. A damping constant $A = 1.5$ was used. The computational domain was meshed with a node spacing of 3 nodes per wavelength.

Selection of the propagation directions of the fundamental solutions

The selection of the directions of propagation of the plane waves is crucial for the FWEM. In this section three different sets of propagation directions were considered to illustrate the importance of this parameter. A propagation angle of 2π ($[0, 2\pi]$), π ($[\frac{\pi}{2}, -\frac{\pi}{2}]$) and $\pi/2$ ($[\frac{\pi}{4}, -\frac{\pi}{4}]$). Sets of 20 evenly distributed propagation directions were considered in all cases. A diagram of the sets of propagation directions is presented in figure 3.39. The subdomains used for these examples were 5 metres long.

Figure 3.39: Propagation directions of the fundamental basis solutions.

The theoretical solution for this problem (Hankel function) is presented in figure 3.40.

Figure 3.40: Resulting sound pressure level (SPL) field, theoretical solution.

The resulting SPL fields computed using propagation angles of 2π , π , and $\pi/2$ are shown in figures 3.41, 3.42 and 3.43 respectively.

Figure 3.41: Resulting SPL field using a propagation angle of 2π .

Figure 3.42: Resulting SPL field using a propagation angle of π .

Figure 3.43: Resulting SPL field using a propagation angle of $\pi/2$.

Plots of the SPL computed with the Hankel function and the FWEM at 0 and 3 metres above the ground level are given in figures 3.44 and 3.45 respectively.

Figure 3.44: Sound pressure level computed at the ground level. — Hankel Function, — FWEM ($\pi/2$), — FWEM (π), — FWEM (2π).

Figure 3.45: Sound pressure level computed at 3 metres above the ground level. — FWEM ($\pi/2$), — FWEM (π), — FWEM (2π), — Hankel Function.

It can be seen from the figures that the source calibration technique, presented in section 3.3.2, gives inaccurate results using propagation angles π and $\pi/2$ (difference in the SPL values for the starting field), however the dissipation effects seem to be correctly modelled. The source calibration works perfectly when the complete range of directions (propagation angle of 2π) was used. It is clear from figures 3.41 and 3.44 that the results obtained using a propagation angle of 2π an error is introduced to the solution at ground level. As the solution advances the error becomes more significant. It is important to point out that this error is not introduced by the forward approximation, but is intrinsic in the WEM.

In the FWEM, a combination of propagation angles can be used. Figure 3.46 shows the computed sound pressure field using a propagation angle of 2π in the first subdomain and $\pi/2$ in the following ones.

Figure 3.46: Sound pressure level (SPL) field using a propagation angle of 2π in the first subdomain and $\pi/2$ in the following ones.

Plots of the SPL computed at elevations of 0 and 3 meters above the ground level with this combination, compared with the theoretical solution are shown in figures 3.47 and 3.48 respectively.

Figure 3.47: Sound pressure level computed at ground level using a propagation angle of 2π in the first subdomain and $\pi/2$ in the following ones.
— Hankel Function, - - FWEM.

Figure 3.48: Sound pressure level computed at 3 metres above the ground level using a propagation angle of 2π in the first subdomain and $\pi/2$ in the following ones. — Hankel Function, - - FWEM.

It is shown in the figures that the combination of the two sets of directions, 2π for the first subdomain and $\pi/2$ in the rest, gives accurate results.

As the calculation proceeds to field positions more remote from the source, where the wave front is almost planar, the angular spread may be reduced. This permits the use of lower nodal spacings in the propagation direction.

This concept was tested using a rectangular computational domain 50 metres high and 400 metres long. For positions at long distances from the source (in this example more than 200 metres from the source) a propagation angle of $\pi/12$ was used (see Fig. 3.49). A mesh spacing of 0.2 nodes per wavelength in the propagation direction was applied. The vertical nodal spacing was kept at 3 nodes per wavelength. An absorbing layer $30 \times \lambda$ deep was used with a damping constant $A = 1.5$.

Figure 3.49: Propagation directions of the fundamental basis solutions in the far field (15°).

The computed solution is presented in figure 3.50. The theoretical solution is shown in figure 3.51 for comparison. Plots of the SPL at elevations of 0 and 3 metres above the ground level are shown in figures 3.52 and 3.53 respectively.

Figure 3.50: Computed sound pressure level (SPL) using propagation angles of 2π for the first subdomain, $\pi/2$ for the next ones until the 200m line (white line) and $\pi/12$ for the remaining subdomains.

Figure 3.51: Sound pressure level (SPL), theoretical solution (Hankel function).

Figure 3.52: Sound pressure level (SPL) computed at ground level.
— Hankel Function, — FWEM.

Figure 3.53: Sound pressure level (SPL) computed at 3 meters above the ground level. — Hankel Function, - - FEM.

The accuracy of the method is apparent from the figures.

In the next section the accuracy of the FEM is tested for a larger computational domain.

Free propagation in large domains

The geometry of the problem is shown in Fig. 3.54.

Figure 3.54: Geometry of the problem.

The source was positioned at point $(x,y) = (0,0)$. The operating frequency was 1000 Hz. The domain was 1 kilometre long and 50 metres high.

An absorbing layer $30 \times \lambda$ deep was applied. This setup leaves an operational altitude of 40 metres. The absorbing layer was implemented using the model in equation 3.18 with a damping constant $A = 1$ for the first segment and $A = 0.6$ for the following ones.

More than 5,000,000 (5,012,505) points would be needed to mesh this domain with a node spacing of 3 nodes per wavelength. To alleviate the computational requirements, the segmentation technique was applied. Five segments of 200 metres long were used. A propagation angle of 2π was used for the first subdomain, $\pi/2$ was the propagation angle used for the rest of the subdomains in the first segment, and an angle of $\pi/12$ for the rest of the segments. A nodal spacing of 3 nodes per wavelength was used for the first segment (1,002,501 nodes were needed for this segment). For the following segments the mesh density in the propagation direction was reduced to 0.2 nodes per wavelength (59,619 nodes were needed for each of these segments). Considering all the segments together the number of nodes needed to discretise the computational domain was reduced from 5,000,000 to less than 1,250,000 nodes. Each subdomain was 5 metres long for the first segment and 20 metres long for the following ones. Each subdomain was formed by 25,551 nodes in the first segment and 6,012 nodes in the following ones. Therefore, the largest size of system matrix to be solved was reduced from $5,000,000 \times 5,000,000$ matrix to a $25,551 \times 25,551$ one.

As a regular mesh was used and there was no parameter change in the propagation direction of the solution, the explicit property of the solution scheme was used after the second subdomain of each segment.

Figure 3.55 show the computed SPL in the complete computational domain. The effect of the absorbing layer in the top of the domain can be seen in this figure. The vertical white lines illustrate the segmentation used to compute the sound field.

Figure 3.55: FWEM, Sound Pressure Level (SPL).

The resulting field in the operational part of the domain is shown in figure 3.56.

Figure 3.56: FWEM, Sound Pressure Level (SPL). Operational part.

Figure 3.57 shows the true vertical-horizontal scale. This shows that the problem has extremely shallow grazing angles on the upper boundary of the domain.

Figure 3.57: True vertical-horizontal scale plot.

A plot of the real part of the sound pressure values obtained with the FWEM and the theoretical solution for the last 15 metres of the computational domain at an elevation 3 metres above ground level is given in figure 3.58. The SPL obtained with the FWEM and the theoretical solution at an elevation 3 metres above ground level is given in figure 3.59.

Figure 3.58: Real part of the pressure values in the last 15 metres, at a line 3 metres above the ground level. — Hankel Function, - - FWEM.

Figure 3.59: Sound pressure level (SPL) computed at a line 3 metres above the ground level. — Hankel Function, - - FWEM.

The solution shows to be very accurate in the entire range. It should be pointed out that an absorbing layer 10 metres deep ($30 \times \lambda$) was used in the calculations.

3.7.2 Diffraction by a wedge

The FWEM is meant to be applied to sound propagation problems when the backscattered field is not of interest. An example of this kind of problem could be the evaluation of the performance of a sound barrier. Most of the time, only the sound field “beyond” the barrier is of interest and the sound reflected by the obstacle back to the source is not of interest.

It was shown in the previous section that the FWEM is highly accurate for modelling free field sound propagation. In this section the accuracy of the FWEM for diffraction problems is tested. The geometry studied is shown in figure 3.60.

Figure 3.60: Geometry of the problem.

The source was positioned at point $(x,y) = (0,0)$. A wedge 4 metres high with an elevation of 45° was positioned at a distance of 10 metres from the source. The operating frequency was 1000 Hz. The domain was 50 metres long and 40 metres high. The top 10 metres ($30 \times \lambda$) were changed into an absorbing layer whilst a loss free domain of height 30 metres was maintained. A damping constant $A = 1.5$ was used. In sections where a regular mesh was used and there was no parameter change in the propagation direction

of the solution and no obstacle was present, the explicit property of the solution scheme was used. It is important to note that the centre of the cone of propagation directions for the fundamental solutions (plane waves) was kept horizontal for the whole domain. It did not follow the shape of the ground (wedge).

The standard WEM was used for validation of the results.

Propagation direction of the fundamental solutions

It was shown in section 3.7.1 that the selection of propagation directions of the fundamental solutions is a very important parameter in the FWEM.

It was shown that a propagation angle of 2π was the best for the first subdomain to get a correct calibration of the source, and $\pi/2$ was the best propagation angle for the following subdomains for free propagation modelling. It is clear that free propagation and diffraction problems involve different directions of propagation of the sound wave fronts. Because of this, the best choice of the propagation directions of the fundamental solutions may not be the same for different problems (*i.e.* free propagation and diffraction effects). Three different sets of propagation directions were tested, 2π ($[0, 2\pi]$), π ($[\frac{\pi}{2}, -\frac{\pi}{2}]$) and $\pi/2$ ($[\frac{\pi}{4}, -\frac{\pi}{4}]$) (see Fig. 3.39). Sets of 20 evenly distributed propagation directions were considered in all cases.

For the following example, the first subdomain was 9.5 metres long. This length was chosen in order to locate the first interface between two subdomains close to the wedge in order to evaluate the effect of the reflections produced by the presence of the wedge on the solution. The following subdomains were 10 metres long so the wedge is completely included in one subdomain. The sound pressure level fields computed with the standard WEM and the FWEM are shown in figures 3.61 and 3.62 respectively.

Figure 3.61: WEM solution, sound pressure level (SPL).

Figure 3.62: FWEM solution, sound pressure level (SPL).

The insertion loss (IL) computed with both methods at 0.1 and 2 meters above the ground level are shown in figures 3.63 and 3.64 respectively.

Figure 3.63: Insertion loss (IL) computed at ground level. — WEM, — — FWEM ($\pi/2$), — — FWEM (π), — — FWEM (2π).

Figure 3.64: Insertion loss (IL) computed at 2 metres above the ground level. — — FWEM ($\pi/2$), — — FWEM (π), — — FWEM (2π), — WEM.

Figure 3.65 shows the insertion loss computed at 2 metres above the ground level for the last 10 metres of the computational domain. In this figure it can be seen that the diffraction effect is a little better modelled by the propagation angle of 2π . This may be due to the fact that in the region of the wedge there are significant components scattered in all directions. In any case the effect seems to be very slight. However, it may become important in long range problems.

Figure 3.65: Insertion loss (IL) computed at 2 metres above the ground level, last 10 metres. — WEM, - - FWEM ($\pi/2$), - - FWEM (π), - - FWEM (2π).

For the previous calculations the subdomains were chosen in such a way that the complete wedge was included in one subdomain. To evaluate the performance of the interfaces between subdomains for modelling the diffraction effect a new subdomain distribution was tested. In the following example the subdomains were 2 metres long, therefore, the surface of the wedge was covered by different subdomains. The insertion loss computed with the WEM and the FWEM are shown in figures 3.66 and 3.67 respectively.

Figure 3.66: Insertion loss computed at ground level. — WEM, - - FWEM (one subdomain), - · - FWEM (many subdomains).

Figure 3.67: Insertion loss computed at 2 metres above the ground level. — WEM, - - FWEM (one subdomain), - · - FWEM (many subdomains).

This example shows that the interfaces between the subdomains do not interfere with the modelling of the diffraction effect.

The results demonstrate that the FWEM is highly accurate for diffraction modelling of sound fields when the wave front deviates considerably from the free field case.

3.7.3 Diffraction by a sound barrier

It was shown in the previous section that the FWEM can accurately model the sound pressure field in the presence of obstacles like a wedge with an elevation of 45° . In the present section the FWEM is tested in the presence of a highly reflective obstacle: a thin vertical sound barrier. The geometry studied is shown in figure 3.68.

Figure 3.68: Geometry of the problem.

The source was positioned at point $(x,y) = (0,0)$. The operating frequency was 1000 Hz. The domain was 50 metres long and 40 metres high. The top 10 metres ($30 \times \lambda$) were changed into an absorbing layer whilst a loss free domain of height 30 metres was maintained. A damping constant $A = 1.5$ was used. In sections where a regular mesh was used and there was no parameter change in the propagation direction of the solution and no obstacle was

present, the explicit property of the solution scheme was used. The sound barrier was 4 metres high and 0.1 metres wide. A very thin barrier was used in order to compare the computed field with the results obtained with the Fresnel Integral Method. The barrier was positioned at 10 metres from the source.

Propagation direction of the fundamental solutions

According to the study presented in the previous section, the best results for diffraction modelling were obtained with a propagation angle of 2π . However, there was little difference in accuracy between the three sets of propagation directions studied. In the case of a vertical barrier more significant components of the sound field may be scattered in all directions, therefore, the difference in the results may become more significant.

To evaluate the accuracy of the method using different sets of propagation directions a propagation angles of 2π ($[0, 2\pi]$), π ($[\frac{\pi}{2}, -\frac{\pi}{2}]$) and $\pi/2$ ($[\frac{\pi}{4}, -\frac{\pi}{4}]$) were tested. Sets of 20 evenly distributed propagation directions were considered in all cases.

The insertion loss due to the sound barrier was the parameter used to assess the accuracy of the results. The results obtained with the FWEM were compared with the solutions obtained with the standard WEM and the Fresnel integral method [75, 89]. Plots of the insertion loss computed at 0.1 and 2 metres above the ground level are given in figures 3.69 and 3.70 respectively. The results show that even though the diffraction effect seems to be correctly modelled (except for the field close to the sound barrier using a propagation angle of $\pi/2$), the resulting amplitudes are inaccurate for the

three sets of propagation angles. The cause of these inaccuracies is explained in the following section.

Figure 3.69: Insertion loss (IL) computed at 0.1 metres above the ground level. — Fresnel Integral Method, - - WEM, — FWEM ($\pi/2$), — FWEM (π), — FWEM (2π).

Figure 3.70: Insertion loss (IL) computed at 2 metres above the ground level. — Fresnel Integral Method, - - WEM, — FWEM ($\pi/2$), — FWEM (π), — FWEM (2π).

Effect of the backscattered sound on the resulting field

To explain the inaccuracy of the results, we have to take a closer look to the way the standard WEM and the FWEM work. In the WEM the solution involves simultaneously both, the forward and the backward propagating sound fields, for the entire computational domain (see Fig. 3.71a). In the FWEM the backscattered field is not “present” until the solution reaches the subdomain that contains the obstacle, the sound barrier in this case (see Fig. 3.71b).

Figure 3.71: (a) Solution scheme for the WEM , (b) Solution scheme for the FWEM (three subdomains).

In the FWEM the sound pressure values of the first subdomain were computed considering the forward advancing sound field and neglecting any backscattered sound field. To compute the solution in the subdomain that includes the sound barrier, the second subdomain in this example, the starting (driving) sound field is given by the precalculated sound values on the column of nodes on the left hand side of the subdomain. The sound pressure values imposed on that column of nodes do not correspond to the “actual” sound field that is produced by both, the forward advancing front and the backscattered field. This incongruence generate the errors in the resulting sound pressure field.

Solution to the problem of the driving pressures and the backscattered sound incongruence

To compute the sound field for a subdomain like the one of figure 3.72, a “forward advancing” driving condition has to be applied to the left hand side of the subdomain instead of a simple Dirichlet boundary condition. To do this, the corresponding forcing vector has to be calculated.

Figure 3.72: Subdomain with sound barrier. ●●● Points with precalculated sound pressure values, — Radiation boundary condition.

This “forward advancing” driving force is no other than the forcing vector that would “produce” the same pressure values (precalculated) on the left hand side of the subdomain.

Consider now the same subdomain but without the obstacle (see Fig. 3.73).

Figure 3.73: “Dirichlet-driven” boundary condition scheme. ●●● Points with precalculated sound pressure values, — Radiation boundary condition.

Using Dirichlet boundary condition in the left hand side boundary of the subdomain, the pressure values for a forward propagating field can be computed as

$$\mathbf{p}_{\text{inc}} = \mathbf{S}_1^+ \cdot \mathbf{f}_{\text{dir}} \quad (3.21)$$

where $\mathbf{p}_{\text{inc}}$ is the array of pressure values for a forward propagating field, $\mathbf{S}_1^+$ is the pseudo-inverse of the stiffness matrix corresponding to the geometry of figure 3.73 and $\mathbf{f}_{\text{dir}}$ is the forcing vector corresponding to the Dirichlet boundary condition.

Once the pressure vector for the forward propagating field is computed, the “forward advancing” driving force can be calculated applying radiation boundary condition to the left hand side boundary of the subdomain (see Fig. 3.74).

Figure 3.74: Boundary condition scheme to compute the ‘forward advancing’ driving force vector. — Radiation boundary condition.

The force vector is given by

$$\mathbf{f}_{\text{rad}} = \mathbf{S}_2 \cdot \mathbf{p}_{\text{inc}} \quad (3.22)$$

where $\mathbf{f}_{\text{rad}}$ is the “forward advancing” driving force vector and $\mathbf{S}_2$ is the stiffness matrix corresponding to the boundary conditions scheme applied in the subdomain of figure 3.74.

The sound pressure values for the actual subdomain (including the obstacle) may now be computed with the following equation:

$$\mathbf{p} = \mathbf{S}_3^+ \cdot \mathbf{f}_{\text{rad}} \quad (3.23)$$

where $\mathbf{S}_3^+$ is the pseudo-inverse of the stiffness matrix for the geometry of

figure 3.75.

Figure 3.75: Boundary condition scheme for the FWEM. “Dirichlet-driven” scheme. — Radiation boundary condition.

The solution procedure presented above was tested to solve the sound barrier diffraction problem of figure 3.68. The first subdomain, containing the sound source, was 2 metres long. The other subdomains were 1 metre long. The propagation angle used in the first subdomain and in the subdomain containing the sound barrier was of 2π . A propagation angle of $\pi/2$ was used for the rest of the subdomains. An absorbing layer $30 \times \lambda$ deep was applied. A damping constant $A = 1.5$ was used.

The sound pressure level field computed with the standard WEM and the FWEM are shown in figures 3.76 and 3.77 respectively.

Figure 3.76: WEM solution, sound pressure level (SPL).

Figure 3.77: FWEM solution, sound pressure level (SPL).

Plots of the insertion loss computed at 0 and 2 metres above the ground level are given in figures 3.78 and 3.79 respectively.

Figure 3.78: Insertion loss computed at 0 metres above the ground level.
- - WEM, — FWEM, — Fresnel Integral Method.

Figure 3.79: Insertion loss computed at 2 metres above the ground level.
- - WEM, — FWEM, — Fresnel Integral Method.

The figures show the accuracy of the results. FWEM is accurate for diffraction modelling, but care must be taken to ensure that a “forward advancing” driving condition is applied to the left hand side of the subdomain instead of a simple Dirichlet boundary condition.

Propagation in large domains

The geometry studied is shown in figure 3.80.

Figure 3.80: Geometry of the problem.

The source was positioned at point $(x,y) = (0,0)$. The operating frequency was 1000 Hz. The domain was 1 kilometre long and 50 metres high. The top 10 metres ($30 \times \lambda$) were changed into an absorbing layer whilst a loss free domain of height 40 metres was maintained. The absorbing layer was formulated using the model in equation 3.18 with a damping constant $A = 1.5$ for the first segment and $A = 0.9$ for the following ones.

More than 5,000,000 points would be needed to mesh this domain with a node spacing of 3 nodes per wavelength. To alleviate the computational requirements, the segmentation technique was applied, 5 segments 200 me-

tres long where used. A nodal spacing of 3 nodes per wavelength was used for the first segment (1,002,421 nodes were needed for this segment). The mesh density was reduced to 0.2 nodes per wavelength in the propagation direction for the following segments (59,619 nodes were needed for each of them). Considering all the segments together, the number of nodes needed to discretise the computational domain was reduced from 5,000,000 to less than 1,250,000 nodes.

A propagation angle of 2π was used in the first subdomain and in the one including the sound barrier, and $\pi/2$ was used in the following subdomains of the first segment. For the following segments a propagation angle of $\pi/12$ was used. Each subdomain was formed by 25,551 nodes in the first segment and 6,012 nodes in the following ones. This means that the largest matrix system to be solved was reduced from a $5,000,000 \times 5,000,000$ matrix to a $25,551 \times 25,551$ one.

The semi-explicit property of the solution scheme was used in the first segment (the explicit property was used from the second subdomain to the one containing the sound barrier, and from the one following it to the end of the segment) and the explicit property was used for the following segments.

Figure 3.81 show the computed sound field in the complete computational domain.

Figure 3.81: FWEM solution, sound pressure level (SPL).

The resulting sound field for the complete operational part of the domain is shown in figure 3.82.

Figure 3.82: FWEM solution, sound pressure level (SPL). Operational part of the domain.

Plots of the insertion loss computed at 0 and 2 metres above the ground level are given in figures 3.83 and 3.84 respectively. The results are compared with the Fresnel Integral Method [75, 89].

Figure 3.83: Insertion loss (IL) computed at ground level. — Fresnel Integral Method, - - FWEM.

Figure 3.84: Insertion loss (IL) computed at 2 metres above the ground level. — Fresnel Integral Method, - - FWEM.

A plot of the results in the first 100 metres is shown in figure 3.85.

Figure 3.85: Insertion loss (IL) computed at 2 metres above the ground level. First 100 metres. — Fresnel Integral Method, - - FWEM.

The accuracy of the method for diffraction modelling is apparent from the figures.

3.7.4 Sound refraction under upwind and downwind atmospheric conditions

In the previous sections modelling of sound propagation in a non-refracting atmosphere was considered. In general, for long distance sound propagation, atmospheric refraction has large effects on the sound pressure field. Therefore, in atmospheric sound propagation modelling it is imperative for the numerical solution to consider refraction effects. In the following examples the accuracy of the method for modelling sound propagation under downwind and upwind conditions was tested.

Atmospheric refraction is mainly caused by vertical gradients of the temperature and the wind speed. Wind and turbulence may have complex effects. In general, wind can be taken into account approximately by treating wind speed as a contribution to the sound speed. The resulting sound speed profile, the *effective sound speed*, can be approximated by the sum of the adiabatic sound speed and the component of the wind velocity in the direction of sound propagation. In general, the effective sound speed approximation is valid in situations in which sound waves travel with relatively small elevation angles [68]. In the atmosphere, the vertical variation of the wind speed and the temperature is larger near the ground surface and decreases with height. For the following examples, a logarithmic sound speed profile [65, 67, 70] was introduced to model the refraction effects:

$$c(z) = \begin{cases} c_0 + b \cdot \ln\left(\frac{z}{z_0}\right), & \text{for } z > z_0 \\ c_0, & \text{for } z \leq z_0 \end{cases} \quad (3.24)$$

where c_0 is the sound speed at ground level, z_0 is called the roughness length of the ground surface and b is a parameter that can be positive or negative. For downwind refraction b is positive and for upwind directions b is negative. This was directly implemented by allowing the wavenumber k to vary accordingly through the field.

A propagation problem including refraction and diffraction effects was chosen to evaluate the performance of the method. A computational domain 30 metres high and 150 metres long is used to show the resulting sound pressure level. The geometry of figure 3.86 was used in these examples. The presence of a sound barrier has the advantage of demonstrating the effect of a refractive atmosphere more clearly.

Figure 3.86: Geometry of the problem.

Sound propagation under downwind refracting condition is considered in the first example. The values used for the parameters b and z_0 of equation 3.24 were $b = 2$ and $z_0 = 0.1$. Figure 3.87 shows the sound pressure level computed with the FWEM.

Figure 3.87: Downwind refracting condition. FWEM solution, sound pressure level (SPL).

Figure 3.87 shows that the sound pressure field is enhanced behind the sound barrier due to the energy refracted back towards the ground.

To asses the accuracy of the method the results obtained with the FWEM and the standard WEM are compared . A computational domain 30 metres high and 50 metres long is used for the comparisons.

The insertion loss computed at 0 and 2 metres above the ground level is shown in figures 3.88 and 3.89 respectively.

Figure 3.88: Downwind refracting condition: Insertion loss (IL) computed at 0 metres above the ground level. — WEM, - - FWEM.

Figure 3.89: Downwind refracting condition: Insertion loss (IL) computed at 2 metres above the ground level. — WEM, - - FWEM.

In figures 3.88 and 3.89 the SPL are higher than the ones of figures 3.78 and 3.79 because of the downwind refraction effects.

Sound propagation under upwind refracting condition is considered in the next example. The values used for the parameters b and z_0 of equation 3.24 were $b = -2$ and $z_0 = 0.1$.

Figure 3.90 shows the sound pressure level computed with the FWEM. It can be seen in the figure the shadow zone behind the sound barrier produced by the upwind refraction.

Figure 3.90: Upwind refracting condition. FWEM solution, sound pressure level (SPL).

The insertion loss computed at 0 and 2 metres above the ground level is shown in figures 3.91 and 3.92 respectively.

Figure 3.91: Upwind refracting condition: Insertion loss (IL) computed at 0 metres above the ground level. — WEM, - - FWEM.

Figure 3.92: Upwind refracting condition: Insertion loss (IL) computed at 2 metres above the ground level. — WEM, - - FWEM.

The shadow zone effect is clear in figure 3.91 as, after some metres (where the shadow zone begins), the insertion loss increases with distance from the sound barrier. This effect is not noticeable yet in figure 3.92 because at 2 metres from the ground level, the shadow zone becomes more evident at further distances from the barrier.

It should be noted that the previous examples are included only to demonstrate the accuracy of the method. The logarithmical sound speed profile used in these examples do not take into account additional wind speed gradients induced by the screen. These screen induced wind-speed gradients may enhance the downward refraction of sound waves [90]. More realistic sound speed profiles can be introduced into the model in a similar way.

Sound refraction and diffraction modelling with the source located above the ground level

In the previous examples the sound source was located at ground level. In the present section the geometry of figure 3.93 is used to examine the accuracy of the method when the source is located above ground level.

Figure 3.93: Geometry of the problem.

The source was positioned at point $(x,y) = (0,1)$. The operating frequency was 1000 Hz. The domain was 50 metres long and 40 metres high. The top 10 metres ($30 \times \lambda$) were changed into an absorbing layer whilst a loss free domain of height 30 metres was maintained. The sound barrier was 4 metres high and 0.1 metres wide. The barrier was positioned at 10 metres from the source.

The sound pressure level fields computed with the standard WEM and the FWEM are shown in figures 3.94 and 3.95 respectively.

Figure 3.94: Source above the ground level: WEM solution, sound pressure level (SPL).

Figure 3.95: Source above the ground level: FWEM solution, sound pressure level (SPL).

The interference pattern characteristic of a source above a reflecting flat surface is clear in the figures. In figure 3.94, the reflections back from the sound barrier are present in the complete area behind it. For the FWEM solution (figure 3.95) these reflections are present only in the subdomain containing the wall.

The insertion loss computed with both methods at 0 and 2 metres above the ground level are shown in figures 3.96 and 3.97 respectively.

Figure 3.96: Source above the ground level: Insertion loss (IL) computed at ground level. — WEM, - - FWEM.

Figure 3.97: Source above the ground level: Insertion loss (IL) computed at 2 metres above the ground level. — WEM, - - FWEM.

The accuracy of the FWEM is apparent from the figures.

The same geometry was used to show the accuracy of the results under the effects of a refractive atmosphere. A computational domain 40 metres high and 150 metres long is used to show the resulting sound pressure level. An absorbing layer 10 metres deep, with damping constant $A = 1.5$ was used. Sound propagation under downwind refracting condition is considered in the first example. The values used for the parameters b and z_0 of equation 3.24 were $b = 2$ and $z_0 = 0.1$. Figure 3.98 shows the sound pressure level computed with the FWEM.

Figure 3.98: Source above the ground level. Downwind refracting condition. FWEM solution, sound pressure level (SPL).

The refraction effects are evident in the figure.

To asses the accuracy of the method the results obtained with the FWEM and the standard WEM are compared . A computational domain 30 metres high and 50 metres long is used for the comparisons.

The insertion loss computed at 0 and 2 metres above the ground level is shown in figures 3.99 and 3.89 respectively. The figures show that FWEM remains highly accurate.

Figure 3.99: Source above the ground level. Downwind refracting condition. Insertion loss (IL) computed at ground level. — WEM, - - FWEM.

Figure 3.100: Source above the ground level. Downwind refracting condition. Insertion loss (IL) computed at 2 metres above the ground level. — WEM, - - FWEM.

For upwind refracting condition the parameters b and z_0 of equation 3.24 were set to $b = -2$ and $z_0 = 0.1$. Figure 3.101 shows the sound pressure level computed with the FWEM.

Figure 3.101: Source above the ground level. Upwind refracting condition. FWEM solution, sound pressure level (SPL).

The insertion loss computed at 0 and 2 metres above the ground level is shown in figures 3.103 and 3.103 respectively.

Figure 3.102: Source above the ground level. Upwind refracting condition. Insertion loss (IL) computed at 0 metres above the ground level.
— WEM, - - FWEM.

Figure 3.103: Source above the ground level. Upwind refracting condition. Insertion loss (IL) computed at 2 metres above the ground level.

— WEM, - - FWEM.

It is clear from the figures that the accuracy of the FWEM is maintained for these complex sound propagation problems.

3.8 Discussion

In this chapter, the implementation of the FWEM for two dimensional problems was presented and the accuracy of the method was tested. The benchmark problems used for testing included not only long range free propagation problems, but also diffraction and refraction effects. No modification to the original implementation is needed to compute the solution for these problems.

It can be seen from the results that the FWEM is highly accurate for modelling sound propagation problems in extensive domains with mesh densities

as low as 0.2 nodes per wavelength in the propagation (horizontal) direction and 3 node per wavelength in the quadrature (vertical) direction.

In the examples presented for large domains the number of nodes needed to discretise the computational domain was reduced from more than 5,000,000 (using a mesh spacing of 3 nodes per wavelength) to less than 1,250,000 nodes (using a combination of 3 and 0.2 nodes per wavelength in the propagation direction). Five “domains” (segments) were used. The sizes of the “actual” meshes used were reduced from 5,000,000 nodes to less than 1,002,501 nodes (1,002,501 nodes for the first segment and 59,619 nodes for the rest). This makes the meshing process much easier to handle.

Each subdomain was 5 metres long for the first segment and 20 metres long for the following ones. Each subdomain was formed by 25,551 nodes in the first segment and 6,012 nodes in the following ones. Therefore, the largest matrix system to be solved was reduced from a $5,000,000 \times 5,000,000$ matrix to a $25,551 \times 25,551$ one. This considerably increases the efficiency of the method.

In sections of the computational domain where there were no obstacles or parameter changes in the advancing direction of the numerical solution, the advancing method became completely explicit. This considerably reduced the computing time.

Although only regular meshes have been used in the calculations, the FWEM may also be formulated using irregular meshes thus enabling modelling around objects of arbitrary shape. An issue which will now arise is the identification of the advancing front.

The only mesh data necessary for the FWEM implementation is the points coordinates and any information to identify the neighbouring points. This flexibility allows the FWEM to be compatible with FE or FD meshes.

The discretization method was shown to be highly stable, and the implementation straightforward.

Chapter 4

Implementation of the three dimensional FWEM

4.1 Introduction

It will be shown that a critical issue for the adaptation of the FWEM to 3D problems is to find an adequate selection of the propagation directions of the fundamental solutions. Another important issue is the implementation of the energy absorbing layer for the radiation condition in order to avoid “artificial” reflections that may contaminate the solution in the operational part of the computational domain.

4.2 Implementation

The 3D FWEM solution scheme follows the same implementation as the 2D strategy. The computational domain (see Fig. 4.1) is divided into a number of subdomains. To compute the sound pressure field in each subdomain a modified version of the WEM [85] is applied.

Figure 4.1: Computational domain.

For the first subdomain, if the solution is advancing to the right, a free radiating condition is imposed on the right-hand side boundary of the local mesh. Neumann, Dirichlet, impedance or radiation boundary conditions may be also applied to the other surfaces as required by the model.

For the following subdomains, a Dirichlet condition is imposed on the left hand side surface of the new subdomain using precalculated values. If an obstacle is present, a volumetric loading should be applied. A free radiating condition is imposed on the opposite boundary of the local mesh. Neumann, Dirichlet, Impedance or Radiation boundary conditions may then be applied to the upper and lower faces as required by the model. Figure 4.2 shows the restraint scheme applicable to each local mesh.

Figure 4.2: Computational domain. FWEM restraint scheme. Green colour represents Dirichlet boundary condition, radiation boundary condition is represented in white. Blue colour represent the surfaces with boundary conditions imposed by the “original” model.

4.3 Specific modelling issues

The FWEM requires modifications to the standard WEM implementation to compute the sound pressure field for each subdomain. These modifications mainly involve the directions of propagation of the fundamental solutions to be used for the field nodes, the propagation directions used for the boundary nodes with radiation condition, and the application of an absorbing layer to prevent “artificial” reflections from the upper and lateral surfaces of the computational domain.

To present these modifications, a benchmark propagation problem consisting in a point source located at ground level in a rectangular computational domain is used. The domain was 10 metres long ($0 < x < 10$), 10 metres wide ($-5 < y < 5$) and 5 metres high ($0 < z < 5$). The sound source was located at point $(x, y, z) = (0, 0, 0)$. The operating frequency used for the calculations in this section was 1000 Hz. A mesh spacing of 3 nodes per wavelength was used. The subdomains used in the examples were 0.5 metre long. The two surfaces in contact with the sound source (the surface $x = 0$ and the ground surface) were assumed perfectly reflective.

4.3.1 Propagation direction of the fundamental solutions

The importance of a correct selection of the propagation directions of the fundamental solutions for the accuracy of the discretization method was shown in the previous chapter for the two dimensional implementation. This must also be considered for the 3D case.

A direction set based on the Buckminster Fuller geodesic dome (using the `Matlab` function “bucky”) was shown to give accurate results for the standard WEM implementation [85]. This set consists in 60 evenly spaced directions (see Fig. 4.3).

Only the directions propagating outwards the computational domain are used for radiation boundary condition in the standard WEM implementation (see Fig. 4.4). The number of wave functions used is dependent on the propagation cone apex angle. In the first instance only vector directions from the original bucky distributions were used.

Figure 4.3: Propagation directions generated by “bucky”, a direction set based on the Buckminster Fuller geodesic dome.

Figure 4.4: Propagation directions for radiation condition using “bucky”.

The result for the SPL computed with the FWEM using the standard WEM implementation for the solution within each subdomain is shown in figure 4.5.

Figure 4.5: SPL computed with the FWEM using the standard WEM implementation in each subdomain.

The inaccuracy of the solution is evident from the figure.

In the second calculation modified sets of propagation directions were used for the FWEM implementation for internal and radiation templates. A number of distributions were explored before the parametric model described

below was adopted. Figure 4.6 presents the parameters used to define the new sets of directions.

Figure 4.6: Diagram of the angles for the selection of the propagation directions. ($\longrightarrow$) Direction of propagation of the solution scheme (or direction of the normal to a surface with radiation boundary condition when defining the directions for radiation condition), ($\longrightarrow$) directions of propagation of the fundamental solutions.

The set of propagation directions used (internally) in this FWEM consisted of six arrays of vectors. The first three arrays had 8 vectors each, with β ranging from 0 to 2π in steps of $2\pi/8$. The value of α was $\pi/12$ for the first array, $2\pi/12$ for the second one, and $3\pi/12$ for the third one. For the next three arrays, β varied from 0 to 2π in steps of $2\pi/16$. The value of α was $4\pi/12$ for the first array, $5\pi/12$ for the second one, and $(6\pi - \pi/3)/12$ for the third one. The set of propagation directions is shown in figures 4.7, 4.8 and 4.9.

Figure 4.7: Propagation directions for the FWEM.

Figure 4.8: Propagation directions for the FWEM. View perpendicular to the propagation direction of the wave front.

Figure 4.9: Propagation directions for the FWEM. View parallel to the propagation direction of the wave front.

The set of propagation directions used in the FWEM for the radiation condition consisted of six arrays of vectors. The first three arrays had 8 vectors each, with β ranging from 0 to 2π in steps of $2\pi/8$. The value of α was $\pi/24$ for the first array, $2\pi/24$ for the second one, and $3\pi/24$ for the third one. For the next three arrays, β varied from 0 to 2π in steps of $2\pi/16$. The value of α was $4\pi/24$ for the first array, $5\pi/24$ for the second one, and $6\pi/24$ for the third one. The set of propagation directions is shown in figures 4.10, 4.11 and 4.12.

Figure 4.10: Propagation directions for radiation condition.

Figure 4.11: Propagation directions for radiation condition. View parallel to the normal of the surface.

Figure 4.12: Propagation directions for radiation condition. View perpendicular to the normal of the surface.

The solution computed with the new set of propagation directions for the fundamental solutions is shown in figure 4.13.

Figure 4.13: SPL computed with the FWEM using the standard WEM implementation in each subdomain.

Figure 4.13 shows an apparent improvement in the solution. Undesired “artificial” reflections from the upper and lateral surfaces of the computational domain are still present in the computed sound field. Again an energy absorbing boundary layer is required to prevent this undesired reflections from contaminating the solution.

4.3.2 Implementation of the energy absorbing layer for radiation boundary condition

For a rectangular computational domain (see Fig. 4.14), the absorbing boundary layer can be simply implemented as a function of the radial distances between the nodes within the absorbing layer and the centerline running along the lower surface of the domain. As a result the operational part of the computational domain is now semi-cylindrical in shape (see Fig. 4.15).

Figure 4.14: Computational domain.

The absorption is progressively applied to the wave number k , once a threshold radius is exceeded, by adding an imaginary term following equation 4.1

$$k_{(\mathbf{r})} = k_0 - jA \left(\frac{\mathbf{r} - \mathbf{r}_T}{\mathbf{r}_M - \mathbf{r}_T} \right)^2 \quad (4.1)$$

where $\mathbf{r}_T$ is the radial distance from the bottom of the absorbing layer to the centerline of the half-cylinder, $\mathbf{r}_M$ is the larger radial distance from the

centerline of the half-cylinder to the outer surfaces of the computational domain, A is the damping constant, and k_0 is the wavenumber that would correspond to the point without absorption.

Figure 4.15: Computational domain with absorbing boundary layer. □ Points laying within the absorbing boundary layer. □ Points in the operational part of the computational domain.

An absorbing boundary layer $4 \times \lambda$ deep (maximum thickness) was used to compute the solution for the free propagation problem presented in the previous section. A damping constant $A = 1.5$ was used. The computed SPL field is shown in figure 4.16.

Figure 4.16: SPL computed with the FWEM applying an absorbing layer.

The figure shows the effect of the absorbing layer which is successfully absorbing the sound before it reaches the original perimeter.

The SPL field for the operational part of the computational domain without boundary layer is shown in figure 4.17.

Figure 4.17: SPL computed with the FWEM without absorbing boundary layer. Operational part of the computational domain.

The solution computed with the boundary layer is shown in figure 4.18.

Figure 4.18: SPL computed with the FWEM applying an absorbing layer. Operational part of the computational domain.

The figures show that the results are considerably improved with the implementation of a boundary layer even though a relatively thin absorbing layer was applied in this example. An absorbing layer more than 7 times deeper was used in the 2D examples presented in the previous chapter). In this simulation reflections are inevitably generated from the new internal boundary because of the sudden acoustic parameter change. For larger domains (in the propagation direction) deeper layers will be necessary.

To assess the accuracy of the sound pressure field computed with the FWEM, the results were compared with the theoretical solution for free field radiation, $P_t(\mathbf{r})$, the Green's function

$$P_t(\mathbf{r}) = \frac{e^{-jkr}}{r}. \tag{4.2}$$

where $\mathbf{r}$ is the position vector of a point in the computational domain and r is the distance from the point to the sound source. The theoretical solution is presented in figure 4.19 for comparison.

Figure 4.19: SPL computed with the theoretical solution.

The resulting SPL for the plane $y = 0$ computed without an absorbing layer, with an absorbing layer and with the theoretical solution, is shown in figures 4.20, 4.21 and 4.22 respectively.

Figure 4.20: SPL computed with the FWEM without absorbing boundary layer for the plane $y = 0$.

Figure 4.21: SPL computed with the FWEM applying an absorbing boundary layer for the plane $y = 0$.

Figure 4.22: SPL computed with the theoretical solution for the plane $y = 0$.

The solution of figure 4.21 presents an improvement over figure 4.20 on account of the reduction of the spurious reflections. Figure 4.23 shows the three solutions for the SPL at a line defined by $(0 < x < 10, y = 0, z = 0.5)$.

Figure 4.23: SPL computed at a line defined by $(0 < x < 10, y = 0, z = 0.5)$.
— Theoretical solution, — FWEM (without absorbing layer), - - FWEM (with absorbing layer).

The figures show a very good agreement between the two results and the theoretical solution. However, it should be noted that this domain is relatively short.

Figures 4.24 and 4.25 show the SPL computed at the plane $z = 1$ (for points $0 < y < 4$) with the FWEM and the theoretical solution respectively.

Figure 4.24: SPL computed at the plane $z = 1$ with the FWEM applying an absorbing boundary layer.

Figure 4.25: SPL computed at the plane $z = 1$ with the theoretical solution.

These figures show that although the results are accurate in the plane $y = 0$, the solution becomes less accurate away from the direct line of the source (in the propagation direction).

In order to get a better evaluation of this 3D FWEM formulation some simulations were then run on a longer domain.

4.4 Numerical results - Long Domain

The FWEM was then applied to three different models. These included a free radiation problem and two diffraction problems. A simple “continuous” sound barrier and a “finite” sound barrier were used for the diffraction problems. In all cases the ground and obstacles were assumed perfectly reflective ($\partial p / \partial \mathbf{n} = 0$). Identical FWEM formulations were used in all three. In sections of the computational domain where there were no obstacles the advancing computation was completely explicit.

The free propagation results were compared with the theoretical solution (Eqn. 4.2). The diffraction results were qualitatively assessed.

4.4.1 Free field propagation

In the previous sections the FWEM was already tested for a free field sound propagation problem. In this section a computational domain two times longer (in the x direction) is used.

A point source was located at ground level in a rectangular computational domain. The computational domain was 20 metres long ($0 < x < 10$), 10 metres wide ($-5 < y < 5$) and 5 metres high ($0 < z < 5$). The sound source was located at point $(x, y, z) = (0, 0, 0)$. The operating frequency used for the calculations in this section was 1000 Hz. A mesh spacing of 3 nodes per wavelength was used, 1,035,351 points were required. Each subdomain was 0.5 metres long, each conformed by approximately 25,000 points. The two surfaces in contact with the sound source (the surface $x = 0$ and the ground surface) were assumed perfectly reflective. An absorbing layer $4 \times \lambda$ deep with a damping constant $A = 1.5$ was applied.

The computed SPL for the operational part of the domain is shown on the iso surface plot of figure 4.26.

Figure 4.26: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph.

It is clear from the figure that, as the solution advanced, the effect of the absorbing boundary layer used in this example becomes insufficient to avoid the undesired “artificial” reflections.

The theoretical solution is shown in figure 4.27 for reference.

Figure 4.27: Theoretical solution for the SPL field in the operational part of the computational domain. Iso-surface graph.

Figure 4.28 shows the SPL computed in the plane $y = 0$.

Figure 4.28: FWEM solution for the SPL field in the plane $y = 0$.

The theoretical solution is shown in figure 4.29.

Figure 4.29: Theoretical solution for the SPL field in the plane $y = 0$.

These figures show how the “artificial” reflections produced by the surfaces with radiation boundary condition affect the accuracy of the solution.

Figure 4.30 shows the SPL computed in the plane $z = 0$.

Figure 4.30: FWEM solution for the SPL field in the plane $z = 0$.

The theoretical solution is shown in figure 4.31.

Figure 4.31: Theoretical solution for the SPL field in the plane $z = 0$.

As it was mentioned in the previous section, for distances “close” to the sound source (*i.e.* less than 10 metres in this example) the solution is less accurate away from the line of the source (in the propagation direction).

As it would be expected, as the solution advances in the propagation direction the inaccuracies become smaller (if the effects of the “artificial” reflections are excluded) as the wavefront curvature decreases. Then the restricted range of directions of the fundamental solutions become better aligned with the propagating wave.

4.4.2 Diffraction by a “continuous ” sound barrier

In the following example, a problem of sound diffraction by a thin sound barrier is considered. The barrier was 2 metres high, the thickness was of 0.1 metres and it was considered infinitely wide. The wall was located at 3.2 metres from the sound source. The surfaces of the barrier and the two surfaces in contact with the source (the surface $x = 0$ and the ground surface) were assumed perfectly reflective. A mesh spacing of 3 nodes per wavelength was used, 1,035,351 points were required. Each subdomain was 0.5 metres long, and contained approximately 25,000 points.

The computed SPL is shown in the iso-surface plot of figure 4.32.

Figure 4.32: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph.

A view in the xz plane is shown in figure 4.33.

Figure 4.33: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph.

The SPL computed at a plane $y = 0$ (centerline of the computational domain in the propagation direction) is shown in figure 4.34.

Figure 4.34: FWEM solution for the SPL field in the operational part of the computational domain. Slice in the plane $y = 0$.

The figures show clear shadow patterns and are consistent with the 2D calculations presented in the previous chapter (see figure 3.77).

4.4.3 Diffraction by a finite sound barrier

Sound diffraction by a “narrow” sound barrier was then considered. In the following example, the barrier was 2 metres high, 0.1 metres thick and 4 metres wide. It was located at 3.2 metres from the sound source. The surfaces of the barrier and the two surfaces in contact with the sound source (the surface $x = 0$ and the ground surface) were assumed perfectly reflective. A mesh spacing of 3 nodes per wavelength was used, 1,035,351 points were required. Each subdomain was 0.5 metres long, and contained approximately 25,000 points.

The computed SPL is shown in the isosurfaces plot of figure 4.35.

Figure 4.35: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph.

A view in the xz plane is shown in figure 4.36. In order to see the diffraction over the upper edge of the barrier, a view of the results in the xz plane for points $-5 < y < 0$ is presented in figure 4.37.

Figure 4.36: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph.

Figure 4.37: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph for points lying between $y = -5$ and $y = 0$.

The results at the plane $y = 0$ are shown in figure 4.38. A view in the xy plane is shown in figure 4.40.

Figure 4.38: FWEM solution for the SPL field in the operational part of the computational domain. Slice in the plane $y = 0$.

Figure 4.39: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces graph.

In order to see the flanking diffraction effects on the sides of the barrier, a view of the results in the xy plane for points $z < 1$ is presented in figure 4.37.

Figure 4.40: FWEM solution for the SPL field in the operational part of the computational domain. Isosurfaces lying below the plane $z = 1$.

The results at the plane $z = 1$ are shown in figure 4.41.

Figure 4.41: FWEM solution for the SPL field in the operational part of the computational domain. Slice in the plane $z = 1$.

The results are extremely encouraging as the expected mix of shadow zones are produced. However, no quantitative validation of these results were carried out in the present work.

4.5 Discussion

Based on the detailed study of the FWEM for 2D sound propagation problems, an implementation for 3D problems was presented. The importance of a correct selection of the propagation directions of the fundamental solutions was demonstrated with a simple free propagation benchmark problem. A set of refined propagation directions was then proposed and a significant improvement on the accuracy of the method was obtained.

An absorbing boundary layer to prevent “artificial” reflections back from the upper and lateral surfaces of the computational domain to affect the solution in the operational part of the domain was implemented. A relatively thin layer was used all of these examples and the contaminating effect of the “artificial” reflections was seen to increase as the solution was advancing. A deeper boundary layer should be applied to increase the efficiency of the boundary layer.

The method was then implemented to successfully model some diffraction effects.

In sections of the computational domain where there were no obstacles the advancing method became completely explicit. This was taken advantage of to considerably reduce the computing time. There are still challenging issues associated with general computational details which are beyond the scope of this thesis but these are common with any large 3D modelling study.

Chapter 5

Conclusions

5.1 Conclusions

5.1.1 Two dimensional implementation

It can be seen from the results that the FWEM offers the basis of a highly accurate discretization method to solve the Helmholtz equation for extensive domains with mesh densities as low as 0.2 nodes per wavelength in the propagation (horizontal) direction and 3 node per wavelength in the quadrature (vertical) direction.

As expected, due to loss of reflections from obstacles and difficulties modelling wave fronts with high curvature, the FWEM gives inaccurate results in the near field. This problem can be circumvented to some extent by using the regular WEM for the near field calculations with which it has a natural interface. The method was tested with sources positioned not only at ground level but also at a higher level while maintaining the accuracy of the results.

The selection of the directions of propagation of the fundamental solutions was studied and some clear conclusions were reached. The propagation angle of the set of directions used in the subdomain that contains the sound source/sources should be 2π . The same angle should be used for any subdomain containing an obstacle. The reason for this is that the sound source and the diffraction effects are better modelled using a propagation angle of 2π . For free propagation, for the first hundred metres approximately, an angle of $\pi/2$ should be used. Once the wave front has become approximately planar the propagation angle may be reduced to $\pi/12$ allowing the mesh spacing in the direction of propagation to be significantly reduced (to 0.2 nodes per wavelength).

If a reflective object or some kind of “secondary source” produces a change in the wave front, the propagation angle should be changed back to $\pi/2$ (a mesh spacing of 3 nodes per wavelength should then be applied). Once the wavefront becomes planar again, the propagation angle may be once more reduced to $\pi/12$ and a mesh spacing of 0.2 nodes per wavelength may be used.

The “improved” radiation condition developed in section 3.3.1 for the advancing front is remarkably robust. The treatment of upper radiation condition is critical but this is a problem with any acoustic calculation in a field with shallow grazing angles and is not specific to this method. The results showed that an absorbing layer approximately $30 \times \lambda$ deep is enough in most cases. A correct selection of the damping constant is essential. Too weak a damping regime would allow some of the sound energy to be reflected back into the operational part of the domain. If, on the other hand, the damping is too severe, the “fast” change in the media conditions would produce reflections from the interface to the absorbing layer. Unfortunately, there is not

a clear analytical method to determine the best choice. In the present work the value for the damping constant was adjusted by a heuristic approach. In the present study, a number of around 5 trials was found to be sufficient to find a suitable value for this parameter (not necessarily optimal). The optimum choice for this parameter varies with frequency.

It should be noted that even a combination of an absorbing layer and a radiation condition cannot *completely* eliminate spurious reflections. However, the results presented in section 4.4 show the computed sound fields to be highly accurate even for very shallow domains.

The importance of using local coordinate systems to reduce numerical noise was also noted. This problem is not specific to the FWEM, and can be attributed to the evaluation of trigonometric functions using large arguments.

Although only regular meshes have been used in the calculations, the FWEM may also be formulated using irregular meshes thus enabling modelling around objects of arbitrary shape. A specific issue which will then arise will be the identification of the advancing front.

In sections of the computational domain where regular meshes are used, the advancing method may be made completely explicit if there are no wavenumber parameter changes in the advancing direction of the numerical solution. A semi-explicit scheme may be used if the changes in the wavenumber parameters are located only in the boundaries.

It was demonstrated that the FWEM required less computational time than the WEM to compute the sound pressure field and that the explicit property enhanced the speed of the method.

The accuracy of the sound field computed with the FWEM was shown to be highly accurate in all the benchmark problems studied. The main purpose of this method, long range free field propagation modelling, was successfully achieved, not only for non-refracting atmospheres but also for refracting problems.

A representative operational frequency of 1000 Hz was used in the examples presented in the present work. The dimensions and “shapes” of the geometries used in the examples were defined arbitrarily. There is no “special” relationship between the dimensions and angles used in the geometries and the sound wavelength. Similar results were obtained in tests carried out by the author for different operational frequencies and different geometries.

When tested for modelling diffraction effects the FWEM produced highly accurate results. Nevertheless, care must be taken when using the FWEM to model sound diffraction in cases where there is significant reflection from the obstacles (*i.e.* diffraction by a vertical barrier). Inaccuracies may arise as, if a Dirichlet loading is applied, the sound pressure values of the starting field used for the solution at the subdomain that contains the obstacle is incongruent with the “actual” field that is affected by the reflections produced. In these cases, the accuracy of the method is maintained using a volumetric rather than a simple Dirichlet loading.

In the examples presented for large domains the number of nodes needed to discretise the computational domain was reduced from more than 5,000,000 (using a mesh spacing of 3 nodes per wavelength) to less than 1,250,000

nodes (using a combination of 3 and 0.2 nodes per wavelength in the propagation direction). Five “domains” (segments) were used. This makes the meshing process much easier to handle. The sizes of the system matrix to be solved were reduced from 5,000,000 degrees of freedom to less than 1,002,501 (1,002,501 nodes for the first segment and 59,619 nodes for the rest).

Each subdomain was 5 metres long for the first segment and 20 metres long for the following ones. Each subdomain was formed by 25,551 nodes in the first segment and 6,012 nodes in the following ones. Therefore, the largest matrix system to be solved was reduced from a $5,000,000 \times 5,000,000$ matrix to a $25,551 \times 25,551$ one.

5.1.2 Three dimensional implementation

The implementation of the FWEM was extended to solve 3D sound propagation problems. The importance of a correct selection of the propagation directions of the fundamental solutions was again demonstrated. Further research should be carried out to establish if greater improvement in the accuracy of the solution is possible.

An absorbing boundary layer to prevent “artificial” reflections back from the upper and lateral surfaces of the computational domain was implemented. In this study a very thin layer (less than $4 \times \lambda$) was used in the examples and the effect of the “artificial” reflections was increasing as the solution was advancing. A deeper boundary layer should be applied to increase the efficiency of the boundary layer. A change on the damping constant to lower values as the solution advance (as implemented for 2D problems) should also be applied.

The method was implemented successfully to model 3D sound diffraction effects.

The largest matrix system to be solved was reduced from $1,035,351 \times 1,035,351$ to $25,000 \times 25,000$ (for each subdomain). This considerably reduced the computational resources needed to compute the solution.

In sections of the computational domain where there were no obstacles or parameter changes in the advancing direction of the numerical solution, the advancing solution method again became completely explicit considerably reducing the computing time.

The only mesh data necessary for the FWEM implementation is the points coordinates and any information to identify the neighbouring points. This flexibility allows the FWEM to be compatible with FE or FD meshes.

5.2 Developments

Only regular meshes have been used in the calculations presented in this thesis. The FWEM may also be formulated using irregular meshes thus enabling modelling around objects of arbitrary shape. The issue will now be the identification of the advancing front. A method to allow the solution scheme to follow the main direction of propagation of the wave fronts should be studied. No problem should arise if the meshing process is carried out in a way that the computational domain is divided into a number of subdomains each one with regular/irregular meshes once the interfaces are clearly defined between them.

The boundary layer implementation requires an increase in the area/volume of the computational domain. If the thickness required for the layer is large, a significant increment of the number of points on the computational domain might be needed. Further research should be carried out in order to obtain an optimal configuration for the absorbing boundary layer and/or to develop an alternative radiation condition implementation.

The importance of a correct selection of the propagation directions of the fundamental solutions was demonstrated. Further research should be carried out to establish if greater improvement in the accuracy of the solution is possible for the 3D implementation.

For the FWEM implementation presented in this thesis, direct solutions of the system of equations $\mathbf{S} \cdot \mathbf{p} = \mathbf{f}$ were used. For 2D problems this solutions are efficient enough not to restrict substantially the are of the subdomains. For 3D problems, the sizes of the subdomains are still highly restricted. Iterative solvers should be implemented to allow the use of larger subdomains. Some efficient iterative solvers have been proposed to compute the solutions to the systems of equations resulting from the WEM discretization [91]. These iterative techniques should be implemented and tested for the FWEM discretization.

5.3 Final conclusions

The FWEM represents a computationally light modelling scheme capable of computing accurate solutions for sound propagation problems where other similar methods, like *i.e.* the WEM, would fail. The FWEM retains many other formulation advantages of the WEM.

The maximum mesh spacing needed to obtain accurate solutions was highly reduced (to 0.2 nodes per wavelength) comparing with other methods, including the highly efficient standard WEM.

The FWEM is suitable for long range propagation problems where the backscattered sound field is not of interest. This type of problems are very common in actual situations, *i.e.* study of noise barrier shielding, airplane noise pollution, etc. However further improvements should be carried out to the method to obtain a generally applicable modelling tool, specially for three dimensional problems.

FWEM offers a mathematically simple yet physically satisfying method of modelling large scale propagation problems. In addition all refinements are based on the tuning of a small set of parameters directly associated with the physics of wave propagation.

Publications and presentations associated with this thesis

L. Barrera Rolla and H.J. Rice, *A forward-advancing wave expansion method for numerical solution of large-scale sound propagation problems*. Journal of Sound and Vibration, 296:406-415, 2006.

L. Barrera Rolla and H.J. Rice, *Application of the Forward Wave Expansion Method (FWEM) to sound propagation in the atmosphere*. In the 8th International Symposium TRANSPORT NOISE AND VIBRATION, St. Petersburg, Russia, 2006, S1-1.

L. Barrera Rolla and H.J. Rice, *Explicit Forward-Advancing Wave Expansion Method for Numerical Solution of Large Scale Sound Propagation Problems*. ISMA2004, Leuven, Belgium, 2004, pp. 1433-1448.

Appendix A

Boundary Element formulation

The basics of the direct boundary element method are presented here. Starting with the derivation of the fundamental solution in free space. Then the boundary integral equations are derived from Green's second identity. The theory and implementation of the indirect BEM can be found in [36–38].

First, the formulation for interior problems is presented, followed by the solution of exterior problems, and finally the solution for scattering problems is derived.

A.1 Fundamental Equations

A.1.1 Helmholtz equation.

Consider the *acoustic wave equation*:

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad (\text{A.1})$$

where p represents the sound pressure, c denotes the sound wave speed in the medium and t is time.

If we look at time harmonic standing waves of frequency ω , function p is defined as follows:

$$p(\mathbf{r}, t) = p(\mathbf{r}) \cdot e^{-j\omega t} \quad (\text{A.2})$$

where $\mathbf{r}$ represents the coordinates of the point in the sound field. $p(\mathbf{r})$ satisfies the *Helmoltz equation*:

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = 0 \quad (\text{A.3})$$

where $k = \omega/c$ is the wave number.

Equation A.3 is the governing differential equation for the time-harmonic linear acoustics and must be satisfied in the acoustic domain. The solution of this equation represents the *spatial part* of the solution of the wave equation (Eqn. A.1). The actual pressure time history is given by:

$$p(\mathbf{r}, t) = \text{Re}[p(\mathbf{r}) \cdot e^{-j\omega t}] \quad (\text{A.4})$$

If there is an harmonic source ($q(\mathbf{r}) \cdot e^{-j\omega t}$) producing sound waves in the

medium, the Helmholtz equation get the form:

$$\nabla^2 p_{(\mathbf{r})} + k^2 p_{(\mathbf{r})} = q_{(\mathbf{r})} \quad (\text{A.5})$$

In the following section we are going to find the fundamental solution for this equation.

A.1.2 Fundamental solution. Green's kernel function.

If we consider a point source located at point q in the acoustic domain, emitting a continuous harmonic signal of frequency ω , the sound field G generated by this source at any point $\mathbf{r}_x$ in a free three dimensional space, obeys the following inhomogeneous wave equation:

$$\nabla^2 G_{(r)} + k^2 G_{(r)} = \delta_{(r)} \quad (\text{A.6})$$

where $r = |\mathbf{r}_q - \mathbf{r}_x|$, denotes the distance between $\mathbf{r}_q$ that represents the coordinates of the source, and $\mathbf{r}_x$, the point of the sound field. Note that this function becomes singular when the distance r becomes zero.

The complex solution of equation A.1 for an harmonic plane wave is given by:

$$G_{(r)} \cdot e^{-j\omega t} = \frac{A}{4\pi r} \cdot e^{-j(kr-\omega t)} + \frac{B}{4\pi r} \cdot e^{-j(kr+\omega t)} \quad (\text{A.7})$$

Now, any function $f(kr - \omega t)$ represents a wave moving away from $r = 0$ towards $r \rightarrow \infty$ as t increases. On the other hand a function $f(kr + \omega t)$ represents a wave travelling in towards $r = 0$ from $r \rightarrow \infty$. The $\delta_{(r)}$ function in equation A.6 represents a disturbance at the origin; physically we expect

waves to propagate outward away from this disturbance and not inward from infinity towards the disturbance. This is our *radiation condition*: there should only be waves moving away from the disturbance at the origin. Thus we take $B = 0$. So we get the *Green's kernel function*, $G_{(r)}$:

$$G_{(r)} = \frac{e^{-jkr}}{4\pi r} \quad (\text{A.8})$$

A.1.3 Direct boundary integral formulation.

Direct boundary integral formulation for exterior problems

Consider a body O of boundary surface S , and assume that the singular point q associated with the fundamental solution $G_{(r)}$ is located in the acoustic domain. Since $G_{(r)}$ is singular at $\vec{r}_q$, a tiny spherical volume V_E of radius R_1 and surface S_E enclosing the singular point is excluded from the domain V , as shown in figure A.1. We also define a surface S_R of radius R_2 that involves V .

Figure A.1: q excluded from an exterior domain.

At any point $\mathbf{r}_x$ in the domain V , bounded by S , S_E and S_R , the Green's

kernel function $G_{(r)}$ is non-singular, since the point at position $\mathbf{r}_q$ is not located in V , and the function satisfy the Helmholtz equation,

$$\nabla^2 G_{(r)} + k^2 G_{(r)} = 0 \quad (\text{A.9})$$

The sound pressure at any point in V satisfy

$$\nabla^2 p_{(\mathbf{r}_x)} + k^2 p_{(\mathbf{r}_x)} = 0 \quad (\text{A.10})$$

and the Sommerfeld radiation condition,

$$\lim_{|\mathbf{r}_x| \rightarrow \infty} |\mathbf{r}_x| \cdot \left(\frac{\partial p_{(\mathbf{r}_x)}}{\partial |\mathbf{r}_x|} + jk p_{(\mathbf{r}_x)} \right) = 0 \quad (\text{A.11})$$

Application of Green's second identity to functions G and p produces

$$\int_{V-V_E} (G_{(r)} \nabla^2 p_{(\mathbf{r}_x)} - p_{(\mathbf{r}_x)} \nabla^2 G_{(r)}) dV = \int_{S+S_E+S_R} (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.12})$$

where the normal vector $\mathbf{n}$ is pointing away from the acoustic domain. From equations A.9 and A.10, we have $\nabla^2 p = -k^2 p$ and $\nabla^2 G = -k^2 G$. Therefore, the left-hand side of equation A.12 is simply zero due to cancellation of the two terms. Therefore,

$$\int_{S+S_E+S_R} (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS = 0 \quad (\text{A.13})$$

It can be shown that, for the points located in the surface, by carrying out the integration in spherical coordinates and taking the limit as R_1 goes to zero,

$$\lim_{R_1 \rightarrow 0} \int_{S_E} G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} dS = 0 \quad (\text{A.14})$$

and

$$\lim_{R_1 \rightarrow 0} \int_{S_E} p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}} dS = p_{(\mathbf{r}_q)} \lim_{R_1 \rightarrow 0} \int_{S_E} \frac{\partial G_{(r)}}{\partial \mathbf{n}} dS = p_{(\mathbf{r}_q)} \quad (\text{A.15})$$

Due to the Sommerfeld radiation condition, any physical solution $p_{(\mathbf{r}_x)}$ can be represented by an equivalent point source of certain strength when the solution in infinity is desired:

$$p \rightarrow A \frac{e^{-jkr}}{r}$$

where A is some constant. For the points located in the surface S_R , since p has the same form than G in the infinity, it can be shown that

$$\lim_{R_2 \rightarrow \infty} \int_{S_R} (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS = 0 \quad (\text{A.16})$$

Substituting equations A.14, A.15 and A.16 into equation A.12, we obtain the *boundary integral* equation:

$$p_{(\mathbf{r}_q)} = \int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.17})$$

This formulation states that, for any pressure field, which satisfies the homogeneous Helmholtz equation (Eqn. A.3) and the Sommerfeld radiation

condition (Eqn. A.11), the pressure at any point $\mathbf{r}_q$ in the unbounded domain $V \setminus S$, is related to the pressure distribution $p(\mathbf{r}_x)$ on the closed surface S_E .

When the point $\mathbf{r}_q$ is located on the surface S , as shown in figure A.2, the surface S_E becomes an hemisphere, then we get

$$\lim_{R_1 \rightarrow 0} \int_{S_E} (G(r) \frac{\partial p(\mathbf{r}_x)}{\partial \mathbf{n}} - p(\mathbf{r}_x) \frac{\partial G(r)}{\partial \mathbf{n}}) dS = \frac{1}{2} p(\mathbf{r}_q) \quad (\text{A.18})$$

and equation A.17 becomes

$$\frac{1}{2} p(\mathbf{r}_q) = \int_S (G(r) \frac{\partial p(\mathbf{r}_x)}{\partial \mathbf{n}} - p(\mathbf{r}_x) \frac{\partial G(r)}{\partial \mathbf{n}}) dS \quad (\text{A.19})$$

Figure A.2: q on the boundary.

If the point $\mathbf{r}_q$ is not located in the domain V , the surface S_E is no longer part of the boundary surface of the domain V , at least for $R_1 \rightarrow 0$. Hence, the

direct boundary integral formulation for any position $\mathbf{r}_q$ outside V becomes

$$\int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS = 0 \quad (\text{A.20})$$

The formulations of equations A.17, A.19 and A.20 for smooth surfaces S may be combined into the general direct boundary integral formulation for exterior acoustic problems,

$$C_{(\mathbf{r}_q)} \cdot p_{(\mathbf{r}_q)} = \int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.21})$$

with

$$C_{(\mathbf{r}_q)} = \begin{cases} 1, & \mathbf{r}_q \in V \setminus S, \\ 0, & \mathbf{r}_q \notin V, \\ \frac{1}{2}, & \mathbf{r}_q \in S \end{cases}$$

Direct boundary integral formulation for interior problems

For interior problems with a bounded domain V , a similar derivation applies as for exterior problems, but only one spherical surface S_E must be defined, as shown in figure A.3.

Figure A.3: q excluded from an interior domain.

In this case we have

$$\int_{S+S_E} (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS = 0 \quad (\text{A.22})$$

where the normal vector $\mathbf{n}$ is pointing away from the acoustic domain. As in the exterior problem we find that

$$\lim_{R_1 \rightarrow 0} \int_{S_E} G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} dS = 0 \quad (\text{A.23})$$

and

$$\lim_{R_1 \rightarrow 0} \int_{S_E} p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}} dS = p_{(\mathbf{r}_q)} \lim_{R_1 \rightarrow 0} \int_{S_E} \frac{\partial G_{(r)}}{\partial \mathbf{n}} dS = p_{(\mathbf{r}_q)} \quad (\text{A.24})$$

Then, for $\mathbf{r}_q$ in the domain V ,

$$p_{(\mathbf{r}_q)} = \int_{S+S_E} (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.25})$$

For $\mathbf{r}_q$ located in the boundary,

$$\frac{1}{2}p(\mathbf{r}_q) = \int_{S+S_E} (G_{(r)} \frac{\partial p_{(r_x)}}{\partial \mathbf{n}} - p_{(r_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.26})$$

and, if $\mathbf{r}_q$ is outside the acoustic domain V

$$0 = \int_{S+S_E} (G_{(r)} \frac{\partial p_{(r_x)}}{\partial \mathbf{n}} - p_{(r_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.27})$$

Finally, as for exterior problems, we can express the boundary integral equation for interior problems as:

$$C_{(\mathbf{r}_q)}^0 \cdot p(\mathbf{r}_q) = \int_S (G_{(r)} \frac{\partial p_{(r_x)}}{\partial \mathbf{n}} - p_{(r_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.28})$$

with

$$C_{(\mathbf{r}_q)}^0 = \begin{cases} 1, & \mathbf{r}_q \in V \setminus S, \\ 0, & \mathbf{r}_q \notin V, \\ \frac{1}{2}, & \mathbf{r}_q \in S \end{cases}$$

The body O can be either a vibrating structure (radiation problem) or a passive obstacle (scattering problem).

Scattering problem

The system studied in section A.1.3 represents a typical scattering problem. An incident sound wave p^i is striking an obstacle O in an unbounded acoustic domain V (see Fig. A.4).

Figure A.4: Scattering problem.

The scattered sound wave is denoted by p^s . The total sound pressure p is the superposition of the scattered sound wave and the incident wave:

$$p = p^i + p^s \quad (\text{A.29})$$

Therefore, p^i should be interpreted as the sound field in the absence of the obstacle O . In most applications, the incident wave may be a plane wave or a spherical wave. For an incident plane wave,

$$p^i = Ae^{-jk\mathbf{r}} \quad (\text{A.30})$$

where A is the amplitude of the incident plane wave and $\mathbf{r}$ is the direction of propagation. For an incident spherical wave,

$$p^i = B \frac{e^{-jkr}}{r} \quad (\text{A.31})$$

where B is related with the strength of the point source, and r is the distance from the source. Both, p^i and p^s satisfy the Helmholtz equation, but only p^s has to satisfy the Sommerfeld radiation condition.

If we apply the exterior integral equation, equation A.21, to p^s in the exterior domain V , we get:

$$C_{(\mathbf{r}_q)} \cdot p_{(\mathbf{r}_q)}^s = \int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}^s}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)}^s \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.32})$$

Applying the interior integral equation, equation A.28, to the incident field, if we keep the same normal direction $\mathbf{n}$ as in equation A.32, the interior integral equation for p^i becomes:

$$C_{(\mathbf{r}_q)}^0 \cdot p_{(\mathbf{r}_q)}^i = \int_S (-G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}^i}{\partial \mathbf{n}} + p_{(\mathbf{r}_x)}^i \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS \quad (\text{A.33})$$

Working with equations A.29 and A.32, and taking into account that $C_{(\mathbf{r}_q)} + C_{(\mathbf{r}_q)}^0 = 1$, we get:

$$C_{(\mathbf{r}_q)} \cdot p_{(\mathbf{r}_q)} = \int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS + p_{(\mathbf{r}_q)}^i \quad (\text{A.34})$$

This is the integral equation for a scattering problem. Notice that the only difference between this integral and equation A.21 is the free term due to the incident wave.

A.2 Boundary Element Implementation.

The implementation of the boundary element method consist of two main steps. The first one is to determine the pressure and normal velocity distribution on the boundary surface. In the second step, according with the results obtained in the first step, the pressure field at any point of the acoustic domain is calculated.

In section A.1 we found the direct boundary integral equation for interior or exterior acoustic problems with a closed boundary S :

$$C_{(\mathbf{r}_q)} \cdot p_{(\mathbf{r}_q)} = \int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS$$

where $G_{(r)}$ is the Green's kernel function

$$G_{(r)} = \frac{e^{-jkr}}{4\pi r} \quad (\text{A.35})$$

For two-dimensional harmonic problems we consider the sound field generated by an infinite line source of amplitude $q = -\delta_{(r)}/(2\pi r)$, the resulting sound field can be expressed by:

$$G_{(r)} = -\frac{j}{4} H_0^{(1)}(kr) \quad (\text{A.36})$$

where $H_0^{(1)}(x)$ is the Hankel function of the first kind and zero order. And

$$\frac{\partial G_{(r)}}{\partial n} = j \frac{k}{4} H_1^{(2)}(kr) \frac{\partial r}{\partial n} \quad (\text{A.37})$$

Applying equations A.36 and A.37 to equation A.17, we get:

$$C_{(\mathbf{r}_q)} p_{(\mathbf{r}_q)} = \int_S \left(-\frac{j}{4} H_0^{(1)}(kr) \cdot \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - j \frac{k}{4} H_1^{(2)}(kr) \cdot \frac{\partial r}{\partial \mathbf{n}} \cdot p_{(\mathbf{r}_x)} \right) ds \quad (\text{A.38})$$

To determine the pressure and normal velocity distribution on the boundary S , the surface is discretised into a number of small subsurfaces ("boundary elements"). We then place $\mathbf{r}_q$ (singular point) at each of nodal points on the boundary successively. This procedure is called *collocation*.

For each collocation point, assuming that the pressure and normal velocity distribution in each element is constant, and applying that $C_{(\mathbf{r}_q)} = 1/2$ when $\mathbf{r}_q$ is in the boundary, equation A.38 becomes:

$$\frac{1}{2}p_{(\mathbf{r}_q)} = \sum_{i=1}^{Ne} -\frac{j}{4}H_{0(kr_i)}^{(2)} \cdot \frac{\partial p_{(\mathbf{r}_{xi})}}{\partial \mathbf{n}} \cdot \Delta S_i - \sum_{i=1}^{Ne} j\frac{k}{4}H_{1(kr_i)}^{(2)} \cdot \frac{\partial r_i}{\partial \mathbf{n}} \cdot p_{(\mathbf{r}_{xi})} \cdot \Delta S_i \quad (\text{A.39})$$

where Ne is the number of elements in the surface, and ΔS_i is the length of each element.

Expressing equation A.39 in matrix notation we get

$$\frac{1}{2} \{p_{(\mathbf{r}_q)}\} = [A] \cdot \left\{ \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} \right\} - [B] \cdot \{p_{(\mathbf{r}_x)}\} \quad (\text{A.40})$$

where

$$[A] = \left[-\frac{j}{4}H_{0(kr)}^{(2)} \cdot \Delta S \right]$$

and

$$[B] = \left[j\frac{k}{4}H_{1(kr)}^{(2)} \cdot \frac{\partial r}{\partial \mathbf{n}} \cdot \Delta S \right]$$

Three types of boundary conditions may occur on the boundary surface S :

$$S = S_p \cup S_{v_n} \cup S_Z \Rightarrow \begin{cases} p = \bar{p} & \text{on } S_p, \\ \frac{1}{jw\rho} \cdot p = \bar{v}_n & \text{on } S_Z, \\ p = \bar{Z}_n \cdot v_n & \text{on } S_Z \end{cases}$$

where $\bar{p}$, $\bar{v}_n$ and $\bar{Z}_n$ are prescribed pressure, normal velocity and normal impedance functions, respectively. Equation A.40 is a set of Ne algebraic equations with $2Ne$ nodal unknowns. By means of the boundary conditions we can find either the pressure value $p_{(\mathbf{r}_{xi})}$ or the normal velocity value

$v_{n(\mathbf{r}_{xi})}$ at each node point. The remaining Ne equations in the boundary element model can be solved using equation A.40.

In the second step of the method the pressure at any position $\mathbf{r}_x$ in the acoustic domain, which is not located on the boundary surface S , is obtained from

$$\{p(\mathbf{r}_x)\} = [A_{field}] \cdot \left\{ \frac{\partial p(\mathbf{r}_q)}{\partial \mathbf{n}} \right\} - [B_{field}] \cdot \{p(\mathbf{r}_q)\} \quad (\text{A.41})$$

where $p(\mathbf{r}_q)$ and $\partial p(\mathbf{r}_q)/\partial \mathbf{n}$ are the surface results from the first step, $r = |\mathbf{r}_q - \mathbf{r}_x|$, denotes the distance between $\mathbf{r}_q$ that represents the coordinates of the boundary point, and $\mathbf{r}_x$, the point of the sound field, and

$$[A_{field}] = \left[-\frac{j}{4} H_{0(kr)}^{(2)} \cdot \Delta S \right]$$

$$[B_{field}] = \left[j \frac{k}{4} H_{1(kr_i)}^{(2)} \cdot \frac{\partial r_i}{\partial n} \cdot \Delta S_i \right]$$

Where the only unknown is the pressure vector for the field points.

Scattering problems

As we saw in section A.4, the boundary integral equation for scattering problems is

$$C_{(\mathbf{r}_q)} \cdot p_{(\mathbf{r}_q)} = \int_S (G_{(r)} \frac{\partial p_{(\mathbf{r}_x)}}{\partial \mathbf{n}} - p_{(\mathbf{r}_x)} \frac{\partial G_{(r)}}{\partial \mathbf{n}}) dS + p_{(\mathbf{r}_q)}^i$$

Following the same procedure than in the previous section, for the points in the boundary we get

$$\frac{1}{2}p(\mathbf{r}_q) = \sum_{i=1}^{Ne} -\frac{j}{4}H_{0(kr_i)}^{(2)} \cdot \frac{\partial p(\mathbf{r}_{xi})}{\partial \mathbf{n}} \cdot \Delta S_i - \sum_{i=1}^{Ne} \frac{j}{4}H_{1(kr_i)}^{(2)} \cdot \frac{\partial r_i}{\partial \mathbf{n}} \cdot p(\mathbf{r}_{xi}) \cdot \Delta S_i + p_{(\mathbf{r}_q)}^i \quad (\text{A.42})$$

where

$$p_{(\mathbf{r}_q)}^i = -\frac{j}{4}H_{0(kr_{sq})}^{(2)}$$

and $r_{sq} = |\mathbf{r}_{sq}| = |\mathbf{r}_s - \mathbf{r}_q|$ is the distance between the singular point and the sound source, In matrix notation

$$\frac{1}{2} \{p(\mathbf{r}_q)\} = [A] \cdot \left\{ \frac{\partial p(\mathbf{r}_x)}{\partial \mathbf{n}} \right\} - [B] \cdot \{p(\mathbf{r}_x)\} + \{p_{(\mathbf{r}_q)}^i\} \quad (\text{A.43})$$

where

$$[A] = \left[-\frac{j}{4}H_{0(kr)}^{(2)} \cdot \Delta S \right]$$

and

$$[B] = \left[j\frac{k}{4}H_{1(kr)}^{(2)} \cdot \frac{\partial r}{\partial \mathbf{n}} \cdot \Delta S \right]$$

The rest of the calculation is identical to the one explained in the previous section. The pressure at any point in the acoustic domain is calculated using equation A.41.

Appendix B

Numerical solutions of the Parabolic Equation

Different discretization techniques have been developed to compute the solution to the parabolic equations derived in section 2.6.

B.1 Split-step algorithm

The first technique developed to solve the Standard PE was the split-step Fourier algorithm. In the implementation of this algorithm for underwater problems an artificial horizontal boundary is introduced below the physical bottom boundary, and a zero boundary condition is assumed there.

The Standard PE (Eqn. 2.11)

$$\frac{\partial \varphi}{\partial r} = \frac{jk_0}{2} (n^2 - 1) + \frac{j}{2k_0} \frac{\partial^2 \varphi}{\partial z^2}$$

can take the following operator form:

$$\frac{\partial \varphi}{\partial r} = j (A_{(r,z)} + B_{(z)}) \varphi$$

where

$$A_{(r,z)} = \frac{k_0}{2} (n^2 - 1)$$

and

$$B_{(z)} = -\frac{1}{2k_0} \frac{\partial^2}{\partial z^2}$$

The second order operator in z , is then represented by the inverse transform of its Fourier transform. The split-step algorithm applied at range $(r + \Delta r)$ is

$$\varphi_{(r+\Delta r)} = e^{\frac{jk_0 \Delta r (n^2 - 1)}{2}} F^{-1} \left(e^{\frac{-j \Delta r s^2}{2k_0}} F(\varphi_{(r_0, z)}) \right)$$

where F is the integral Fourier transform and s is the integration variable. The forward and inverse Fourier transforms are approximated by the fast Fourier transform (FFT) in z .

The split-step algorithm is efficient for a certain class of problems. The use of the fast Fourier transform requires periodic boundary condition in z .

B.2 Finite-difference solution

B.2.1 Standard PE

Lee *et al.* [47] presented an implicit-finite difference (IFD) formulation to solve the Standard PE for underwater acoustic wave propagation problems. This method has been adapted to atmospheric acoustic problems by West *et al.* [66].

Equation 2.11 can be expressed in the form

$$\frac{\partial \varphi}{\partial r} = a \frac{\partial^2 \varphi}{\partial z^2} + b_m \varphi \quad (\text{B.1})$$

where

$$a = \frac{j}{2k_0} \quad \text{and} \quad b_m = \frac{jk_0}{2} (n_m^2 - 1), \quad m = 1 \text{ to } M$$

M is the total number of vertical mesh points, and $n_m = c_0/c_m$, where c_m is the sound speed in layer m . Assuming that $\partial \varphi / \partial z$ is approximately linear over the interval z_{m-1} to z_{m+1} , then

$$\left(\frac{\partial^2 \varphi}{\partial z^2} \right)_{z_m} = \frac{1}{\Delta z} \left[\left(\frac{\partial \varphi}{\partial z} \right)_{z_{m+1/2}} - \left(\frac{\partial \varphi}{\partial z} \right)_{z_{m-1/2}} \right]$$

If φ is linear over the separate intervals z_{m-1} to z_m and z_m to z_{m+1} , then

$$\begin{aligned} \left(\frac{\partial \varphi}{\partial z} \right)_{z_{m+1/2}} &= \left(\frac{\varphi_{z_{m+1}} - \varphi_{z_m}}{\Delta z} \right) \\ \left(\frac{\partial \varphi}{\partial z} \right)_{z_{m-1/2}} &= \left(\frac{\varphi_{z_m} - \varphi_{z_{m-1}}}{\Delta z} \right) \\ \left(\frac{\partial \varphi}{\partial z} \right)_{z_m} &= \frac{\varphi_{z_{m+1}} - 2\varphi_{z_m} + \varphi_{z_{m-1}}}{(\Delta z)^2} \end{aligned}$$

The solution for φ at the next range step (or finite difference “level”) $r + \Delta r$ and at one $z = z_m$ must incorporate the solution for φ at the distance r for three heights $z = z_{m-1}$, z_m and z_{m+1} . This results in a marching solution. Given an initial field at some range r the field at the advanced range $r + \Delta r$ may be computed by solving linear equations. The same procedure may be

applied to compute the field values for the next range $r + 2\Delta r$, etc. until the solution is computed for the hole range. This requires a tridiagonal matrix formulation.

The matrix form of equation B.1 is

$$\frac{\partial}{\partial r} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_M \end{pmatrix} = \left[\frac{j}{2k_0\Delta z^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 \end{pmatrix} + \begin{pmatrix} b_1 & & & & \\ & b_2 & & & \\ & & b_3 & & \\ & & & \ddots & \\ & & & & b_M \end{pmatrix} \right. \\ \left. \times \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_M \end{pmatrix} + \frac{j}{2k_0\Delta z^2} \begin{pmatrix} \varphi_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]$$

where φ_m represents function φ evaluated at each vertical mesh point z_m , where z_M is the top of the atmosphere and z_0 is the ground surface. Only the nonzero elements are indicated in the matrices. The term φ_0 , that represents the field at the ground level z_0 , is obtained as a linear combination of φ_1 and φ_2 (see Section B.2.3):

$$\varphi_0 = \sigma_1\varphi_1 + \sigma_2\varphi_2$$

then we get

$$\frac{\partial \mathbf{g}_{(r)}}{\partial r} = (a\mathbf{T} + \mathbf{D}) \mathbf{g}_{(r)} \quad (\text{B.2})$$

where $\mathbf{g}_{(r)}$ is a column vector with the values $\varphi_{(r)}$, $\mathbf{T}$ and $\mathbf{D}$ are

$$\mathbf{T} = \begin{pmatrix} -2 + \sigma_1 & 1 + \sigma_2 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_M \end{pmatrix}$$

a is now

$$a = \frac{j}{2k_0\Delta z^2}$$

Integration of equation B.2 for range r to $r + \Delta r$ gives

$$\mathbf{g}_{(r+\Delta r)} - \mathbf{g}_{(r)} = \int_r^{r+\Delta r} (a\mathbf{T} + \mathbf{D}) \mathbf{g}_{(r)} dr$$

Provided this interval is small, $(a\mathbf{T} + \mathbf{D})$ will be approximately constant so that the above equation can be written

$$\mathbf{g}_{(r+\Delta r)} - \mathbf{g}_{(r)} = (a\mathbf{T} + \mathbf{D}) \int_r^{r+\Delta r} \mathbf{g}_{(r)} dr$$

The Crank-Nicholson approximation can now be used, allowing the integral to be replaced by Δr times the average value of the integral over the Δr

interval. This gives

$$\mathbf{g}_{(r+\Delta r)} - \mathbf{g}_{(r)} = (a\mathbf{T} + \mathbf{D}) \frac{\mathbf{g}_{(r+\Delta r)} + \mathbf{g}_{(r)}}{2} \Delta r$$

Rearranging,

$$\left(\mathbf{I} - \frac{\Delta r}{2} (a\mathbf{T} + \mathbf{D}) \right) \mathbf{g}_{(r+\Delta r)} = \left(\mathbf{I} + \frac{\Delta r}{2} (a\mathbf{T} + \mathbf{D}) \right) \mathbf{g}_{(r)}$$

or

$$\mathbf{M}_2 \cdot \mathbf{g}_{(r+\Delta r)} = \mathbf{M}_1 \cdot \mathbf{g}_{(r)}$$

where

$$\mathbf{M}_1 = \mathbf{I} + \frac{\Delta r}{2} (a\mathbf{T} + \mathbf{D})$$

and

$$\mathbf{M}_2 = \mathbf{I} - \frac{\Delta r}{2} (a\mathbf{T} + \mathbf{D})$$

where $\mathbf{I}$ is the identity matrix. $\mathbf{M}_1$ and $\mathbf{M}_2$ are known since $\mathbf{T}$ and $\mathbf{D}$ depend only on known constants (including the mesh dimensions Δr , Δz and k) and all the values of $\mathbf{g}$ at the range r for the heights z_m determined in the previous calculation step. Then, the forward propagating field is obtained at a given range from the field at a previous range and appropriated boundary conditions at the top and bottom of the domain, in other words the solution is marched in range.

The required acoustic pressure can be found by multiplying this solution by the exponential “carrier” factor in equation 2.7. This method is called the Crank-Nicholson PE (CNPE).

B.2.2 Wide-angle CNPE

The derivation of Claerbout's wide-angle PE (Eqn. 2.16) was presented in section 2.6.3. This equation differs from the Standard PE (Eqn. 2.12) by the factor $(1 + Q/4)$ on the left hand side. In Section B.2.1 we defined (see Eqn. B.2):

$$(a\mathbf{T} + \mathbf{D}) \equiv jk_0 \frac{Q}{2} \quad (\text{B.3})$$

Equation 2.16 can be written as

$$\left(\mathbf{I} + \frac{(a\mathbf{T} + \mathbf{D})}{2jk_0} \right) \frac{\partial \varphi}{\partial r} = jk_0 \frac{(a\mathbf{T} + \mathbf{D})}{2} \varphi \quad (\text{B.4})$$

Applying the Crank Nicholson approximation, equation B.4 becomes

$$\left(\mathbf{I} + \frac{\mathbf{E}}{2ik_0} \right) (\mathbf{g}_{(r+\Delta r)} - \mathbf{g}_{(r)}) = \mathbf{E} \frac{\Delta r}{2} (\mathbf{g}_{(r+\Delta r)} + \mathbf{g}_{(r)})$$

where

$$\mathbf{E} \equiv (a\mathbf{T} + \mathbf{D})$$

Rearranging,

$$\left(\mathbf{I} - \mathbf{E} \frac{\Delta r}{2} + \frac{\mathbf{E}}{2ik_0} \right) \mathbf{g}_{(r+\Delta r)} = \left(\mathbf{I} + \mathbf{E} \frac{\Delta r}{2} + \frac{\mathbf{E}}{2ik_0} \right) \mathbf{g}_{(r)} \quad (\text{B.5})$$

The matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are then

$$\mathbf{M}_1 = \mathbf{I} + \mathbf{E} \frac{\Delta r}{2} + \frac{\mathbf{E}}{2ik_0} \quad , \quad \mathbf{M}_2 = \mathbf{I} - \mathbf{E} \frac{\Delta r}{2} + \frac{\mathbf{E}}{2ik_0}$$

$\mathbf{M}_1$ and $\mathbf{M}_2$ have been modified only by the addition of the term $\mathbf{E}/2ik_0$.

Equation B.5 can be solved using an LU transformation.

B.2.3 Ground impedance boundary condition

First-order accurate case

At the ground surface $z = z_0$, we have

$$Z_s = \frac{p_0}{u_z} = \frac{-p_0}{\frac{1}{iw\rho_s} \frac{\partial p}{\partial z}} \quad (\text{B.6})$$

where ρ_s is the air density just above the ground.

The derivative $\frac{\partial p}{\partial z}$ can be written in finite form:

$$\frac{\partial p}{\partial z} \simeq \frac{p_1 - p_0}{z_1 - z_0}$$

Substituting in Eqn(B.6) and rearranging gives

$$\frac{p_0}{p_1} = \frac{1}{1 - iw \frac{\rho_s}{Z_s} (z_1 - z_0)} \quad (\text{B.7})$$

This condition is imposed at each range increment.

Second-order accurate case

We take an expansion for p

$$p = p_0 + \frac{\Delta p_0}{\Delta z} z + \frac{\Delta^2 p_0}{2\Delta z^2} z(z - \Delta z) \quad , \quad \frac{dp}{dz} = \frac{\Delta p_0}{\Delta z} + \frac{\Delta^2 p_0}{2\Delta z^2} (2z - \Delta z)$$

and using the central finite difference formula for $\Delta^2 p_0 / \Delta z^2$ used in Section

B.2.1:

$$\frac{dp}{dz} = \frac{p_1 - p_0}{\Delta z} + \frac{p_2 - 2p_1 + p_0}{2\Delta z^2} (2z - \Delta z)$$

At $z=0$

$$\frac{dp}{dz} = \frac{4p_1 - 3p_0 - p_2}{2\Delta z}$$

Applying the above relationship in equation B.6 we get

$$Z_s = \frac{-2iw\rho_s\Delta zp_0}{4p_1 - 3p_0 - p_2}$$

or

$$p_0 = \frac{4p_1 - p_2}{3 - 2iw\frac{\rho_s}{Z_s}\Delta z}$$

B.3 Finite-element discretization

Gilbert and White introduced a linear finite element discretization technique for sound propagation through a nonturbulent atmosphere overlying a flat locally reacting ground surface [67]. With this approach it is possible to use a nonuniform vertical grid spacing. Vertical density variations are also included in the solution by replacing $\partial^2/\partial z^2$ with $\rho(\partial/\partial z)(1/\rho)(\partial/\partial z)$ in the equations.

This finite element approach is based on Claerbout's PE (Eqn. 2.16)

$$\left(1 + \frac{Q}{4}\right) \frac{\partial \varphi}{\partial r} = jk_0 \frac{Q}{2} \varphi$$

Applying the Crank-Nicolson approximation it becomes

$$\left(1 + \frac{Q}{4}\right) (\varphi_{(r+\Delta r)} - \varphi_{(r)}) = jk_0 \frac{Q}{2} \left(\frac{\varphi_{(r+\Delta r)} + \varphi_{(r)}}{2} \cdot \Delta r \right)$$

rearranging

$$\left[\left(1 + \frac{Q}{4}\right) - jk_0 \frac{Q}{4} \Delta r \right] \varphi_{(r+\Delta r)} = \left[\left(1 + \frac{Q}{4}\right) + jk_0 \frac{Q}{4} \Delta r \right] \varphi_{(r)} \quad (\text{B.8})$$

In the linear finite elements method the interval $[z_o, z_M]$ is splitted into M

intervals defined by $M + 1$ points given by z_m . We first expand φ in terms of linear finite element basis functions, $h_m(z)$ given by

$$h_m(z) = \begin{cases} 0 & \text{if } z < z_{m-1}, \\ \frac{z - z_{m-1}}{z_m - z_{m-1}} & \text{if } z_{m-1} \leq z < z_m, \\ 1 - \frac{z - z_m}{z_{m+1} - z_m} & \text{if } z_m \leq z < z_{m+1}, \\ 0 & \text{if } z \geq z_{m+1}. \end{cases}$$

The linear approximated functions (“hat” functions), evaluated in the base $h_m(z)$, are given by

$$\varphi(r, z) = \sum_{m=0}^M \varphi(r, z_m) h_m(z)$$

and

$$\varphi(r + \Delta r, z) = \sum_{m=0}^M \varphi(r + \Delta r, z_m) h_m(z)$$

To obtain the matrix representation, the hat function expansion is substituted into Eqn(B.8):

$$\left[\left(1 + \frac{Q}{4} \right) - jk_0 \frac{Q}{4} \Delta r \right] \cdot \sum_{m=1}^M \varphi(r + \Delta r, z_m) h_m(z) = \left[\left(1 + \frac{Q}{4} \right) + jk_0 \frac{Q}{4} \Delta r \right] \cdot \sum_{m=1}^M \varphi(r, z_m) h_m(z)$$

Applying the definition of Q and rearranging we get

$$\begin{aligned} & \left[\left(1 + \frac{n^2 - 1}{4} \right) - jk_0 \frac{n^2 - 1}{4} \Delta r + \left(\frac{1 - jk_0 \Delta r}{4k_0^2} \right) \frac{\partial^2}{\partial z^2} \right] \cdot \sum_{m=1}^M \varphi(r + \Delta r, z_m) h_m(z) = \\ & \left[\left(1 + \frac{n^2 - 1}{4} \right) + jk_0 \frac{n^2 - 1}{4} \Delta r + \left(\frac{1 + jk_0 \Delta r}{4k_0^2} \right) \frac{\partial^2}{\partial z^2} \right] \cdot \sum_{m=1}^M \varphi(r, z_m) h_m(z) \end{aligned}$$

We introduce the vertical density dependence into the equations, this yields in a wave equation that contains the operator $\rho(\partial/\partial z)(1/\rho)(\partial/\partial z)$ instead

of $\partial^2/\partial z^2$. Making this replacement and dividing by ρ in the above equation we obtain

$$\left[\left(1 + \frac{n^2 - 1}{4} \right) \frac{1}{\rho} - jk_0 \frac{n^2 - 1}{4\rho} \Delta r + \left(\frac{1 - jk_0 \Delta r}{4k_0^2} \right) \left(\frac{\partial}{\partial z} \frac{1}{\rho} \frac{\partial}{\partial z} \right) \right] \cdot \sum_{m=1}^M \varphi_{(r+\Delta r, z_m)} h_m(z) =$$

$$\left[\left(1 + \frac{n^2 - 1}{4} \right) \frac{1}{\rho} + jk_0 \frac{n^2 - 1}{4\rho} \Delta r + \left(\frac{1 + jk_0 \Delta r}{4k_0^2} \right) \left(\frac{\partial}{\partial z} \frac{1}{\rho} \frac{\partial}{\partial z} \right) \right] \cdot \sum_{m=1}^M \varphi_{(r, z_m)} h_m(z)$$

Then, we multiply by $h_{g(z)}$ and integrate over z . The integration requires computation of integrals of the form

$$I_{mg}^1 = \int h_{m(z)} \frac{1}{\rho_{(z)}} h_{g(z)} dz$$

$$I_{mg}^2 = \int h_{m(z)} \frac{1}{\rho_{(z)} c_{(z)}^2} h_{g(z)} dz$$

$$I_{mg}^3 = \int h_{m(z)} \left(\frac{\partial}{\partial z} \frac{1}{\rho_{(z)}} \frac{\partial}{\partial z} \right) h_{g(z)} dz = \int h_{m(z)} \frac{\partial}{\partial z} \left(\frac{1}{\rho_{(z)}} \frac{\partial h_{g(z)}}{\partial z} \right) dz$$

From the definition of the basis $h_{m(z)}$, we have that $I_{mg}^1 = I_{mg}^2 = 0$ unless $m = g, g \pm 1$. That is, the I matrices are tridiagonal. for the same reason, the limits of integration are:

$$\begin{array}{lll} z_{g-1} & \text{to } z_{g+1} & \text{for } I_{gg}^1 \quad \text{and } I_{gg}^2, \\ z_{g-1} & \text{to } z_g & \text{for } I_{gg-1}^1 \quad \text{and } I_{gg-1}^2, \quad \text{and} \\ z_g & \text{to } z_{g+1} & \text{for } I_{gg+1}^1 \quad \text{and } I_{gg+1}^2 \end{array}$$

We take $\rho_{(z)}^{-1}$ and $\rho_{(z)}^{-1} c_{(z)}^{-2}$ to be linear functions of z between the grid points.

Hence, between z_g and z_{g+1} we approximate this functions as

$$\rho_{(z)}^{-1} = \rho_{(z_m)}^{-1} + (z - z_m) \left(\frac{\rho_{(z_{m+1})}^{-1} - \rho_{(z_m)}^{-1}}{z_{m+1} - z_m} \right)$$

and

$$\rho_{(z)}^{-1} c_{(z)}^{-2} = \rho_{(z_m)}^{-1} c_{(z_m)}^{-2} + (z - z_m) \left(\frac{\rho_{(z_{m+1})}^{-1} c_{(z_{m+1})}^{-2} - \rho_{(z_m)}^{-1} c_{(z_m)}^{-2}}{z_{m+1} - z_m} \right)$$

Inserting this linear approximations into the corresponding integrals gives

$$I_{gg}^1 = \frac{1}{12} \left[(z_g - z_{g-1}) \rho_{(z_{g-1})}^{-1} + 3(z_{g+1} - z_{g-1}) \rho_{(z_g)}^{-1} + (z_{g+1} - z_g) \rho_{(z_{g+1})}^{-1} \right]$$

$$I_{gg}^2 = \frac{1}{12} \left[(z_g - z_{g-1}) \rho_{(z_{g-1})}^{-1} c_{(z_{g-1})}^{-2} + 3(z_{g+1} - z_{g-1}) \rho_{(z_g)}^{-1} c_{(z_g)}^{-2} + (z_{g+1} - z_g) \rho_{(z_{g+1})}^{-1} c_{(z_{g+1})}^{-2} \right]$$

and

$$I_{gg\pm 1}^1 = \pm \frac{1}{12} \left[\rho_{(z_g)}^{-1} + \rho_{(z_{g+1})}^{-1} \right] (z_{g\pm 1} - z_g)$$

$$I_{gg\pm 1}^2 = \pm \frac{1}{12} \left[\rho_{(z_g)}^{-1} c_{(z_g)}^{-2} + \rho_{(z_{g+1})}^{-1} c_{(z_{g+1})}^{-2} \right] (z_{g\pm 1} - z_g)$$

In order to compute I_{jg}^3 we first integrate by parts to obtain

$$I_{mg}^3 = (z_{g\pm 1} - z_g) \left(\rho_{(z)}^{-1} h_{m(z)} \frac{\partial h_{g(z)}}{\partial z} \right) \Big|_{z_a}^{z_b} - \int_{z_a}^{z_b} \frac{\partial h_{m(z)}}{\partial z} \rho_{(z)}^{-1} \frac{\partial h_{g(z)}}{\partial z} dz$$

where z_a and z_b are the limits of integration discussed earlier. Since the product of h_m and $\partial h_{g(z)}/\partial z$ is zero at z_a and z_b we have

$$I_{mg}^3 = - \int_{z_a}^{z_b} \frac{\partial h_{m(z)}}{\partial z} \rho_{(z)}^{-1} \frac{\partial h_{g(z)}}{\partial z} dz$$

Inserting the approximation of $\rho_{(z)}^{-1}$ obtained earlier we get

$$I_{mm\pm 1}^3 = \pm \left(\frac{1}{2} \right) \left(\frac{\rho_{(z_m)}^{-1} + \rho_{(z_{m\pm 1})}^{-1}}{z_{m\pm 1} - z_m} \right)$$

$$I_{mm}^3 = - (I_{mm+1}^3 + I_{mm-1}^3)$$

This leads to a tridiagonal matrix equation of the form

$$\mathbf{M}^- \mathbf{g}_{(r+\Delta r, z_m)} = \mathbf{M}^+ \mathbf{g}_{(r, z_m)}$$

The matrices $\mathbf{M}^\pm$ are given by (see Ref. [68]):

$$\mathbf{M}^\pm = \begin{bmatrix} M_{0,1}^\pm & M_{1,1}^\pm & & & & \\ M_{-1,2}^\pm & M_{0,2}^\pm & M_{1,2}^\pm & & & \\ & M_{-1,3}^\pm & M_{0,3}^\pm & M_{1,3}^\pm & & \\ & & \ddots & \ddots & \ddots & \\ & & & M_{-1,M-1}^\pm & M_{0,M-1}^\pm & M_{1,M-1}^\pm \\ & & & & M_{-1,M}^\pm & M_{0,M}^\pm \end{bmatrix}$$

where the quantities $M_{m,g}^\pm$ are given by

$$M_{m,g}^\pm = \sum_{u=1}^3 c_u^\pm I_{m,m+g}^u$$

for $m = -1, 0, 1$ and $m = 1, 2, \dots, M$. The coefficients $c_u^\pm$ are given by

$$\begin{aligned} c_1^\pm &= \frac{3}{4} \mp \frac{1}{4} j k_0 \Delta r \\ c_2^\pm &= \frac{1}{4} \pm \frac{1}{4} j k_0 \Delta r \\ c_3^\pm &= \frac{c_2^\pm}{k_0^2} \end{aligned}$$

On each range step, the equations are solved using Gauss elimination.

Although the computations with linear finite elements is more complicated than the finite-difference technique, there are two main advantages that justify the added effort. First, nonuniform grid spacing is easily incorporated.

The second advantage is that, since the acoustic field is defined between grid points, small-scale vertical variations in sound speed and density can be integrated over in the computations. Thus, the grid spacing is determined by the acoustical wavelength and not by small-scale variation in the medium.

B.4 Split-step Padé solution

B.4.1 Padé approximations [1]

Let $f_{(z)}$ be a power series

$$f_{(z)} = \sum_{j=0}^{\infty} c_j z^j \quad (\text{B.9})$$

A Padé approximant is a rational fraction

$$[L/M] = \frac{a_0 + a_1 z + \cdots + a_L z^L}{b_0 + b_1 z + \cdots + b_M z^M} \quad (\text{B.10})$$

which has a Maclaurin expansion which agrees with $f_{(z)}$ as far as possible.

There are $L+1$ numerator coefficients and $M+1$ denominator coefficients. For definiteness we take $b_0 = 1$. So there are $L+1$ independent numerator coefficients and M independent denominator coefficients, making $L+M+1$ unknown coefficients in all. Normally the $[L/M]$ ought to fit the power series B.9 through the orders $1, z, z^2, \dots, z^{L+M+1}$. In the notation of formal power series,

$$\sum_{i=0}^{\infty} c_i z^i = \frac{a_0 + a_1 z + \cdots + a_L z^L}{b_0 + b_1 z + \cdots + b_M z^M} + O(z^{L+M+1})$$

Example, for

$$f_{(z)} = 1 - \frac{1}{2}z + \frac{1}{3}z^2 + \cdots.$$

we have

$$[1/0] = 1 - \frac{1}{2}z = f_{(z)} + O(z^2),$$

$$[0/1] = \frac{1}{1-\frac{1}{2}z} = f_{(z)} + O(z^2),$$

$$[1/1] = \frac{1+\frac{1}{6}z}{1-\frac{2}{3}z} = f_{(z)} + O(z^3).$$

B.4.2 Split-step Padé implementation [2–4]

“Polynomial” Padé approximation

To obtain the split-step Padé solution, we solve equation 2.10 analytically before applying a Padé approximation. Given the field over all depth at the arbitrary range r , and assuming that the operator $\sqrt{Q+1}$ is slowly varying with range, the solution of equation 2.10 at the range $r + \Delta r$ is

$$\varphi_{(r+\Delta r, z)} = e^{ik_0(\sqrt{Q+1}-1)\Delta r} \varphi_{(r, z)}$$

The next and major idea of this method is to approximate this exponential operator by a second-order Padé expansion, which yields [3, 4]

$$e^{ik_0\Delta r\sqrt{Q+1}} \approx \frac{1 + a_1Q + a_2Q^2}{1 + b_1Q + b_2Q^2}$$

where the coefficients are deduced from a fourth-order Taylor series:

$$\begin{aligned} a_1 &= \frac{3+ik_0\Delta r}{4} \\ a_2 &= \frac{(ik_0\Delta r)^2+6ik_0\Delta r+3}{48} \\ b_1 &= \frac{3-ik_0\Delta r}{4} \\ b_2 &= \frac{(ik_0\Delta r)^2-6ik_0\Delta r+3}{48} \end{aligned}$$

Finally, the marching algorithm to obtain $\varphi_{(r+\Delta r,z)}$ from $\varphi_{(r,z)}$ is expressed as

$$\begin{aligned} \left[1 + b_1 \left(\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2 - 1 \right) + b_2 \left(\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2 - 1 \right)^2 \right] \varphi_{(r+\Delta r,z)} = \\ \left[1 + a_1 \left(\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2 - 1 \right) + a_2 \left(\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2 - 1 \right)^2 \right] \varphi_{(r,z)} \end{aligned}$$

The numerical scheme deduced from the above equation leads to a linear system with pentadiagonal matrices, solved at each step with a standard LU decomposition method.

“First power” Padé approximation [52, 61, 72, 92]

The implementation presented above is difficult to implement because powers of Q are involved. An approximation that involves only first powers is

based on the following Padé series [51]:

$$\sqrt{1+Q} - 1 = \sum_{j=1}^n \frac{a_{j,n}Q}{1+b_{j,n}Q} + O(x^{2n+1})$$

where n is the number of terms in the Padé expansion and

$$\begin{aligned} a_{j,n} &= \frac{2}{2n+1} \sin^2 \left(\frac{j\pi}{2n+1} \right) \\ b_{j,n} &= \cos^2 \left(\frac{j\pi}{2n+1} \right) \end{aligned}$$

Going back to equation 2.10:

$$\frac{\partial \varphi}{\partial r} = ik_0 \left(\sqrt{Q+1} - 1 \right) \cdot \varphi$$

applying the above approximation we get

$$\frac{\partial \varphi}{\partial r} = ik_0 \sum_{j=1}^n \frac{a_{j,n}Q}{1+b_{j,n}Q} \cdot \varphi \quad (\text{B.11})$$

which we refer to as PE_n . Equation B.11 can be solved with the method of alternating directions [93].

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