Applications of Complex Numbers and Quaternions: Historical Remarks, with a Note on Clifford Algebra

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Applications and Logicism

A central tenet of Frege's philosophy of mathematics was that the applications of a theory such as that of the natural numbers (arithmetic) or the real numbers (analysis) should be implicitly prefigured in the logicist definition of the objects of the theory in question. The natural numbers should be defined as objects apt for representing the cardinality property of a concept (how many objects fall under it); the real numbers should be defined as objects apt for representing the ratio or proportion of signed (relational) quantities. While detailed individual applications are not written into the definition, since they cannot all be foreseen, present and future applications must in principle be catered for. It cannot be the case that after the definition is complete, a new application comes along which does not fit the definition and must be, as Frege says, "patched on from outside". Call this the *applicability constraint* of logicist definitions.

In the one logicist definition he completed, that for the natural numbers, and the one he outlined without completion, that for the real numbers, Frege's projected definitions do seem—modulo the inconsistency of his logic—to satisfy the applicability constraint.¹ When considering how to extend Frege's ideas in neo-logicist fashion to other mathematical systems, the question arises as to whether the applicability constraint is one which can in every case be fulfilled, whether in some cases a pure theory has to be constructed for its own sake, irrespective of the possibility or mode of its application.

It is natural to continue Frege's work by considering more general concepts of number: after the real numbers, the complex numbers beckon, while after them a plausible next step is the quaternions. In each case however, the applications of the pure theory are less straightforward than in the case of the naturals and the reals,

¹ In Neo-Fregean Foundations for Real Analysis: Some Reflections on Frege's Constraint, *Notre Dame Journal of Formal Logic* **41**, (2000), 317–334, Crispin Wright argues that the constraint does not apply even to the (neo-logicist) reconstruction of the reals.

raising the possibility that the applicability constraint is too restrictive, and that the Frege was in a sense “lucky” to have started with theories where the applications were obvious and well known.

Complex Numbers

From the point of view of application, the situation with complex numbers is interestingly different from that of the reals. Non-mathematical applications of the real numbers are legion, but it took until the later nineteenth century for acceptable modern axiomatizations to be found for them: essentially those of (in order of publication) Méray, Cantor and Dedekind. By contrast, complex numbers, though they had been around since the sixteenth century, had been safely lodged as respectable entities next to the reals since 1833 via Sir William Rowan Hamilton’s simple ordered pair representation, and had been known in their geometric guise since Caspar Wessel’s brilliant discovery of 1797 and the more widely publicised work of Argand. Yet at that time their interest was wholly within pure mathematics, first as solutions for polynomial equations, and later as the materials of complex analysis. The first serious application for complex numbers emerged relatively recently, in the work of the electrical engineering genius Charles Proteus Steinmetz (1865–1923) who made power transmission by alternating current a practical prospect.² In 1893 Steinmetz gave a lecture on how to use complex numbers in analogues to Kirchoff’s Laws for DC circuits to calculate impedance in AC circuits.³ Impedance is the AC analogue of resistance in DC circuits. The impedance of a device is defined as the ratio of phasor voltage to phasor current across the device; it consists of a real part called the *resistance* and an imaginary part called the *reactance*. Steinmetz’s discovery enabled engineers to design AC equipment by calculation instead of trial-and-error, and was one contributory factor in AC winning the so-called “War of Currents” for Westinghouse against Thomas Edison’s DC.⁴

² R. R. Kline, *Steinmetz: Engineer and Socialist*. Johns Hopkins U. P., 1992.

³ Complex Quantities and their Use in Electrical Engineering. *Proceedings of the International Electrical Conference*, AIEE Proceedings, Chicago, 1893, 33–74.

⁴ Edison’s campaign to expose the danger of AC sunk to such levels that he had animals—including an elephant—electrocuted in demonstrations, and although against capital punishment, persuaded legislatures to adopt AC for execution by electric chair, a procedure he wanted to be called ‘westinghousing’.

Steinmetz's discovery was the first serious application for complex numbers, and it is notable that he had a dual mathematical and engineering training. However it is doubtful that the use of complex numbers here is *indispensable* as distinct from highly convenient. Similar remarks apply to other applications of complex numbers, such as the use of conformal mapping in complex analysis for two-dimensional boundary value problems.⁵

To date by far the most important application of complex numbers, and the one where they can most plausibly be regarded as indispensable, is in quantum physics. The most widespread mathematical formulation of quantum mechanics, due to Paul Dirac and Johann von Neumann, takes the possible states of a physical system to be represented by unit vectors in a complex vector space, the *state space*, the nature of which varies according to the system. The space is determined up to a complex number of norm 1, called the *phase* factor.⁶ Each physical quantity or *observable* for a system is defined by a Hermitian (linear self-adjoint) operator, whose eigenvalues are all real, and which represent potentially measured or observed values of the observable. Heisenberg's Uncertainty Principle is encoded in the circumstance that operators corresponding to certain observables do not commute. The evolution of the system through time is governed by the Schrödinger equation, whose Hamiltonian operator represents the total energy at a time. The success of the Schrödinger equation and associated methods depends essentially and inescapably on the state vectors' being complex.

There appears however to be no fully satisfactory explanation as to *why* this is so: physicists generally content themselves with declaring that it *is* so. The nearest to an explanation that I can summon invokes the fact that non-collapsed quantum states of a system *interfere* with one another, giving rise to such phenomena as two-slit interference patterns. The patterns are different from those predicted by classical physics with its wholly real-valued observables, and the way in which they are observed to differ tells us that the difference between quantum states which accounts for this requires complex quantities corresponds to a *phase* difference among quantum states, that is, differences which manifest themselves in cyclical, sine-like variations such as are typical of waves. For this to happen the state vectors must be complex and their interference must affect the probabilities of eigenstates' being observed. The crucial factor is then not the probability itself, which is a real number in the interval

⁵ A. Jeffrey, *Complex Analysis and Applications*, London: Taylor & Francis, 2nd ed. 2006, ch. 3.

⁶ Not phases themselves but their *differences* are physically significant.

$[0,1]$, but a complex number called a *probability amplitude* whose real magnitude is the probability. The wavelike interference of quantum states is represented by the complex scalar inner product of their state vectors, which is the probability amplitude.

The periodic oscillatory pattern of real-valued sine and cosine functions, rendering them apt to represent wave phenomena, is a side-effect of the remarkable fact that the exponential function, so manifestly real when defined on the reals, reveals itself as beautifully and periodically oscillatory when extended to the complex numbers. The sense one has when observing this is that periodic wave phenomena must in some sense be driven by the complex exponential (whose differential is equal to its value) rather than that the complex exponential is a convenient tool for representing wobbly phenomena. It is as if God is causing the fundamental variables of the system to rotate in complex space, the real manifestation for us being only the wobbly real variables. The use of complex numbers in quantum physics extends further into the more complex and adequate machinery of quantum field theory.

The moral for abstractionist approaches to the complex numbers is moderately sobering. Suppose we expect our abstractionist construction of the complex numbers to fulfil the applicability constraint. No one could have foreseen the applications of complex numbers. They are a pure-mathematical invention and their interest and beauty is largely a pure mathematician's. Abstractionist reconstructions will inevitably therefore have a strong air of me-tooism about them until such time as a deeper understanding of the rationale for complex number application is attained.

Quaternions

Paradoxically, although quaternions are mathematically more abstruse and complicated than complex numbers, their application story is much more easily told. Quaternions arose out of Hamilton's attempt to extend the complex numbers and their connection with planar geometry to the geometry of three dimensions. His inability to do so with triples famously led to a decade of frustration and breakfast-time confessions to his young sons, and to his even more famous *Aha-Erlebnis* of 16 October 1843 that not two but three imaginary units are required for multiplication to work, leading to a piece of graffitory vandalism on Dublin's Broom Bridge. By the evening of that day Hamilton had enlisted quaternions in the representation of rotations in 3-space, and thereby made his first, fatal and unreversed error. Because he treated

rotations conically rather than spherically, and by analogy with the simpler planar case considered imaginary units to correspond to rotations of $\pi/2$ instead of the correct π , he made the angle of rotation ϕ the argument of the cosine and sine functions used to describe how rotations combine. This error never clearly emerged in Hamilton's lifetime, but that did not impede him or his champion, the combative Scottish mathematician Peter Guthrie Tait, from writing many extremely long and today almost unreadable books extolling the virtues of the quaternion, which Tait thought qualified Hamilton to be ranked alongside Euclid. Quaternions are mathematically extremely interesting, being the largest division algebra, and were liberating algebraically in that they presented the first example of a number-like system in which multiplication is non-commutative. In applications in geometry and mechanics however, despite Hamilton's and Tait's impassioned advocacy, they largely failed to live up to their early promise. Lord Kelvin wrote in 1892 that

Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell.⁷

James Clerk Maxwell had made some use of quaternions in his *Treatise on Electricity and Magnetism*, but he was not thoroughly convinced, and ceased using them after 1873. His widely noted use of them and their thereby exposed shortcomings ironically prompted independent work by the American Josiah Willard Gibbs and the Englishman Oliver Heaviside which resulted in what we now know as *vector calculus*. Both Gibbs and Heaviside were, like Steinmetz, equally mathematicians and physical scientists. Heaviside, whose irony could be eponymous, wrote that

Quaternions furnish a uniquely simple and natural way of treating *quaternions*. Observe the emphasis.⁸

while his opponent Tait wrote, with equal firmness but less irony

Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on *vector analysis*, a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann.⁹

⁷ Quoted from S. Altmann, *Rotations, Quaternions and Double Groups*, Oxford: Clarendon, 1986, p. 9.

⁸ Quoted from Paul J. Nahin, *Oliver Heaviside: the Life, Work, and Times of an Electrical Genius of the Victorian Age*, New York: IEEE Press, 1987, p. 187.

⁹ Quoted from Altmann, *op. cit.*, p. 17.

In the Great Quaternionic War, vectors won out over quaternions,¹⁰ and now every physicist learns vector theory while quaternion theory has been (until very recently) a stagnant backwater. Occasional uses of quaternions in formulating relativity and other physical theories may best be described as quixotic.

The irony is that the correct method for representing the combination of rotations in 3-space, which quaternions *are* a uniquely natural and simple way of representing, was already in print in 1840, had Hamilton but known it. In an elegant paper¹¹ the French socialist banker Benjamin Olinde Rodrigues had formulated to perfection the rule for combining rotations using not the rotation angle ϕ but the half-angle $\frac{1}{2}\phi$. Of course Rodrigues did not invent quaternions, and unlike Hamilton he lapsed into obscurity, but unlike Hamilton he did get rotation right. When a sphere with fixed centre is rotated about one axis by an angle, and then about another axis by another angle, the resulting orientation of the sphere can be expressed as a rotation from the original position about a third axis by a third angle. This had been proved geometrically by Euler in 1775. If the axis of rotation is given by a unit quaternion $\mathbf{u}$ with no real or scalar part,¹² $\mathbf{u} = i\mathbf{x} + j\mathbf{y} + k\mathbf{z}$, and the rotation angle is ϕ , then representing the rotation by a quaternion (sum of a scalar and a vector)

$$\mathbf{q} = \cos \frac{1}{2}\phi + \mathbf{u} \sin \frac{1}{2}\phi$$

allows us to express the product of two rotations (the third rotation) by the quaternion product. Rodrigues's combination rule for rotations conforms exactly to that for quaternion multiplication.

Quaternionic representation of rotations using Rodrigues's formula was rediscovered by the Oxford materials scientist Simon Altmann and publicised in his wonderful book *Rotations, Quaternions and Double Groups*,¹³ but unbeknown to Altmann at that time, quaternions were already being lined up to do duty in the theory of rotations in space, in more than one sense. The standard representation of rotations in three dimensions, for example in studying aircraft orientation, is by means of angles to three standard axes, the so-called Euler angles, called yaw, pitch and roll in

¹⁰ For a fascinating account of the quaternion/vector controversy see M. J. Crowe, *A History of Vector Analysis*, Notre Dame: University of Notre Dame Press, 1967.

¹¹ Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ses déplacements considérés indépendamment des causes qui peuvent les produire. *J. de Mathématiques Pures et Appliquées* **5** (1840), 380–440. They don't write titles like that any more.

¹² Called a *pure quaternion* or *vector* by Hamilton, who thereby coined the latter term.

¹³ Altmann, *op. cit.*

aeronautic parlance. Working with these can be messy and does not always give unique answers: like rotation matrices, representation by Euler angles has *singularities*. A practical analogue of the singularity issue is shown by the problem affecting gyroscopic inertial orientation devices known as *gimbal lock*.¹⁴ During the Apollo 11 landing and later during the crucial recovery manoeuvring on Apollo 13 the spacecraft's IMU (Inertial Measurement Unit) attitude instrument, consisting of three gyroscopes suspended within three concentric gimbals, came close to gimbal lock, which would have caused data on spacecraft orientation to be catastrophically lost. Later a fourth gimbal was fitted to Apollo IMUs, which while not making gimbal lock impossible made it extremely improbable.¹⁵

When Rockwell were designing the space shuttle's orbiter spacecraft in the early 1980s the decision was taken to use not Euler angles but quaternions for guidance, navigation, and control, and indeed quaternions found application in most aspects of shuttle software. There are two reasons: one is that calculation with quaternions is efficient: a quaternion has four components whereas a rotation matrix has nine; and the other is that unique solutions are always forthcoming: the quaternion representation is singularity-free. The orbiter's attitude is always represented by a quaternion. Since the shuttle system is the most complex mobile artefact ever produced, this is a uniquely prestigious technological application.

A more recent and perhaps even more surprising application of quaternions is in the code running computer games. Fast action games involving moving players in a virtual spatiotemporal environment, who turn and track, as in dogfight games, require very efficient singularity-free algorithms, and it turns out again that quaternions fit the bill.¹⁶ Whether this would have pleased the gravely serious Hamilton is a moot point.

Thus it seems that quaternions do after all have a serious part to play in applications, albeit one which is less widely useful than that of vectors. In these quaternion applications however there are flaws from the point of view of the abstractionist. Firstly, for rotations, only unit quaternions (ones with modulus 1) are required, not the whole algebra. Addition is not needed except for the components. The situation is analogous to that for complex numbers and plane rotations: only the unit complex numbers (the unit circle) are needed, and they need only be multiplied. If

¹⁴ <http://www.hq.nasa.gov/alsj/gimbals.html>

¹⁵ Ironically the earlier Gemini craft already had them.

¹⁶ J. B. Kuipers, *Quaternions and Rotations Sequences: A Primer with Applications to Orbits, Aerospace, and Virtual Reality*. Princeton: Princeton University Press, 1999.

arbitrary non-zero quaternions are used, the factor introduced by the modulus is not needed and simply gets in the way. Secondly, unlike the complex number case, each rotation is represented by *two* quaternions $\mathbf{q}$ and $-\mathbf{q}$. So when quaternions are used to represent rotations, only part of the algebraic structure is used: the rest is excess. In this sense we are still looking for a full-blown application for the whole quaternion algebra. I incline to believe that one will not be forthcoming. Quaternions may be algebraically *too rich* for a directly depictive application of the full structure. There might conceivably be applications in 4-dimensional space that required the full quaternion structure, but we shouldn't hold our breath waiting for them.

Postscript: Clifford Algebra

We have portrayed quaternion and vector algebra in the traditional fashion, and as perceived by their respective proponents at the time, as antagonistic rivals for the favour of those wishing to use such methods in application: let's call them the *clients*. In the twentieth century, these clients voted with their feet, *en masse*, for vectors, consigning quaternions seemingly to the dustbin of history. The algebraic sophistication of quaternions appeared *de trop*. Despite their latter-day use in the theory of rotations, they signally failed to live up to expectations. However the pendulum may be about to swing away from vector algebra and analysis as the method of choice for geometric applications. Another invention of mathematics' miraculous nineteenth century may be poised to replace vectors as the tool of choice for clients. Ironically, it is one which constitutes a compromise between vectors and quaternions, indeed manages to subsume both, and seems likely also to reduce the role of complex numbers in applications. It is known as geometric algebra, or, after its inventor, the English mathematician–philosopher William Kingdon Clifford (1845–1879), *Clifford Algebra*.

Clifford studied the works of both Hamilton and Hermann Gunther Grassmann, and combined their advances. The algebraic signature of Clifford algebra is its notion of *geometric product*. Whereas quaternions have their own (non-commutative) product, vectors traditionally have two products, the *scalar* (dot, or inner) product $\mathbf{a} \cdot \mathbf{b}$, whose value for any pair of vectors is a real number or scalar, and the *vector* (cross, or outer) product $\mathbf{a} \times \mathbf{b}$, whose value for any pair of vectors is

another vector, orthogonal to $\mathbf{a}$ and $\mathbf{b}$. The cross product is anti-commutative: $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$, and is only well-defined in three dimensions. Grassmann however defined another product, the *outer* (exterior, or wedge) product, $\mathbf{a} \wedge \mathbf{b}$, which works in arbitrary dimensions. Clifford interpreted this outer product as defining not a vector but a directed *area*, known as a *bivector*. He also, following a late idea of Grassmann, defined a *geometric product* by the apparently absurd device of adding the two products: $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$.¹⁷ By this means both the scalar and outer product can be separately defined, and vectors can also multiply not just scalars and other vectors but multivectors of any dimension, which can also multiply one another. The algebra of multivectors allows sums and a product with an inverse, as in Hamilton, and, in brief, can be employed as a single tool subsuming vectors, quaternions, tensors, and spinors in all their applications. The role of imaginary numbers in two-dimensional geometry is now taken more naturally by bivectors. Clifford's now redubbed *geometric algebra*, resurrected and extended to a geometric calculus from the 1960s by the American mathematician David Hestenes, has acquired a small but enthusiastic following, and its manifold virtues, such as being able to express Maxwell's theory of electromagnetism in a single equation, look fair to make it the mathematical tool of choice for future clients.¹⁸

Conclusion

Both complex numbers and quaternions were invented for reasons of pure mathematics, and applications came later. In the case of complex numbers these are few but one is highly significant; in the case of quaternions the few applications are useful but dispensable, while an application of the full algebraic structure of quaternions is still outstanding. It is stretching credulity to suppose that anything like a direct abstractionist approach to complex numbers or quaternions would have caused them to be uncovered, and it is not clear how a *post hoc* reconstruction can add to their intelligibility. Quaternions lost out in the local war with vector analysis, but vectors may eventually be replaced in their turn by the more powerful and flexible

¹⁷ Recall Hamilton's conception of a quaternion as the sum of a scalar and a vector.

¹⁸ For representative modern works on geometric algebra see D. Hestenes, *New Foundations for Classical Mechanics*, Dordrecht: Kluwer, 2nd ed., 1999, and C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge: Cambridge University Press, 2003.

Clifford algebras, which were developed with a view to applications from the outset. It seems to me that vector spaces and Clifford algebras, along with groups, all form classes of widely applicable but non-categorically determined structures whose roles in applications it would be important for abstractionists to consider. Most neo-logicist effort has gone into explicating the core structures of pure mathematics. Perhaps it is now time for attention to turn to the mathematics that earns its living by application.