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Haldane limits via Lagrangian embeddings

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Abstract

In the present paper we revisit the so-called Haldane limit, i.e. a particular continuum limit, which leads from a spin chain to a sigma model. We use the coherent state formulation of the path integral to reduce the problem to a semiclassical one, which leads us to the observation that the Haldane limit is closely related to a Lagrangian embedding into the classical phase space of the spin chain. Using this property, we find a spin chain whose limit produces a relativistic sigma model with target space the manifold of complete flags $U(3)/U(1)^3$. We discuss possible other future applications of Lagrangian/isotropic embeddings in this context.

1 Introduction

Spin chains and sigma models are the two colossi of two-dimensional physics. Both have a long and rich history, although the sigma models are a much younger, and hence less understood, subject. By their very definition spin chains are finite-dimensional objects, and therefore many problems related to them are more easily formulated and in many cases may be solved numerically (although the simplicity of formulating a problem does not mean it is always easy to solve it analytically). On the other hand, sigma models are examples of interacting quantum field theories and share the common drawbacks of the latter — namely, that generically they can only be formulated in the framework of perturbation theory, which leaves much to be desired: in each of the terms one encounters infinities, which have to be renormalized, and, as if it were not bad enough, it is not known whether the perturbation series can be made to converge by any reasonable means. The situation improved with the advent of integrable methods in two-dimensional quantum field theory. It turned out that these difficulties can be bypassed, if not resolved, if the theory at hand possesses an infinite number of commuting conserved “charges”. Even more importantly, it was shown that the various continuum limits of the spin chains produce sigma model-like actions, and therefore the spin chains may serve as natural

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regularizers, which preserve the symmetries of the sigma models (for an important example of such approach see [1]). However, for a given spin chain there may exist different inequivalent continuous limits, and the models which arise as a result will differ substantially. For example, for the most common case of the $SU(2)$ spin chain with interactions of the form $\vec{S}_i \vec{S}_{i+1}$ the long-range fluctuations over the ferromagnetic vacuum are described by the so-called Heisenberg ferromagnet model (see Section 5.2 below), which is a splendid model apart from being relativistic. An important result of [2] was that the continuum limit around the antiferromagnetic configuration produces a *relativistic* sigma model with target space the sphere S^2 (though with a “theta-angle” $\theta = \pi m$, m being the integer characterizing the representation in which each site of the chain transforms).

The interest to the spin chains and sigma models especially increased after the advent of the AdS/CFT correspondence (for a review see [3, 4]). One of the examples of the AdS/CFT correspondence relates a supersymmetric conformal quantum field theory in a three-dimensional spacetime to a string sigma model with target space $AdS_4 \times \mathbb{CP}^3$. Investigating this sigma model and generalizing earlier results of [5], we found [6] that a particular low-energy limit of this model produces a standard $\mathbb{CP}^3$ action, where the bosonic fields interact with a single Dirac fermion. The Lagrangian of that model may be written as follows:

$$\mathcal{L} = \eta^{\alpha\beta} \overline{\mathcal{D}_\alpha z^j} \mathcal{D}_\beta z^j + i \bar{\psi} \gamma^\alpha \widehat{\mathcal{D}}_\alpha \psi + \frac{g}{4} (\bar{\psi} \gamma^\alpha \psi)^2, \quad (1)$$

where index j runs from 1 to 4, $\mathcal{D}_\alpha = \partial_\alpha - i \mathcal{A}_\alpha$, $\widehat{\mathcal{D}}_\alpha = \partial_\alpha + 2i \mathcal{A}_\alpha$. $\mathcal{A}_\alpha$ is a $U(1)$ gauge field without a kinetic term — it can be integrated out to provide the conventional Fubini-Study form of the action. Besides, in (1) the z^j fields are restricted to lie on the $S^7 \subset \mathbb{C}^4$:

$$\sum_{j=1}^4 |z^j|^2 = R^2 \quad (2)$$

The model defined by (1), (2) has two coupling constants, the “radius” R and the four-fermion coupling g . It is not known whether this model is integrable, but the methods developed in this paper may, with a bit of luck, lead to a resolution of this question in the future.

In the paper we will argue that the generalized Haldane limits of the models described by Hamiltonians (3) and (4) (see below) are relativistic sigma models with target spaces $\mathbb{CP}^N$ and $\mathcal{F}_3$ (flag manifold) respectively. The $\mathbb{CP}^N$ sigma model was obtained in this manner for the first time by Affleck [7] and, in a three-dimensional setting by Read and Sachdev [8]. To our knowledge, the $SU(N)$ flag sigma model has not been obtained yet in this manner (see, however, [9], whose authors obtained a real flag manifold $O(3)/Z_2^3$, and [10]¹ for related constructions). In the present paper we will also develop a general

¹I would like to thank K.Zarembo for pointing out this reference to me.

framework for the “Haldane limits” and explain with the help of two examples (Observations 1 and 2 below) that they are closely related to the isotropic embeddings of certain manifolds.

The paper is organized as follows. In Section 2 we introduce the Hamiltonians of the spin chains that we will be analyzing. In Section 3 we recall the coherent state formalism and in Section 4 we apply it, as a pedagogical exercise and an introduction to what follows, to a quantum mechanical model with phase space the sphere S^2 . Section 5 is dedicated to a generalization of this discussion to the spin chain setup. In particular, in 5.1 we build a path integral for the XXX spin chain and in 5.2 we analyze the continuum limit around the ferromagnetic vacuum. Sections 6 and 7 are at the heart of the paper — there we analyze the expansion around the *antiferromagnetic* configuration. First, in Section 6.1, we rephrase in our language the results of Affleck related to the spin chain described by the Hamiltonian (3). Then in Sections 6.2 and 7 we generalize this result to the case of the antiferromagnetic configuration of the Hamiltonian (4). In particular, in Section 6.2 we find out what the antiferromagnetic vacuum of the spin chain (4) looks like classically. Section 7.1 is devoted to the expansion of the action around this vacuum configuration. As a result we obtain the $SU(3)$ flag sigma model action (72), which is one of the main results of the present paper. In Section 7.2 we elaborate on what the most general $SU(N)$ -invariant metric on a flag manifold looks like. In Section 8 we discuss possible applications of the flag sigma model in condensed matter physics, in particular with regards to the so-called “trimerization”. The discussion in Section 9 is dedicated to an overview of possible ways and directions of extending this line of research in the future. The paper contains four appendices. In Appendix A we give the basic definitions of the permutation and trace operators. Appendix B is dedicated to a general discussion of the Landau-Lifshitz models, some examples of which are encountered in the main text of the paper. In Appendix C we prove that the flag manifold $\mathcal{F}_N$ may be isometrically embedded into the product $(\mathbb{CP}^{N-1})^{\times N}$ as a Lagrangian submanifold. Last but not least, in Appendix D the reader will find a detailed derivation of the sigma-model action (72) from the spin chain “Lagrangian”.

2 The $SU(N+1)$ spin chains

In the following we will frequently encounter the complex projective space $\mathbb{CP}^N$ viewed as a homogeneous space of $SU(N+1)$. For this reason we prefer to write $SU(N+1)$ instead of the arguably more easily readable symbol $SU(N)$. In this paper we consider two families of spin chains having $SU(N+1)$ global symmetry. Members of each family are parametrized by the representation in which each site of the spin chain transforms. Moreover, we have no intention to elaborate on the most general situation possible, but rather wish to present to the reader a couple of clear and representative examples. For

this reason the representations considered in this paper will be symmetric powers of the fundamental (and/or anti-fundamental) representation $\text{Sym}(V^{\otimes m})$. The simplest representatives of the two families are defined by the following Hamiltonians (in the following L is the length of the spin chain):

$$H_1 = \sum_{i=1}^L \text{Tr}_{i,i+1}, \quad (3)$$

where $\text{Tr}_{i,i+1}$ is the trace operator, and

$$H_2 = \sum_{i=1}^L (P_{i,i+1} + \frac{1}{2} P_{i,i+2}), \quad (4)$$

where $P_{i,i+1}$ is the permutation operator. Both $\text{Tr}_{i,i+1}$ and $P_{i,i+1}$ act on the product of two $N+1$ -dimensional vector spaces $\mathbf{C}^{N+1} \otimes \mathbf{C}^{N+1}$. Nevertheless, there is an important difference between Hamiltonians H_1 and H_2 , which will play a role in the foregoing discussion. It comes from the requirement that the spin chain should possess $SU(N+1)$ symmetry. For this to be the case, the action of the group on the vector spaces — sites of the spin chains — is different for (3) and (4). In the spin chain (3) the consecutive sites transform in contragradient representations (say, site i in the fundamental V , and sites $i \pm 1$ in the antifundamental V^*). In the spin chain (4) the representations at each site should be the same, and we take it to be the fundamental V .

It is well-known (and easy to check) that the operators Tr and P entering the above Hamiltonians have simple matrix representations. Recall that in the Lie algebra $u(N+1)$ there is an invariant Killing scalar product, which is usually denoted by κ . Let λ_n be a basis of Hermitian generators in the fundamental representation, and $\bar{\lambda}_n$ — the conjugate matrices, generating the antifundamental representation. Then

$$\sum_{n,m} \kappa_{nm} \lambda_n \otimes \lambda_m = P \quad \text{and} \quad (5)$$

$$\sum_{n,m} \kappa_{nm} \lambda_n \otimes \bar{\lambda}_m = \text{Tr}. \quad (6)$$

The generalization of the Hamiltonians (3) and (4) to the case where the sites are in symmetric powers $\text{Sym}V^{\otimes m}$ of the fundamental representation is as follows. Take the generators $\Lambda_n^{(i)}(m)$ in these particular representations at each site i (clearly, $\Lambda_n^{(i)}(1) = \lambda_n^{(i)}$) and build the following analog of the permutation operator: $\hat{\mathbb{P}}^{(i)} = \frac{1}{m} \sum_n \Lambda_n^{(i)}(m) \otimes \Lambda_n^{(i+1)}(m)$. The sum of this density over all sites i gives the generalization of the permutation operator relevant to our discussion. It is not difficult to convince oneself, however, that this is *not* the same as the permutation operator on $\text{Sym}(V^{\otimes m}) \otimes \text{Sym}(V^{\otimes m})$. Indeed, the generators $\Lambda_n(m)$ (we omit the index i for the moment) are the restrictions to the relevant

subspace (necessary due to the symmetrization) of the following:

$$\Lambda_n(m) = \sum_{s=1}^m 1 \otimes \dots \otimes \underset{\substack{\uparrow \\ \text{s-th place}}}{\lambda_n} \otimes \dots \otimes 1, \quad (7)$$

where there are m factors and λ_n stands in position s . This means that $\hat{\mathbb{P}}^{(i)}$ looks as follows:

$$\hat{\mathbb{P}}^{(i)} = \frac{1}{m} \sum_n \Lambda_n^{(i)}(m) \otimes \Lambda_n^{(i+1)}(m) = \frac{1}{m} \sum_{s,t=1}^m \sum_n (1 \otimes \dots \otimes \underset{\substack{\uparrow \\ \text{s-th place}}}{\lambda_n} \otimes \dots \otimes 1) \otimes (1 \otimes \dots \otimes \underset{\substack{\uparrow \\ \text{t-th place}}}{\lambda_n} \otimes \dots \otimes 1). \quad (8)$$

We know what the sum over n in the above expression is (for fixed s and t) — it is the permutation operator between site s of the first factor in $\text{Sym}(V^{\otimes m}) \otimes \text{Sym}(V^{\otimes m})$ and site t in the second factor. Thus,

$$\hat{\mathbb{P}}^{(i)} = \frac{1}{m} \sum_{s,t=1}^m P_{s(i), t(i+1)} \quad (9)$$

The summation is necessary essentially to ensure that after permuting individual sites of a symmetrized product it remains symmetric.

Example. Let us give a brief example. Suppose $m = 2$. Then $\hat{\mathbb{P}}$ acts as follows on a tensor $T_{ij|kl}$ ($i, j, k, l = 1..N+1$), symmetric in the first two and last two indices respectively:

$$\hat{\mathbb{P}}(T_{ij|kl}) = \frac{1}{2}(T_{kj|il} + T_{ik|jl} + T_{lj|ki} + T_{il|kj}) \quad (10)$$

Upon the introduction of a “generating function” $T = \sum_{i,j,k,l} T_{ij|kl} z_i z_j w_k w_l$ the action of the operator $\hat{\mathbb{P}}$ on $T_{ij|kl}$ may be dually interpreted as an action on $z_i z_j w_k w_l$ defined by the consistency requirement $\sum \hat{\mathbb{P}}(T_{ij|kl}) z_i z_j w_k w_l = \sum T_{ij|kl} \hat{\mathbb{P}}(z_i z_j w_k w_l)$. In this case we obtain:

$$\begin{aligned} \hat{\mathbb{P}}(z_i z_j w_k w_l) &= \frac{1}{2}(z_k z_j w_i w_l + z_i z_k w_j w_l + z_l z_j w_i w_k + z_i z_l w_j w_k) = \\ &= \frac{1}{2}(z_k w_l + z_l w_k)(z_j w_i + z_i w_j), \end{aligned} \quad (11)$$

the second expression vividly demonstrating the symmetry $i \leftrightarrow j$ and $k \leftrightarrow l$, which therefore proves that the action of the operator is well-defined.

We will need the results of this lengthy and pedantic discussion in Section 5 to determine how $\hat{\mathbb{P}}^{(i)}$ acts on the coherent states defined in what follows.

Remark. For the case of $SU(2)$ the fundamental and anti-fundamental representations are equivalent. This means that there exists a matrix C such that

$$\bar{\lambda}_n = -C \lambda_n C^{-1}, \quad n = 1, 2, 3. \quad (12)$$

Let us construct a matrix $\hat{C} = 1 \otimes C \otimes 1 \otimes \dots$ with a total of L factors, 1's standing in the sites of the fundamental representation and C 's in the sites of the anti-fundamental one. Then we see that $-P$ and Tr are conjugate to each other modulo the 4×4 identity matrix I_4 (which is the generator of $u(1) \simeq u(2)/su(2)$):

$$\text{Tr} = I_4 - \hat{C}P\hat{C}^{-1}. \quad (13)$$

3 Coherent states

It is a fact from algebraic geometry that any holomorphic line bundle over $\mathbf{CP}^N$ is a (perhaps inverse) power of the tautological bundle². The line bundles over $\mathbf{CP}^N$ are thus denoted by $\mathcal{O}(m)$, where $m \in \mathbf{Z}$ is the degree, or Chern number, of the bundle.

The coherent states are by definition particular sections of these line bundles $\mathcal{O}(m)$. For a given m there's a very explicit realization of the vector space of such sections $\Gamma(\mathcal{O}(m))$ — these are the homogeneous polynomials of degree m of the $N + 1$ homogeneous coordinates of $\mathbf{CP}^N$. There's a natural scalar product on $\Gamma(\mathcal{O}(m))$. It is given by the following formula:

$$(f, g) = \int \overline{f(z)} g(z) \frac{d\mu(z, \bar{z})}{\left(\sum_{i=1}^{N+1} |z_i|^2\right)^m}, \quad (14)$$

where $d\mu$ is the volume form on $\mathbf{CP}^N$, which in homogeneous coordinates can be described as follows. Denote by $\tilde{\omega}$ the following holomorphic N -form:

$$\tilde{\omega} = \epsilon_{i_1, i_2, \dots, i_{N+1}} z_{i_1} dz_{i_2} \wedge \dots \wedge dz_{i_{N+1}} \quad (15)$$

and by $\bar{\tilde{\omega}}$ the conjugate one. Then the volume form looks as follows (notice that it is $U(N + 1)$ -invariant by construction):

$$(d\mu)_{\mathbf{CP}^N} = \omega_{FS}^N = \frac{i^N \tilde{\omega} \wedge \bar{\tilde{\omega}}}{\left(\sum_{k=1}^{N+1} z_k \bar{z}_k\right)^{N+1}} \quad (16)$$

It is a fact, characteristic of the projective space, that if one rescales all the z 's by a *function*, the Fubini-Study form ω_{FS} as well as the volume form $d\mu$ will not change. An important consequence of this is that the integrand in (14) does not change under such rescalings. This means in particular, that one may choose a convenient “gauge”, for example we can set $z_{N+1} = 1 = \bar{z}_{N+1}$ (in what follows we will mainly use this “gauge”). In

²Sometimes this fact is formulated as the statement that the Picard group has a single generator: $\text{Pic}(\mathbf{CP}^N) \simeq \mathbf{Z}$.

this case the volume form (16) simplifies:

$$(d\mu)_{\mathbb{CP}^N} = \frac{dz_1 \wedge \dots \wedge dz_N \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_N}{(1 + \sum_{k=1}^N z_k \bar{z}_k)^{N+1}} \quad (17)$$

For a given vector $|w\rangle$ of the representation $\hat{V}$ of the group G coherent states are by definition [11] an overcomplete basis for this representation, which is formed by the vectors in the orbit of the group G , containing $|w\rangle$. If V is the fundamental representation of $SU(N+1)$, then the space $\text{Sym}(V^{\otimes m})$ may be identified with the space of all polynomials of degree m on $N+1$ variables z_i , $i = 1 \dots N+1$. In order for the orbit to be as simple as possible we will take $|w\rangle = z_1^m$ — one of the highest weights — as element of the orbit³. In this case the orbit is $\mathbb{CP}^N$ — the smallest available one. Any element of the orbit may be written as follows⁴:

$$\phi_{\bar{v}}(z) = (\sum_i z_i \bar{v}_i)^m \quad (18)$$

The fact that the system is (over)complete means that the following fundamental identity is valid:

$$(\text{partition of unity}) \quad \int d\mu(v, \bar{v}) \frac{\phi_{\bar{v}}(z) \phi_v(\bar{z})}{(\phi_{\bar{v}}, \phi_{\bar{v}})} = 1 \quad (19)$$

4 The quantum sphere S^2

Let us consider in detail the case of $su(2)$ ⁵. When written using a single inhomogeneous coordinate, the coherent states are:

$$\phi_{\bar{v}}(z) = (1 + z\bar{v})^m \quad (20)$$

The kernel of a generic operator $\hat{A}$ may be obtained using a standard construction

$$A(z, \bar{v}) = \frac{(\hat{A}\phi_{\bar{v}}, \phi_{\bar{z}})}{(\phi_{\bar{v}}, \phi_{\bar{z}})} \quad (21)$$

Let us now regard the basis vectors $1, z$ as eigenvectors of the operator σ_3 with eigenvalues $1, -1$ respectively. Then the kernel of $\hat{A} = \hat{T}_3 = \sum_{s=1}^m 1 \otimes \dots \otimes \underset{\substack{\uparrow \\ \text{s-th place}}}{\sigma_3} \otimes \dots \otimes 1$ is

$$T_3(q, \bar{v}) = \frac{(m(1 - z\bar{v})(1 + z\bar{v})^{m-1}, (1 + z\bar{q})^m)}{(1 + z\bar{v}, 1 + z\bar{q})} = m \frac{1 - q\bar{v}}{1 + q\bar{v}} \quad (22)$$

³For a very clear exposition of these properties see [12] and [13].

⁴This notation is borrowed from [14], who used coherent states of this sort to describe the quantization of a sphere $S^2 \sim \mathbb{CP}^1$ — the simplest homogeneous Kähler (symplectic) manifold.

⁵General results along a similar line of reasoning were obtained in [15]

We will now present the derivation of the matrix elements (between coherent states) of the “evolution operator” $\hat{U} = e^{-i\alpha\hat{T}_3}$. Of course, in this finite-dimensional case this is merely a pedagogical exercise, since clearly the action of $\hat{U}$ on a coherent state simply gives

$$\hat{U}\phi_{\bar{v}} = (e^{-i\alpha} + e^{i\alpha}z\bar{v})^m, \quad (23)$$

so the corresponding matrix element is easily calculated:

$$U(q, \bar{v}) = \left(\frac{e^{-i\alpha} + e^{i\alpha}q\bar{v}}{1 + q\bar{v}} \right)^m \quad (24)$$

As is standard in path integral calculations [16], in order to write a path integral representation for a matrix element $U(q, \bar{v})$, we need to know the matrix elements of the generator (22), and then we need to split the “time” interval α into K subintervals of length $\frac{\alpha}{K}$ and use the formula

$$\hat{U} = \lim_{K \rightarrow \infty} (1 - \frac{i\alpha}{K}\hat{T}_3)^K \quad (25)$$

Let us write down a path integral for (25), using (19) and denoting $\hat{t} = 1 + \frac{i\alpha}{K}\hat{T}_3$:

$$\begin{aligned} U(q, \bar{y}) &= \frac{(\hat{U}\phi_{\bar{y}}, \phi_{\bar{q}})}{(\phi_{\bar{y}}, \phi_{\bar{q}})} = \\ &= \lim_{K \rightarrow \infty} \int \prod_{i=1}^{K-1} d\mu(z_i, \bar{z}_i) \tau(q, \bar{z}_{K-1}) \tau(z_{K-1}, \bar{z}_{K-2}) \dots \tau(z_2, \bar{z}_1) \tau(z_1, \bar{y}) \times \\ &\quad \times \frac{(\phi_{\bar{y}}, \phi_{\bar{z}_1})(\phi_{\bar{z}_1}, \phi_{\bar{z}_2}) \dots (\phi_{\bar{z}_{K-2}}, \phi_{\bar{z}_{K-1}})(\phi_{\bar{z}_{K-1}}, \phi_{\bar{q}})}{(\phi_{\bar{y}}, \phi_{\bar{q}})(\phi_{\bar{z}_1}, \phi_{\bar{z}_1}) \dots (\phi_{\bar{z}_{K-1}}, \phi_{\bar{z}_{K-1}})} \end{aligned} \quad (26)$$

To complete the derivation we use the following formulas:

$$\begin{aligned} \tau(z_{k+1}, \bar{z}_k) &= 1 - m \frac{i\alpha}{K} \frac{1 - z_k \bar{z}_{k+1}}{1 + z_k \bar{z}_{k+1}}, \\ (\phi_{\bar{z}_k}, \phi_{\bar{z}_{k+1}}) &= (1 + z_k \bar{z}_{k+1})^m \end{aligned}$$

Then (26) takes the form

$$U(q, \bar{y}) = \lim_{K \rightarrow \infty} \int \prod_{i=1}^{K-1} d\mu(z_i, \bar{z}_i) \left(\frac{1 + z_{K-1}\bar{y}}{1 + q\bar{y}} \right)^m \prod_{j=0}^{K-2} \left(\frac{1 + z_j \bar{z}_{j+1}}{1 + z_{j+1} \bar{z}_{j+1}} \right)^m \prod_{j=0}^{K-1} \left(1 - m \frac{i\alpha}{K} \frac{1 - z_j \bar{z}_{j+1}}{1 + z_j \bar{z}_{j+1}} \right), \quad (27)$$

where $z_0 = q$ and $\bar{z}_K = \bar{y}$. We now want to “take the limit” in the formula (27), assuming that $z_{i+1} - z_i \sim \frac{1}{K}$ (for a justification of this procedure see [17]). In order to do it we need to write the factors $\frac{1 + z_j \bar{z}_{j+1}}{1 + z_{j+1} \bar{z}_{j+1}}$ in the following form:

$$\frac{1 + z_j \bar{z}_{j+1}}{1 + z_{j+1} \bar{z}_{j+1}} = \left(1 - \frac{(z_{j+1} - z_j) \bar{z}_{j+1}}{1 + z_{j+1} \bar{z}_{j+1}} \right) \simeq 1 - \frac{1}{K} \frac{\dot{z}_{j+1} z_{j+1}}{1 + z_{j+1} \bar{z}_{j+1}}, \quad j = 0, 1, \dots, K-2. \quad (28)$$

Then we obtain

$$U(q, \bar{y}) = \int_{\substack{z(0)=q \\ \bar{z}(1)=\bar{y}}} \prod_{t \in [0,1]} d\mu(z(t), \bar{z}(t)) \left(\frac{1+z(1)\bar{y}}{1+q\bar{y}} \right)^m \exp \left(-m \int_0^1 dt \frac{\dot{z}\bar{z}}{1+z\bar{z}} - m i \alpha \int_0^1 dt \frac{1-z\bar{z}}{1+z\bar{z}} \right) \quad (29)$$

Let us elaborate on what the two terms in the exponent

$$\mathcal{L} = i \frac{\dot{z}\bar{z}}{1+z\bar{z}} - \alpha \frac{1-z\bar{z}}{1+z\bar{z}} \quad (30)$$

in (29) are. The first one $j = \frac{dz\bar{z}}{1+z\bar{z}}$ is the “Kähler current”, in other words the connection in a fibre bundle over $\mathbf{CP}^1$, whose derivative produces the Kähler form: $dj = \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2} = \omega_{FS}$. The second term $H = \frac{1-z\bar{z}}{1+z\bar{z}}$ is the Hamiltonian. z and $\bar{z}$ are the stereographic coordinates on the sphere $\mathbf{CP}^1$, and the equations of motion following from the Lagrangian (30) describe the rotation of the sphere around its z -axis (the one orthogonal to the plane of the stereographic projection). Let us write out the e.o.m. which follow from the Lagrangian (30):

$$i\dot{z} = -2\alpha z, \quad i\dot{\bar{z}} = 2\alpha \bar{z}. \quad (31)$$

We see that these are nothing but the equations of harmonic oscillations. In fact with a particular choice of coordinates the Lagrangian (30) may be turned exactly into the canonical Lagrangian of the harmonic oscillator, but this is not necessary for our purposes. Solving the equations with the prescribed initial conditions $z(0) = q$, $\bar{z}(1) = \bar{y}$, we obtain $z(t) = q e^{2i\alpha t}$, $\bar{z}(t) = \bar{y} e^{-2i\alpha(t-1)}$. Plugging this into the exponent of the path integral (29), we get $e^{-m i \alpha}$. The term $\frac{1+z(1)\bar{y}}{1+q\bar{y}}$ in front of the exponent produces $\frac{1+q\bar{y}e^{2i\alpha}}{1+q\bar{y}}$, and as a result we get

$$U(q, \bar{y}) = \left(\frac{e^{-i\alpha} + q\bar{y}e^{i\alpha}}{1+q\bar{y}} \right)^m, \quad (32)$$

which, as we know from (24), is the right answer.

5 Path integral for the spin chain

Similarly to what we did in (29), we now want to derive a path integral expression for the evolution operator of the spin chain $\hat{U} = e^{i\alpha \hat{H}}$, $\hat{H}$ now being one of the Hamiltonians (3)-(4). Thus, we pass from the simple $\mathfrak{su}(2)$ case to the $\mathfrak{su}(3)$, or even $\mathfrak{su}(N+1)$ model. Before actually considering the Hamiltonians (3)-(4) we will start with a typical but simpler example of the XXX Hamiltonian

$$H_{XXX} = \sum_{i=1}^L P_{i,i+1} \quad (33)$$

and its generalizations to the symmetric powers of the fundamental representation, indexed by m as before. These generalizations are obtained by replacing in (33) the permutation operator P with $\hat{\mathbb{P}}$, defined in (8).

5.1 The XXX chain

In order to build the path integral we first need to know the matrix elements of the Hamiltonian itself, which essentially means that we need to know the matrix elements of the operator $\hat{\mathbb{P}}$. This operator acts in the tensor product $\text{Sym}(\mathbf{C}^{N+1})^{\otimes m} \otimes \text{Sym}(\mathbf{C}^{N+1})^{\otimes m}$, and as an (overcomplete) basis in this space we will use the tensor product of the coherent state bases in each factor, i.e.

$$|\bar{a}, \bar{b}\rangle = (1 + v \circ \bar{a})^m (1 + w \circ \bar{b})^m \quad (34)$$

Here and below $\circ$ means a simple contraction (scalar product): $x \circ \bar{y} = \sum_{i=1}^N x_i \bar{y}_i$. As it follows from the discussion of Section 2, the action of $\hat{\mathbb{P}}$ on such a state is given by

$$\hat{\mathbb{P}}[(1 + a \circ \bar{z})^m (1 + b \circ \bar{w})^m] = m (1 + a \circ \bar{z})^{m-1} (1 + b \circ \bar{z}) (1 + b \circ \bar{w})^{m-1} (1 + a \circ \bar{w}). \quad (35)$$

Therefore the kernel of $\hat{\mathbb{P}}$ is easily found to be

$$\mathbb{P}(\bar{y}_1, \bar{y}_2; q_1, q_2) = \frac{\langle \bar{q}_1, \bar{q}_2 | \hat{\mathbb{P}} | \bar{y}_1, \bar{y}_2 \rangle}{\langle \bar{q}_1, \bar{q}_2 | \bar{y}_1, \bar{y}_2 \rangle} = \left[\frac{(1 + q_1 \circ \bar{y}_2)(1 + q_2 \circ \bar{y}_1)}{(1 + q_1 \circ \bar{y}_1)(1 + q_2 \circ \bar{y}_2)} \right]^m \quad (36)$$

Now we can essentially repeat the steps from the previous Section. The only difficulty is notational and it comes from the fact that in this case, as opposed to the previous example, we essentially have two “space-time” directions: one “time” or α -direction, and the second the direction, in which the spin chain is extended. As a consequence, our variables z will now carry two indices: $z_{a,i}$, where a is the time index, and i enumerates the sites of the spin chain (see Fig. 1).

The integrand will again split into two terms: one which may loosely be called the “kinetic” term and the second one being the Hamiltonian:

$$\mathbb{U} = \lim_{K \rightarrow \infty} \int \prod_{a,i} d\mu(z_{a,i}, \bar{z}_{a,i}) \times \mathbb{I}_{\text{kin}} \times \mathbb{I}_{\text{H}} \quad (37)$$

The kinetic term is local in the spin chain index i and has a simplest (nearest-neighbor, or first-order) nonlocality, which is a general feature, since in the continuum

Fig. 1. The space-time ‘ $i - a$ ’ lattice.

limit it should lead to a 1-form:

$$\mathbb{I}_{\text{kin}} = \prod_{a,i} \left(\frac{1 + z_{a,i} \circ \bar{z}_{a+1,i}}{1 + z_{a+1,i} \circ \bar{z}_{a+1,i}} \right)^m \quad (38)$$

On the other hand, the Hamiltonian term has a first-order nonlocality in the spin-chain direction, but also has a first-order nonlocality in the time direction, since the matrix elements of the Hamiltonian entering the integral are always of the form

$$\frac{1}{K} \langle z_{a+1,i} | \hat{H} | z_{a,j} \rangle \quad (39)$$

This latter nonlocality will not play a role, since, as explicitly shown in (39), the contribution of such matrix element always comes with a damping factor $\frac{1}{K}$, and the nonlocality being of order $\frac{1}{K}$ as well enters only subleading terms. In any case, the contribution of the Hamiltonian may be written as

$$\mathbb{I}_H = \prod_{a,i} \left(1 + m \frac{i\alpha}{K} \frac{1 + z_{a,i} \circ \bar{z}_{a+1,i+1}}{1 + z_{a,i} \circ \bar{z}_{a+1,i}} \frac{1 + z_{a,i+1} \circ \bar{z}_{a+1,i}}{1 + z_{a,i+1} \circ \bar{z}_{a+1,i+1}} \right) \quad (40)$$

We may now exponentiate these expressions and take the limit $K \rightarrow \infty$, thus obtaining a continuous time variable t :

$$\mathbb{U} = \int \prod_{t \in [0,1]} d\mu(z(t), \bar{z}(t)) \prod_i \left(\frac{1 + z_i(1) \circ \bar{z}_i(1)}{1 + z_i(0) \circ \bar{z}_i(1)} \right)^m \exp(i\mathcal{S}), \text{ where} \quad (41)$$

$$\mathcal{S} = m \int_0^1 dt \sum_i \left(i \frac{\dot{z}_i \circ \bar{z}_i}{1 + z_i \circ \bar{z}_i} + \alpha \frac{1 + z_i \circ \bar{z}_{i+1}}{1 + z_i \circ \bar{z}_i} \frac{1 + z_{i+1} \circ \bar{z}_i}{1 + z_{i+1} \circ \bar{z}_{i+1}} \right) \quad (42)$$

with boundary conditions $z_i(0) = q_i$, $\bar{z}_i(1) = \bar{y}_i$. The nontrivial question is how to take the continuous limit in the spin chain direction, indexed by “ i ”, — there are several inequivalent ways to do it. It is well-known that the XXX spin chain has two “vacua”, i.e. the states with minimal and maximal energy. They also correspond to the extremal values of the spin: the vacuum with spin zero (or least possible spin in case the length of the chain does not allow for zero spin) is called antiferromagnetic, whereas the state with maximal spin (proportional to L — the length of the chain) is called ferromagnetic. Which one of these states is the true vacuum depends, of course, on the sign of the Hamiltonian.

5.2 Ferromagnetic limit

The ferromagnetic limit is especially simple. It corresponds to the case where the z ’s at the neighboring sites are very close to each other, that is $z_{i+1} - z_i \sim \frac{1}{L}$ (we remind the reader that L is the length of the spin chain, i.e. the number of sites). The first term in

(42) then simply produces

$$\int dt \sum_i \left(i \frac{\dot{z}_i \circ \bar{z}_i}{1 + z_i \circ \bar{z}_i} \right) \rightarrow L \int dt \int_{-1/2}^{1/2} dx i \frac{\dot{z}(t, x) \circ \bar{z}(t, x)}{1 + z(t, x) \circ \bar{z}(t, x)}, \quad (43)$$

whereas the expression in the second term can be rewritten in the same spirit:

$$\begin{aligned} & \frac{1 + z_i \circ \bar{z}_{i+1}}{1 + z_i \circ \bar{z}_i} \frac{1 + z_{i+1} \circ \bar{z}_i}{1 + z_{i+1} \circ \bar{z}_{i+1}} = \\ & = 1 - \sum_i \frac{\Delta z_i \circ \Delta \bar{z}_i}{(1 + z_i \circ \bar{z}_i)(1 + z_{i+1} \circ \bar{z}_{i+1})} + \frac{1}{2} \sum_i \frac{[(z_i \circ \Delta \bar{z}_i) \Delta z_i - (\Delta z_i \circ \Delta \bar{z}_i) z_i] \circ [\bar{z}_i + \bar{z}_{i+1}]}{(1 + z_i \circ \bar{z}_i)(1 + z_{i+1} \circ \bar{z}_{i+1})}, \end{aligned} \quad (44)$$

and in the continuum limit the last two terms reduce to

$$\frac{1}{L} \int dx \left(-\frac{\partial_x z \circ \partial_x \bar{z}}{(1 + z \circ \bar{z})^2} + \frac{(z \circ \partial_x \bar{z})(\partial_x z \circ \bar{z}) - (z \circ \bar{z})(\partial_x z \circ \partial_x \bar{z})}{(1 + z \circ \bar{z})^2} \right) = -\frac{1}{L} \int dx \left(\frac{\partial_x z \circ \partial_x \bar{z}}{1 + z \circ \bar{z}} - \frac{(z \circ \partial_x \bar{z})(\partial_x z \circ \bar{z})}{(1 + z \circ \bar{z})^2} \right) \quad (45)$$

One immediately recognizes that the integrand in (45) is the Fubini-Study metric (written in the inhomogeneous coordinates $z_{N+1} = \bar{z}_{N+1} = 1$). Hence, the full action has the form

$$\mathcal{S} = m \int_0^1 dt \int_{-1/2}^{1/2} dx \left[L i \frac{\dot{z}(t, x) \circ \bar{z}(t, x)}{1 + z(t, x) \circ \bar{z}(t, x)} - \frac{1}{L} \left(\frac{\partial_x z \circ \partial_x \bar{z}}{1 + z \circ \bar{z}} - \frac{(z \circ \partial_x \bar{z})(\partial_x z \circ \bar{z})}{(1 + z \circ \bar{z})^2} \right) \right] \quad (46)$$

or after the rescaling $x \rightarrow \frac{1}{L}x$

$$\mathcal{S} = m \int_0^1 dt \int_{\mathbb{R}} dx \left[i \frac{\dot{z}(t, x) \circ \bar{z}(t, x)}{1 + z(t, x) \circ \bar{z}(t, x)} - \left(\frac{\partial_x z \circ \partial_x \bar{z}}{1 + z \circ \bar{z}} - \frac{(z \circ \partial_x \bar{z})(\partial_x z \circ \bar{z})}{(1 + z \circ \bar{z})^2} \right) \right] \quad (47)$$

Such non-relativistic sigma-models are known as Landau-Lifshitz models (see Appendix B). The target space of the model we have described is, obviously, $\mathbf{CP}^N$. The simplest example corresponds to $N = 1$, i.e. when the target space is a usual 2-sphere. In this case the model is also known as the classical Heisenberg ferromagnet, and it is customary to use the unit three-vector $\vec{n}$ instead of the complex coordinates $z, \bar{z}$ (the two parametrizations are related via the stereographic projection: $n_1 + i n_2 = \frac{2z}{1 + z\bar{z}}$, $n_3 = \frac{1 - z\bar{z}}{1 + z\bar{z}}$). Then the e.o.m., which follows from Lagrangian (47), is:

$$\frac{\partial \vec{n}}{\partial t} = \vec{n} \times \frac{\partial^2 \vec{n}}{\partial x^2}. \quad (48)$$

Before concluding this Section let us return for a moment back to the expression (41) and observe that, besides the action $\mathcal{S}$ in the exponent, it also includes a prefactor

$\prod_i \left(\frac{1+z_i(1)\bar{z}_i(1)}{1+z_i(0)\bar{z}_i(1)} \right)^m$, which in the continuum limit becomes

$$\begin{aligned} \prod_i \left(\frac{1+z_i(1)\bar{z}_i(1)}{1+z_i(0)\bar{z}_i(1)} \right)^m &\rightarrow \exp \left[mL \int_0^1 dx \left[\frac{1+z(t=1,x)\bar{z}(t=1,x)}{1+z(t=0,x)\bar{z}(t=1,x)} \right] \right] = \\ &= \exp \left[mL \int_0^1 dx \left[\frac{1+z(t=1,x)\bar{y}(x)}{1+q(x)\bar{y}(x)} \right] \right], \end{aligned} \quad (49)$$

where in the last expression we took into account the boundary conditions. We will not need this expression in what follows, but it should not be overlooked.

6 The antiferromagnetic limit

The antiferromagnetic limit is much more difficult to analyze. The main idea is that in this case the z -variables on neighboring sites are no longer close to each other. Let us first elaborate on the case of the sphere, that is $N = 1$, which was for the first time explored in [2]. In this case it is intuitively clear that the antiferromagnetic limit corresponds to the case where the spins on the neighboring sites have opposite directions, i.e. $\vec{n}_{i+1} \simeq -\vec{n}_i$. In terms of the complex coordinates used above this may be written as $z_{i+1} \simeq -\frac{1}{\bar{z}_i}$, or, using homogeneous coordinates, as $z_1^{(i+1)} = \bar{z}_2^{(i)}, z_2^{(i+1)} = -\bar{z}_1^{(i)}$. Such simple explanation is due to the fact that on the sphere there exists the antipodal involution, that is the involution, which is an orientation-reversing isometry. The antipodal involution on the sphere is unique. On the other hand, this is no longer so for $\mathbf{CP}^N$ with $N \geq 2$. This is the reason why it is not immediately obvious, how one can extend the $\mathbf{CP}^1$ analysis to a higher-dimensional projective space. The answer crucially depends on the particular Hamiltonian at hand. The first model to be successfully analyzed was (3), so let us now recall how this was accomplished.

6.1 The construction of Affleck

Affleck [7] considered a generalized Haldane limit for the spin chain with Hamiltonian (3). In order to rephrase his results in our language one should follow the steps of the previous Section to obtain the following action in the t -continuum limit:

$$\mathcal{S} = m \int_0^1 dt \sum_i \left(i \frac{\dot{z}_i \circ \bar{z}_i}{z_i \circ \bar{z}_i} + \alpha \frac{z_i \circ z_{i+1}}{z_i \circ \bar{z}_i} \frac{\bar{z}_i \circ \bar{z}_{i+1}}{z_{i+1} \circ \bar{z}_{i+1}} \right) \quad (50)$$

The difference between the second terms in (42) and (50) precisely reflects the difference between P and Tr operators entering the corresponding Hamiltonians. The minimum of the Hamiltonian $H = -\sum_i \frac{z_i \circ z_{i+1}}{z_i \circ \bar{z}_i} \frac{\bar{z}_i \circ \bar{z}_{i+1}}{z_{i+1} \circ \bar{z}_{i+1}}$ is clearly reached for $z_{i+1} = \bar{z}_i$. The important

observation is that for such configuration the first term in (50) turns into a full derivative, since on every two neighboring sites $i \frac{\dot{z}_i \circ \bar{z}_i}{z_i \circ \bar{z}_i} + i \frac{\dot{z}_{i+1} \circ \bar{z}_{i+1}}{z_{i+1} \circ \bar{z}_{i+1}} = i \frac{d}{dt} (\log(z_i \circ \bar{z}_i))$. There is a simple but fundamental explanation of this fact. In order to formulate it let us diverge for a moment to a slightly more general setup.

First of all, it is obvious that the direct product of two symplectic manifolds is a symplectic manifold. Indeed, let $\mathcal{M}_1, \mathcal{M}_2$ be endowed with the respective symplectic forms ω_1, ω_2 . Then the natural symplectic form on $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is $\omega = \omega_1 + \omega_2$. There's a canonical symplectic form on $\mathbf{CP}^N$, namely the Fubini-Study form, which looks as follows:

$$\omega_{FS} = i \left(\frac{da_i \wedge d\bar{a}_i}{\bar{a}_j a_j} - \frac{da_i \bar{a}_i \wedge d\bar{a}_k a_k}{(\bar{a}_j a_j)^2} \right). \quad (51)$$

The indices in this formula run from 1 to $N+1$. According to the above remark, the symplectic form on the direct product of several $\mathbf{CP}^N$'s is the sum of the respective Fubini-Study forms. We may now formulate the following

Observation 1. The embedding of $\mathbf{CP}^N$ into $\mathbf{CP}^N \times \mathbf{CP}^N$, defined by the map $z \rightarrow (z, \bar{z})$, is Lagrangian.

Let us now expand the action (50) around the “vacuum” $z_{i+1} = \bar{z}_i$. The variables z_{i+1} and z_{i+2} are expressed in terms of z_i in the following fashion:

$$z_{i+1} = \bar{z}_i + \frac{1}{L} \bar{\tau}_i, \quad z_{i+2} = z_i + \frac{1}{L} z'_i \quad (52)$$

For convenience we introduce also the projector to the subspace of $\mathbf{C}^{N+1}$ orthogonal to the vector z_i :

$$\Pi_i = \frac{\mathbf{I}}{|z_i|^2} - \frac{\bar{z}_i \otimes z_i}{|z_i|^4} \quad (53)$$

Then the terms in the Hamiltonian have the following expansions:

$$\frac{|z_i \circ z_{i+1}|^2}{|z_i|^2 |z_{i+1}|^2} \simeq \frac{1}{L^2} \tau_i \circ \Pi_i \circ \bar{\tau}_i, \quad \frac{|z_{i+1} \circ z_{i+2}|^2}{|z_i|^2 |z_{i+1}|^2} \simeq \frac{1}{L^2} \bar{\tau}_i \circ \Pi_i \circ \bar{\tau}_i \quad \text{with} \quad \bar{\tau}_i = \tau_i - z'_i \quad (54)$$

The terms in the “kinetic energy” are expanded as follows:

$$i \frac{\dot{z}_i \circ \bar{z}_i}{z_i \circ \bar{z}_i} + i \frac{\dot{z}_{i+1} \circ \bar{z}_{i+1}}{z_{i+1} \circ \bar{z}_{i+1}} = i \frac{1}{L} [\tau_i \circ \Pi_i \circ \dot{\bar{z}}_i - \bar{\tau}_i \circ \Pi_i \circ \dot{z}_i] + \text{full derivative} \quad (55)$$

Thus, the action (50) acquires the following form:

$$\mathcal{S} = m \int_0^1 dt \sum_i \left(i \frac{1}{L} [\tau_i \circ \Pi_i \circ \dot{\bar{z}}_i - \bar{\tau}_i \circ \Pi_i \circ \dot{z}_i] + \frac{1}{L^2} [\tau_i \circ \Pi_i \circ \bar{\tau}_i + (\tau_i - z'_i) \circ \Pi_i \circ (\bar{\tau}_i - \bar{z}'_i)] \right) \quad (56)$$

Now we simply need to “integrate out” the fields $\tau, \bar{\tau}$. Upon setting $\tau, \bar{\tau}$ equal to their stationary values we also pass to the continuum limit with respect to the “ i ” index. This

leads to the following expression:

$$\mathcal{S} = m \int_0^1 dt \int_{\mathbb{R}} dx \left[\frac{1}{2} \partial_\mu z(x, t) \circ \Pi(z, \bar{z}) \circ \partial_\mu \bar{z}(x, t) - \frac{i}{2} \epsilon_{\mu\nu} \partial_\mu z(x, t) \circ \Pi(z, \bar{z}) \circ \partial_\nu \bar{z}(x, t) \right] \quad (57)$$

Clearly, the first term is the standard action of the $\mathbf{CP}^N$ sigma model, whereas the second term is the pull-back to the worldsheet of the Kähler form. The second term is topological and corresponds to the theta-angle $\theta = \pi m \bmod 2\pi$.

6.2 Antiferromagnetic configuration of the Hamiltonian (4)

We now want to move forward from the Hamiltonian (3) and find the sigma model which arises upon taking the continuum limit around the antiferromagnetic “vacuum” of the spin chain (4). First of all, completely parallel to the discussion of the XXX spin chain in Section 5.1 above, we can write a path integral expression for the evolution operator of the spin chain (4). Similarly to (42), the action appearing in the exponent in the integrand of the path integral has the following form:

$$\mathcal{S} = m \int_0^1 dt \sum_i \left(i \frac{\dot{z}_i \circ \bar{z}_i}{z_i \circ \bar{z}_i} + \alpha \underbrace{\left(\frac{z_i \circ \bar{z}_{i+1}}{z_i \circ \bar{z}_i} \frac{z_{i+1} \circ \bar{z}_i}{z_{i+1} \circ \bar{z}_{i+1}} + \frac{1}{2} \frac{z_i \circ \bar{z}_{i+2}}{z_i \circ \bar{z}_i} \frac{z_{i+2} \circ \bar{z}_i}{z_{i+2} \circ \bar{z}_{i+2}} \right)}_{\equiv \mathcal{H}} \right) \quad (58)$$

In this formula each of the variables z_i has an additional (hidden) index, which takes three possible values corresponding to the fundamental representation of $SU(3)$. We emphasize that the Hamiltonian (4) is interesting for us only in the case of $SU(3)$ symmetry — in the $SU(N)$ case one should pick a different Hamiltonian. We claim that in the case of (58) the antiferromagnetic vacuum configuration is when the z -vectors on any 3 neighboring sites are orthogonal to each other. First of all, this is consistent with what we had for the $SU(2)$ case above, since the equation $1 + \bar{z}_1 z_2 = 0$ arising in that case has a unique solution $z_2 = -\frac{1}{\bar{z}_1}$, which is the antipodal involution discussed above. When $N = 3$ we need to take three consecutive sites and impose orthogonality conditions on the three z -vectors z_1, z_2, z_3 sitting at these sites⁶:

$$z_1 \circ \bar{z}_2 = z_2 \circ \bar{z}_3 = z_1 \circ \bar{z}_3 = 0. \quad (59)$$

The submanifold of $(\mathbf{CP}^2)^{\times 3}$ described by (59) is known as the flag manifold $\mathcal{F}_3$ (The index 3 points out that this manifold is a homogeneous space of $SU(3)$. Flag manifolds for the group $SU(N)$ are introduced in Appendix C.). We’re now going to elaborate on this simplest nontrivial example.

⁶Here we use homogeneous coordinates.

6.2.1 The $SU(3)$ case.

Let us first reexamine the l.h.s. of (43). It is clear that the different continuum limit, described by (59), will no longer produce the r.h.s. of (43). Thus, the question is what will arise in the continuum limit. The discussion above indicates that it is natural to first focus on arbitrary 3 consecutive sites. Then the kinetic term in the discretized Lagrangian is the pull-back J_t of the following 1-form (hereafter we employ the homogeneous coordinates):

$$J = i \frac{dz_1 \circ \bar{z}_1}{z_1 \circ \bar{z}_1} + i \frac{dz_2 \circ \bar{z}_2}{z_2 \circ \bar{z}_2} + i \frac{dz_3 \circ \bar{z}_3}{z_3 \circ \bar{z}_3} \quad (60)$$

This is the Kähler current on the product $\mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2$, and its divergence gives the Kähler (symplectic) form:

$$dJ = \Omega. \quad (61)$$

We claim that on the submanifold $\mathcal{F}_3$, described by (59), this 2-form is zero. We may even formulate a slightly more general

Observation 2. The embedding $\mathcal{F}_3 \hookrightarrow (\mathbb{CP}^2)^{\times 3}$ and more generally $\mathcal{F}_N \hookrightarrow (\mathbb{CP}^{N-1})^{\times N}$ is Lagrangian.

The proof of this statement for $N = 3$ is presented in Appendix C, and for the moment let us focus on the consequences of this fact. It follows that

$$J|_{\mathcal{F}_3} = df, \quad (62)$$

f being a function⁷, so the integral $\int_0^1 J_t = f(1) - f(0)$ reduces to the boundary term. We ignore this term in the present discussion.

7 The continuum limit

The “potential energy” term $\mathcal{H}$ in (58) is equal to zero if we impose the background configuration (59): $z_i \circ \bar{z}_{i+1} = 0$. Moreover, since $0 \leq \mathcal{H} \leq 2$, one immediately sees that the ferromagnetic and antiferromagnetic vacua saturate respectively the maximum and minimum of its possible values. In view of the fact that in the following we will build an expansion around the antiferromagnetic vacuum, from this observation we deduce an important consequence, namely that this expansion must start with a quadratic term (at least), i.e. there is no linear term.

Let us assume that the number of sites of our spin chain is a factor of 3 (this is only needed for simplicity, and it does not play a big role for a sufficiently long spin chain). In this case we split the spin chain into $\hat{L}$ segments of length 3 and focus for the moment on just one of these segments, which is link number k in the chain.

⁷In fact, $f = i \log(\epsilon_{abc} z_1^a z_2^b z_3^c)$.

7.1 The expansion around the “vacuum” configuration

On each of the three sites we have a three-dimensional complex vector z . Let us form a 3×3 matrix of these vectors, which we denote by Z . The antiferromagnetic configuration corresponds to the case where the three vectors are mutually orthogonal. Now we need to take the fluctuations into account, and in order to build the sought for expansion we will employ the so-called QR decomposition of a matrix. The QR decomposition theorem says that an arbitrary matrix Z may be decomposed into a product of a unitary matrix U and an upper triangular one B_+ :

$$Z = U \circ B_+ \quad (63)$$

This statement is equivalent to the Gram-Schmidt orthogonalization theorem. Let us parametrize B_+ in the following way:

$$B_+ = \begin{pmatrix} 1 & \frac{1}{L}x_k & \frac{1}{L}y_k \\ 0 & 1 & \frac{1}{L}z_k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_k & 0 & 0 \\ 0 & b_k & 0 \\ 0 & 0 & c_k \end{pmatrix} \quad (64)$$

If we denote the columns of the matrix U as (u_1, u_2, u_3) , the decomposition (63) says that

$$z_{1,k} = a_k u_{1,k}, \quad z_{2,k} = b_k (u_{2,k} + \frac{1}{L}x_k u_{1,k}), \quad z_{3,k} = c_k (u_{3,k} + \frac{1}{L}y_k u_{1,k} + \frac{1}{L}z_k u_{2,k}) \quad (65)$$

The hypothesis of the existence of a continuum limit implies that $u_{1,k}$, $u_{2,k}$, $u_{3,k}$ vary mildly with k , in other words we may approximate

$$u_{i,m+1} = u_{i,m} + \frac{1}{L}u'_{i,m} + \dots \quad (66)$$

Fig. 1. Explanation of the various terms calculated in (68).

Let us introduce the quantity

$$\mathcal{H}_{m,n}^{i,j} = \frac{|z_{i,m} \circ \bar{z}_{j,n}|^2}{|z_{i,m}|^2 |z_{j,n}|^2}, \quad (67)$$

which is the density of the Hamiltonian $\mathcal{H}$ from (58), if the indices i, j, m, n change in a particular range. Indeed, we need to calculate $\mathcal{H}_{m,n}^{i,j}$ for nearest- and next-to-nearest

neighbor sites, using the formulas (65)-(66) and keeping only the leading orders of $\frac{1}{L^2}$ (see Fig.2 for an explanation of what these terms stand for):

$$\begin{aligned}
 \mathcal{H}_{k,k}^{1,2} &= \frac{|z_1 \circ \bar{z}_2|^2}{|z_1|^2 |z_2|^2} \simeq \frac{1}{L^2} |x_k|^2, & \mathcal{H}_{k,k}^{2,3} &= \frac{|z_2 \circ \bar{z}_3|^2}{|z_2|^2 |z_3|^2} \simeq \frac{1}{L^2} |z_k|^2, & \mathcal{H}_{k,k}^{1,3} &= \frac{|z_1 \circ \bar{z}_3|^2}{|z_1|^2 |z_3|^2} \simeq \frac{1}{L^2} |y_k|^2 \\
 \mathcal{H}_{k-1,k}^{3,1} &= \frac{|z_{3,k-1} \circ \bar{z}_{1,k}|^2}{|z_{3,k-1}|^2 |z_{1,k}|^2} = \frac{|(u_{3,k-1} + \frac{1}{L} y_{k-1} u_{1,k-1} + \frac{1}{L} z_{k-1} u_{2,k-1}) \circ \bar{u}_{1,k}|^2}{|u_{3,k-1} + \frac{1}{L} y_{k-1} u_{1,k-1} + \frac{1}{L} z_{k-1} u_{2,k-1}|^2 |u_{1,k}|^2} \simeq \\
 &\simeq \frac{1}{L^2} |-u'_{3,k} \circ \bar{u}_{1,k} + y_k|^2 \\
 \mathcal{H}_{k-1,k}^{3,2} &= \frac{|z_{3,k-1} \circ \bar{z}_{2,k}|^2}{|z_{3,k-1}|^2 |z_{2,k}|^2} = \frac{|(u_{3,k-1} + \frac{1}{L} y_{k-1} u_{1,k-1} + \frac{1}{L} z_{k-1} u_{2,k-1}) \circ (\bar{u}_{2,k} + \frac{1}{L} \bar{x}_k \bar{u}_{1,k})|^2}{|u_{3,k-1} + \frac{1}{L} y_{k-1} u_{1,k-1} + \frac{1}{L} z_{k-1} u_{2,k-1}|^2 |u_{2,k} + \frac{1}{L} x_k u_{1,k}|^2} \simeq \\
 &\simeq \frac{1}{L^2} |-u'_{3,k} \circ \bar{u}_{2,k} + z_k|^2 \\
 \mathcal{H}_{k-1,k}^{2,1} &= \frac{|z_{2,k-1} \circ \bar{z}_{1,k}|^2}{|z_{2,k-1}|^2 |z_{1,k}|^2} = \frac{|(u_{2,k-1} + \frac{1}{L} x_{k-1} u_{1,k-1}) \circ \bar{u}_{1,k}|^2}{|u_{2,k-1} + \frac{1}{L} x_{k-1} u_{1,k-1}|^2 |u_{1,k}|^2} \simeq \frac{1}{L^2} |-u'_{2,k} \circ \bar{u}_{1,k} + x_k|^2
 \end{aligned} \tag{68}$$

Hence, the Hamiltonian $H_2 = \sum_i (P_{i,i+1} + \frac{1}{2} P_{i,i+2})$ produces a contribution

$$\mathcal{H} = \frac{1}{L^2} \left[\sum_k \left(|x_k|^2 + \frac{1}{2} |y_k|^2 + |z_k|^2 + |-u'_{3,k} \circ \bar{u}_{1,k} + y_k|^2 + \frac{1}{2} |-u'_{3,k} \circ \bar{u}_{2,k} + z_k|^2 + \frac{1}{2} |-u'_{2,k} \circ \bar{u}_{1,k} + x_k|^2 \right) \right] \tag{69}$$

Let us now turn to the kinetic term (60) and see what it produces to the leading order in $\frac{1}{L}$. A simple calculation reveals that

$$J_t = \left(i \frac{\dot{z}_1 \circ \bar{z}_1}{z_1 \circ \bar{z}_1} + i \frac{\dot{z}_2 \circ \bar{z}_2}{z_2 \circ \bar{z}_2} + i \frac{\dot{z}_3 \circ \bar{z}_3}{z_3 \circ \bar{z}_3} \right)_{x=y=z=0} - \frac{i}{L} (x u_1 \circ \dot{u}_2 + y u_1 \circ \dot{u}_3 + z u_2 \circ \dot{u}_3 - \text{c.c.}) + \dots \tag{70}$$

But we have proved above that the first part of this expression is in fact a full derivative:

$$\left(i \frac{\dot{z}_1 \circ \bar{z}_1}{z_1 \circ \bar{z}_1} + i \frac{\dot{z}_2 \circ \bar{z}_2}{z_2 \circ \bar{z}_2} + i \frac{\dot{z}_3 \circ \bar{z}_3}{z_3 \circ \bar{z}_3} \right)_{x=y=z=0} = \frac{df}{dt} \tag{71}$$

and, as such, can be omitted up to boundary terms. Next we combine (69) and (70) and “integrate out” the auxiliary variables x_k, y_k, z_k . This is done in the Appendix D in a detailed way, and for the moment we write out the resulting sigma model action:

$$\begin{aligned}
 \mathcal{S} &= m \int dt dx \left(\frac{\sqrt{2}}{3} [|u_1 \circ \partial_\mu \bar{u}_2|^2 + |u_1 \circ \partial_\mu \bar{u}_3|^2 + |u_2 \circ \partial_\mu \bar{u}_3|^2] - \right. \\
 &\quad \left. - \frac{i}{3} \epsilon_{\mu\nu} [(u_1 \circ \partial_\mu \bar{u}_2)(\bar{u}_1 \circ \partial_\nu u_2) + 2(u_1 \circ \partial_\mu \bar{u}_3)(\bar{u}_1 \circ \partial_\nu u_3) + (u_2 \circ \partial_\mu \bar{u}_3)(\bar{u}_2 \circ \partial_\nu u_3)] \right).
 \end{aligned} \tag{72}$$

The first line in (72) is the so-called normal metric on the flag space $\mathcal{F}_3$ (see Section 7.2), whereas the second line is the pull-back to the worldsheet of $\omega|_{\mathcal{F}}$, where $\omega|_{\mathcal{F}}$ is the restriction to the flag of the closed 2-form ω from (107) and one should keep in mind that $u_{1,2,3}$ are subject to the orthonormality conditions. A rather interesting property

of the action (72) is that it contains a θ -term of a special form. Indeed, a priori there is space for two θ -angles, since the corresponding cohomology group $H^2(\mathcal{F}_3, \mathbb{R}) \simeq \mathbb{R}^2$ is two-dimensional⁸, so it is an interesting question, why it is the particular form ω that appears in this context.

7.2 Metrics on flag manifolds

In this Section we will show that the metric (72) on $\mathcal{F}_3$ that we have obtained is not the most general metric compatible with the symmetries of the flag manifold⁹. This is rather obvious from the beginning, since it is clear that we can multiply each of the three terms in the first line of (72) by an arbitrary (positive) constant without breaking the $SU(3)$ symmetry. Below we present the most general $SU(N)$ -invariant metric on $\mathcal{F}_N$, but in many cases one is interested in the special metrics possessing certain properties, such as being Kähler or Einstein — the reader can find a rather detailed discussion of these properties in [18]. In particular, it follows from this work that the normal metric, entering the action (72) above, is Einstein but not Kähler (although the flag manifold is a Kähler manifold meaning that it *can* be equipped with a Kähler metric).

Let us start from the obvious fact that, since $U(1)^N \subset U(N)$ the Lie algebra $\mathfrak{u}(N)$ naturally splits

$$\mathfrak{u}(n) = \underbrace{\mathfrak{u}(1) \oplus \dots \oplus \mathfrak{u}(1)}_{n \text{ times}} \oplus \mathfrak{h} \equiv \mathfrak{t} \oplus \mathfrak{h} \quad (73)$$

This is in fact nothing but the decomposition of a Lie algebra into a Cartan subalgebra and the associated root space $\mathfrak{h} = \bigoplus_{\text{all roots } \alpha} E_\alpha$, E_α being the subspace corresponding to a given root. $\mathcal{F}_3$ is not a symmetric space, i.e. $[\mathfrak{h}, \mathfrak{h}] \not\subset \mathfrak{t}$ — this is obvious, since the commutator of two “roots” in general produces another root¹⁰ $[E_\alpha, E_\beta] \subset E_{\alpha+\beta}$ and not a Cartan generator.

Next we review the standard “coset” construction of metrics. One generally takes a group element $g \in U(N)$ and builds a current

$$J = g^{-1} dg \in \mathfrak{u}(N). \quad (74)$$

⁸There is in fact a simple choice for the de-Rham representatives of this cohomology group. Denote by $\tilde{\Omega}_i = \Omega_i|_{\mathcal{F}_3}$, $i = 1, 2, 3$ the restrictions to the flag manifold of the three Fubini-Study forms. The form $\tilde{\Omega} = \sum_{i=1}^3 \theta_i \tilde{\Omega}_i$ is closed. Since, as discussed above, $\sum_i \tilde{\Omega}_i = 0$, the form $\tilde{\Omega}$ is parametrized by a vector $(\theta_1, \theta_2, \theta_3) \bmod (1, 1, 1)$, which therefore determines a two-dimensional space. As one can check, it is the space of non-exact closed two-forms.

⁹I am grateful to A.Gerasimov and S.Shatashvili for pointing out to me that there is a family of metrics on $\mathcal{F}_N$.

¹⁰With the exception of $\mathfrak{su}(2)$ where there’s only one positive and negative root. In this case $\mathcal{F}_2 \simeq SU(2)/U(1) \simeq S^2$, i.e. a sphere, which clearly is a symmetric space.

The action of the stabilizer $T = U(1)^N$ of the coset on J is as follows:

$$h \circ J = (gh)^{-1} d(gh) = h^{-1} J h + h^{-1} dh \quad (75)$$

The last term in this expression belongs to $\mathfrak{t}$. Thus, the transformation of $\pi(J)$ — the projection of J on $\mathfrak{h} \subset \mathfrak{u}(N)$ — is particularly simple:

$$h \circ \pi(J) = h^{-1} \pi(J) h, \quad (76)$$

or in other words $\mathfrak{t}$ is represented on $\pi(J)$. One is often accustomed to writing the metric on a quotient space G/T (with Lie algebra splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$) simply as $\text{tr}(\pi(J)^2)$, but an important fact to realize is that this is only unique, when the representation of T on $\mathfrak{h}$ is *irreducible*. This is not so for the case at hand (and indeed an irreducible (over $\mathbf{C}$) representation of $U(1)$ must be one-dimensional — this is also true in our situation, since the N groups $U(1)$ commute).

It is easy to see that each component j_{ab} , $a \neq b$ of the matrix J furnishes an irreducible representation of $U(1)^N$ with the weight

$$(0, \dots, \underset{\substack{\uparrow \\ \text{a-th place}}}{1}, \dots, \underset{\substack{\uparrow \\ \text{b-th place}}}{-1}, \dots, 0), \quad (77)$$

which means that for all $a > b$ the quantity $j_{ab} j_{ba} = |j_{ab}|^2$ is an invariant. This means precisely that the generic metric can be written as follows:

$$ds^2 = \sum_{a>b} C_{ab} |j_{ab}|^2 \quad \text{with} \quad C_{ab} > 0, \quad (78)$$

the latter requirement needed for the nondegeneracy of the metric.

8 The mass gap and relation to trimerization

We now pass to a particular application of the general theory. Recently there has been interest in the spin chain models exhibiting the phenomenon of “trimerization”, or even more generally “ N -merization”. This means that the N neighboring sites on a spin chain form a bond — in other words, the N fundamental representations of $SU(N)$ combine into a singlet. This could well be called “baryonization”, if one prefers the elementary particle terminology. When $N = 2$, the phenomenon is known as dimerization and has been well studied (see [8], for instance). There is even a model, which exhibits exact dimerization of the ground state — the so-called Majumdar-Ghosh model, and recently generalizations to the N -merized case have been introduced [19].

The authors of [20] considered a spin chain, which is directly relevant to our discus-

sion in this paper. It is described by the following Hamiltonian:

$$\tilde{H} = J_1 \sum_i (\cos \theta (\mathbf{S}_i \mathbf{S}_{i+1}) + \sin \theta (\mathbf{S}_i \mathbf{S}_{i+1})^2) + J_2 \sum_i (\cos \theta (\mathbf{S}_i \mathbf{S}_{i+2}) + \sin \theta (\mathbf{S}_i \mathbf{S}_{i+2})^2) \quad (79)$$

Here $\mathbf{S}$ are the $su(2)$ spin operators written in the vector $\mathbf{3}$ representation.

One can check that the operator $P = (\mathbf{S}_i \mathbf{S}_{i+1}) + (\mathbf{S}_i \mathbf{S}_{i+1})^2$ is simply a permutation in the tensor product $\mathbf{C}^3 \otimes \mathbf{C}^3$. It means that for $J_2 = \frac{1}{2}J_1 \equiv \frac{1}{2}J$ and $\theta = \frac{\pi}{4}$ the above $SU(2)$ -invariant Hamiltonian reduces to the $SU(3)$ -invariant $\tilde{H} = J \sum_i (P_{i,i+1} + \frac{1}{2}P_{i,i+2})$, which is nothing but our Hamiltonian (4). Figure 3 reproduces a picture from the paper [20], representing the phase diagram of the model (79), which was obtained numerically.

Fig. 3. Phase diagram of the Hamiltonian (79) [20].

from our analysis, since the σ -model has a nonzero θ -term for the fundamental representation of $SU(3)$, i.e. $m = 1$. It would be very interesting to investigate this analytically for the flag sigma model described by the action (72), but so far we have not been able to advance in this direction. The reason for this is that the usual method for this kind of calculations — the $1/N$ expansion — does not work in this case, since the model (72) is more of a matrix model than a vector model. Indeed, the dynamical degrees of freedom are the N vectors with N components each. Essentially the problems we encounter along the $1/N$ expansion route are similar to the ones that arise in the principal chiral model [21].

9 Discussion

In this paper we found a spin chain, whose excitations near the antiferromagnetic vacuum are described by the sigma-model with target space the manifold of complete flags¹¹

¹¹B here stands for the Borel subgroup of $GL(3, \mathbf{C})$.

In particular, one can see that for the prescribed values of the parameters we get to the bottom of the trimerized phase, close to a triple vertex of the phase diagram. The numerical calculations performed by the authors of [20] also suggest that, similarly to the Haldane gap of the integer spin $\vec{S}_i \vec{S}_{i+1}$ chain, the trimerized phase has a gap, too, however for our values of the parameters the gap is either very small or zero. This also seems plausible

$\mathcal{F}_3 \simeq GL(3, \mathbf{C})/B \simeq U(3)/U(1)^3$ (with a special S_3 -symmetric choice of metric). We believe that the flag sigma model captures the low-energy dynamics over the trimerized vacuum configuration (or over the N -merized vacuum in the case of $SU(N)$) of a particular spin chain. This seems to be consistent with the recent results in condensed matter [20, 19].

Besides these applications, we hope our approach can be generalized to other situations. In particular, it opens the possibility of searching for various Lagrangian embeddings and building the corresponding spin chains. It is also worth emphasizing, that although in this paper we were dealing solely with the Lagrangian embeddings, this is in fact too strong a requirement, and in principle an isotropic embedding is sufficient for our argumentation (An embedding $M \hookrightarrow (N, \Omega)$ is said to be isotropic, if $\Omega|_M = 0$ — that is, no restriction on the dimension of M is imposed). Such situations have been widely explored in the symplectic geometry literature [22] and it would be interesting to understand, what they mean in the spin chain setup.

Another interesting question has to do with the integrability properties of the spin chains/sigma models. As explained in the introduction, this was actually the main motivation of our work — to establish integrability/non-integrability of various $\mathbf{CP}^N$ models with fermions (model (1) in particular). It has been known for a while now that the bosonic $\mathbf{CP}^N$ model is not integrable, and its “would-be” solitons are confined [23]. On the other hand, once you add the fermions, the situations can change dramatically, and the solitons can in some cases become liberated [24].

As we discussed above, the $\mathbf{CP}^N$ sigma model has been obtained from an alternating spin chain in [7]. It is also likely that the fermionic $\mathbf{CP}^N$ models can be obtained in a similar spirit (target-space supersymmetric $\mathbf{CP}^{N|M}$ models have been obtained in [25]). The main difficulty, however, is that in general these spin chains for arbitrary values of representation (or spin) S are not integrable¹². Thus, the question splits into two:

- 1) whether one can find a family (parametrized by S) of integrable spin chains, giving rise to the desired sigma model in the continuum limit **and**
- 2) if not, how one can find traces of integrability properties of the continuum sigma model in the spin chain (even if the spin chain itself is not integrable).

In our opinion, it would be very interesting to carry out this program for the $\mathbf{CP}^N$ models, in particular to see on the level of the spin chain why the bosonic $\mathbf{CP}^N$ model and the $\mathbf{CP}^N$ model with fermions differ so much.

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¹²I am grateful to F.Smirnov for drawing my attention to this fact.

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A Definitions.

Let V be a vector space. Its tensor square is denoted by $W = V \otimes V$. The permutation operator $P \in \text{End}(W)$ is defined in an obvious way: $P(a \otimes b) = b \otimes a$. The definition of the trace operator Tr is slightly less obvious. Let s be the “scalar product” in V , in other words a $\mathbf{C}$ -linear nondegenerate symmetric operator $s : W \rightarrow \mathbf{C}$ ¹³. We also fix the scalar product on W : $(a \otimes b, c \otimes d) = (a, c)(b, d) = s(a \otimes c)s(b \otimes d)$. Since $s \in W^*$, using the scalar product on W we find a unique dual element $s^* \in W$. In more detail, its defining property is $(s^*, c \otimes d) = s(c \otimes d) = (c, d)$. The trace operator, which is an element of $\text{End}(W) \simeq W \otimes W^*$, is defined as follows: $\text{Tr} = s^* \otimes s$. This means that $\text{Tr}(a \otimes b) = (a, b)s^*$ and, for instance, $(\text{Tr}(a \otimes b), c \otimes d) = (a, b)(c, d)$.

B Landau-Lifshitz models.

Landau-Lifshitz models are Hamiltonian systems of a particular type¹⁴. The main dynamical variable is a function $m(x)$, taking values in a Lie algebra $\mathfrak{g}$ (corresponding to a Lie group which we denote by G). $m(x)$ can be written in component notation with respect to a basis of generators $\{\tau_a\}$ of the Lie algebra: $m(x) = \sum m^a(x)\tau_a$. The variable x may take values in $\mathbb{R}$, although a more complete mathematical theory exists for the case where $x \in S^1$. The Poisson structure is defined as follows:

$$\{m^a(x), m^b(y)\} = i f_c^{ab} \delta(x - y) m^c(x), \quad (80)$$

where f_c^{ab} are the structure constants of the Lie algebra, and $\delta(x - y)$ is a delta-function, which in the case of a circle S^1 should be understood as a mod 2π delta-function. The Poisson structure is naturally continued to arbitrary functions of the fundamental variables $m(x)$.

The Landau-Lifshitz Hamiltonian is defined as follows:

$$H_{LL} = \frac{1}{2} \int dx \text{tr}(\partial_x m)^2. \quad (81)$$

One can then easily show that Hamilton’s equations, which follow from (80) and (81),

¹³Note that this convention is slightly different from the usual one, when s is taken to be semilinear. This is done so that we can realize the trace operator as a matrix acting on vectors, which is in the spirit of the spin chain.

¹⁴In this Section we mainly follow the exposition of [26], slightly adapting it to our needs.

have the following form:

$$\partial_t m = [m, \partial_x^2 m]. \quad (82)$$

The important point to realize is that (82) is an equation on a function m taking values in a vector space $\mathfrak{g}$ (which is also a Lie algebra of course), which means that it does not satisfy any additional constraints. However, it is easy to see that the time evolution of eq. (82) preserves the following local quantities

$$l_k(x) = \text{tr}(m(x)^k), \quad k = 1, 2, 3, \dots \quad (83)$$

Note that only a finite number of $l_k(x)$ are independent quantities. It follows that the motion actually takes place on proper submanifolds of $\mathfrak{g}$. These submanifolds are parametrized by the values of $l_k(x)$, and on each such manifold the Poisson bracket (80) induces a symplectic form, so that each such manifold is symplectic. This is in contrast to the original manifold (vector space) $\mathfrak{g}$, which was Poisson but not symplectic (in fact there are plenty of cases when it is odd-dimensional). The symplectic submanifolds mentioned above are also called symplectic leaves of the corresponding Poisson structure.

A particularly interesting case is when $l_k(x)$ are constants, i.e. when they do not depend on x . If one chooses particular values for these constants, one obtains the various coadjoint orbits of the group G , and the equation (82) then naturally reduces to these particular orbits. For example, for the Lie algebra $\mathfrak{g} = \mathfrak{so}_3$ the only nontrivial orbit is the two-sphere, corresponding to $l_2(x) = R^2 > 0$, and in this case (82) turns into the Heisenberg ferromagnet equation (48).

C Proof that the embedding $\mathcal{F}_N \hookrightarrow (\mathbb{CP}^{N-1})^{\times N}$ is Lagrangian and isometric.

In this Section we first explain what the manifold of complete flags is, and then we show that it is possible to embed the complete flag $\mathcal{F}_N = U(N)/U(1)^N$ into $(\mathbb{CP}^{N-1})^N$ isometrically and, moreover, as a Lagrangian submanifold.

C.1 The manifold of complete flags $\mathcal{F}_N$

Let V be a vector space over $\mathbb{C}$. A complete flag (or simply a flag in what follows) is by definition a sequence $V_1, \dots, V_m = V$ of vector subspaces of V , with the property $V_i \subset V_{i+1}$ and $\dim V_i = i$. A flag may be given by an (ordered) sequence of m linearly independent vectors $(v_1, \dots, v_m)$. Here v_1 generates V_1 , (v_1, v_2) generate V_2 , etc. The group $GL(m)$ acts transitively on the space of such flags. Indeed, denote the matrix $(v_1, \dots, v_m)$ by F_1 and $(w_1, \dots, w_m)$ by F_2 . Since the vectors in both sets are linearly independent, the two matrices are nondegenerate. Thus, we can pick $g = F_1^{-1}F_2 \in GL(m)$, which brings one flag

to the other one: $F_2 = F_1 g$. However, some of the matrices g preserve a given flag. After a moment's thought one realizes that these are the upper triangular matrices $g \in B$ (where B stands for the Borel subgroup, which we identify with the upper triangular matrices). Indeed, for upper triangular g the right action $(v_1, \dots, v_m) \circ g$ multiplies v_1 by a scalar, thus preserving V_1 , produces a linear combination of v_1 and v_2 instead of the original v_2 , thus leaving V_2 unaltered, etc. This means that the flag space $\mathcal{F}$ is isomorphic as a complex manifold to $GL(m)/B$.

In what follows we will view the space $U(N)/U(1)^N$ as the space of N ordered lines in $\mathbf{C}^N$, orthogonal with respect to the scalar product $(a, b) \equiv \sum_{i=1}^N \bar{a}_i b_i$. One can check that this is equivalent to the definition given above.

C.2 The embedding is Lagrangian.

Let y be a point of the complete flag, that is $y \equiv \{x_1, \dots, x_N\}$ is a set of N complex vectors (in $\mathbf{C}^N$, i.e. with N components each) representing the N orthogonal lines. This means, in particular, that each of the vectors is defined up to multiplication by a complex number, or in other words $y \in (\mathbf{CP}^{N-1})^{\times N}$. It follows that there's an embedding $\mathcal{F}_N \hookrightarrow (\mathbf{CP}^{N-1})^{\times N}$. Our statement is as follows: this embedding is Lagrangian. The latter statement comprises two facts:

- 1) the 2-form induced on $\mathcal{F}_N$ by this embedding is identically zero **and**
- 2) the dimension of $\mathcal{F}_N$ is half the dimension of $(\mathbf{CP}^{N-1})^{\times N}$.

Clearly, the second statement is easier to verify. Indeed, since $\mathcal{F}_N = U(N)/U(1)^N$, its dimension over the real numbers is $\dim \mathcal{F}_N = N^2 - N$, whereas $\dim_{\mathbf{R}} \mathbf{CP}^{N-1} = 2(N-1)$, and it follows that $\dim [(\mathbf{CP}^{N-1})^{\times N}] = 2(N-1)N = 2 \dim \mathcal{F}_N$.

Let us now prove the first statement for the special case of $N = 3$. It is straightforwardly generalized to the case of arbitrary N . When $N = 3$ we have three vectors, which we will call a, b, c . The orthonormality conditions for these vectors looks as follows:

$$a_i \bar{b}_i = 0, \quad a_i \bar{c}_i = 0, \quad b_i \bar{c}_i = 0. \quad (84)$$

The symplectic form on the ambient space $(\mathbf{CP}^2)^{\times 3}$ is

$$\Omega = \omega_{FS}(a) + \omega_{FS}(b) + \omega_{FS}(c), \quad (85)$$

meaning that in each term the Fubini-Study form should be taken in the a, b or c coordinates. An important thing to realize is that the form Ω being zero on the flag $\mathcal{F}_3$ means that it gives zero when acting on any two vector fields tangent to $\mathcal{F}_3$: $\Omega(t_1, t_2) = 0$ for $t_{1,2} \in \Gamma(T\mathcal{F}_3)$. This is what we are going to prove, since choosing an explicit parametriza-

tion for the flag manifold is not a very easy enterprise. In practice we will use an over-complete system of tangent vectors, which may be written as

$$v^n(a, b, c) = w^n(a) + w^n(b) + w^n(c), \quad (86)$$

where

$$w^n(x) = (\lambda^n)_{ij} \bar{x}^i \frac{\partial}{\partial \bar{x}^j} - (\lambda^n)_{ji} x^i \frac{\partial}{\partial x_j}, \quad n = 1 \dots 9 \quad (87)$$

and λ^n are the Gell-Mann matrices and the identity matrix. Since there are nine tangent vectors to a six-dimensional flag manifold $\mathcal{F}_3$, not all of them are linearly independent. This does not come as any surprise, since some of these vector fields (or perhaps their linear combinations) act along the orbit of the denominator $U(1)^3$ of the coset $U(3)/U(1)^3$.

It is easy to check that the vector fields (86-87) annihilate the defining conditions (84) of the flag manifold, which means precisely that they are tangent to it. As explained above, we will now evaluate the two-form Ω on these vectors, i.e. we will calculate $\Omega(v^n, v^m)$ and prove that this is zero.

Using (85), we obtain

$$\Omega(v^n, v^m) = \omega_{FS}(w^n(a), w^m(a)) + \omega_{FS}(w^n(b), w^m(b)) + \omega_{FS}(w^n(c), w^m(c)) \quad (88)$$

The Fubini-Study form (51) has two terms. Let us evaluate each of them separately, starting from the second one:

$$\begin{aligned} \frac{da_i \bar{a}_i \wedge d\bar{a}_k a_k}{(\bar{a}_j a_j)^2} (w^n(a), w^m(a)) &= \\ &= \frac{1}{(\bar{a}_j a_j)^2} [da_i(w^n) \bar{a}_i d\bar{a}_j(w^m) a_j - da_i(w^m) \bar{a}_i d\bar{a}_j(w^n) a_j] = \\ &= \frac{1}{(\bar{a}_j a_j)^2} [-(\bar{a} \lambda^n a)(\bar{a} \lambda^m a) - (m \leftrightarrow n)] = 0 \end{aligned}$$

The first term in the Fubini-Study form produces the following after evaluation:

$$\begin{aligned} \frac{da_i \wedge d\bar{a}_i}{\bar{a}_j a_j} (w^n(a), w^m(a)) &= \frac{1}{(\bar{a}_j a_j)} [da_k(w^n) d\bar{a}_k(w^m) - da_k(w^m) d\bar{a}_k(w^n)] = \\ &= \frac{1}{(\bar{a}_j a_j)} [-\bar{a} \lambda^n \lambda^m a - (m \leftrightarrow n)] = \frac{1}{(\bar{a}_j a_j)} \bar{a} [\lambda^m, \lambda^n] a \end{aligned}$$

Summing this over a , b and c , we obtain:

$$\Omega(v^n, v^m) = \frac{i}{(\bar{a}_j a_j)} \bar{a} [\lambda^m, \lambda^n] a + \frac{i}{(\bar{b}_j b_j)} \bar{b} [\lambda^m, \lambda^n] b + \frac{i}{(\bar{c}_j c_j)} \bar{c} [\lambda^m, \lambda^n] c \quad (89)$$

Since a, b, c are orthogonal to each other,

$$\frac{\bar{a} \otimes a}{(a, a)} + \frac{\bar{b} \otimes b}{(b, b)} + \frac{\bar{c} \otimes c}{(c, c)} = I_3 \quad (90)$$

is a unit matrix, and it follows from (89-90) that

$$\Omega(v^n, v^m) = i \operatorname{tr}([\lambda^m, \lambda^n]) = 0 \quad (91)$$

The proof is complete.

C.3 The embedding is isometric.

Similarly to the case of symplectic structures, which was discussed in Section 6.1, the metric on the direct product of two manifolds is naturally the sum of metrics on each factor. Our second claim is that the very same embedding $\mathcal{F}_N \hookrightarrow (\mathbb{CP}^{N-1})^{\times N}$ is isometric.

We start the proof by considering the metric entering the action (72):

$$ds^2 = |u_1 \circ d\bar{u}_2|^2 + |u_1 \circ d\bar{u}_3|^2 + |u_2 \circ d\bar{u}_3|^2, \quad \text{where} \quad (92)$$

$$u_i \circ \bar{u}_j = \delta_{ij} \quad (93)$$

It can clearly be rewritten as follows:

$$ds^2 = \frac{1}{2} (du_1 [\bar{u}_2 \otimes u_2 + \bar{u}_3 \otimes u_3] d\bar{u}_1 + du_2 [\bar{u}_1 \otimes u_1 + \bar{u}_3 \otimes u_3] d\bar{u}_2 + du_3 [\bar{u}_1 \otimes u_1 + \bar{u}_2 \otimes u_2] d\bar{u}_3) \quad (94)$$

Now we apply (90) to each of the terms in the square brackets to obtain:

$$ds^2 = \frac{1}{2} (du_1 (I - \bar{u}_1 \otimes u_1) d\bar{u}_1 + du_2 (I - \bar{u}_2 \otimes u_2) d\bar{u}_2 + du_3 (I - \bar{u}_3 \otimes u_3) d\bar{u}_3), \quad (95)$$

where I is the 3×3 identity matrix. It is clear that each term of the form $du (I - \bar{u} \otimes u) d\bar{u}$ is the Fubini-Study metric written in the “semi-gauge” $|u| = 1$. It follows immediately that the metric (92) is the sum of three Fubini-Study metrics restricted by the conditions (93):

$$g_{\mathcal{F}_3} = \text{Restriction of } \frac{1}{2} (g_{FS}^{(1)} + g_{FS}^{(2)} + g_{FS}^{(3)}). \quad (96)$$

The proof is complete.

D Derivation of the σ -model action with the θ -term.

Let us slightly generalize the discussion from the body of the paper and consider the more

general Hamiltonian $\hat{H}_2 = \sum_{i=1}^L (P_{i,i+1} + a P_{i,i+2})$, where ‘ a ’ is arbitrary. The derivation of the coherent-state Lagrangian, which corresponds to it, is identical to the one of Section

7.1. It produces the following result:

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{L^2}(1+a) \sum (|x_k|^2 + |y_k|^2 + |z_k|^2) - \\
 & -\frac{1}{L} x_k (\dot{u}_{1,k} \circ \dot{u}_{2,k} - a \frac{1}{L} \bar{u}'_{2,k} \circ u_{1,k}) + \frac{1}{L} \bar{x}_k (\dot{\bar{u}}_{1,k} \circ \dot{u}_{2,k} + a \frac{1}{L} u'_{2,k} \circ \bar{u}_{1,k}) - \\
 & -\frac{1}{L} y_k (\dot{u}_{1,k} \circ \dot{u}_{3,k} - \frac{1}{L} \bar{u}'_{3,k} \circ u_{1,k}) + \frac{1}{L} \bar{y}_k (\dot{\bar{u}}_{1,k} \circ \dot{u}_{3,k} + \frac{1}{L} u'_{3,k} \circ \bar{u}_{1,k}) - \\
 & -\frac{1}{L} z_k (\dot{u}_{2,k} \circ \dot{u}_{3,k} - a \frac{1}{L} \bar{u}'_{3,k} \circ u_{2,k}) + \frac{1}{L} \bar{z}_k (\dot{\bar{u}}_{2,k} \circ \dot{u}_{3,k} + a \frac{1}{L} u'_{3,k} \circ \bar{u}_{2,k}) - \\
 & -\frac{1}{L^2} \sum (|u'_{3,k} \circ \bar{u}_{1,k}|^2 + a |u'_{3,k} \circ \bar{u}_{2,k}|^2 + a |u'_{2,k} \circ \bar{u}_{1,k}|^2)
 \end{aligned} \tag{97}$$

In order to perform the Gaussian integration over x_k, y_k, z_k we use the simple identity

$$A|w|^2 + Bw + C\bar{w} = A(w + \frac{C}{A})(\bar{w} + \frac{B}{A}) - \frac{BC}{A} \tag{98}$$

The integration effectively reduces to dropping the terms in brackets. For the x_k, y_k, z_k variables the corresponding A, B, C coefficients are summarized in the table below:

	A	B	C= $\bar{B}$
x_k	$-\frac{1+a}{L^2}$	$-\frac{1}{L}(\dot{u}_{1,k} \circ \dot{u}_{2,k} - a \frac{1}{L} \bar{u}'_{2,k} \circ u_{1,k})$	$\frac{1}{L}(\dot{\bar{u}}_{1,k} \circ \dot{u}_{2,k} + a \frac{1}{L} u'_{2,k} \circ \bar{u}_{1,k})$
y_k	$-\frac{1+a}{L^2}$	$-\frac{1}{L}(\dot{u}_{1,k} \circ \dot{u}_{3,k} - \frac{1}{L} \bar{u}'_{3,k} \circ u_{1,k})$	$\frac{1}{L}(\dot{\bar{u}}_{1,k} \circ \dot{u}_{3,k} + \frac{1}{L} u'_{3,k} \circ \bar{u}_{1,k})$
z_k	$-\frac{1+a}{L^2}$	$-\frac{1}{L}(\dot{u}_{2,k} \circ \dot{u}_{3,k} - a \frac{1}{L} \bar{u}'_{3,k} \circ u_{2,k})$	$\frac{1}{L}(\dot{\bar{u}}_{2,k} \circ \dot{u}_{3,k} + a \frac{1}{L} u'_{3,k} \circ \bar{u}_{2,k})$

Summing the values of $-\frac{BC}{A}$ over x, y, z and adding the result to the last line of (97) we obtain¹⁵:

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{1+a} (|u_{1,k} \circ \dot{u}_{2,k}|^2 + a^2 \frac{1}{L^2} |\bar{u}'_{2,k} \circ u_{1,k}|^2) - \frac{ia}{1+a} \frac{1}{L} \epsilon_{\mu\nu} (u_{1,k} \circ \partial_\mu \bar{u}_{2,k}) (\bar{u}_{1,k} \circ \partial_\nu u_{2,k}) + \\
 & + \frac{1}{1+a} (|u_{1,k} \circ \dot{u}_{3,k}|^2 + \frac{1}{L^2} |\bar{u}_{1,k} \circ u'_{3,k}|^2) - \frac{i}{1+a} \frac{1}{L} \epsilon_{\mu\nu} (u_{1,k} \circ \partial_\mu \bar{u}_{3,k}) (\bar{u}_{1,k} \circ \partial_\nu u_{3,k}) + \\
 & + \frac{1}{1+a} (|u_{2,k} \circ \dot{u}_{3,k}|^2 + a^2 \frac{1}{L^2} |\bar{u}_{2,k} \circ u'_{3,k}|^2) - \frac{ia}{1+a} \frac{1}{L} \epsilon_{\mu\nu} (u_{2,k} \circ \partial_\mu \bar{u}_{3,k}) (\bar{u}_{2,k} \circ \partial_\nu u_{3,k}) \\
 & - \frac{1}{L^2} \sum (|u'_{3,k} \circ \bar{u}_{1,k}|^2 + a |u'_{3,k} \circ \bar{u}_{2,k}|^2 + a |u'_{2,k} \circ \bar{u}_{1,k}|^2)
 \end{aligned} \tag{99}$$

Interestingly, the coefficients in front of $|u'_{3,k} \circ \bar{u}_{1,k}|^2, |u'_{3,k} \circ \bar{u}_{2,k}|^2, |u'_{2,k} \circ \bar{u}_{1,k}|^2$ in fact turn out to be equal due to a nice cancellation $a^2 - a(1+a) = 1 - (1+a) = -a$:

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{1+a} (|u_{1,k} \circ \dot{u}_{2,k}|^2 - a \frac{1}{L^2} |\bar{u}'_{2,k} \circ u_{1,k}|^2) - \frac{ia}{1+a} \frac{1}{L} \epsilon_{\mu\nu} (u_{1,k} \circ \partial_\mu \bar{u}_{2,k}) (\bar{u}_{1,k} \circ \partial_\nu u_{2,k}) + \\
 & + \frac{1}{1+a} (|u_{1,k} \circ \dot{u}_{3,k}|^2 - a \frac{1}{L^2} |\bar{u}_{1,k} \circ u'_{3,k}|^2) - \frac{i}{1+a} \frac{1}{L} \epsilon_{\mu\nu} (u_{1,k} \circ \partial_\mu \bar{u}_{3,k}) (\bar{u}_{1,k} \circ \partial_\nu u_{3,k}) + \\
 & + \frac{1}{1+a} (|u_{2,k} \circ \dot{u}_{3,k}|^2 - a \frac{1}{L^2} |\bar{u}_{2,k} \circ u'_{3,k}|^2) - \frac{ia}{1+a} \frac{1}{L} \epsilon_{\mu\nu} (u_{2,k} \circ \partial_\mu \bar{u}_{3,k}) (\bar{u}_{2,k} \circ \partial_\nu u_{3,k})
 \end{aligned} \tag{100}$$

¹⁵Note the convention for the epsilon-symbol: $\epsilon_{01} = 1$, i.e. for example $\epsilon_{\mu\nu} \partial_\mu x \partial_\nu y = \dot{x} y' - \dot{y} x'$.

Of course, each of the first terms in the three lines above can be brought to a canonical relativistic-invariant form by a rescaling of the space variable. It is of importance, however, that the coefficients of the three epsilon-terms are not all equal. We want these three terms to be the pull-back of a closed form, so that we can treat them as a topological contribution — this is a nontrivial condition: in fact, it fixes the value of a . Let us see how this happens.

First of all, let us introduce notations for the three 2-forms which we encountered above:

$$u \equiv i(u_1 \circ d\bar{u}_2) \wedge (\bar{u}_1 \circ du_2) \quad (101)$$

$$v \equiv -i(u_1 \circ d\bar{u}_3) \wedge (\bar{u}_1 \circ du_3) \quad (102)$$

$$w \equiv i(u_2 \circ d\bar{u}_3) \wedge (\bar{u}_2 \circ du_3) \quad (103)$$

Then the three epsilon-terms in (100) are the pull-back of the following 2-form:

$$\omega = \frac{1}{1+a} (v - a u - a w) \quad (104)$$

Let us start from the fact that the Fubini-Study form of each $\mathbf{CP}^2$ factor is closed. Let us recall that

$$\Omega_1 = i \left(du_1 \wedge d\bar{u}_1 - (du_1 \circ \bar{u}_1) \wedge (d\bar{u}_1 \circ u_1) \right) \quad (105)$$

and similar expressions hold for Ω_2 and Ω_3 . Now let us use the identity $u_1 \otimes \bar{u}_1 + u_2 \otimes \bar{u}_2 + u_3 \otimes \bar{u}_3 = I_3$. Using this, we may rewrite the restriction of Ω_1 to the flag manifold $\mathcal{F}_3$ in the following way:

$$\Omega_1|_{\mathcal{F}_3} = i \left((du_1 \circ \bar{u}_2) \wedge (d\bar{u}_1 \circ u_2) + (du_1 \circ \bar{u}_3) \wedge (d\bar{u}_1 \circ u_3) \right) = u - v \quad (106)$$

External differentiation and restriction to a submanifold are commutative operations, therefore this restricted form is a closed 2-form on $\mathcal{F}_3$. Analogously $\Omega_2|_{\mathcal{F}_3} = w - u$, $\Omega_3|_{\mathcal{F}_3} = v - w$. In particular, we see that the sum $(\Omega_1 + \Omega_2 + \Omega_3)|_{\mathcal{F}_3} = 0$ is zero, as expected.

Our statement is that the form ω is expressible in terms of Ω_i , and therefore closed, only for $a = \frac{1}{2}$. Indeed, in the (u, v, w) Euclidean vector space the forms Ω_i lie in a plane orthogonal to $(1, 1, 1)$. On the other hand, ω is represented by the vector $(a, -1, a)$, and this vector is orthogonal to $(1, 1, 1)$ only for $a = \frac{1}{2}$. For this value of a we obtain

$$\omega = \frac{1}{1 + \frac{1}{2}} \frac{1}{2} (\Omega_3 - \Omega_1) = \frac{1}{3} (\Omega_3 - \Omega_1). \quad (107)$$

Upon rescaling the space variable $x \rightarrow \frac{1}{\sqrt{2}L}x$ we obtain the following action:

$$\begin{aligned} \mathcal{S} = m \int dt dx \left(\frac{\sqrt{2}}{3} [|u_1 \circ \partial_\mu \bar{u}_2|^2 + |u_1 \circ \partial_\mu \bar{u}_3|^2 + |u_2 \circ \partial_\mu \bar{u}_3|^2] - \right. \\ \left. - \frac{i}{3} \epsilon_{\mu\nu} [(u_1 \circ \partial_\mu \bar{u}_2) (\bar{u}_1 \circ \partial_\nu u_2) + 2(u_1 \circ \partial_\mu \bar{u}_3) (\bar{u}_1 \circ \partial_\nu u_3) + (u_2 \circ \partial_\mu \bar{u}_3) (\bar{u}_2 \circ \partial_\nu u_3)] \right). \end{aligned} \quad (108)$$

References

- [1] L. Faddeev and N. Reshetikhin, “Integrability Of The Principal Chiral Field Model In (1+1)-Dimension,” *Annals Phys.*, vol. 167, p. 227, 1986.
- [2] F. Haldane, “Nonlinear field theory of large spin Heisenberg antiferromagnets. Semiclassically quantized solitons of the one-dimensional easy Axis Neel state,” *Phys.Rev.Lett.*, vol. 50, pp. 1153–1156, 1983.
- [3] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.*, “Review of AdS/CFT Integrability: An Overview,” 2010.
- [4] G. Arutyunov and S. Frolov, “Foundations of the $AdS_5 \times S^5$ Superstring. Part I,” *J.Phys.A*, vol. A42, p. 254003, 2009.
- [5] L. F. Alday and J. M. Maldacena, “Comments on operators with large spin,” *JHEP*, vol. 0711, p. 019, 2007.
- [6] D. Bykov, “The worldsheet low-energy limit of the $AdS_4 \times CP^3$ superstring,” *Nucl.Phys.*, vol. B838, pp. 47–74, 2010.
- [7] I. Affleck, “The Quantum Hall Effect, Sigma Models At Theta = Pi And Quantum Spin Chains,” *Nucl.Phys.*, vol. B257, p. 397, 1985.
- [8] N. Read and S. Sachdev, “Some Features Of The Phase Diagram Of The Square Lattice $Su(N)$ Antiferromagnet,” *Nucl.Phys.*, vol. B316, pp. 609–640, 1989.
- [9] S. Rao and D. Sen, “Nonlinear field theory of a frustrated Heisenberg spin chain,” *Nucl.Phys.*, vol. B424, pp. 547–566, 1994.
- [10] S. Randjbar-Daemi, A. Salam, and J. Strathdee, “Generalized spin systems and sigma models,” *Phys.Rev.*, vol. B48, pp. 3190–3205, 1993.
- [11] A. Perelomov, *Generalized coherent states and their applications*. Springer, 1986.
- [12] W. Fulton and J. Harris, *Representation theory. A first course*. Springer, 1st, ed., 1991.

- [13] S. Murray and C. Sämann, “Quantization of Flag Manifolds and their Supersymmetric Extensions,” *Adv.Theor.Math.Phys.*, vol. 12, pp. 641–710, 2008.
- [14] F. Berezin, “General Concept of Quantization,” *Commun.Math.Phys.*, vol. 40, pp. 153–174, 1975.
- [15] A. Alekseev, L. Faddeev, and S. L. Shatashvili, “Quantization of symplectic orbits of compact Lie groups by means of the functional integral,” *J.Geom.Phys.*, vol. 5, pp. 391–406, 1988.
- [16] L. Faddeev and A. Slavnov, “Gauge Fields. Introduction To Quantum Theory,” *Front.Phys.*, vol. 50, pp. 1–232, 1980.
- [17] J. Zinn-Justin, *Path Integrals in Quantum Mechanics*. Oxford University Press, 2004.
- [18] A. Arvanitoyeorgos, “New invariant Einstein metrics on generalized flag manifolds,” *Trans. Amer. Math. Soc.*, vol. 337, no. 2, pp. 981–995, 1993.
- [19] S. Rachel and M. Greiter, “Exact models for trimerization and tetramerization in spin chains,” *Phys. Rev. B*, vol. 78, pp. 134415–+, Oct. 2008.
- [20] P. Corboz, A. M. Läuchli, K. Totsuka, and H. Tsunetsugu, “Spontaneous trimerization in a bilinear-biquadratic $S=1$ zig-zag chain,” *Phys. Rev. B*, vol. 76, p. 220404, Dec. 2007.
- [21] A. Polyakov, *Gauge fields and strings*. Harwood Academic Publishers, 1987.
- [22] A. Weinstein, “Neighborhood classification of isotropic embeddings,” *J. Differential Geom.*, vol. 16, no. 1, pp. 125–128, 1981.
- [23] A. D’Adda, P. Di Vecchia, and M. Lüscher, “Confinement and Chiral Symmetry Breaking in CP^{n-1} Models with Quarks,” *Nucl.Phys.*, vol. B152, pp. 125–144, 1979.
- [24] R. Koberle and V. Kurak, “Deconfinement In The CP^{N-1} Model By Massless Fermions,” *Phys.Rev.Lett.*, vol. 58, pp. 627–628, 1987.
- [25] C. Candu, V. Mitev, T. Quella, H. Saleur, and V. Schomerus, “The Sigma Model on Complex Projective Superspaces,” *JHEP*, vol. 1002, p. 015, 2010.
- [26] V. Arnold and B. Khesin, *Topological methods in hydrodynamics*. Springer, 1998.